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Macsyma Computation of Local Minimal Realization of Dynamical Systems of which Generating Power Series are Finite

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We present here a package of *Macsyma* programs, allowing the manipulation of words, and noncommutative power series over some finite alphabet.

On the basis of the works of M.Fliess and C.Reutenauer, concerning local realization of nonlinear dynamical systems, we present an algorithm allowing the computation of the local and minimal realization of finite generating power series. We describe that algorithm in the computer algebra system *Macsyma*.

1. Introduction

We present here a package of *Macsyma* programs, allowing the manipulation of words, and noncommutative power series over some finite alphabet X.

This package contains, in particular, an implementation of *shuffle product* of two noncommutative polynomials, *left* and *right remainder* of a noncommutative polynomial by another noncommutative polynomial, production (up to some fixed degree) of the *Lyndon basis* of the *free Lie algebra* $\mathcal{L}ie\langle X\rangle$, and the *canonical* "Poincaré-Birkhoff-Witt" *basis* of noncommutative polynomials over X.

As a development, we present a package of some programs which computes the local and minimal realization of dynamical systems of which generating power series are finite. The first version of this package was implemented with computer algebra system *Macsyma* on the Bull computer DPS 8 under the operating system Multics. The actual version is implemented with computer algebra system *Macsyma* on the workstation Sun 3/80 under the operating system Unix.

Our program can deal with all polynomials but, since it uses the shuffle product, its capacity is limited by the core size of the computer. It is able to treat polynomials up to degree five.

Now, we have a mean which allows us to implement and manipulate rational series. A continuation of this work is the realization of rational series.

We use Lyndon basis as $\mathcal{L}ie\langle X\rangle$–basis. Lyndon words are used to compute the local coordinate system.

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2. Definitions and notations

Let X be a finite and totally ordered set called *alphabet*. The elements of X are called *letters* and the elements of the *free monoid* X^* generated by X are called *words*. The empty word is denoted by ε.

We define a noncommutative power series S over $X = \{x_0, ..., x_{m-1}\}$ with coefficients in $\mathbb{R}$ as a mapping from X^* into $\mathbb{R}$. We denote by $\langle S|w\rangle$ the image of w by S. $\langle S|w\rangle$ is also called the *coefficient* of w in S. The formal power series is denoted by the following formal sum :

$$S = \sum_{w \in X^*} \langle S|w\rangle w.$$

The set of all formal power series is denoted by $\mathbb{R}\langle\langle X\rangle\rangle$ and is an algebra for the *Cauchy product*.

The subset of X^* defined as follows :

$$\text{supp}(S) = \{w \in X^* | \langle S|w\rangle \neq 0\}$$

is called the *support* of the power series S.

A power series which has finite support is called a *polynomial*. The set of all polynomials is a sub-algebra of $\mathbb{R}\langle\langle X\rangle\rangle$ denoted by $\mathbb{R}\langle X\rangle$.

Let $\mathcal{L}ie\langle X\rangle$ be the *free Lie algebra* generated by X in which the *Lie-brackets* are defined by :

$$[x_i, x_j] = x_i x_j - x_j x_i.$$

Lie polynomials are the linear combinations of *Lie words*. Lie words are either elements of X or brackets of Lie words.

2.1. REMAINDERS OF A NONCOMMUTATIVE POWER SERIES

Let $S \in \mathbb{R}\langle\langle X\rangle\rangle$ be a noncommutative power series and let $u \in X^*$ be a word. We define and denote $S \triangleright u$ (resp. $u \triangleleft S$) the *right* (resp. *left*) *remainder of the power series* S by the word u as follows (Jacob & Oussous, 1987) :

$$S \triangleright u = \sum_{w \in X^*} \langle S|w\rangle w \triangleright u, \qquad \left(\text{resp. } u \triangleleft S = \sum_{w \in X^*} \langle S|w\rangle u \triangleleft w\right),$$

where

$$w \triangleright u = \begin{cases} v, & if \quad w = uv, \\ 0, & otherwise, \end{cases} \qquad \left(\text{resp. } u \triangleleft w = \begin{cases} v, & if \quad w = vu, \\ 0, & otherwise. \end{cases}\right)$$

Properties $\forall S \in \mathbb{R}\langle\langle X\rangle\rangle, \forall u, v \in X^*$,

$$\begin{aligned} S \triangleright (uv) &= (S \triangleright u) \triangleright v, \quad \left(\text{resp. } (uv) \triangleleft S = u \triangleleft (v \triangleleft S)\right), \\ (u \triangleleft S) \triangleright v &= u \triangleleft (S \triangleright v). \end{aligned}$$

Remark 2.1 $\forall S \in I\!R\langle\langle X\rangle\rangle, \quad \varepsilon \triangleleft S = S \triangleright \varepsilon = S.$

This definition can be extended to polynomials as follows: Let $P = \sum_{v \in X^*} \langle P|v\rangle v$ be a polynomial then:

$$S \triangleright P = \sum_{w,v \in X^*} \langle S|w\rangle \langle P|v\rangle w \triangleright v,$$

$$\left(\text{resp. } P \triangleleft S = \sum_{w,v \in X^*} \langle S|w\rangle \langle P|v\rangle v \triangleleft w\right).$$

2.2. LIE-HANKEL MATRIX

Definition 2.1 *The Lie-Rank of a formal power series $S \in I\!R\langle\langle X\rangle\rangle$ is defined by :*

$$\mathcal{LR}(S) = \dim\Big(S \triangleright \mathcal{L}ie\langle X\rangle\Big)$$
$$= \dim\Big[span\{S \triangleright P | P \in \mathcal{L}ie\langle X\rangle\}\Big],$$

where "$S \triangleright P$" means the right remainder of S by the Lie polynomial P. Recall that $S \triangleright [x, y] = S \triangleright xy - S \triangleright yx$.

Definition 2.2 *Let $S \in I\!R\langle\langle X\rangle\rangle$. We define the Lie-Hankel matrix associated with S as the infinite array, denoted by $\mathcal{LH}_S$, of which the lines are indexed by some totally ordered basis of $\mathcal{L}ie\langle X\rangle$ and the columns are indexed by X^* (sorted for lexicographic by length order) such that :*

$$\mathcal{LH}_S(P_i, w) \;=\; \langle S|P_i w\rangle \;=\; \langle S \triangleright P_i | w\rangle.$$

We show easily that :

$$\mathcal{LR}(S) = Rank(\mathcal{LH}_S).$$

2.3. SHUFFLE PRODUCT

Let $u, v, u', v' \in X^*$ be words, $x, y \in X$ be letters. We define (Jacob & Oussous, 1987) the *shuffle product* of u and v recursively as follows :

$$\begin{cases} u \sqcup\!\sqcup \varepsilon = \varepsilon \sqcup\!\sqcup u = u, \\ u \sqcup\!\sqcup v = x(u' \sqcup\!\sqcup v) + y(u \sqcup\!\sqcup v'), \quad \text{if } u = xu' \text{ and } v = yv'. \end{cases}$$

This product is *commutative* and *associative* and can be extended to formal power series by setting, for $S, T \in I\!R\langle\langle X\rangle\rangle$:

$$S \sqcup\!\sqcup T = \sum_{u,v \in X^*} \langle S|u\rangle \langle T|v\rangle u \sqcup\!\sqcup v.$$

3. Lyndon words and Lie basis

3.1. ORDER OVER WORDS AND CONJUGATION CLASSES

Let X be a finite and totally ordered set. The " lexicographical order" on X^* (Lothaire, 1983) is the total order defined as follows :
$\forall u, v \in X^*$, $u < v$ if and only if

$$\begin{cases} \text{either} & \exists w \neq \varepsilon \text{ such that } uw = v, \\ \text{or} & \exists x, y, z \in X^* \text{ and } a, b \in X \text{ such that } u = xay,\ v = xbz \text{ and } a < b. \end{cases}$$

With this order, we have the following properties:

$$\begin{cases} (i) & \forall w \in X^*, & u < v & \Longleftrightarrow & wu < wv. \\ (ii) & \text{If } v \notin uX^*, \text{ then} & u < v & \Longrightarrow & \forall w, z \in X^*\ uw < vz. \end{cases}$$

A word u is a *factor* of a word v if

$$\exists\, x, y \in X^* \quad \text{such that} \quad v = xuy.$$

If $x = \varepsilon$ (resp. $y = \varepsilon$), we say that u is a *left* (resp. *right*) *factor* of v, *proper* if $y \neq \varepsilon$ (resp. $x \neq \varepsilon$).

Two words u and v are said to be *conjugate* if

$$\exists\, x, y \in X^* \quad \text{such that} \quad u = xy \quad \textit{and} \quad v = yx.$$

3.2. LYNDON WORDS

The definition and properties of Lyndon words can be found in (Lothaire, 1983; Melançon & Reutenauer, 1987; Duval, 1988).

Definition 3.1 *A word $w \in X^*$ is a Lyndon word if it satisfies one of the equivalent following statements:*

(i) it is strictly less than any of its conjugate,

(ii) it is strictly less than any of its proper right factors.

Let us denote L, as the set of Lyndon words over X.

Properties

1. Let $w \in L \backslash X$ and m be its *longest proper right factor* in L. Let $l \in L$ such that $w = lm$ and $l < lm < m$. Then the couple $\sigma(w) = (l, m)$ is called the *standard factorization* of w.
   Examples:
$$\begin{aligned} \sigma(x_0 x_1^3) &= (x_0 x_1^2, x_1) \\ \sigma(x_0 x_1^2) &= (x_0 x_1, x_1) \\ \sigma(x_0^2 x_1^2) &= (x_0, x_0 x_1^2) \\ &\neq (x_0^2 x_1, x_1). \end{aligned}$$

2. $w \in L$ if and only if

$$\begin{cases} \text{either} & w \in X, \\ \text{or} & w = lm \text{ with } l, m \in L \text{ and } l < m. \end{cases}$$

The last property gives us an algorithm to construct the Lyndon words up to some given degree. *Macsyma* implementations of that algorithm have been given by Oussous(1988).

3.3. LYNDON BASIS

For further details, see Viennot(1978). We consider on $\mathcal{L}ie\langle X\rangle$ the *Lyndon basis* (called also *Chen–Fox–Lyndon* basis) which is recursively defined as follows:

$$\begin{cases} c(x) &= x \quad \text{for} \quad x \in X, \\ c(w) &= [c(l), c(m)], \quad \text{for } w \in L\backslash X, \text{ such that } \sigma(w) = (l, m), \end{cases}$$

where the brackets are the Lie one. This definition gives us an algorithm to construct the *Lyndon* basis of the free Lie algebra $\mathcal{L}ie\langle X\rangle$.

4. Dynamic systems and generating power series

We consider a system in the following form:

$$\begin{cases} \dot{q}(t) &= \sum_{i=0}^{m-1} u_i(t) Y_i(q) \quad \text{with } u_0(t) \equiv 1, \\ y(t) &= h(q(t)), \end{cases} \tag{1}$$

where q belongs to a *connected* $\mathbb{R}$*-analytic variety* Q, the Y_i's are *analytic vector fields*, and h, a $\mathbb{R}$*-analytic function* called *observation*, defined in a neighbourhood of the given *initial state* $q(0)$. The inputs, $u_1, ..., u_{m-1}$, are *real and piecewise continuous.*

Each *input* u_i, is associated with a *letter* x_i, $(0 \le i \le m-1)$. The set of all the letters, $X = \{x_0, x_1, ..., x_{m-1}\}$ will be called the *control alphabet*. Let X^* be the free–monoid generated by X.

For each word $w \in X^*$, we denote by Y_w the *differential operator* associated with it, and defined as follows:

$$\begin{cases} Y_\varepsilon &= Identity, \\ Y_w &= Y_{i_1} \circ Y_{i_2} ... \circ Y_{i_k} \quad \text{if } w = x_{i_1} x_{i_2} ... x_{i_k}, \end{cases}$$

where the Y_{i_j}'s are vector fields and "$\circ$" means the composition operation.

The *action* of the differential operator Y_w over an analytic function f defined on Q is denoted by $Y_w \circ f$.

For short enough time and inputs, the output y of the system (1) is given by the *Peano–Baker Formula*, called also *Fliess Fundamental Formula* (Fliess, 1983; Reutenauer, 1986):

$$y(t) = \sum_{w \in X^*} (Y_w \circ h)_{|_{q(0)}} \int_0^t \delta_u w, \tag{2}$$

where $|_{q(0)}$ means the evaluation in $q(0)$, and $\int_0^t \delta_u w$ is the *iterated integral* defined recursively as follows:

$$\begin{cases} \text{if} \quad w = \varepsilon, & \text{then} \quad \int_0^t \delta_u \varepsilon = 1, \\ \text{if} \quad w = x_i \in X, & \text{then} \quad \int_0^t \delta_u x_i = \int_0^t u_i(\tau)d\tau, \\ \text{if} \quad w = v x_i, & \text{then} \quad \int_0^t \delta_u w = \int_0^t \left(\int_0^\tau \delta_u v\right) u_i(\tau)d\tau. \end{cases}$$

The Input/Output behaviour of system (1) is completely defined by its *generating power series* in the noncommutative variables $x_0, x_1, ..., x_{m-1}$, given by the formula (Fliess, 1983):

$$g = \sum_{w \in X^*} \langle g|w \rangle w = \sum_{w \in X^*} (Y_w \circ h)_{|_{q(0)}} w. \tag{3}$$

Thus, the output $y(t)$, given by (2), can be written:

$$y(t) = \sum_{w \in X^*} \langle g|w \rangle \int_0^t \delta_u w. \tag{4}$$

5. Realization of dynamic systems

According to Fliess(1983), the problem of local realization can be expressed as follows:

> *Let an Input/Output behaviour, given by its generating power series. Is there a differential system like (1) which has the same generating power series? In the positive case, describe it.*

5.1. DIFFERENTIALLY PRODUCED POWER SERIES

Definition 5.1 *The formal power series $g \in \mathbb{R}\langle\langle X \rangle\rangle$ is produced differentially if and only if there exist :*

1. *an integer $r \in \mathbb{N}$,*
2. *an homomorphism $\mathcal{Y}$ from X^* into differential operators algebra over the commutative algebra $\mathbb{R}[[q_1, ..., q_r]]$ such that $\forall x_i \in X, Y_i = \mathcal{Y}(x_i)$ is a vector field,*
3. *a commutative power series $h \in \mathbb{R}[[q_1, ..., q_r]]$,*

such that:

$$\forall w \in X^*, \quad \langle g|w \rangle = (\mathcal{Y}(w) \circ h)_{|_0}, \tag{5}$$

where $|_0$ means the evaluation in $q_1 = ... = q_r = 0$.

The couple $(\mathcal{Y}, h)$ is called a *differential representation* of g, of dimension r.

From (3) it is obvious that:

g is the generating power series of a system like (1) if and only if g is produced differentially.

Thus:

The study of local realizations
is equivalent to
the study of differential representations

5.2. FLIESS THEOREM

Theorem 5.1 *The power series $g \in \mathbb{R}\langle\langle X\rangle\rangle$ is produced differentially if and only if its Lie–Rank, r, is finite. In this case, r is equal to the smallest dimension of all its differential representations.*

If $(\mathcal{Y}, h)$ and $(\mathcal{Y}', h')$ are two differential representations of dimension r of g, then it exists a continuous automorphism φ of $\mathbb{R}[\![q_1 ..., q_r]\!]$ such that:

$$\forall w \in X^*,\ \forall k \in \mathbb{R}[\![q_1, ..., q_r]\!], \quad h' = \varphi(h) \ \text{ and } \ \varphi(Y_w \circ k) = Y'_w \circ \varphi(k).$$

(Y_w (resp. Y'_w) means the image of w by $\mathcal{Y}$ (resp. $\mathcal{Y}'$)).

The realization $(\mathcal{Y}, h)$, unique up to isomorphism, is said *minimal* or *reduced.*

The *observation algebra* is the complete sub–algebra $\mathcal{O}_{q(0)}$ of $\mathcal{C}^\omega(Q)$ in a neighborhood of $q(0)$, stable for the analytic vector fields actions. We can associate (Fliess, 1983; Jacob & Oussous, 1989) with each system (1) a map:

$$\sigma : \mathcal{O}_{q(0)} \longmapsto \mathbb{R}\langle\langle X\rangle\rangle,$$

defined by:

$$\forall h \in \mathcal{O}_{q(0)}, \qquad \sigma(h) = \sum_{w \in X^*} (Y_w \circ h)_{|_{q(0)}} w. \tag{6}$$

Therefore, $\sigma(h)$ is the generating power series of the system (1) related to the observation h.

The map σ is a morphism which translates the observation algebra structure in the formal power series algebra (with Shuffle product) structure.

Lemma 5.1 *σ is a linear map, that is:* $\forall h, k \in \mathcal{O}_{q(0)}, \quad \forall \alpha, \beta \in \mathbb{R}$,

$$\sigma(\alpha h + \beta k) = \alpha\sigma(h) + \beta\sigma(k).$$

Lemma 5.2 *Let $h, k \in \mathcal{O}_{q(0)}$, be two observations. We have:*

$$\sigma(h \cdot k) = \sigma(h) \sqcup\!\sqcup\ \sigma(k).$$

The vector fields, $Y_0, ..., Y_{m-1}$, are given by the following formula (Fliess, 1983; Jacob & Oussous, 1988; Reutenauer, 1986) :

$$Y_i = \sum_{j=1}^{r} \theta_i^j(q_1, ..., q_r)\frac{\partial}{\partial q_j}, \quad \theta_i^j(q_1, ..., q_r) \in \mathbb{R}[\![q_1, ..., q_r]\!]. \tag{7}$$

Remark 5.1 Let k be an observation over $q_1, ..., q_r$. We denote by $\sigma(k)$ the corresponding generating power series. Then

$$\begin{aligned}\forall w \in X^*, \quad \forall v \in X^*, \quad \langle \sigma(Y_w \circ k)|v\rangle &= [Y_v \circ (Y_w \circ k)]_{|0} \\ &= (Y_{vw} \circ k)_{|0} \\ &= \langle \sigma(k)|vw\rangle \\ &= \langle w \triangleleft \sigma(k)|v\rangle.\end{aligned}$$

Then
$$\sigma(Y_w \circ k) = w \triangleleft \sigma(k). \tag{8}$$

We saw that the *action* of the differential operator Y_w over the observation k is equivalent to the left remainder of the generating power series $\sigma(k)$ by the word w.

5.3. LOCAL COORDINATES

We consider the *sub-Lie-algebra* of $\mathcal{L}ie\langle X\rangle$, of codimension r, defined by (Jacob & Oussous, 1988; Reutenauer, 1986) :

$$\mathcal{A}(g) = \{ P \in \mathcal{L}ie\langle X\rangle \mid g \triangleright P = 0 \}$$

$\mathcal{A}(g)$ is generated by the Lie polynomials which annihilate the power series g.

We set:

$$\mathcal{V}(\mathcal{A}(g)) = \{ Q \in \mathbb{R}\langle\langle X\rangle\rangle \mid Q \triangleright \mathcal{A}(g) = 0 \}.$$

Let $(P_i)_{i\geq 1}$ be Lie polynomials such that $P_1, ..., P_r$ is a basis of $\mathcal{L}ie\langle X\rangle$ modulo $\mathcal{A}(g)$, and $P_{r+1}, ..., P_n, ...$ is a basis of $\mathcal{A}(g)$. We define the polynomials $Q_1, ..., Q_r$ without constant term such that:

$$\begin{cases} \langle Q_j \triangleright P_i|\varepsilon\rangle = \delta_{ij} & \text{for } i \leq r, \\ Q_j \in \mathcal{V}(\mathcal{A}(g)). \end{cases} \tag{9}$$

where $\langle Q_j \triangleright P_i|\varepsilon\rangle$ is the coefficient of ε in the right remainder of Q_j by the Lie polynomial P_i.

We know that, according to Melançon & Reutenauer(1987) and Radford(1979), the Lyndon words are a transcendance basis of the shuffle algebra $\mathbb{R}\langle X\rangle$. Otherwise, we have the following important relation:

$$\langle l_j \triangleright P_i|\varepsilon\rangle = \delta_{ij}$$

where l_j is a Lyndon word which corresponds to the element P_j of a Lyndon basis.

For us, the problem of the local minimal realization can be expressed as follows:
Let g be a *finite* generating power series. Then

- To compute the observation h, we proceed as follows :
  - We compute the *Lie-Rank* of g which is denoted by r.

- We construct some noncommutative polynomials, $Q_1, ..., Q_r$, without constant term, as a linear combination of the shuffles of the Lyndon words and which verify the relation (9).
- We express g as a linear combination of the shuffles of the polynomials, $Q_1, ..., Q_r$.

With this expression of g, we associate the observation h (analytic function over local coordinates, $q_1, ..., q_r$).

- In order to define the homomorphism $\mathcal{Y}$, we compute the vector fields, $Y_0, ..., Y_{m-1}$, given by (7), which are the respective images of the letters, $x_0, ..., x_{m-1}$, by the homomorphism $\mathcal{Y}$.
  If we consider q_j as an observation then, under (8), $\sigma(Y_i \circ q_j) = x_i \triangleleft \sigma(q_j)$. Let $Q_j = \sigma(q_j)$ denote the generating power series corresponding to q_j, then

$$\sigma(\theta_i^j) = \sigma(Y_i \circ q_j) = x_i \triangleleft Q_j.$$

Hence, we have a way to compute $\theta_i^j(q_1, ..., q_r)$.

The couple $(\mathcal{Y}, h)$ is the differential representation of the polynomial g of dimension

6. The program

Our program is composed of two parts : the first part computes the local minimal realization and the second part allows us to verify that this realization is correct.

The first part consists of one main procedure which has two arguments and calls five procedures.

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(1) LYN_COMP(S, m) : the principal procedure. It has two arguments:

- S : a noncommutative polynomial (as a generating power series of a dynamical analytic system).
- m : a cardinal of the alphabet in which S is written.
- Output :
  - the system of *vector fields*, $\{Y_0, Y_1, ..., Y_{m-1}\}$,
  - the *observation* H.

  This procedure calls five procedures which are described below.

(1.1) LIE_HAN_MAT(S, m) : has the same arguments as LYN_COMP.

- Output :
  - LieHanMat : the Lie–Hankel matrix associated with S.
  - r : the Lie–Rank of S.

(1.2) LOC_COORD(Mat) : has one argument :

- *Mat* : is the matrix constructed by the above procedure.
- Output :
  - a system of r local coordinates, $\{Q_1, ..., Q_r\}$, which are the polynomials constructed over the Lyndon words.

(1.3) EXPRESS($Pol, Tab, ztab$) : has three arguments.

- *Pol* : is a noncommutative polynomial.
- *Tab* : is an array of Lie polynomials.
- *ztab* : is an array of noncommutative polynomials $(Q_1, ..., Q_r)$.
- Output :
  - *Pol* : is a commutative polynomial over $Q_1, ..., Q_r$, with shuffle product.

This procedure is used to compute the observation and the coefficients of vector fields.

(1.4) ACTION_X(Z) : has one argument.

- Z : is an array of local coordinates.
- Output :
  - A two-dimensional array of coefficients (noncommutative polynomials) of vector fields.

This procedure computes, for each local coordinate, the left remainder by each letter.

(1.5) VECT_FIELD($Matx$) : has one argument.

- *Matx* : is a two–dimensional array of coefficients of vector fields produced by the above procedure.
- Output :
  - The system of vector fields $\{Y_0, ..., Y_{m-1}\}$.

The second part consists of one main procedure which has three arguments.

(2) VERIFY(Tab, H, n) : has three arguments :

- Tab : a two–dimensional array produced by VECT_FIELD and enclosing the coefficients of vector fields,
- H : a commutative polynomial (the observation),
- n : the Lie–Rank of S.
- Output :
  - The generating power series associated with this differential system.

Polynomials	Realizations	CPU(msec)
$x_0x_1 + x_1x_0$	$Q_1 = x_0$; $Q_2 = x_1$ $h = q_1q_2$ $Y_0 = \frac{\partial}{\partial q_1}$ $Y_1 = \frac{\partial}{\partial q_2}$	6967
$x_0x_1 + x_0x_1x_0$	$Q_1 = x_0$; $Q_2 = x_0x_1$ $Q_3 = x_0^2.x_1$ $h = -2q_3 + q_1q_2 + q_2$ $Y_0 = \frac{\partial}{\partial q_1}$ $Y_1 = \frac{q_1^2}{2}\frac{\partial}{\partial q_3} + q_1\frac{\partial}{\partial q_2}$	21100
$x_0x_1 + x_1x_0 + x_0x_1^2 + x_1^2x_0$	$Q_1 = x_0$; $Q_2 = x_1$ $Q_3 = x_0x_1$; $Q_4 = x_0x_1^2$ $h = 2q_4 - q_2q_3 + \frac{q_1q_2^2}{2} + q_1q_2$ $Y_0 = \frac{\partial}{\partial q_1}$ $Y_1 = q_3\frac{\partial}{\partial q_4} + q_1\frac{\partial}{\partial q_3} + \frac{\partial}{\partial q_2}$	29017
$x_0 + x_0^4 + x_0x_1x_0^2 - x_0x_1^2 - x_0^3x_1 + x_0x_1^3$	$Q_1 = x_0$; $Q_2 = x_0x_1$ $Q_3 = x_0^2x_1$; $Q_4 = x_0x_1^2$ $Q_5 = x_0x_1^3 + 2x_0^3x_1$ $h = q_5 - q_4 - 2q_1q_3 + \frac{q_1^2q_2}{2} + \frac{q_1^4}{24} + q_1$ $Y_0 = \frac{\partial}{\partial q_1}$ $Y_1 = (q_4 + \frac{q_1^3}{3})\frac{\partial}{\partial q_5} + q_2\frac{\partial}{\partial q_4} + \frac{q_1^2}{2}\frac{\partial}{\partial q_3} + q_1\frac{\partial}{\partial q_2}$	98550

Table 1. Table of examples

7. Examples

We give in the table 1 some examples of polynomials with their realizations and the approximative CPU time used. The q_i's are the local coordinates ($\sigma(q_i) = Q_i$), h is the observation function and the Y_j's are the vector fields. The CPU time can be reduced if the Lyndon words shuffles are stored in the memory.

8. Conclusion

Our software consists of a package of *Macsyma* programs. Those programs are stored in independent files and can be used separatly by any *Macsyma* users. The source codes of both the first and the actual version are published in (Oussous 1989).

This program deals with finite generating power series. We hope that we can extend it to deal with rational generating power series.

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