Morphisms for the Maximum Weight Ideal Problem

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A notion of morphism for the Maximum Weight Ideal Problem is defined and applied to solve the Edge-Isoperimetric Problem on the graphs of regular solids and their products. © 2000 Academic Press

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1. INTRODUCTION

1.1. The Edge-Isoperimetric Problem

Given a graph $G = (V, E, \partial)$ having vertex-set $V$, edge-set $E$, and boundary-function $\partial: E \to (\frac{1}{2})$ which identifies the pair of vertices incident to each edge, we let

$$\Theta(S) = \{ e \in E : \partial(e) = \{ v, w \}, v \in S \land w \notin S \}. $$

Then given $k \in \mathbb{Z}^+$, the (External) Edge-Isoperimetric (E-I) Problem is to minimize $|\Theta(S)|$ over all $S \subseteq V$ such that $|S| = k$.

1.2. Variants

There are several variants of the External E-I Problem. They are all equivalent for regular graphs but each provides an insight.

1.2.1. Internal Edges

For $S \subseteq V$, let

$$I(S) = \{ e \in E : \partial(e) = \{ v, w \}, v \in S \land w \in S \}. $$


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The members of $\mathcal{I}(S)$ are \textit{internal edges} of $S$. Maximizing $|\mathcal{I}(S)|$ over all $S \subseteq V$, $|S| = k$, is called the \textit{(Internal) Edge-Isoperimetric Problem}. If $G$ is regular of degree $\delta$, then

$$|\mathcal{I}(S)| = \delta |S| - 2 |\mathcal{I}(S)|,$$

so

$$|\mathcal{I}(S)| = \frac{1}{2} (\delta |S| - |\mathcal{I}(S)|),$$

and for $|S| = k$, fixed, minimizing $|\mathcal{I}(S)|$ is equivalent to maximizing $|\mathcal{I}(S)|$ (on regular graphs).

1.2.2. \textit{Incident Edges}

Another variant is defined by

$$\partial^*(S) = \{ e \in E : \partial(e) = \{ v, w \}, w \notin S \text{ or } w \in S \},$$

the set of edges incident to members of $S \subseteq V$. Minimizing $|\partial^*(S)|$ over all $S \subseteq V$, $|S| = k$ is called the \textit{(Incident) Edge-Isoperimetric Problem}. For all graphs $\partial^*(S) = \mathcal{I}(S) + \mathcal{I}(S)$ ("\+$" representing disjoint union) so

$$|\partial^*(S)| = |\mathcal{I}(S)| + |\mathcal{I}(S)| = \frac{1}{2} (\delta |S| + |\mathcal{I}(S)|)$$

for regular graphs and in that case, the Incident E-I Problem is equivalent to the External and Internal E-I Problems.

1.2.3. \textit{Shadows}

In the face-lattice, $\mathcal{F}$, of a convex polytope, or more generally in any ranked poset, $\partial : \mathcal{F} \to 2^\mathcal{F}$ is defined by

$$\partial(x) = \{ y \in \mathcal{F} : y \prec x \}\text{ ("y \prec x" means that x covers y)}$$

and extended to $2^\mathcal{F} = \{ S \subseteq \mathcal{F} \}$ by

$$\partial(S) = \bigcup_{x \in S} \partial(x).$$

The \textit{Minimum Shadow (MS) Problem} is to minimize $|\partial(S)|$ over all $S \subseteq \mathcal{F} = \{ x \in \mathcal{F} : r(x) = r \}$, $|S| = k$. Note that the Incident E-I Problem is the MS Problem for the dual of the face-lattice of its graph.
For graphs which are not regular, the Internal, External, and Incident E-I Problems are distinct. However, all of the graphs in this paper are regular so we shall consider the three as interchangeable.

1.3. A Summary of the Theory of Stabilization (from [7])

If \( G = (V, E, \partial) \) is a finite graph embedded in \( \mathbb{R}^d \), \( d \)-dimensional Euclidean space, and \( \mathcal{R} \) is a reflection which acts as a symmetry of \( G \), then

**Definition 1.** \( \mathcal{R} \) is called stabilizing if for all \( e \in E \), \( \partial(e) = \{v, w\} \), if \( v \) and \( w \) are on opposite sides of the fixed hyperplane of \( \mathcal{R} \), then \( \mathcal{R}(v) = w \).

**Definition 2.** If \( \mathcal{R} \) is stabilizing for \( G \), \( p \in \mathbb{R}^d \) is not fixed by \( \mathcal{R} \) and \( R \subseteq V \) with

\[
S_0 = \{ v \in S : \|v - p\| > \|\mathcal{R}(v) - p\| \& \mathcal{R}(v) \notin S \}
\]

then

\[
\text{Stab}_{\mathcal{R}, p}(S) = S - S_0 + \mathcal{R}(S_0).
\]

**Theorem 1.** \( \forall S, T \subseteq V, \)

1. \( |\text{Stab}_{\mathcal{R}, p}(S)| = |S| \),
2. \( |\Theta(\text{Stab}_{\mathcal{R}, p}(S))| \leq |\Theta(S)| \) and
3. If \( S \subseteq T \) then \( \text{Stab}_{\mathcal{R}, p}(S) \leq \text{Stab}_{\mathcal{R}, p}(T) \).

**Proof.** Parts 1 and 3 follow directly from the definition of \( \text{Stab}_{\mathcal{R}, p} \). Part 2 is based upon the observation that for any edge in \( \Theta(\text{Stab}_{\mathcal{R}, p}(S)) \) but not in \( \Theta(S) \), there is a corresponding edge (its image under \( \mathcal{R} \)) in \( \Theta(S) \) but not in \( \Theta(\text{Stab}_{\mathcal{R}, p}(S)) \). There are two ways this can happen depending on which side of the fixed hyperplane the edge lies. It cannot penetrate the fixed hyperplane because of Definition 1 above.

Parts 1, 2 and 3 of Theorem 1 above show that stabilization is a discrete analog of Steiner symmetrization. Any set operation which has 1.1 and 1.2 has been called a **Steiner operation** (see [7]) and with 1.3 it is continuous.

Given \( G \) and stabilizing reflections \( \mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{k-1} \) and \( p \in \mathbb{R}^d \) not fixed by any \( \mathcal{R}_i \), define a transformation \( T_j : 2^V \rightarrow 2^V, j = 0, 1, \ldots, \) by

\[
T_0 = I, \text{ the identity, and}
\]

\[
T_{j+1} = \text{Stab}_{\mathcal{R}_j(p), p} \circ T_j.
\]

**Theorem 2.** There exists an integer \( j_0 \) such that for all \( j \geq j_0 \), \( T_{j+1} = T_j \).
Definition 3. A set $S \subseteq V$ such that $\text{Stab}_{p_i}S = S$ for $i = 0, 1, \ldots, k-1$ is called stable.

Theorems 1 and 2 show that in minimizing $|\Theta|$ over $2^V$, we need only consider stable sets. But how can we tell which sets are stable and which are not?

Definition 4. Let

$$\mathcal{C}(V; R; p) = \{(v, w) \in V \times V : R(v) = w \land \|v - p\| < \|w - p\|\}.$$  

Then the stability order, $\mathcal{C}(V; R_0, R_1, \ldots, R_{k-1}; p)$, is defined to be the transitive closure of $\bigcup_{i=0}^{k-1} \mathcal{C}(V; R_i; p)$.

Definition 5. If $S \subseteq \mathcal{P}$, a partially ordered set, then $S_0 = \{x \in \mathcal{P} : \exists y \in S \land x \leq y\}$. If $S = S_0$, then $S$ is called an ideal (or lower set or down-set; see [4]). Note that

$$\overline{(S)} = S,$$

so for any $S \subseteq \mathcal{P}$, $\overline{S}$ is an ideal, the ideal generated by $S$.

Theorem 3. A set $S \subseteq V$ is stable iff it is an ideal in $\mathcal{C}(V; R_0, R_1, \ldots, R_{k-1}; p)$.

Note that every edge of any regular convex polytope is perpendicularly bisected by the fixed hyperplane of a reflective symmetry. The ends of the edge are therefore comparable in its stability order. Thus if we define

$$A(v) = |\{w \in V : \exists e \in E, \bar{e}(v) = \{v, w\} \land w <_{\mathcal{C}} v\}|$$

then

$$|I(S)| = \sum_{v \in S} A(v).$$

This same representation of $|I(S)|$ for stable sets is valid in any graph where every vertex is comparable (in the stability order) to its neighbors.

The EIP on the graphs of regular convex polytopes has thus been transformed to maximizing a sum of weights $A(v)$ over all ideals in the poset $\mathcal{C}$. But this only constitutes real progress if we can facilitate the calculation of $\mathcal{C}$. If all of the reflective symmetries of $G$ are stabilizing, as they are for the regular solids, then the group they generate is a Coxeter group (see [3]). As Coxeter showed, the fixed hyperplanes of the reflections in a
Coxeter group partition $\mathbb{R}^d$ into connected components, called chambers, which are simplices if the group is irreducible. The chamber containing $p$ is called the fundamental chamber. Each connected component of $\mathcal{C}$ will have exactly one vertex in the fundamental chamber, its minimum element. In [8] the connected stability orders of irreducible Coxeter groups are identified as the quotients of the Bruhat order of that group by its parabolic subgroups.

A reflection whose fixed hyperplane bounds the fundamental chamber is called a basic reflection. Coxeter showed that the basic reflections deserve their name by forming a minimal generating set for the group. There are $d$ of them ($d$ being the dimension of the space in which the graph is represented) if the group is finite, and $d+1$ if it is infinite. The Hasse diagram of the stability order of $G$ with respect to the basic reflections, called the weak stability order, is particularly easy to calculate since it is just $\bigcup_{i=0}^{k} \mathcal{C}(V; R_i; p)$ (with $k = d$ or $d+1$ as noted above). The Matsumoto–Verma Exchange Property (see [8]) implies that the weak and strong stability orders of $G$ have a rank function, $\ell$, (called length) which is the same for both. Altogether, these observations give us a very simple 2-step process for constructing the Hasse diagram of the stability order of $G$ (Let $\mathcal{C}_\ell = \{ v \in V : \ell(v) = \ell \}$):

1. Generate the Hasse diagram of the weak stability order
   (a) Begin with the unique vertex, $v_0$, in the fundamental chamber: $\mathcal{C}_{0} = \{ v_0 \}$,
   (b) Extend from $\mathcal{C}_\ell$ to $\mathcal{C}_{\ell+1}$ by applying each basic reflection to each member of $\mathcal{C}_\ell$. The result will either be in $\mathcal{C}_{\ell-1}$ or $\mathcal{C}_{\ell+1}$, so we need only eliminate those we know to be in $\mathcal{C}_{\ell-1}$ to get those in $\mathcal{C}_{\ell+1}$.

2. Examine all pairs $(v, w)$, $v \in \mathcal{C}_\ell$ and $w \in \mathcal{C}_{\ell+1}$, to see if $v <_G w$, i.e. if there exists a reflective symmetry, $R$, such that $R(v) = w$. Those for which it does, complete the Hasse diagram of $\mathcal{C}$.

1.4. A Summary of the Theory of Compression (from [7])

Another Steiner operation (see the remarks following Theorem 1), arises from product decompositions of a graph. It is called compression.

**Definition 6.** A graphs, $G = (V, E, \partial)$, is said to have nested solutions if there exists a numbering, $\eta : V \rightarrow \{ 1, \ldots, |V| \}$, one-to-one and onto, such that $\forall k, S_k(\eta) = \{ v \in V : \eta(v) \leq k \}$ minimizes $|I(S)|$ over all $S \subseteq V$ with $|S| = k$. Note that (with $S_k(\eta)$ shortened to $S_k$), $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{|V|} = V$ and $|S_k| = k$. 
Definition 7. If $H = (V_H, E_H, \partial_H)$ has nested solutions and $J = (V_J, E_J, \partial_J)$ is any graph, then we define the compression operators for their product

$H \times J = (V_H \times V_J, E_H \times E_J \cup V_H \times E_J, \partial_H \times V_J \cup V_H \times \partial_J)$

by

$$\text{Comp}(S) = \bigcup_{w \in V_J} S_{k_w} \times \{w\},$$

where $S \subseteq V_{H \times J}$ and $k_w = |S \cap (V_H \times \{w\})|$.

Theorem 4. \(\forall S, T \subseteq V\),

1. \(|\text{Comp}(S)| = |S|\),
2. \(|\Theta(\text{Comp}(S))| \leq |\Theta(S)|\) and
3. \(S \subseteq T\ implies\ \text{Comp}(S) \subseteq \text{Comp}(T)\).

Note the similarity between Theorem 4 and Theorem 1. From this point on the theory of compression is very similar to that for stabilization as summarized in the previous section: If a graph has multiple product decompositions, $V_G \cong V_{H_i} \times V_{J_i}$, $i = 1, 2, ...$, with $H_i$ having nested solutions, and they are consistent (i.e., all of the partial orders on $V_G \cong V_{H_i} \times V_{J_i}$ induced by the total order $\eta_i$ on $H_i$, $i = 1, 2, ...$, are contained in some total order on $V_G$), then compositions of the compression operators in repeated cycles will eventually be constant. The sets in the range of this constant operator are fixed by all of the compression operations so are called compressed and we need only solve the E-I Problem for compressed sets. The transitive closure of the union of all those partial orders (on $V_G \cong V_{H_i} \times V_{J_i}$, $i = 1, 2, ...$) is a partial order whose ideals are exactly the compressed sets. That partial order is called the compressability order of $G$. In general, calculating compressibility orders is difficult, but the only case we shall consider in this paper is the case when $G$ has just one nontrivial product decomposition and both of its factors have nested solutions. In that case it is easy to see that the compressability order is just the product (order) of the two total orders.

The similarity between stabilization and compression also carries over to weights: If $V_G \cong V_H \times V_J$ and $H$ has nested solutions then

$$|I(S_k(\eta))| = \max_{S \subseteq V_H, |S| = k} |I(S)|$$
and

\[ A_H(h) = \max_{S \subseteq V_H} |I(S)| - \max_{S \subseteq V_H, |S| = h} |I(S)| \]

may be thought of as a weight on \( H \) (totally ordered by \( \eta \)) which adds up to \( |I(S_k(\eta))| \) on the initial segment (ideal) of cardinality \( k \). But if \( J \) (or any number of other factors) also has nested solutions we have \( A_J(j) \) and then \( A_{H \times J} \) is defined by

\[ A_{H \times J}(h, j) = A_H(h) + A_J(j). \]

Then for any compressed set \( S \subseteq V_G \) (ideal of the compressibility order)

\[ |I(S)| = A_{H \times J}(S) = \sum_{(h, j) \in S} A_{H \times J}(h, j). \]

This additivity, first noticed by Bezrukov [2], makes \( |I(S)| \) the easiest form of the boundary functional to work with.

### 2. MWI-MORPHISMS

We have summarized the methods for systematically reducing the E-I problem on a graph having stabilizing symmetry, or a graph which is a product of factors having nested solutions, to the problem of maximizing the weight of any ideal in a certain partial order on the vertex-set of the graph. This problem has been called the Maximum Weight Ideal (MWI) Problem. Now we show how the MWI Problem may in turn be reduced. But first we must extend the definition of a weighted poset. This kind of completion of a structure is often necessary in order to facilitate the definition of morphisms for it.

#### 2.1. List-Weighted Posets

**Definition 8.** Let \( \mathcal{P} = (P, \leq) \) be a partially ordered set (poset), \( P \) being a set and \( \leq \) a partial order relation on \( P \). A function \( |\cdot|: P \to \mathbb{Z}_+ \), called a *cardinality function*, extends to \( 2^P \) by additivity, i.e., \( |S| = \sum_{x \in S} |x| \) for \( S \subseteq P \). A function

\[ A: \{(x, i): x \in P \& 1 \leq i \leq |x|\} \to \mathbb{Z}_+, \]
called a list-weighting of \( P \), also extends by additivity and we let
\[
A(x) = \sum_{i=1}^{[x]} A(x, i).
\]

A list-weighted poset is then a triple, \((P, |\cdot|, A)\) consisting of a poset, \( P \), a cardinality function, \(|\cdot|\), and a list-weighting \( A \).

**Definition 9.** Given a poset, \( P \), and \( S \subseteq P \) let
\[
\overline{S} = \{ y \in P : \exists x \in S \& x \leq y \}.
\]
If \( S = \overline{S} \), then \( S \) is called a filter.

\[
\overline{\overline{S}} = S
\]
so \( \overline{S} \) is a filter, the filter generated by \( S \). (See Definition 5.) Ideal and filter are dual concepts. Also, the complement of an ideal is a filter and vice versa.

**Definition 10.** An ideal \((I, F, c)\), of a list-weighted poset, \((P, |\cdot|, A)\), consists of
1. An ideal \( I \) and a filter \( F \) of \( P \) such that
   a) \( I \cup F = P \) and
   b) \( I \cap F \) is an antichain (set of incomparables; see [4]), and
2. A function \( c : I \cap F \to \mathbb{Z}_+ \) such that \( \forall x \in I \cap F, 0 < c(x) < |x| \).

**Definition 11.** The cardinality of \((I, F, c)\) is then given by
\[
|I(F, c)| = \sum_{x \in I - F} |x| + \sum_{x \in I \cap F} c(x).
\]
This is intended to extend the notion of ideal to list-weighted posets: \( x \in I \) is at least partially filled and \( x \in F \) is at most partially filled. Thus the contribution, \( c(x) \), of \( x \in I \cap F \) to the cardinality of the ideal, is strictly between 0 and \( |x| \). If \( x \in I - F \) then \( x \) is necessarily totally filled.

**Definition 12.** The weight of an ideal \((I, F, c)\) is given by
\[
A(I, F, c) = \sum_{x \in I - F} A(x) + \sum_{x \in I \cap F} \sum_{i=1}^{c(x)} A(x, i).
\]
Definition 13. \((I, F, c) \subseteq (I', F', c')\) means that \(I \subseteq I',\ F \subseteq F\ & \ \forall x \in I \wedge F', \ c(x) \leq c'(x)\).

Definition 14. The set of all ideals of \((\mathcal{P}, \cdot\cdot, A)\), partially ordered by \(\subseteq\), will be denoted \(\mathcal{I}(\mathcal{P}, \cdot\cdot, A)\).

The triple \((I, M, c), M\) a set of maximal elements of \(I\), also characterizes \((I, F, c) \in \mathcal{I}(\mathcal{P}, \cdot\cdot, A)\) with \(M = F \cap I\), since \(F = M + (P - I)\). However, we chose Definition 10 since it takes advantage of duality; the triple \((F, I, \cdot\cdot - c)\) defining the filter dual to \((I, F, c)\).

Definition 15. The Maximum Weight Ideal (MWI) Problem on \((\mathcal{P}, \cdot\cdot, A)\), given by \(k \in \mathbb{Z}_+\), is then to compute

\[
\text{MWI}(\mathcal{P}, \cdot\cdot, A; k) = \max_{(I, F, c) \in \mathcal{I}(\mathcal{P}, \cdot\cdot, A)} A(I, F, c).
\]

Every list-weighted poset corresponds to a weighted poset, each element of the former being replaced by a chain of elements in the latter. An ideal in the former then becomes an ideal in the latter (note that Definition 10.1.b is invoked). Thus the MWI Problem for list-weighted posets is essentially the same as that for weighted posets.

2.2. The Main Definitions

There are two properties which we would like any notion of morphism for a problem like the MWI Problem to have. It should be

1. Comprehensive, i.e., covering a variety of applications, and
2. Effective, i.e., easily verified.

Unfortunately, there does not seem to be one definition of MWI-morphism which has them both, so we make two definitions. The first is comprehensive and the second, a special case of the first, is effective.

Definition 16. Let \((\mathcal{P}, \cdot\cdot, A)\) and \((\mathcal{Q}, \cdot\cdot, A)\) be list-weighted posets and \(\varphi: P \rightarrow Q\) a function. Then \(\varphi\) is an MWI-morphism, \(\varphi: (\mathcal{P}, \cdot\cdot, A) \rightarrow (\mathcal{Q}, \cdot\cdot, A)\), if

1. \(\varphi\) is order-preserving: \(\forall x \leq_{\mathcal{P}} y, \ \varphi(x) \leq \varphi(y)\),
2. \(\varphi\) is weight-preserving: \(\forall x \in \mathcal{P}\) and \(0 \leq i \leq c(x)\),

\[
\text{MWI}(\varphi^{-1}(x), \cdot\cdot, A; i) = A_{\mathcal{Q}}(x, \{1, \ldots, i\})
\]

where \(\varphi^{-1}(x)\) inherits its partial order, cardinality and weight from \((\mathcal{P}, \cdot\cdot, A)\)
Remark 1. If \((I, F, c) \in \mathfrak{A}(\emptyset, |\cdot|, \Delta)\) then, by Definition 16.2, \(\forall x \in I \cap F, \exists (I_x, F_x, c_x) \in \mathfrak{A}(\varphi^{-1}(x), |\cdot|, \Delta)\) such that

\[|(I_x, F_x, c_x)| = c(x)\]

and

\[A(I_x, F_x, c_x) = A(x, \{1, \ldots, c(x)\}).\]

This implies that

\[(a) \quad \varphi^{-1}(F - I) + \bigcup_{x \in I \cap F} I_x, \varphi^{-1}(F - I) + \bigcup_{x \in I \cap F} F_x, \bigcup_{x \in I \cap F} c_x \in \mathfrak{A}(\mathcal{P}, |\cdot|, \Delta),\]

\[(b) \quad ||\varphi^{-1}(F - I) + \bigcup_{x \in I \cap F} I_x, \varphi^{-1}(F - I) + \bigcup_{x \in I \cap F} F_x, \bigcup_{x \in I \cap F} c_x|| = ||(I, F, c)||\]

and

\[(c) \quad A(\varphi^{-1}(I - F) + \bigcup_{x \in I \cap F} I_x, \varphi^{-1}(I - F) + \bigcup_{x \in I \cap F} F_x, \bigcup_{x \in I \cap F} c_x) = A(I, F, c).\]

Thus the set of numbers of which \(\text{MWI}(Q, |\cdot|, 2; k)\) is the maximum is contained in the set of which \(\text{MWI}(P, |\cdot|, 2; k)\) is the maximum and

\[\text{MWI}(\mathcal{P}, |\cdot|, \Delta; k) \supseteq \text{MWI}(\emptyset, |\cdot|, \Delta; k).\]

3. \(\forall (I, F, c) \in \mathfrak{A}(\mathcal{P}, |\cdot|, \Delta) \exists (I', F', c') \in \mathfrak{A}(\emptyset, |\cdot|, \Delta)\) such that

\[(a) \quad ||I', F', c'|| = ||(I, F, c)||,\]

and

\[(b) \quad A(I', F', c') \supseteq A(I, F, c).\]

Theorem 5 (The Fundamental Lemma). If \(\varphi: (\mathcal{P}, |\cdot|, \Delta) \rightarrow (\emptyset, |\cdot|, \Delta)\) is an \(\text{MWI}\)-morphism, then \(\forall k,\)

\[\text{MWI}(\mathcal{P}, |\cdot|, \Delta; k) = \text{MWI}(\emptyset, |\cdot|, \Delta; k);\]

i.e., the \(\text{MWI}\) Problem on \((\mathcal{P}, |\cdot|, \Delta)\) is reducible to that on \((\emptyset, |\cdot|, \Delta)\).

Proof. The theorem follows from Remark 1 and Definition 16.3.

Example 1. If \((\mathcal{P}, |\cdot|, \Delta)\) is any list-weighted poset and \(Q = \{1\}\), a singleton set, then the unique function \(\varphi: P \rightarrow Q\) determines an \(\text{MWI}\)-morphism with the list-weight on \(\emptyset\) defined by

\[A(1; k) = \text{MWI}(\mathcal{P}, |\cdot|, \Delta; k) - \text{MWI}(\mathcal{P}, |\cdot|, \Delta; k - 1).\]

Example 2. If \((\mathcal{P}, |\cdot|, \Delta)\) is any list-weighted poset having nested solutions with respect to the total extension \(\eta: \mathcal{P} \rightarrow \{1 < 2 < \cdots < |P|\} = \emptyset,\)
then \( \varphi: P \to Q \) determines an MWI-morphism with the list-weighted on \( \mathcal{A} \) defined by

\[
A(k; 1) = \text{MWI}(\mathcal{P}, |\cdot|, A; k) - \text{MWI}(\mathcal{P}, |\cdot|, A; k - 1).
\]

Definition 16 and its attendant Fundamental Lemma represent the achievement of a long-standing personal goal to extend the notion of Steiner operation to many-one functions. For a Steiner operation (see [7]), such as stabilization or compression, the underlying set is fixed. In Definition 16, \( Q \) may differ from \( P \) and \( \varphi: P \to Q \) may be many-to-one. This shows the way for novel reductions of MWI problems. We might also point out that the methods of [6] are implicitly based on MWI-morphisms.

Definition 16 is not effective however, because of the existential quantifier in part 3. To be effective, there must be an efficient way to verify Definition 16.3. There seem to be several qualitatively different ways in which that can be realized, but the following (Definition 18) is the one we have found most useful.

**Definition 18.** Let \((\mathcal{P}, |\cdot|, A)\) and \((\mathcal{Q}, |\cdot|, A)\) be list-weighted posets and \( \varphi: P \to Q \) a function. Then \( \varphi \) is a **skeletal MWI-morphism**, \( \varphi: (\mathcal{P}, |\cdot|, A) \to (\mathcal{Q}, |\cdot|, A) \), if

1. \( \varphi \) is order-preserving (same as Definition 16.1),
2. \( \varphi \) is weight-preserving (same as Definition 16.2),
3. \( \forall x \leq_A y; \)
   
   (a) \( \forall y' \in \varphi^{-1}(y), \exists x' \in \varphi^{-1}(x) \) such that \( x' < y' \) and
   
   (b) \( \forall x' \in \varphi^{-1}(x), \exists y' \in \varphi^{-1}(y) \) such that \( x' < y' \).
Remark 3. This implies that if \((I, F, c) \in \mathcal{J}(\mathcal{P}, |.|, A)\) then \(\varphi(I)\) is an ideal (by 3(a)) and \(\varphi(F)\) a filter in \(\mathcal{J}\) (by 3b). Also \(\varphi(I) \cup \varphi(F) = Q\) (since \(\varphi\) is onto and \(I \cup F = P\)). From past experience, we need an additional condition, local in the sense that it just depends on pairs, \(x < y\), which allow us to transform \(\varphi(I, F, c)\) to \((I', F', c') \in \mathcal{J}(\mathcal{P}, |.|, A)\) maintaining the same cardinality and without decreasing weight.

4. \(\forall x < y; \forall j, 0 < j < |y|; \forall i \geq \text{MinShadow}(x, y; i)\) either
   
   (a) \(i + j \leq |x|\) and
   
   \[
   \text{MWI}(\varphi^{-1}(x), |.|, A; i) + \text{MWI}(\varphi^{-1}(y), |.|, A; j) \leq \text{MWI}(\varphi^{-1}(x), |.|, A; i + j),
   \]
   
   or
   
   (b) \(i + j > |x|\) and
   
   \[
   \text{MWI}(\varphi^{-1}(x), |.|, A; i) + \text{MWI}(\varphi^{-1}(y), |.|, A; j) \leq A(x) + \text{MWI}(\varphi^{-1}(x), |.|, A; i + j - |x|).
   \]

Theorem 6. Definitions 18.3 and 18.4 imply Definition 16.3; i.e., a skeletal MWI-morphism is an MWI-morphism.

Proof. Given \((I, F, c) \in \mathcal{J}(\mathcal{P}, |.|, A)\) and recalling Remark 3, let \(I' = \varphi(I), F' = \varphi(F)\), and for \(x' \in I' \cap F'\) let

\[
c'(x') = \sum_{s \in I' \setminus F' \atop \varphi(s) = x'} |x| + \sum_{s \in I' \setminus F' \atop \varphi(s) = x'} c(x).
\]

Then \(0 < c'(x') < |x'|\) and the triple \((I', F', c')\) satisfies all the requirements for a list-weighted ideal (Definition 10) except 10.1.b (that \(I' \cap F'\) be an antichain). Such a triple is called a quasi ideal. If \((I', F', c')\) is not an antichain then \(\exists x', y' \in I' \cap F'\) with \(x' < y', x'\) minimal and \(y'\) maximal (in \(I' \cap F'\)). Apply Definition 18.4 to \(x', y'\) with \(j = c'(y')\) and \(i = c'(x')(\geq \text{MinShadow}(x', y'; j)\) by definition of the \(\text{MinShadow}\) function). If \(i + j < |x'|\), then let \(I'' = I' - \{y'\}, F'' = F'\), and

\[
c''(x') = \begin{cases} 
  c'(x') & \text{if } x \neq x' \\
  i + j & \text{if } x = x'. 
\end{cases}
\]

If \(i + j > |x'|\), then let \(I'' = I', F'' = F' - \{x'\}\), and

\[
c''(x') = \begin{cases} 
  c'(x) & \text{if } x \neq y' \\
  i + j - |x'| & \text{if } x = y'. 
\end{cases}
\]
Or if $i + j = |x'|$, then let $I'' = I' - \{y'\}$, $F'' = F' - \{x'\}$, and

$$c''(x) = c'(x) \quad \forall x \in I'' \cap F''.$$

In any case $(I'', F'', c'')$ is still a quasi-ideal and at least one step closer to being an ideal, $|(I'', F'', c'')| = |(I', F', c')| = |(I, F, c)|$. If $(I', F', c') \in \mathcal{I}(A, |.|, A)$ we are done. If not, we can repeat the process, again eliminating at least one of a pair of comparables from $I' \cap F'$. $(I', F', c')$ may no longer be the image of some member of $\mathcal{P}(\mathcal{P}, |.|, A)$, the way that $(I', F', c')$ was, but after choosing $x'' < y''$ (as we did $x' < y'$) and $j = c''(y'')$ and $i = c''(x'')$, we still get $i \geq \text{MinShadow}(x'', y''; j)$ by the monotonicity of $\text{MinShadow}(x; y; j)$ in $j$ and transitivity. The process must terminate in a most $|I \cap F|$ steps.

2.3. Some Properties of MinShadow

**Theorem 7.** If $\varphi: (\mathcal{P}, |.|, A) \rightarrow (\mathcal{P}, |.|, A)$ is a skeletal MWI-morphism then

1. If $x < y$ then $\forall j, \text{MinShadow}(x, y; j) = 0$,
2. If $x < y$ then $\text{MinShadow}(x, y; 1) > 0$ and $\text{MinShadow}(x, y; |y|) = |x|$.

3. EXAMPLES AND APPLICATIONS

Compression can give solutions of the E–I Problem on graphs $G = G_1 \times G_2 \times \cdots \times G_d$, of unlimited size but has no effect for irreducible graphs $(d = 1)$ and is not very effective for $d = 2$. If $G$ is irreducible but highly symmetric, or if $d = 2$ and $G_1 = G_2$, stabilization can be a useful tool, but there are many interesting regular graphs for which neither compression or stabilization nor the two together are enough to achieve a solution. Those are the cases for which MWI-morphisms were made.

3.1. Examples

Our examples have been chosen small enough that the solution may be calculated by hand without further reduction but large enough that the MWI-morphisms are not totally trivial.

3.1.1. Dodecahedron

Figure 1 shows the stability order of the dodecahedron. Each letter, $x$, labels a vertex of the dodecahedron and the adjacent number is its
weight, $A(x)$. With $\mathcal{J} = \{ A < B < C < D < E < F \}$, Table I represents $\varphi$. To facilitate the verification of Definition 11.3, note that $\forall X \varphi^{-1}(X)$ has a maximum element and a minimum element (in this case $\varphi^{-1}(X)$ is totally ordered). The third column gives the list-weight of $X$.

Table II represents the $\text{MinShadow}$ function for $\varphi$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$\varphi^{-1}(X)$</th>
<th>$A(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>${a, b, c, d, f}$</td>
<td>$(0, 1, 1, 1, 2)$</td>
</tr>
<tr>
<td>$B$</td>
<td>${e, g, h}$</td>
<td>$(1, 1, 2)$</td>
</tr>
<tr>
<td>$C$</td>
<td>${i, j}$</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td>$D$</td>
<td>${k, l}$</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td>$E$</td>
<td>${m, n, p}$</td>
<td>$(1, 2, 2)$</td>
</tr>
<tr>
<td>$F$</td>
<td>${o, q, r, s, t}$</td>
<td>$(1, 2, 2, 2, 3)$</td>
</tr>
</tbody>
</table>
Last, for each $X < Y$, $j$ and $i \geq \text{MinShadow}(X, Y; j)$, Table III gives $M = \text{MWI}(X; i) + \text{MWI}(Y; j)$ and $N = \text{MWI}(X; i + j)$ or $O = A(X) + \text{MWI}(Y; i + j - |X|)$. Since the value appearing in the rightmost column is always at least that in the previous column, we have shown that $\varphi$ is a skeletal MWI-morphism. From that, and the fact that $\mathcal{I}$ and $\varphi^{-1}(X)$, $\forall X \in \mathcal{I}$, are totally ordered, we can conclude that the dodecahedron has nested solutions with respect to the ordering

$$(a, b, c, d, f, e, g, h, i, j, k, l, m, p, o, q, r, s, t)$$

of its vertices.

### 3.1.2. $BS_4$

The bubblesort graph, $BS_n$, may be defined as the Cayley graph of the symmetric group, $S_n$, with respect to the consecutive transpositions,
\{(i, i+1): 1 \leq i < n\}. \(S_n\) is a Coxeter group and the theory of stabilization applies to the edge-isoperimetric problem on \(BS_n\) (see [7] for details).

Another way of looking at \(BS_n\) is as the graph of the permutohedron, the convex polytope generated by the set
\[
\{(\pi(1), \pi(2), ..., \pi(n))\colon \pi \in S_n\}.
\]
Because \(\forall \pi \in S_n, \sum_{i=1}^{n} \pi(i) = \sum_{j=1}^{n} j = \binom{n+1}{2}\), the permutohedron is only \((n-1)\)-dimensional. As early as 1911, Schoute noted that the 3-dimensional permutohedron is isomorphic to the snub-octahedron (the octahedron with each vertex sliced off to make a square face, see [11, pp. 17, 18]) which means that its symmetry group is that of the octahedron (and cube). All the reflections of that larger (order 48 rather than 4! = 24) Coxeter group are stabilizing for \(BS_4\) and the resulting stability order is shown in Fig. 2.

The letters represent 4-permutations according to Table IV. With \(\mathcal{Q} = \{A < B < C < D < E < F\}\), Table V represents \(\varphi\). We leave it to the reader to verify that this \(\varphi\) gives a skeletal MWI-morphism, proving that \(BS_4\) has nested solutions and that the optimal total extension is given by alphabetic order.

3.1.3. The 24-Cell

The stability order of the (graph of) the 24-cell (see [3]) is given in Fig. 3.

With \(\mathcal{Q} = \{A < B < C < D < E < F < G\}\), Table VI represents \(\varphi\). We again leave it to the reader to verify that this \(\varphi\) gives a skeletal MWI-morphism,
proving that (the graph of) the 24-cell has nested solutions and that the optimal total extension is

\[ a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x. \]

3.1.4. \( \mathbb{Z}_5 \times \mathbb{Z}_5 \)

It is easy to see that \( \mathbb{Z}_5 \), the graph of the pentagon, has nested solutions for the E–I Problem and that the maximum values for \( |I(S)| \) and the resulting \( D \)-sequence are given by Table VII.

For \( \mathbb{Z}_5 \times \mathbb{Z}_5 \), the compressibility order is the product order, with the weights summarized in Table VIII. (See Section 1.4). In addition, interchanging \( i_1 \) and \( i_2 \) gives a reflective symmetry which adds the relations \( (i_1, i_2) < (i_2, i_1) \) if \( i_1 < i_2 \). If we assume that \( \mathbb{Z}_5 \times \mathbb{Z}_5 \) also has nested solutions then we can find the optimal total extension by locally maximizing the weight of successive augmentations. This leads us to the numbering shown in Table IX. To prove that the initial segments,

\[ S_k(\eta) = \{(i_1, i_2) \in \mathbb{Z}_5 \times \mathbb{Z}_5 : \eta(i_1, i_2) \leq k\}, \quad 0 \leq k \leq 25, \]

of this numbering are optimal, we define an MWI-morphism with \( \varphi = \{A < B < C\} \) and \( \varphi : \mathbb{Z}_5 \times \mathbb{Z}_5 \rightarrow Q \) given in Table X. Verifying that this \( \varphi \) does determine a skeletal MWI-morphism involves an extra step at this point, compared to previous examples. \( \varphi^{-1}(A) \) and \( \varphi^{-1}(C) \) are not totally

### TABLE IV

<table>
<thead>
<tr>
<th>( a )</th>
<th>( i )</th>
<th>( q )</th>
<th>( 3124 )</th>
</tr>
</thead>
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<td>( j )</td>
<td>( r )</td>
<td>( 4132 )</td>
</tr>
<tr>
<td>( c )</td>
<td>( k )</td>
<td>( s )</td>
<td>( 3241 )</td>
</tr>
<tr>
<td>( d )</td>
<td>( l )</td>
<td>( t )</td>
<td>( 4231 )</td>
</tr>
<tr>
<td>( e )</td>
<td>( m )</td>
<td>( u )</td>
<td>( 3412 )</td>
</tr>
<tr>
<td>( f )</td>
<td>( n )</td>
<td>( v )</td>
<td>( 4312 )</td>
</tr>
<tr>
<td>( g )</td>
<td>( o )</td>
<td>( w )</td>
<td>( 3421 )</td>
</tr>
<tr>
<td>( h )</td>
<td>( p )</td>
<td>( x )</td>
<td>( 4321 )</td>
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</tbody>
</table>

### TABLE V

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \varphi^{-1}(X) )</th>
<th>( \Lambda(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( {a, b, c, d} )</td>
<td>( (0, 1, 1, 2) )</td>
</tr>
<tr>
<td>( B )</td>
<td>( {e, f, g, h} )</td>
<td>( (1, 1, 1, 2) )</td>
</tr>
<tr>
<td>( C )</td>
<td>( {i, j, k, l} )</td>
<td>( (1, 2, 1, 2) )</td>
</tr>
<tr>
<td>( D )</td>
<td>( {m, n, o, p} )</td>
<td>( (1, 2, 1, 2) )</td>
</tr>
<tr>
<td>( E )</td>
<td>( {q, r, s, t} )</td>
<td>( (1, 2, 2, 2) )</td>
</tr>
<tr>
<td>( F )</td>
<td>( {u, v, w, x} )</td>
<td>( (1, 2, 2, 3) )</td>
</tr>
</tbody>
</table>
FIG. 3. 3-Stability order of the 24-cell.

TABLE VI

<table>
<thead>
<tr>
<th>X</th>
<th>( \varphi^{-1}(X) )</th>
<th>( \mathcal{A}(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>{a, b, c, d, e, g}</td>
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</tr>
<tr>
<td>B</td>
<td>{f, h, i}</td>
<td>{2, 3, 4}</td>
</tr>
<tr>
<td>C</td>
<td>{r, k}</td>
<td>{3, 4}</td>
</tr>
<tr>
<td>D</td>
<td>{l, m}</td>
<td>{4, 4}</td>
</tr>
<tr>
<td>E</td>
<td>{n, o}</td>
<td>{4, 5}</td>
</tr>
<tr>
<td>F</td>
<td>{p, q, s}</td>
<td>{4, 5, 6}</td>
</tr>
<tr>
<td>G</td>
<td>{r, t, n, v, w, x}</td>
<td>{4, 5, 6, 7, 8}</td>
</tr>
</tbody>
</table>

TABLE VII

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max_{x \in \mathcal{A}}</td>
<td>R(S)</td>
<td>)</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \mathcal{A}(k) )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
ordered so we must solve the MWI Problem on each of them. That may be accomplished by another MWI-morphism $\varphi': \varphi^{-1}(A) \to \{1 < 2 < 3 < 4 < 5\}$ defined by $\varphi'((i_1, i_2)) = i_2$. There is a small hitch in verifying Definition 16 for $\varphi'$. The same problem comes up later so let us examine it in this transparent example: $\varphi'$ is not a skeletal MWI-morphism because Definition 18.4.a fails for $x = 1$, $y = 5$ and $i = j = 1$. The ideals of $\mathbb{Z}_5 \times \mathbb{Z}_5$ containing $(1, 5)$ as a maximal element and not containing $(2, 1)$ cannot be reduced as in the proof of Theorem 2. However, there is only one such ideal, $\{1\} \times \mathbb{Z}_5$, $|\{1\} \times \mathbb{Z}_5| = 5$ and $A(\{1\} \times \mathbb{Z}_5) = 5$ whereas $|S_{(1)}(\eta)| = 5$ and $A(S_{(1)}(\eta)) = 5$ also. Thus with $S' = S_{(1)}(\eta)$ we complete the proof that $\varphi'$ is an MWI-morphism. This proves that $\varphi^{-1}(A)$ has nested solutions and that the optimal numbering is $\eta$ restricted to $\varphi^{-1}(A)$. The weights on $\varphi^{-1}(C)$ only differ from those on $\varphi^{-1}(A)$ by a constant so we get the same result there too.

3.2. Applications

In the first application, the pairwise product of Petersen graphs, solution by hand was not possible before the introduction of MWI-morphisms but is easy by computer [2]. The second application, to the 600-vertex regular solid in four dimensions had already used 100 h of CPU time on a
computer without result (we underestimated the number of stable sets) but the calculation using MWI-morphisms took about 30 s.

3.2.1. The Product of Petersen Graphs

A diagram of the Petersen graph, $P$, is shown in Fig. 4. Among graph theorists it has a reputation as a universal counterexample and there is even a book devoted to its lore [9].

In a recent paper [2], S. L. Bezrukov and R. Elsässer present a solution of the edge-isoperimetric problem for $P^d$, the $d$-fold product of Petersen graph. Their proof is by induction on $d$. Since $P$ has only 10 vertices and girth (length of the smallest cycle) 5, it is easy to show that the initial segments

$$S_k(\eta) = \{ v \in V : \eta(v) \leq k \},$$

of the numbering function, $\eta : V_P \to \mathbb{Z}_{10} = \{ 1, 2, ..., 10 \}$, shown in Fig. 1, are solutions of the edge-isoperimetric problem. (See Table XI.) By compression, (see Sect. 1.4) the E-I Problem on $P \times P$ reduces to the MWI Problem on $\mathbb{Z}_{10} \times \mathbb{Z}_{10}$ with the weight, $\Delta(i_1, i_2) = \Delta(i_1) + \Delta(i_2)$, summarized in Table XII.
TABLE XI

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max_{x \in \mathbb{P}_2 \setminus {0,1}}</td>
<td>H(x)</td>
<td>$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>$d(k)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

and stabilization (see Sect. 1.3) gives the additional relations $(i_1, i_2) < (i_2, i_1)$ if $i_1 < i_2$. Assuming that $P \times P$ has nested solutions, we find the optimal numbering by optimizing locally as in Section 3.1.4. (See Table XIII.) Bezrukov and Elsaesser used a computer to evaluate the weights of all $\binom{20}{10} = 184760$ ideals and verify that initial segments of their numbering are indeed optimal. For $d > 2$ compression is much more powerful and has the paradoxical effect of reducing the number of cases which must be considered to about 50 (independent of $d$, remarkably). The qualitative differences between $d = 1$, $d = 2$, and $d > 2$ are inherent in the theory of compression but especially beautifully exemplified by the product of Petersen graphs. Our initial goal for the study of MWI-morphisms was to present a humanly verifiable proof of the $d = 2$ case of the Bezrukov–Elsaesser theorem.

Looking at $\eta$, it seems natural to define

$$Q = \{A < B < C < D < E < F\}$$

and $\varphi: \mathbb{Z}_{10} \times \mathbb{Z}_{10} \to Q$ as shown in Table XIV. $\eta$, restricted to the first two columns, $\{1\} \times \mathbb{Z}_{10} + \{2\} \times \mathbb{Z}_{10}$, may be shown optimal by another MWI-morphism, $\varphi': \{1\} \times \mathbb{Z}_{10} + \{2\} \times \mathbb{Z}_{10} \to Q' = \{A' < B'\}$ defined as shown in Table XV. The list-weights were calculated previously in the $\mathbb{Z}_5 \times \mathbb{Z}_5$ example. There are 25 cases to consider; all but two satisfy Definition 18.4 and those

TABLE XII

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>9</th>
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<tbody>
<tr>
<td>$\mathcal{A}$</td>
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<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>$\mathcal{I}_1$</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

MWI-MORPHISMS
TABLE XIII
The Bezrukov-Elsaesser Numbering of \( P \times P \)

<table>
<thead>
<tr>
<th>( P \times P )</th>
<th>10</th>
<th>19</th>
<th>20</th>
<th>30</th>
<th>49</th>
<th>50</th>
<th>69</th>
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<tr>
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<td>18</td>
<td>29</td>
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<td>71</td>
<td>81</td>
<td>82</td>
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</tr>
</tbody>
</table>

\( \eta \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
<table>
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<th></th>
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</tbody>
</table>

are easily rectified as in the \( \mathbb{Z}_5 \times \mathbb{Z}_5 \) example. \( \eta \)'s restrictions to the other pairs of columns, \( \varphi^{-1}(C) \), \( \varphi^{-1}(D) \), and \( \varphi^{-1}(F) \), are also optimal since their list-weights differ from that on \( \varphi^{-1}(A) \) by a constant.

Then the \( \text{MinShadow} \) function, whose nontrivial domain is of cardinality 184, must be calculated, and the really tedious part by hand, the calculation of

\[
\text{MWI}(X; i) + \text{MWI}(Y; j)
\]

for all \( X < Y \) and \( i \geq \text{MinShadow}(X; Y; j) \); define \( \phi(X; j) \) for comparison with \( \text{MWI}(X; i) + \text{MWI}(Y; j) - |X| \). There are about 1200 such calculations. All but 41 satisfy Definition 18.4. Of those that fail, most involve zero or one ideal, as in the \( \mathbb{Z}_5 \times \mathbb{Z}_5 \) example. The only ones that involve more have \( X = B, Y = F, j = 4 \), and \( 3 \leq i \leq 5 \). Those ideals all contain \( \mathbb{Z}_2 \times \mathbb{Z}_{10} \cup \mathbb{Z}_{10} \times \mathbb{Z}_2 \), which has \( |\mathbb{Z}_2 \times \mathbb{Z}_{10} \cup \mathbb{Z}_{10} \times \mathbb{Z}_2| = 36 \) and \( \Delta(\mathbb{Z}_2 \times \mathbb{Z}_{10} \cup \mathbb{Z}_{10} \times \mathbb{Z}_2) = 76 \). Their additional elements are an ideal in \( (\mathbb{Z}_8 - \mathbb{Z}_2) \times (\mathbb{Z}_5 - \mathbb{Z}_2) \), a

Table XIV

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \varphi^{-1}(X) )</th>
<th>( \Delta(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>{1} \times \mathbb{Z}<em>{10} + {2} \times \mathbb{Z}</em>{10}</td>
<td>(1, 1, 1, 2, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4)</td>
</tr>
<tr>
<td>( B )</td>
<td>{3} \times \mathbb{Z}_{10}</td>
<td>(1, 2, 2, 2, 3, 2, 3, 3, 3, 4)</td>
</tr>
<tr>
<td>( C )</td>
<td>{4} \times \mathbb{Z}<em>{10} + {5} \times \mathbb{Z}</em>{10}</td>
<td>(1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 4, 4, 5)</td>
</tr>
<tr>
<td>( D )</td>
<td>{6} \times \mathbb{Z}<em>{10} + {7} \times \mathbb{Z}</em>{10}</td>
<td>(1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 4, 4, 5)</td>
</tr>
<tr>
<td>( E )</td>
<td>{8} \times \mathbb{Z}_{10}</td>
<td>(2, 3, 3, 3, 4, 4, 4, 4, 5)</td>
</tr>
<tr>
<td>( F )</td>
<td>{9} \times \mathbb{Z}<em>{10} + {10} \times \mathbb{Z}</em>{10}</td>
<td>(2, 3, 3, 4, 3, 4, 4, 4, 5, 3, 4, 4, 5, 4, 5, 5, 5, 5, 6)</td>
</tr>
</tbody>
</table>
6 × 3 rectangle which has \( \binom{9}{3} = 84 \) ideals, still nontrivial. If we show that 76 + MWI((\( \mathbb{Z}_8 \oplus \mathbb{Z}_2 \)) × (\( \mathbb{Z}_5 \oplus \mathbb{Z}_2 \))), \( A; k \leq 18 \), then we are done (\( S' = S_{36 + k}(\eta) \)). This may be accomplished with yet another MWI-morphism, \( \varphi': (\mathbb{Z}_8 \oplus \mathbb{Z}_2) \times (\mathbb{Z}_5 \oplus \mathbb{Z}_2) \rightarrow Q = \{ A'' < B'' < C'' \} \) defined as shown in Table XVI, which is even skeletal.

This proof of the optimality of the Bezrukov–Elsässer numbering involves about 1600 steps but is better than the brute force method which Bezrukov and Elsässer used by a factor of more than 100.

### 3.2.2. The 600-Vertex

In about 1840 Schläfli defined and catalogued all regular \( d \)-dimensional solids (convex polytopes) for all \( d \geq 1 \). For every \( d \) they include the simplex ((\( d + 1 \)) vertices at distance 1 from each other), the cube (\( 2^d \) vertices, the \( d \)-tuples of 0’s and 1’s), the crosspolytope (\( 2^d \) vertices, the \( d \)-tuples of 0’s and ±1’s with exactly one nonzero entry) and that is all there are for \( d \geq 5 \). For \( d = 1 \) there is essentially only one regular solid, an interval. For \( d = 2 \) the simplex is a triangle and the cube and crosspolytope degenerate to the square, but there are also infinitely many “exceptional” ones, the regular \( n \)-gons for \( n \geq 5 \). For \( d = 3 \) there are the three “standard” ones, the simplex (aka the tetrahedron), the crosspolytope (aka the octahedron) and the cube (which might also be called the hexahedron) plus two exceptional ones, the dodecahedron and icosahedron. And for \( d = 4 \) there are the standard three plus three exceptional ones. The smallest of the exceptional 4-dimensional regular solids has 24 vertices and 24 octahedral faces. The next has 120 vertices and 600 tetrahedral faces, and the last has 600 vertices and 120 dodecahedral faces (see [3] for additional information).

The Edge–Isoperimetric Problem has been solved for the graph of every regular solid in all dimensions except one, the 600-vertex solid in four dimensions.

### Table XV

<table>
<thead>
<tr>
<th>( X' )</th>
<th>( \varphi^{-1}(X') )</th>
<th>( A(X') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A' )</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_5 )</td>
<td>(0, 1, 1, 2, 1, 2, 1, 2, 2, 3)</td>
</tr>
<tr>
<td>( B' )</td>
<td>( \mathbb{Z}_2 \times (\mathbb{Z}_4 - \mathbb{Z}_5) )</td>
<td>(1, 2, 2, 3, 2, 3, 2, 3, 3, 4)</td>
</tr>
</tbody>
</table>

### Table XVI

<table>
<thead>
<tr>
<th>( X'' )</th>
<th>( \varphi''^{-1}(X'') )</th>
<th>( A(X'') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A'' )</td>
<td>( { 3 } \times (\mathbb{Z}_3 - \mathbb{Z}_2) )</td>
<td>(2, 2, 3)</td>
</tr>
<tr>
<td>( B'' )</td>
<td>( { 4, 5 } \times (\mathbb{Z}_5 - \mathbb{Z}_2) )</td>
<td>(2, 3, 2, 3, 3, 4)</td>
</tr>
<tr>
<td>( C'' )</td>
<td>( { 6, 7, 8 } \times (\mathbb{Z}_5 - \mathbb{Z}_3) )</td>
<td>(2, 3, 3, 2, 3, 3, 4, 3, 4)</td>
</tr>
</tbody>
</table>
For the three standard families the (more general) Minimum Shadow Problem has been solved using compression, and for the dodecahedron, icosahedron, 24-vertex, and 120-vertex the E-I Problem has been solved by stabilization [5, 1]. The calculations for the dodecahedron, icosahedron, and 24-vertex were all done by hand but the 120-vertex required a computer since it has 883 stable sets (ideals in the stability order (see Section 1.3)). In 1979 it required several hours of CPU time on the largest computer on the UCR campus to achieve this so the 600-vertex problem seemed out of reach. Recently Bezrukov pointed out to us that computers are much faster now so it might be possible to finish off the 600-vertex (Berenguer and I [1] had estimated that it had no more than $2^{25} \approx 3.4 \times 10^7$ stable sets). However, after spending 100 h of CPU time on it and generating more than $10^9$ stable sets we gave up. From that calculation we now estimate that there are on the order of $10^{16}$ stable sets, which seems to put a brute force solution beyond reach. However, applying an MWI-morphism to the stability order of the 600-vertex reduces it to less than a million list-weighted ideals, easily doable on a PC. D. Dreier and I will report on details of the calculation in another paper.

REFERENCES

AFTERWORD

In looking back, I feel very fortunate to have been a postdoc at the Rockefeller University with Gian-Carlo Rota. I received my Ph.D. degree
in June 1965 from the University of Oregon, having written my thesis in Fourier analysis with Victor Shapiro, a student of Zygmund. But I had also worked summers at the Jet Propulsion Laboratory and Bell Labs solving combinatorial problems. I liked combinatorics, what I knew of it, but was dubious that it could sustain me through a career. I knew what real mathematics was from my academic work. Combinatorics, fostered mainly in industry to solve problems in communication engineering, was relatively undeveloped. My fear was that I might look back at the end and see that I had just applied the same small bag of tricks over and over. When I joined G.-C. at RU in September '65, my apprehensions about combinatorics were quickly allayed. He knew more mathematics, more mathematicians, more history of mathematics, and more about mathematics as a human activity than anyone I had ever met. And he was committed to bringing academic respectability to combinatorics. On long afternoon walks from the Rockefeller campus (at 65th and York in Manhattan) he laid out his vision of “combinatorial theory” (as he called it): Combinatorics is “applicable mathematics,” but must not be confused with Applied Mathematics, which is mainly the study of differential equations and their physical interpretations. Algebra is the language of modern mathematics. In such a diffuse and immature subject (as combinatorics) our main task is to identify its fundamental problems and the methods appropriate for their solution. At that time he was focusing on geometric lattices, a cryptomorph of the concept of matroid (a term which he avoided whenever possible). An impressive theory had already been erected by Whitney, Tutte, and others and G.-C. wished to augment it with his incidence algebras and Critical Problem, an extension of the Coloring Problem for graphs.

Unfortunately, despite considerable effort, I was never able to contribute to G.-C.’s ambitious projects. The one question of his which I did make progress on was whether there is an analog of Sperner’s theorem for the lattice of partitions [3]. However, I had absorbed his optimistic view of the future of combinatorics and its relationship to classical mathematics. After he returned to MIT in the summer of ’67 we interacted more as colleagues than as teacher–student and I found that he was not only a good expositor but a good listener. As I developed my own projects, he sometimes made connections to the literature which I found invaluable. When I first presented the theory of stabilization to him at one of our get-togethers (about ’76), he remarked that it reminded him of Steiner symmetrization and referred me to Polya and Szegö’s book [2]. I had noticed the analogy between certain constrained combinatorial optimization problems and the classical isoperimetric problem of Greek geometry much earlier [1], but G.-C.’s observation took the connection to another level.

I regard the concept of MWI-morphism introduced in this paper as a capstone in my own work on combinatorial optimization. As I have tried
to show, it reflects Gian-Carlo Rota’s influence and support. It is the most fitting offering I can make to his memory.

Let me close with my favorite recollection of G.-C. It occurred at the end of one of our afternoon walks around Manhattan’s upper east side. He noticed a Spanish restaurant which had just opened and invited me to try it with him. He loved continental cuisine and wines, so such an occasion became another opportunity for him to add to my education. The decor was impressive and there was a classical guitarist but the food was not very memorable. We did have a nice bottle of Rioja wine and afterward G.-C. ordered a Spanish brandy, “Carlos Primero,” with his coffee. While we waited, he informed me that Spanish brandy was “sherryized” by being aged in old sherry barrels and that it came in five grades. The lowest was Fundador and the rest were named after Spanish kings with the quality of the king reflecting the quality of the brandy. Carlos Primero was the best. The brandy and coffee were served by a uniformed waiter from a silver tray. However, when G.-C. tasted the brandy a look of disappointment crossed his face. He turned to the waiter and said “This is not what I ordered. This is not Carlos Primero.” “But of course it is,” the waiter insisted. G.-C. took another sip from the snifter and pronounced it to be Fundador. “No, no, it’s Carlos Primero, I am sure,” the waiter replied. “All right then,” G.-C. said with finality, “show me the bottle.” The waiter disappeared for several minutes and when he returned, stopped about ten feet short of our table and flashed the bottle. It was Fundador!

REFERENCES