Journal of Differential Equations **172**, 59–82 (2001)
doi:10.1006/jdeq.2000.3869, available online at http://www.idealibrary.com on IDEAL®

Uniqueness of Classical and Nonclassical Solutions for Nonlinear Hyperbolic Systems

Paolo Baiti

Università di Padova, Via G. B. Belzoni 7, Padova 35131, Italy; and Dipartimento di Matematica ed Informatica, Università di Udine, Via delle Scienze 206, Udine 33100, Italy
E-mail: baiti@dimi.uniud.it

Philippe G. LeFloch

Centre de Mathématiques Appliquées and Centre National de la Recherche Scientifique, UMR 7641, Ecole Polytechnique, 91128 Palaiseau, France
E-mail: lefloch@cmap.polytechnique.fr

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Benedetto Piccoli

S.I.S.S.A., Via Beirut 4, Trieste 34014, Italy
E-mail: piccoli@sissa.it

Received July 26, 1999; revised February 16, 2000

In this paper we establish a general uniqueness theorem for nonlinear hyperbolic systems of partial differential equations in one-space dimension. First of all we introduce a new notion of *admissible solutions* based on prescribed sets of *admissible discontinuities* Φ and *admissible speeds* ψ. Our definition unifies in a single framework the various notions of entropy solutions known for hyperbolic systems of conservation laws, as well as for systems in nonconservative form. For instance, it covers the nonclassical (undercompressive) shock waves generated by a vanishing diffusion-dispersion regularization and characterized via a kinetic relation. It also covers Dal Maso, LeFloch, and Murat's definition of weak solutions of nonconservative systems.

Under certain natural assumptions on the prescribed sets Φ and ψ and assuming the existence of a $\mathbf{L}^1$-continuous semi-group of admissible solutions, we prove that, for each Cauchy datum at $t=0$, there exists at most one admissible solution to the Cauchy problem depending $\mathbf{L}^1$-continuously upon the initial data. In particular, our result shows the uniqueness of the $\mathbf{L}^1$-continuous semi-group of admissible solutions.

In short, this paper proves that supplementing a hyperbolic system with the "dynamics" of elementary discontinuities characterizes at most one $\mathbf{L}^1$-continuous and admissible solution. © 2001 Academic Press

Key Words: hyperbolic system; entropy solution; kinetic relation; semi-group of solutions; uniqueness.

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1. INTRODUCTION

This paper is motivated by a recent activity concerning the uniqueness of entropy solutions for hyperbolic systems of conservation laws. The aim of this paper is to extend the previous results in this direction to general solutions of general hyperbolic systems of the form

$$\partial_t u + A(u)\, \partial_x u = 0, \qquad u(x, t) \in \mathbf{R}^N, \tag{1.1}$$

where $A(u)$ is a $N \times N$ matrix depending smoothly upon u. This work is also part of a series of papers by the authors [1, 3, 4] devoted to nonclassical entropy solutions of hyperbolic systems.

The recent research focuses on the Cauchy problem for a hyperbolic system of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x, t) \in \mathbf{R}^N, \tag{1.2}$$

$$u(0) = u_0. \tag{1.3}$$

This problem generally does not admit smooth solutions and, therefore, weak solutions in the sense of distributions are considered. Solutions in the class **BV** of (bounded) functions with bounded variation are sought and, for the sake of uniqueness, an entropy criterion for admissibility is added. (See Lax [20, 21] and Smoller [28].) When (1.2) is strictly hyperbolic and admits only genuinely nonlinear or linearly degenerate characteristic fields, Glimm [15] established the existence of an entropy solution for the Cauchy problem (1.2)–(1.3). Non-genuinely nonlinear fields were treated by Liu [24] and Ancona and Marson [2], and nonconservative systems by LeFloch and Liu [23]. The notion of weak solutions proposed by DalMaso, LeFloch and Murat [13] was necessary in [23] as (1.1) is written in a nonconservative form, and the notion of a solution in the sense of distributions has to be extended.

Until recently, only partial results on uniqueness and continuous dependence were available for the entropy solutions of (1.1). In the new approach developed by Bressan (see [6] for a review), uniqueness for (1.2)–(1.3) is tackled by considering a whole semi-group of solutions

$$(t, u_0) \mapsto S(t)\, u_0$$

depending Lipschitz continuously upon (t, u_0), in the sense that for some constant $L > 0$

$$\| S(t)\, u_0 - S(s)\, v_0 \|_{\mathbf{L}^1(\mathbf{R})} \leqslant L\, |t - s| + L\, \| u_0 - v_0 \|_{\mathbf{L}^1(\mathbf{R})} \tag{1.4}$$

for all $t, s \geqslant 0$ and for all u_0, v_0 in a (suitably large) class of functions with bounded total variation.

Relying on the existence of a semi-group satisfying (1.4)—established by Bressan, Crasta and Piccoli [8] and Liu and Yang [25], and next, with much simpler proofs, by Bressan, Liu and Yang [12] and Hu and LeFloch [18]—several results about the uniqueness for (1.2)–(1.3) were obtained in [5, 10, 9, 11]. The attention therein was restricted to strictly hyperbolic and genuinely nonlinear systems of conservation laws and to entropy solutions in the sense of the Lax shock admissibility inequalities [20]. In particular, Bressan and LeFloch [10] proved that any solution of the Cauchy problem (1.2)–(1.3) must coincide with the semi-group solution $S(t)\,u_0$, under the assumption that it has *Tame Variation.* (See [10] for the definition.) This assumption holds for instance for solutions obtained by the Glimm scheme or the wave front tracking algorithm. Bressan and Goatin [9] extended the result in [10] to functions having *Tame Oscillation* and we will rely here on this weaker property (see Section 2). In a more recent result, Bressan and Lewicka [11] introduced another condition, the *Locally Bounded Variation* property, which we will also investigate here.

In the present paper, we deal with general solutions that need coincide with the entropy solutions in the sense of Lax or, more generally, in the sense of Liu [24] and we consider a hyperbolic system of equations of the general form (1.1), that not necessarily can be rewritten in conservative form. We are particularly interested in non-genuinely nonlinear systems and in the so-called nonclassical, undercompressive solutions constructed in Jacobs, McKinney and Shearer [19] and Hayes and LeFloch [16, 17]: these solutions are generated by a vanishing diffusion-dispersion regularization and characterized via a kinetic relation.

First of all we will introduce a new concept of *admissible solutions*, in which each shock wave connecting a left-hand state u_- to a right-hand state u_+ and with speed λ must belong to a given set of *admissible jumps*, i.e.,

$$(u_-, u_+) \in \Phi \subset \mathbf{R}^N \times \mathbf{R}^N, \tag{1.5}$$

and propagates with an *admissible speed*,

$$\lambda = \psi(u_-, u_+),$$

where the function $\psi\colon \Phi \to \mathbf{R}$ is also prescribed. We refer to Section 2 for the rigorous definition. Our notion of admissible solutions (Definitions 2.3 and 2.4) includes in the same framework all of the previous notions of weak solutions.

Our purpose here is solely to identify the minimal set of assumptions on Φ and ψ under which the Cauchy problem for (1.1) has at most one solution depending continuously on the data. In other words we show that, to this end, it is sufficient to supplement the hyperbolic system (1.1) simply with the "dynamics" of elementary jumps.

Section 2 also contains the statement of the main results, Theorems 2.5 and 2.8, while the proof of the uniqueness result is postponed to Section 3. Theorem 2.5 establishes a sharp tangency estimate between two solutions leaving from the same initial data. Theorem 2.8 shows the uniqueness of an admissible solution of the Cauchy problem depending $\mathbf{L}^1$-continuously on the initial data. The proof given in Section 3 cannot rely on the conservation property nor on the standard Lax entropy inequalities, as was used in previous works. In Sections 4 and 5, we conclude with a few remarks.

2. ADMISSIBLE SOLUTIONS AND MAIN RESULTS

Consider the quasilinear system of partial differential equations (1.1), where $A(u)$ needs not to be the Jacobian matrix of a vector-valued mapping. All values u under consideration belong to a fixed ball $\mathcal{U} := \mathcal{B}(0, \delta) \subset \mathbf{R}^N$ with center 0 and radius $\delta > 0$. We assume that, for each u, the matrix $A(u)$ is hyperbolic, that is, it admits N real and distinct eigenvalues

$$\lambda_1(u) < \cdots < \lambda_N(u)$$

and basis of left- and right-eigenvectors $l_j(u)$, $r_j(u)$, $1 \leqslant j \leqslant N$, respectively. The normalization

$$l_i(u) \cdot r_j(u) = \delta_{ij}, \qquad u \in \mathcal{U} \tag{2.1}$$

will be used below. Denote by λ_∞ a (possibly large) upper bound for all wave speeds:

$$\sup_{1 \leqslant j \leqslant N,\, u \in \mathcal{U}} |\lambda_j(u)| < \lambda_\infty.$$

Definition 2.1 (See [9]). A function $u: \mathbf{R} \times \mathbf{R}_+ \to \mathcal{U}$ is said to be in the *Class* $\mathcal{K}$ if it satisfies the following conditions: u is a bounded function of bounded variation (**BV**) in both (x, t), the function $x \mapsto u(x, t)$ has bounded variation on $\mathbf{R}$ for each $t \geqslant 0$, and $u \in \mathrm{Lip}(\mathbf{R}_+, \mathbf{L}^1(\mathbf{R}))$. Moreover u satisfies the following *Tame Oscillation Property*: for every point (ξ, τ)

$$\limsup_{x \to \xi,\, t \to \tau+} |u(x, t) - u(\xi \pm, \tau)| \leqslant K\, |u(\xi +, \tau) - u(\xi -, \tau)| \tag{2.2a}$$

and

$$\lim_{\substack{x \to \xi\pm,\, t \to \tau+ \\ |x-\xi| > \lambda_\infty (t-\tau)}} u(x, t) = u(\xi\pm, \tau). \tag{2.2b}$$

Here $K > 0$ is a fixed constant and $u(\xi\pm, \tau)$ are the left- and right-traces of the **BV** function $x \mapsto u(x, \tau)$.

DEFINITION 2.2 (See [11]). A function $u: \mathbf{R} \times \mathbf{R}_+ \to \mathscr{U}$ belongs to the *Class* $\mathscr{K}'$ if it satisfies the same assumptions as in Class $\mathscr{K}$ above but the tame oscillation property is replaced with the following *Locally Bounded Variation Property*:

along every space-like Lipschitz curve $t = \gamma(x)$ such that

$$|\gamma'(x)| \leqslant 1/\lambda_\infty \text{ almost everywhere,} \tag{2.3}$$

the total variation of u is locally bounded.

We notice that all of these properties hold for solutions generated by the Glimm scheme or by the front tracking algorithm, for instance. Recall that a function u of bounded variation in the two variables (x, t) admits a decomposition of the form ([14, 29], for instance)

$$\mathbf{R} \times \mathbf{R}_+ = \mathscr{C}(u) \cup \mathscr{J}(u) \cup \mathscr{I}(u). \tag{2.4}$$

Here $\mathscr{C}(u)$ is the set of all (Lebesgue) points of approximate continuity of u (in the $\mathbf{L}^1$-norm), $\mathscr{J}(u)$ is the set of all points of approximate jump (in the $\mathbf{L}^1$-norm) and the set of "irregular points" $\mathscr{I}(u)$ has zero one-dimensional Hausdorff measure $\mathscr{H}_1$. Moreover for each point $(x, t) \in \mathscr{J}(u)$, one can define a left- and a right-approximate limit $u_\pm(x, t)$ and a discontinuity speed $\sigma_u(x, t)$. More precisely, for every $(x, t) \in \mathscr{J}(u)$ defining

$$U(y, s) := \begin{cases} u_-(x, t) & \text{if} \quad y < x + \sigma_u(x, t)(s-t), \\ u_+(x, t) & \text{if} \quad y > x + \sigma_u(x, t)(s-t), \end{cases}$$

one has

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \int_{t-\rho}^{t+\rho} \int_{x-\rho}^{x+\rho} |u(y, s) - U(y, s)| \, dy \, ds = 0.$$

Moreover the traces $u_\pm$ are $\mathscr{H}_1$-measurable functions. Changing the function u on a set of zero Lebesgue measure only, we can always assume that $u(x, t)$ coincides with its approximate limit at all points of approximate continuity. Finally the restriction of u to its set of approximate continuity points $\mathscr{C}(u)$ is measurable and bounded with respect to the Borel measure $\partial_x u$.

We shall also say that a point (x, t) is a *forward regular point for u* if either it is a (Lebesgue) point of approximate continuity for u in the set $\mathbf{R} \times [t, +\infty)$, or there exist $u_\pm(x, t)$ and a discontinuity speed $\sigma_u(x, t)$ such that

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \int_t^{t+\rho} \int_{x-\rho}^{x+\rho} |u(y, s) - U(y, s)| \, dy \, ds = 0.$$

It is not hard to check that, for both Definitions 2.1 and 2.2, one has

$$u_\pm(x, t) = u(x\pm, t) \qquad \text{for all } (x, t) \in \mathscr{J}(u).$$

In particular, in our case, it may well happen that some or all points $(x, 0)$ lying on the x-axis are forward regular points. This will be needed in the proof of Theorem 2.5.

To define the notion of admissible solution, we assume that a set of *admissible jumps*

$$\Phi \subset \mathscr{U} \times \mathscr{U}$$

is prescribed together with a family of *admissible speeds*

$$\psi: \Phi \to [-\lambda_\infty, \lambda_\infty].$$

It is assumed that the mapping ψ satisfies the following *consistency property*:

(C1) There exists $C > 0$ such that for each $(u_-, u_+) \in \Phi$ and each $j \in \{1, ..., N\}$, we have

$$|(\psi(u_-, u_+) - \lambda_j(u_-)) \, l_j(u_-) \cdot (u_+ - u_-)| \leqslant C \, |u_+ - u_-|^2. \qquad (2.5)$$

This condition ensures that, as the wave strength $|u_+ - u_-|$ tends to zero, the corresponding discontinuity connecting u_- to u_+ is "asymptotic" to the propagating discontinuity $u_+ = u_- + \alpha \, r_j(u_-)$, $\psi(u_-, u_+) = \lambda_j(u_-)$ (for some $\alpha \in \mathbf{R}$), which corresponds to a solution of the linear hyperbolic system

$$\partial_t u + A(u_-) \, \partial_x u = 0. \qquad (2.6)$$

DEFINITION 2.3 Let $\Phi \subset \mathcal{U} \times \mathcal{U}$ be a set of admissible jumps and $\psi: \Phi \to [-\lambda_\infty, \lambda_\infty]$ be a family of speeds satisfying (2.5). A function u in the class $\mathcal{K}$ (or in the class $\mathcal{K}'$) is called a (Φ, ψ)-*admissible solution* of (1.1) iff:

(1) The equality

$$\partial_t u + A(u)\, \partial_x u = 0 \qquad \text{on the set } \mathcal{C}(u) \tag{2.7}$$

holds between bounded Borel measures restricted to the set of points of approximate continuity of u; precisely for all Borelian $B \subset \mathcal{C}(u)$ we have $\int_B \partial_t u + \int_B A(u)\, \partial_x u = 0$.

(2) For each point of approximate jump $(x, t) \in \mathcal{J}(u)$, the limits $u_\pm := u(x\pm, t)$ and the speed $\sigma := \sigma_u(x, t)$ satisfy

$$(u_-, u_+) \in \Phi, \qquad \sigma = \psi(u_-, u_+). \tag{2.8}$$

DEFINITION 2.4 A function u in the class $\mathcal{K}$ (or in the class $\mathcal{K}'$) is called a (Φ, ψ)-*admissible solution of* (1.1)–(1.3) if it is a (Φ, ψ)-admissible solution and $u(0, x) = u_0(x)$ almost everywhere.

To illustrate Definition 2.3, we describe several examples of sets Φ and speeds ψ. Suppose first that $N = 1$ and consider the scalar equation

$$\partial_t u + a(u)\, \partial_x u = 0. \tag{2.9}$$

This equation can always be rewritten in conservative form, namely

$$\partial_t u + \partial_x f(u) = 0, \qquad f(u) = \int^u a(v)\, dv. \tag{2.10}$$

First of all, one may define the speed ψ in agreement with the Rankine–Hugoniot relation associated with the conservative form (2.10), that is,

$$\psi(u_-, u_+) := \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{2.11}$$

However this choice is somehow arbitrary if no conservative form of (2.9) were specified in the first place. One may as well define

$$\psi(u_-, u_+) := \frac{h(u_+) - h(u_-)}{g(u_+) - g(u_-)}, \tag{2.12}$$

where the functions g, h: $\mathbf{R} \to \mathbf{R}$ are given such that

$$h'(u) = g'(u)\, a(u), \qquad u \in \mathbf{R}. \tag{2.13}$$

Observe that both choices satisfy the consistency property (2.5). As a matter of fact, the speed (2.12) corresponds also to the standard Rankine–Hugoniot relation, but for *another* conservative form of (2.9), i.e.,

$$\partial_t g(u) + \partial_x h(u) = 0. \tag{2.14}$$

For the equivalence between the *smooth* solutions of (2.10) and (2.14), one should impose the restriction $g'(u) > 0$, for instance.

However, the speed needs *not* to correspond to any conservative form of (2.9). In particular, it needs *not* to be a symmetric function in (u_-, u_+). This may be relevant in certain physical applications, if *different* conservative forms of (2.9) are necessary in *different* ranges of values u. For example, suppose we are given two conservative forms of (2.9), like (2.14), associated with two pairs (g_1, h_1), (g_2, h_2) of conservative variables and flux-functions satisfying the condition (2.13). Then define the discontinuity speed by

$$\psi(u_-, u_+) := \begin{cases} \dfrac{h_1(u_+) - h_1(u_-)}{g_1(u_+) - g_1(u_-)} & \text{for} \quad u_- < u_+, \\[2ex] \dfrac{h_2(u_+) - h_2(u_-)}{g_2(u_+) - g_2(u_-)} & \text{for} \quad u_- > u_+. \end{cases} \tag{2.15}$$

The second ingredient in Definition 2.3, the set Φ, determines which discontinuities are admissible. An obvious choice is to use the standard entropy criterion [27]:

$$\Phi := \{(u_-, u_+) \text{ that satisfy the Oleinik entropy inequalities}\}. \tag{2.16}$$

Another choice is to include in Φ a subset of the jumps satisfying the Oleinik criterion while allowing some jumps that violate it. This allows us to recover the definition of *nonclassical entropy solutions* of Hayes and LeFloch [16]. Therein an algebraic condition —the *kinetic relation*—is imposed on the "undercompressive" shocks only. Nonclassical shocks appear naturally in vanishing diffusion-dispersion limits of (2.10). (See also Section 4.)

The above observations easily extend to systems of equations. When (1.1) can be written in a conservative form,

$$\partial_t u + \partial_x f(u) = 0, \qquad A(u) = Df(u), \tag{2.17}$$

one may define the speed ψ directly from the Rankine–Hugoniot relation associated with this conservative form:

$$-\psi(u_-, u_+)(u_+ - u_-) + f(u_+) - f(u_-) = 0. \tag{2.18}$$

However more general choices are possible, that may or may not be associated with such a conservative form, as was already pointed out when $N = 1$. One can also take Φ to be the set of shocks satisfying the classical Lax entropy criterion [20] when all of the characteristic fields are genuinely nonlinear, and more generally the Liu entropy criterion [24] when the system is not genuinely nonlinear.

Therefore, Definition 2.3 covers all the standard notions of entropy solutions in the class **BV**. It also includes the weak solutions to nonconservative systems in the sense of DalMaso, LeFloch and Murat [13] and the nonclassical shocks of systems of conservation laws in the sense introduced by Hayes and LeFloch [17].

The main results of this paper concerning the uniqueness for the Cauchy problem for (1.1) are stated now. Throughout, it is assumed that Φ and ψ are prescribed and satisfy the condition (2.5).

THEOREM 2.5 (Tangency Condition). *Let u and v be two (Φ, ψ)-admissible solutions of* (1.1) *in the class $\mathscr{K}$ (or $\mathscr{K}'$). Denote by $\tilde{\mathscr{I}}$ the projection on the t-axis of the set $\mathscr{I}(u) \cup \mathscr{I}(v)$ of all irregular points of u or v.*

Let $\tau \notin \tilde{\mathscr{I}}$ be such that there exists a constant $C(\tau)$ so that for all h small enough

$$\frac{1}{h} \int_{\tau}^{\tau+h} \mathbf{TV}(u(t))\, dt \leqslant C(\tau). \tag{2.19}$$

If $u(\tau) = v(\tau)$ then

$$\liminf_{h \to 0+} \frac{1}{h} \|u(\tau+h) - v(\tau+h)\|_{\mathbf{L}^1(\mathbf{R})} = 0. \tag{2.20}$$

For instance, if τ is a Lebesgue point of the function $t \mapsto \mathbf{TV}(u(t))$, then the assumption of the theorem holds with the constant $C(\tau) = 1 + \mathbf{TV}(u(\tau))$. In particular the assumption is satisfied for every τ whenever the uniform bound $\mathbf{TV}(u(t)) \leqslant C'$ for all t is assumed.

The estimate (2.20) is to be compared with the rough estimate (valid for every τ)

$$\begin{aligned}
&\|u(\tau+h) - v(\tau+h)\|_{\mathbf{L}^1(\mathbf{R})} \\
&\quad \leqslant \|u(\tau+h) - u(\tau)\|_{\mathbf{L}^1(\mathbf{R})} + \|u(\tau) - v(\tau)\|_{\mathbf{L}^1(\mathbf{R})} + \|v(\tau+h) - v(\tau)\|_{\mathbf{L}^1(\mathbf{R})} \\
&\quad \leqslant C\, h,
\end{aligned}$$

which is immediate from the assumptions $u(\tau)=v(\tau)$ and $u, v \in \mathrm{Lip}(\mathbf{R}_+, \mathbf{L}^1(\mathbf{R}))$. The interest of (2.20) will become clear shortly after we introduce a definition.

Remark 2.6. It is interesting to observe that only the values $t \geqslant \tau$ are relevant in the statement of Theorem 2.5. Indeed, consider two functions u, v which are defined and are admissible solutions on the set $\mathbf{R} \times [\tau, +\infty)$, such that (x, τ) is a forward regular point of both u and v for every $x \in \mathbf{R}$ and u satisfies (2.19). Assume that, if (x, τ) is a point of jump, then $(u_-(x, \tau), u_+(x, \tau)) \in \Phi$ and $\sigma(u_-(x, \tau), u_+(x, \tau)) = \psi(u_-(x, \tau), u_+(x, \tau))$. Then the conclusion of Theorem 2.5 still holds at time τ for u and v.

We emphasize that knowing the solution of the Riemann problem is *not* necessary in Theorem 2.5, as the assumption $\tau \notin \widetilde{\mathscr{I}}$ excludes the points of interaction of waves.

Definition 2.7. A *Lipschitz continuous semi-group of* (Φ, ψ)*-admissible solutions* $S^{\Phi,\psi}\colon \mathscr{D} \times [0, \infty[\to \mathscr{D}$ defined on a closed subset $\mathscr{D}$ of $\mathbf{L}^1(\mathbf{R})$ satisfies the following properties.

(1) For all $t, s \geqslant 0$ we have the semi-group property $S^{\Phi,\psi}(s) \circ S^{\Phi,\psi}(t) = S^{\Phi,\psi}(s+t)$.

(2) For some $c > 0$, every function u_0 in $\mathbf{L}^1(\mathbf{R})$ such that $\mathbf{TV}(u_0) \leqslant c$ lies in $\mathscr{D}$.

(3) For some $L > 0$ and for every $u_0, v_0 \in \mathscr{D}$ and every $t, s \geqslant 0$,

$$\|S^{\Phi,\psi}(t)\, u_0 - S^{\Phi,\psi}(s)\, v_0\|_{\mathbf{L}^1(\mathbf{R})} \leqslant L\, |t-s| + L\, \|u_0 - v_0\|_{\mathbf{L}^1(\mathbf{R})}. \tag{2.21}$$

(4) $S^{\Phi,\psi}(t)\, u_0$ (belongs to the class $\mathscr{K}$ or $\mathscr{K}'$ and) is a (Φ, ψ)-admissible solution of (1.1)–(1.3).

Theorem 2.8 (Uniqueness of solutions). *Consider a pair of admissible jumps and speeds* Φ, ψ *satisfying the property* **(C1)**. *When treating solutions in the class* $\mathscr{K}'$ *we also assume that*:

(C2) *For every* $u \in \mathscr{U}$ *there exist two sequences of points* $v_\nu, w_\nu \in \mathscr{U}$ *converging to* u *as* $\nu \to \infty$ *and such that* $(v_\nu, w_\nu) \in \Phi$.

Assume that a Lipschitz continuous semi-group $S^{\Phi,\psi}\colon \mathscr{D} \times [0, \infty[\to \mathscr{D}$ *of* (Φ, ψ)*-admissible solutions of* (1.1) *exists, and satisfies the following* Shock Compatibility *condition*:

(SC) *If* u_0 *is a function having a single jump* $(u_-, u_+) \in \Phi$, *then* $S^{\Phi,\psi}(t)\, u_0$ *is that single shock wave connecting* u_- *to* u_+ *and propagating* *with the admissible speed* $\psi(u_-, u_+)$. *Let* u *be a* (Φ, ψ)*-admissible solution of* (1.1)–(1.3) *in the class* $\mathcal{K}$ *or in the class* $\mathcal{K}'$. *Then for all* $t \geqslant 0$ *we have* $u(t) = S^{\Phi, \psi}(t)\, u_0$. *In particular,* (Φ, ψ)*-admissible solutions to* (1.1)–(1.3) *are unique.*

The proof of Theorem 2.8 is based on Theorem 2.5 and on the following estimate due to Bressan [5, 7]: for all $u \in \mathrm{Lip}(\mathbf{R}_+, \mathbf{L}^1(\mathbf{R}))$

$$\|u(T) - S^{\Phi, \psi}(T)\, u_0\|_{\mathbf{L}^1(\mathbf{R})} \leqslant L \int_0^T \liminf_{h \to 0} \frac{1}{h} \|u(\tau + h) - S^{\Phi, \psi}(h)\, u(\tau)\|_{\mathbf{L}^1(\mathbf{R})}\, d\tau. \tag{2.22}$$

Estimate (2.20) shows that the integrand of the right-hand side of (2.22) vanishes almost everywhere.

3. PROOFS OF THEOREMS 2.5 AND 2.8

We will need the following two lemmas: Lemma 3.1 provides a control of the space averages of a function by its averages in both space and time. Lemma 3.2 provides a control of the $\mathbf{L}^1$-norm of a function from the sole knowledge of its integrals on arbitrary intervals.

LEMMA 3.1 *Let* w *be in* $\mathrm{Lip}(\mathbf{R}_+, \mathbf{L}^1(\mathbf{R}))$ *with Lipschitz constant* $L > 0$. *Then for each* $h > 0$ *we have*

$$\frac{1}{h} \int_{-h}^{h} |w(h)|\, dx \leqslant \sqrt{2L} \left(\frac{1}{h^2} \int_0^h \int_{-h}^{h} |w|\, dx\, dt \right)^{1/2}, \tag{3.1}$$

whenever the right-hand side is less than L.

Proof. For each $h, h' > 0$ we have

$$\frac{1}{h} \int_{-h}^{h} |w(h)|\, dx \leqslant \frac{1}{h} \int_{-h}^{h} |w(h')|\, dx + \frac{L}{h} |h - h'|.$$

Averaging with respect to $h' \in (h - \varepsilon h, h)$, we obtain

$$\frac{1}{h} \int_{-h}^{h} |w(h)|\, dx \leqslant \frac{1}{\varepsilon h^2} \int_{h - \varepsilon h}^{h} \int_{-h}^{h} |w|\, dx\, dt + \frac{L}{h} \frac{\varepsilon h}{2} \leqslant \frac{1}{\varepsilon h^2} \int_0^h \int_{-h}^{h} |w|\, dx\, dt + \frac{L\varepsilon}{2}, \tag{3.2}$$

provided $h - \varepsilon h > 0$ that is $\varepsilon < 1$. Taking in the right-hand side of (3.2) the optimal value for ε, we choose

$$\varepsilon = \sqrt{\frac{2}{L}} \left(\frac{1}{h^2} \int_0^h \int_{-h}^h |w| \, dx \, dt \right)^{1/2}.$$

By assumption, the right-hand side of (3.1) is less than L and this ensures that $\varepsilon < 1$, hence the conclusion follows from (3.2). ∎

LEMMA 3.2 *For each function w in $\mathbf{L}^1((a, b), \mathbf{R}^N)$ we have*

$$\int_a^b |w(x)| \, dx = \sup_{a < x_1 < x_2 < \cdots < b} \sum_{k = 1, 2, ...} \left| \int_{x_k}^{x_{k+1}} w(x) \, dx \right|. \tag{3.3}$$

Proof. The result is obvious if w is piecewise constant. The general case follows by approximation of w (in the $\mathbf{L}^1$-norm) by a piecewise constant function. ∎

Proof of Theorem 2.5. Let u and v be two (Φ, ψ)-admissible solutions of (1.1), satisfying

$$u(\tau) = v(\tau)$$

for some $\tau \geq 0$ and belonging to the class $\mathcal{K}$ (we address later the changes necessary to deal with the class $\mathcal{K}'$). Assume that $\tau \notin \tilde{\mathcal{I}}$, the projection of $\mathcal{I}(u) \cup \mathcal{I}(v)$ on the t-axis, and satisfies (2.19). It is sufficient to prove the statement for a bounded interval $[-R, R]$, since the functions u and v are integrable at infinity.

Fix $\varepsilon > 0$. Let $\xi'_1, \xi'_2, \cdots, \xi'_p \in [-R, R]$ be the (finite) set of all large jumps in $u(\tau)$ such that

$$|u(\xi'_k +, \tau) - u(\xi'_k -, \tau)| \geq \varepsilon, \qquad k = 1, 2, ..., p. \tag{3.4}$$

Since $(\xi'_k, \tau) \notin \mathcal{I}(u)$ by assumption, we have $(\xi'_k, \tau) \in \mathcal{J}(u)$ (see (2.4) for the definition). Then the pair $(u_k^-, u_k^+) := (u(\xi'_k -, \tau), u(\xi'_k +, \tau))$ belongs to Φ and therefore the speed $\psi(u_k^-, u_k^+)$ is well-defined. In that situation we define for all $t \geq \tau$ and all x

$$u_k^\sharp(x, t) = \begin{cases} u_k^- & \text{if} \quad x - \xi'_k < \psi(u_k^-, u_k^+)(t - \tau), \\ u_k^+ & \text{if} \quad \psi(u_k^-, u_k^+)(t - \tau) < x - \xi'_k. \end{cases}$$

Nearby the point $(\xi'_k, \tau) \in \mathscr{J}(u)$, the function $u^\sharp_k$ is a good approximation of the solutions u, v. Namely by definition of a point of approximate jump and for h sufficiently small, we have

$$\frac{1}{h^2} \int_\tau^{\tau+h} \int_{\xi'_k - \lambda_\infty h}^{\xi'_k + \lambda_\infty h} |u(x, t) - u^\sharp_k(x, t)| \, dx \, dt \leqslant \frac{\varepsilon^2}{2L\lambda_\infty \, p^2}, \qquad k = 1, ..., p,$$

where L is a Lipschitz constant of $u - u^\sharp_k$, for every k. Relying on Lemma 3.1 with $w(x, t) := (u - u^\sharp_k)(\xi'_k + \lambda_\infty x, \tau + t)$, we deduce that

$$\frac{1}{h} \int_{\xi'_k - \lambda_\infty h}^{\xi'_k + \lambda_\infty h} |u(x, \tau + h) - u^\sharp_k(x, \tau + h)| \, dx \leqslant \frac{\varepsilon}{p}, \qquad k = 1, ..., p, \tag{3.5}$$

for all h sufficiently small. (Indeed one can always take $\varepsilon/p < L$ so that the assumption in Lemma 3.1 holds.) Since $v(\tau) = u(\tau)$, then $v^\sharp_k = u^\sharp_k$ and the function v satisfies the same estimate (3.5). This implies that

$$\begin{aligned}
&\frac{1}{h} \sum_{k=1}^{p} \int_{\xi'_k - \lambda_\infty h}^{\xi'_k + \lambda_\infty h} |u(x, \tau + h) - v(x, \tau + h)| \, dx \\
&\quad \leqslant \frac{1}{h} \sum_{k=1}^{p} \int_{\xi'_k - \lambda_\infty h}^{\xi'_k + \lambda_\infty h} |u(x, \tau + h) - u^\sharp_k(x, \tau + h)| \, dx \\
&\qquad + \frac{1}{h} \sum_{k=1}^{p} \int_{\xi'_k - \lambda_\infty h}^{\xi'_k + \lambda_\infty h} |u^\sharp_k(x, \tau + h) - v(x, \tau + h)| \, dx \\
&\quad \leqslant 2\,\varepsilon
\end{aligned} \tag{3.6}$$

for all h sufficiently small.

Next we also define an approximation adapted to points of approximate continuity in $\mathscr{C}(u)$ and to points where the jump in $u(\tau)$ is less than ε. Choose $\rho > 0$ such that

$$\rho < \tfrac{1}{2} \min_{k \neq m} |\xi'_k - \xi'_m|$$

and

$$\begin{aligned}
\mathbf{TV}(u(\tau); (a, b)) \leqslant \varepsilon \quad &\text{for every interval such that } b - a < 2\,\rho, \\
&(a, b) \cap \{\xi'_1, ..., \xi'_p\} = \varnothing.
\end{aligned} \tag{3.7}$$

Select finitely many points ξ_l $(l = 1, ..., q)$ to obtain a covering of

$$[-R, R] \backslash \{\xi'_1, ..., \xi'_p\}$$

by intervals of the form $(\xi_l - \rho, \xi_l + \rho)$, $l = 1, ..., q$. We also assume that each point of the x-axis belongs to at most two such intervals. Set

$$I_l(h) = (\xi_l - \rho + \lambda_\infty h,\ \xi_l + \rho - \lambda_\infty h).$$

It can be checked that (3.7) together with the tame oscillation assumption (2.2) imply that the oscillation of u and v is small, i.e.,

$$|u(x, \tau + h) - u(\xi_l +, \tau)| \leqslant (K+2)\,\varepsilon, \qquad x \in I_l(h), \tag{3.8}$$

for h sufficiently small, the same being true for v. (See [9] for this argument.)

We are ready to derive the second main estimate, in the regions where the solutions have small oscillations. For each $l = 1, ..., q$, denote by $u_l^\flat$ the solution of the linear hyperbolic problem

$$\begin{aligned} \partial_t u_l^\flat + A(u(\xi_l, \tau))\, \partial_x u_l^\flat &= 0, \\ u_l^\flat(\tau) &= u(\tau). \end{aligned} \tag{3.9}$$

To simplify the notation fix l and set $A^\flat := A(u(\xi_l, \tau))$, $\lambda_j^\flat := \lambda_j(u(\xi_l, \tau))$, $l_j^\flat := l_j(u(\xi_l, \tau))$. Define Γ_l to be the set of points (x, t) such that

$$x \in (\xi_l - \rho + \lambda_\infty(t-\tau),\ \xi_l + \rho - \lambda_\infty(t - \tau)), \qquad \tau \leqslant t \leqslant \tau + h.$$

For each $j = 1, ..., N$, multiplying the equation in (3.9) by $l_j^\flat$, we find

$$\partial_t(l_j^\flat \cdot u_l^\flat) + \lambda_j^\flat\, \partial_x(l_j^\flat \cdot u_l^\flat) = 0. \tag{3.10}$$

We shall derive the equation satisfied by u, for instance. Define the matrix-valued function $\tilde{A}$ $\mathscr{H}_1$-almost everywhere by

$$\tilde{A}(x, t) = \begin{cases} A(u(x, t)) & \text{if } \ (x, t) \in \mathscr{C}(u), \\ A(u(x-, t)) & \text{if } \ (x, t) \in \mathscr{J}(u). \end{cases}$$

Since $\partial_x u(\mathscr{I}(u)) = \mathscr{H}_1(\mathscr{I}(u)) = 0$, the product $\tilde{A}\, \partial_x u$ now makes sense as a pointwise product between a Borel function and a Radon measure, and the function u solves the equation

$$\partial_t u + \tilde{A}\, \partial_x u = \mu, \tag{3.11}$$

where μ is a bounded measure concentrated on the set $\mathscr{J}(u)$. Recall now that the jump part Jw of the Radon measure Dw, where w: $\mathbf{R}^2 \mapsto \mathbf{R}$ is a **BV** function, satisfies [14, 29]

$$Jw(B) = \iint_{B \cap \mathscr{J}(w)} (w_+ - w_-)\, v_w\, d\mathscr{H}_1(x, t)$$

for every Borel set B, where $\nu_w = (\nu_w^t, \nu_w^x)$ is the interior normal to the set $B \cap \mathcal{J}(w)$ at the point (x, t). Let $A = (A_{ik})$ and $\mu = (\mu_1, ..., \mu_N)$ where each μ_i is a Radon measure. Since u is bounded, for every Borel set B and every $i = 1, ..., N$, we have

$$\mu_i(B) = \iint_{B \cap \mathcal{J}(u)} \Big((u_i(x+, t) - u_i(x-, t))\, \nu_u^t + \sum_k A_{ik}(u(x-, t))\, (u_k(x+, t) - u_k(x-, t))\, \nu_u^x \Big)\, d\mathcal{H}_1(x, t)$$

and, in vectorial form,

$$\begin{aligned} \mu(B) &= \int_{B \cap \mathcal{J}(u)} \Big((u(x+, t) - u(x-, t)), \\ &\quad A(u(x-, t))\, (u(x+, t) - u(x-, t)) \Big) \cdot \nu_u\, d\mathcal{H}_1(x, t) \\ &= \iint_{B \cap \mathcal{J}(u)} \Big(A(u(x-, t))\, (u(x+, t) - u(x-, t)) \\ &\quad - \sigma_u(x, t)(u(x+, t) - u(x-, t)) \Big)\, d\mathcal{H}_1(x, t). \end{aligned}$$

Since u is a (Φ, ψ) admissible solution, the shock speed σ_u is determined by the function ψ and so

$$\mu(B) = \iint_{B \cap \mathcal{J}(u)} (A(u(x-, t)) - \psi(u(x-, t), u(x+, t))\, Id)\, (u(x+, t) - u(x-, t))\, d\mathcal{H}_1(x, t), \tag{3.12}$$

where Id is the $N \times N$ identity matrix.

We rewrite (3.11) in the form

$$\partial_t u + A^\flat\, \partial_x u = (A^\flat - \tilde{A})\, \partial_x u + \mu.$$

Multiply it by $l_j^\flat$ for $j = 1, ..., N$ we obtain

$$\partial_t(l_j^\flat \cdot u) + \lambda_j^\flat\, \partial_x(l_j^\flat \cdot u) = l_j^\flat \cdot (A^\flat - \tilde{A})\, \partial_x u + l_j^\flat \cdot \mu. \tag{3.13}$$

Comparing (3.10) and (3.13), we finally obtain an equation for the function $u - u_l^\flat$:

$$\partial_t(l_j^\flat \cdot (u - u_l^\flat)) + \lambda_j^\flat\, \partial_x(l_j^\flat \cdot (u - u_l^\flat)) = l_j^\flat \cdot (A^\flat - \tilde{A})\, \partial_x u + l_j^\flat \cdot \mu. \tag{3.14}$$

Observe that the coefficients in the right-hand side of (3.14) satisfy

$$|(A^\flat - \tilde{A})(x, t)| \leqslant C\, |u(\xi_l, \tau) - u(x-, t)|. \tag{3.15}$$

Now, if $B \subset \Gamma_l$ is a Borel set, using (3.8), (3.11)–(3.15) and condition (C1), we get

$$\begin{aligned}
&|(\ell(l_j^\flat \cdot \mu)(B)| \\
&\quad = O(1) \iint_{B \cap \mathscr{J}(u)} |l_j^- \cdot (A(u(x-, t)) - \psi(u(x-, t), u(x+, t))\, Id) \\
&\qquad (u(x+, t) - u(x-, t))|\, d\mathscr{H}_1(x, t) \\
&\qquad + O(1) \iint_{B \cap \mathscr{J}(u)} |l_j^- - l_j^\flat|\, |u(x+, t) - u(x-, t)|\, d\mathscr{H}_1(x, t) \\
&\quad = O(1)\, \varepsilon \int_{B \cap \mathscr{J}(u)} |u(x+, t) - u(x-, t)|\, d\mathscr{H}_1(x, t) \\
&\quad = O(1)\, \varepsilon \int_\tau^{\tau+h} \mathbf{TV}(u(t); (B \cap \mathscr{J}(u))_t)\, dt,
\end{aligned} \tag{3.16}$$

where $l_j^- := l_j(u(x-, t))$, E_t denotes the t-section of a set $E \subset \mathbf{R}^2$, and $O(1)$ is a constant depending only on the system under consideration and not on a particular solution.

For each $j = 1, ..., N$ and each ξ', ξ'' in $I_l(h)$, define the region $\Gamma^j_{\xi', \xi''}$

$$\xi' + (t - \tau - h)\, \lambda_j^\flat \leqslant x \leqslant \xi'' + (t - \tau - h)\, \lambda_j^\flat \qquad \tau \leqslant t \leqslant \tau + h,$$

which is contained in Γ_l. By approximating the characteristic function of $\Gamma^j_{\xi', \xi''}$ by a sequence of C_0^∞ functions and then by passing to the limit, we obtain that

$$\begin{aligned}
&\int_\tau^{\tau+h} \int_{\xi' + (t-\tau-h)\lambda_j^\flat}^{\xi'' + (t-\tau-h)\lambda_j^\flat} \partial_t(l_j^\flat \cdot (u - u_l^\flat)) + \lambda_j^\flat\, \partial_x(l_j^\flat \cdot (u - u_l^\flat)) \\
&\qquad = \int_{\xi'}^{\xi''} l_j^\flat \cdot (u - u_l^\flat)(x, \tau + h)\, dx,
\end{aligned}$$

hence integrating (3.14) over $\Gamma^j_{\xi', \xi''}$, with some abuse of notation we get

$$\begin{aligned}
&\int_{\xi'}^{\xi''} l_j^\flat \cdot (u - u_l^\flat)(x, \tau + h)\, dx \\
&\qquad = \int_\tau^{\tau+h} \int_{\xi' + (t-\tau-h)\lambda_j^\flat}^{\xi'' + (t-\tau-h)\lambda_j^\flat} \left(l_j^\flat \cdot (A^\flat - \tilde{A})\, \partial_x u + l_j^\flat \cdot \mu \right).
\end{aligned}$$

Using (3.15)–(3.16) we arrive at the estimate

$$\left| \int_{\xi'}^{\xi''} l_j^\flat \cdot (u - u_l^\flat)(x, \tau + h)\, dx \right|$$
$$\leqslant O(1)\, \varepsilon \int_\tau^{\tau+h} \mathbf{TV}(u(t); (\xi' + (t - \tau - h)\, \lambda_j^\flat, \xi'' + (t - \tau - h)\, \lambda_j^\flat))\, dt. \tag{3.17}$$

In view of Lemma 3.2 and summing (3.17) over finitely many intervals we find

$$\int_{I_l(h)} |l_j^\flat \cdot (u - u_l^\flat)(x, \tau + h)|\, dx \leqslant O(1)\, \varepsilon \int_\tau^{\tau+h} \mathbf{TV}(u(t); I_l(t - \tau))\, dt.$$

This estimate also holds for the solutions v and therefore

$$\int_{I_l(h)} |l_j^\flat \cdot (u - v)(x, \tau + h)|\, dx \leqslant O(1)\, \varepsilon \int_\tau^{\tau+h} \mathbf{TV}(u(t); I_l(t - \tau))\, dt,$$

for each $j = 1, ..., N$. Since

$$|u - v| \leqslant O(1) \sum_{j=1}^{N} |l_j^\flat \cdot (u - v)|$$

we conclude that

$$\int_{I_l(h)} |(u - v)(x, \tau + h)|\, dx \leqslant O(1)\, \varepsilon \int_\tau^{\tau+h} \mathbf{TV}(u(t); I_l(t - \tau))\, dt. \tag{3.18}$$

By the assumption (2.19), we know that

$$\int_\tau^{\tau+h} \mathbf{TV}(u(t))\, dt \leqslant h\, C(\tau) \tag{3.19}$$

for all small h.

We now combine the estimates (3.18) for $l = 1, ..., q$ and (3.19) and obtain

$$\sum_{l=1}^{q} \int_{\xi_l - \rho + \lambda_\infty h}^{\xi_l + \rho - \lambda_\infty h} |(u - v)(x, \tau + h)|\, dx$$
$$\leqslant O(1)\, \varepsilon \int_\tau^{\tau+h} \mathbf{TV}(u(t); [-R + \lambda_\infty (t - \tau), R - \lambda_\infty (t - \tau)])\, dt$$
$$\leqslant O(1)\, \varepsilon h\, C(\tau). \tag{3.20}$$

Combining (3.6) and (3.20) we have established that for each $\varepsilon > 0$

$$\int_{-R}^{R} |(u - v)(x, \tau + h)| \, dx \leqslant O(1) \, \varepsilon \, h \, (1 + C(\tau)),$$

provided h be sufficiently small, hence (2.20) holds.

For the case when u, v belong to the class $\mathscr{K}'$, note that Lemmas 3 and 4 in [11] hold. It follows that there are only finitely many points ξ'_k, $k = 1, ..., r$ in $[-R, R]$ such that the oscillation of u in a forward neighborhood of (ξ'_k, τ) is greater than ε. At these points we will use estimates (3.5)–(3.6). At the remaining points ξ, (3.16) can be replaced by the following

$$|(l_j^\flat \cdot \mu)(B)| = O(1) \sup_{(x, s) \in B} |u(x, s) - u(\xi, \tau)| \int_{\tau}^{\tau + h} \mathbf{TV}(u(t); (B \cap \mathscr{J}(u))_t) \, dt, \tag{3.21}$$

where now B is a Borel set contained in Γ_l with $\xi_l = \xi$. Moreover, as in [11], for $h > 0$ sufficiently small it follows that

$$\sup_{(x, s) \in \Gamma_l} |u(x, s) - u(\xi, \tau)| \leqslant 2\varepsilon + \mathbf{TV}(u(\tau); (\xi - \rho, \xi + \rho)) \leqslant 3\varepsilon.$$

Hence (3.20) still holds and we conclude as before. This proves the desired result. ∎

Proof of Theorem 2.8. Let u be a (Φ, ψ) admissible solution of the Cauchy problem (1.1)–(1.3). We want to show that $u(t)$ coincides with $w(t) := S^{\Phi, \psi}(t) \, u_0$. Consider any $\tau \geqslant 0$ with $\tau \notin \mathscr{I}(u)$ such that it is a Lebesgue point of the function $t \mapsto \mathbf{TV}(u(t))$, and define

$$v(t) := S^{\Phi, \psi}(t - \tau) \, u(\tau), \qquad \text{if} \quad t \geqslant \tau.$$

We claim that necessarily (ξ, τ) is a forward regular point of v for every $\xi \in \mathbf{R}$.

First, assume that the (Φ, ψ) admissible solutions belong to the class $\mathscr{K}$. Consider any point of continuity ξ of the function $x \mapsto u(x, \tau)$. Then the Tame Oscillation property (2.2) gives

$$\limsup_{x \to \xi, \, t \to \tau+} |v(x, t) - u(\xi \pm, \tau)| = 0,$$

which clearly implies that (ξ, τ) is a point of approximate continuity for the function v.

Next, consider any point of jump ξ of the function $x \mapsto u(x, \tau)$. By the assumption of regularity on u and since u is an admissible solution, the limits $u_{\pm} := u(\xi \pm, \tau)$ yield a pair in Φ. Call $u^{\#}$ the single propagating jump having the speed $\psi(u_-, u_+)$. By the assumption **(SC)** of the theorem, we have the property

$$S^{\Phi, \psi}(h)\, u^{\#}(\tau) = u^{\#}(\tau + h),$$

for all h. Using the Lipschitz continuity of the semigroup, we see that for $\rho = \lambda_\infty h$

$$\begin{aligned}
&\frac{1}{h\rho} \int_\tau^{\tau+h} \int_{\xi - \rho + \lambda_\infty(t-\tau)}^{\xi + \rho - \lambda_\infty(t-\tau)} |v(x, t) - u^{\#}(x, t)|\, dx\, dt \\
&\quad = \frac{1}{h\rho} \int_\tau^{\tau+h} \int_{\xi - \rho + \lambda_\infty(t-\tau)}^{\xi + \rho - \lambda_\infty(t-\tau)} |S^{\Phi, \psi}(t-\tau)\, u(\tau) - S^{\Phi, \psi}(t-\tau)\, u^{\#}(\tau)|\, dx\, dt \\
&\quad \leqslant \frac{L}{\rho} \int_{\xi-\rho}^{\xi+\rho} |u(\tau) - u^{\#}(\tau)|\, dx \to 0,
\end{aligned} \tag{3.22}$$

since ξ is a point of jump of the function $x \mapsto u(x, \tau)$. Thus (ξ, τ) is a point of approximate jump for the function v.

Now, assume that the (Φ, ψ) admissible solutions belong to the class $\mathscr{K}'$ and satisfy the condition (C2). By the $\mathbf{L}^1$-continuity of the semigroup and (C2), it follows that if $w(x) \equiv \bar{w}$ with $\bar{w} \in \mathscr{U}$ then $S^{\Phi, \psi}(h)\, w = \bar{w}$. Hence, if (ξ, τ) is a point of continuity for the function u then as in (3.22) we obtain

$$\begin{aligned}
&\frac{1}{h\rho} \int_\tau^{\tau+h} \int_{\xi - \rho + \lambda_\infty(t-\tau)}^{\xi + \rho - \lambda_\infty(t-\tau)} |v(x, t) - u(\xi, \tau)|\, dx\, dt \\
&\quad \leqslant \frac{L}{\rho} \int_{\xi-\rho}^{\xi+\rho} |u(x, \tau) - u(\xi, \tau)|\, dx \to 0
\end{aligned}$$

since ξ is a point of continuity for the function $x \mapsto u(x, \tau)$. The case when (ξ, τ) is a point of jump for the function u can be treated as above. This completes the proof that (ξ, τ) is a forward regular point of v for every $\xi \in \mathbf{R}$.

Recalling Remark 2.6, we can apply Theorem 2.5 to u and v at time τ and we get

$$\liminf_{h \to 0} \frac{1}{h} \|u(\tau + h) - S^{\Phi, \psi}(h)\, u(\tau)\|_{\mathbf{L}^1(\mathbf{R})} = 0.$$

This proves that the integrand on the right-hand side of (2.22) vanishes at a.e. τ. Thus we have $u(T) := S^{\Phi,\psi}(T)\, u_0$ for every $T > 0$. ∎

Remark 3.3. In [13] the authors introduced a notion of solution for the nonconservative systems (1.1). This relies on the choice of a suitable family of Lipschitz continuous paths $s \mapsto \phi(s; u^-, u^+)$ connecting u^- to u^+. We want to show that this definition coincides with our for some choices of Φ and ψ. Indeed, it is sufficient to choose Φ to be equal to set of couples (u, v) for which $v \in S_i(u)$ for some i, where $S_i(u)$ is the i-th shock-curve, and $\psi(u^-, u^+)$ is the corresponding shock speed (see [13]). Notice that if u is a solution in the sense of [13], then it is also a (Φ, ψ)-admissible solution. So, if the ball $\mathcal{U}$ is sufficiently small and there exists a (Φ, ψ)-admissible semigroup, then by uniqueness the two definitions must coincide.

It is worth noticing that not all the (Φ, ψ)-admissible solutions can be recovered in this way. Indeed, if $A = Df$ for some f, then the shock waves in the definition [13] must satisfy the classical Rankine-Hugoniot condition. If the speed $\psi(u_-, u_+)$ does not satisfy this last condition (as for example in (2.12)), then the shock (u_-, u_+) is not admissible in the sense of [13].

Remark 3.4. The proofs in the paper can be generalized to the case that the matrix $A = A(u, x)$ in (1.1) depends (smoothly) on the state variable x. The set of admissible set Φ and the admissible speed ψ may depend (smoothly) on x as well. Note that in this case there are no self-similar solutions of the Riemann problems. All the estimates for approximate jump points do not change. But the condition (SC) of Theorem 2.8 should be replaced by a *local integral estimate* involving the single shock.

4. NONCLASSICAL SHOCKS AND THE RIEMANN PROBLEM

In the definition and in the proof of uniqueness of (Φ, ψ)-admissible solutions, we need not define directly the solutions of the Riemann problems. However, it is a consequence of our assumptions and of the definition of an admissible semigroup that:

LEMMA 4.1. *If a (Φ, ψ)-admissible semigroup exists, so does the solution of every Riemann problem (u_l, u_r), for all pairs of states $u_l, u_r \in \mathcal{U}$. This solution is unique and depends continuously on u_l, u_r in the $\mathbf{L}^1_{loc}$-norm.*

We emphasize that, given an arbitrary pair (Φ, ψ), we do *not* expect in general a semigroup of (Φ, ψ)-admissible solutions to exist. In particular, this is certainly the case when the Riemann problems admit several solutions.

For instance, assume (1.1) admits a conservative form like (1.2). If the system is neither genuinely nonlinear nor linearly degenerate, it was observed in [16, 17] that, given a left-hand state u_l, the i-th shock curve of points u_r such that (u_l, u_r) satisfies a single entropy inequality, form a two-parameter set $S_i(u_l)$. It follows from this result, that if we define $\Phi := \{(u_l, u_r); u_l \in \mathcal{U}, u_r \in S_i(u_l) \text{ for some } i\}$ and ψ to be the standard speed given by the Rankine–Hugoniot relations, then the solution of the Riemann problems is not unique and a semigroup of (Φ, ψ)-admissible solutions (satisfying (SC)) cannot exist.

Similarly, in the case of a cubic non-convex conservation law [16], we can define Φ to be the set of pairs (u_l, u_r) which can be connected by a shock-wave obtained as limit of traveling waves of the regularized equations

$$\partial_t u + \partial_x u^3 = \varepsilon\, \partial_{xx} u + \gamma \varepsilon^2\, \partial_{xxx} u, \qquad \varepsilon \to 0 \text{ with } \gamma \text{ fixed},$$

and take ψ given by the Rankine–Hugoniot relations. For $u_l > \frac{2}{3}\sqrt{\frac{2}{\gamma}}$, it happens that $(u_l, u_r) \in \Phi$ iff $u_r \in [-u_l/2, u_l) \cup \{\varphi(u_l)\}$, where $\varphi(u_l) = -u_l + \frac{1}{3}\sqrt{\frac{2}{\gamma}} \leqslant -u_l/2$. Hence the Riemann problem (u_l, u) with $u \in (\varphi(u_l), -u_l/2)$ admits two (Φ, ψ)-admissible solutions, namely, one given by a classical shock to $-u_l/2$ followed by a rarefaction to u, the other by a (nonclassical) shock to $\varphi(u_l)$ followed by a (classical) shock connecting to u. Hence, also in this case a semigroup of (Φ, ψ)-admissible solutions satisfying (SC) can not exist.

The importance of the Riemann solvers is well-recognized for the (classical) genuinely nonlinear and strictly hyperbolic systems of conservation laws. Indeed, the construction of the solutions of (1.2)–(1.3), by the wave front tracking algorithm or the Glimm scheme, relies strongly on a prescribed Riemann solver. In addition, for classical shocks, it was possible to construct a Lipschitz continuous semigroup of solutions, by a careful analysis of the interactions of waves [8, 12, 18].

In several situations of interest (for instance, classical shocks of conservative systems), we can thus select a set Φ such that there exists an (SC) semigroup of (Φ, ψ)-admissible solutions. Constructing Φ is based on both

- necessary conditions: the basic requirement is the existence and continuous dependence in $\mathbf{L}^1_{loc}$ of the solution of Riemann problems. (But this is generally not sufficient for the existence of a Lipschitz continuous semigroup.)

- and physical considerations: for instance, the Riemann solver may be derived by adding vanishing diffusion-dispersion terms taking into higher-order modeling effects, neglected at the level of the hyperbolic model. In that situation, the *kinetic relation* and the *nucleation criterion* for nonclassical solutions impose restrictions on the set of admissible waves to obtain a unique Riemann solver.

The results in the present paper apply to nonclassical shocks and phase boundaries, in the sense considered in [1, 3, 16, 17, 22, 26] and the references therein. To establish that a solution—constructed by wave front tracking, say—is admissible, one need to derive uniform local convergence in order to carry over to the limiting solution the properties imposed on the approximate solutions. (The latter are precisely constructed by combining piecewise constant, admissible solutions.)

In the example of the scalar nonconvex equation, we can consider two choices of the set: either Φ contains only the jumps (u_l, u_r) for which $u_r \in [-u_l/2, u_l)$, or else contains those jump for which $u_r \in [-\frac{1}{3}\sqrt{\frac{2}{\gamma}}, u_l) \cup \{\varphi(u_l)\}$. We recover in the first case the classical Liu–Oleinik solutions [24, 27], in the second one the nonclassical solutions in [1, 3, 16].

5. CONCLUDING REMARKS

The existence of a continuous semi-group of entropy admissible solutions is known for several choices of sets Φ and functions ψ (see [2, 8]). The existence under more general assumptions will be addressed in a follow-up paper [4]. For the existence of a semi-group it is necessary that Φ be "large enough", specifically a minimal assumption is:

Any Riemann problem admits at least one admissible solution

made of a combination of both rarefaction waves and shock waves in Φ propagating with an admissible speed ψ. On the other hand, Theorem 2.8 requires that the set Φ be "small enough" and precisely provides the minimal set of assumptions.

It would be interesting to extend Theorem 2.5 by bypassing the assumption of the existence of the continuous semi-group of solutions. Our uniqueness theorem also applies to hyperbolic-elliptic systems, provided the solutions do not enter the elliptic region. Such models arise in material science to describe the dynamics of phase transitions in solids, see [22, 26].

ACKNOWLEDGMENTS

This work was partially supported by the European TMR Project HCL ERBFMRXCT960033. P.G.L. was also supported in parts by the Centre National de la Recherche Scientifique and a Faculty Early Career Development award (CAREER) from the National Science Foundation.

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