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Attractors for nonlinear reaction–diffusion systems in unbounded domains via the method of short trajectories

Maurizio Grasselli [a,*], Dalibor Pražák [b], Giulio Schimperna [c]

[a] Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, I-20133 Milano, Italy
[b] Katedra matematické analýzy, Matematicko-fyzikální fakulta Univerzity Karlovy, Sokolovská 83, CZ-186 75, Czech Republic
[c] Dipartimento di Matematica, Università di Pavia, Via Ferrata, 1, I-27100 Pavia, Italy

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ABSTRACT

We consider a nonlinear reaction–diffusion equation on the whole space $\mathbb{R}^d$. We prove the well-posedness of the corresponding Cauchy problem in a general functional setting, namely, when the initial datum is uniformly locally bounded in L^2 only. Then we adapt the short trajectory method to establish the existence of the global attractor and, if $d \leqslant 3$, we find an upper bound of its Kolmogorov's ε-entropy.

1. Introduction

The asymptotic behavior of solutions to reaction–diffusion equations has been the object of a large number of investigations. In particular, the existence of global and exponential attractors and their fractal dimension have been carefully studied in the case of bounded domains (see, e.g., [17] and references therein). However, unbounded domains are also rather interesting for applications. In this case, the dynamics can exhibit a more complex behavior characterized, for instance, by travelling waves connecting constant equilibria or by a continuum of space periodic equilibria (and much more, as shown in [27]). The lack of standard compact injections requires new ideas and the choice of the

* Corresponding author.
E-mail addresses: maurizio.grasselli@polimi.it (M. Grasselli), prazak@karlin.mff.cuni.cz (D. Pražák), giusch04@unipv.it (G. Schimperna).

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topology becomes crucial in order not to exclude interesting invariant solutions. In the pioneering papers on the existence of global attractors [1,4] weighted norms were introduced and used. Since then, many contributions have followed (see, e.g., [2,3,7,8,11,16,18–20,23] and references therein) making use of weighted or not phase spaces and under various assumptions on the reaction (and, possibly, convective) terms. However, as noticed in [4], the global attractor can be noncompact (but just locally compact) and infinite-dimensional. Actually, this is the more realistic case where the richness of the dynamics is preserved. On the other hand, it is still possible to give a quantitative estimate of the thickness of the global attractor by means of the so-called Kolmogorov's ε-entropy. Estimates of this quantity were proven in [26] under quite general assumptions (see also [9] for a generalization which accounts for convection and [27] for a careful analysis of related spatially chaotic phenomena). Nevertheless, in the existing literature the phase spaces consist of functions which are (locally) rather smooth (i.e., strong solutions, at least) so that very general reaction terms are allowed. Besides, the diffusion effects are usually expressed by a linear operator (see, however, [14,24,25] and references therein).

On the contrary, here we want to consider (suitably defined) weak solutions so that the resulting dynamical system acts on a very large phase space (for instance, globally bounded initial data are admissible). Moreover, we want to handle nonlinear diffusion operators. On account of these goals, establishing the existence of the global attractor through standard methods is not trivial because higher-order regularity of solutions is hard to prove. This is the reason why an appropriate adaptation of the short trajectory approach (see [15]) seems particularly effective in the present case. More precisely, we use a somewhat simplified weighted space setting along the lines of [26] to analyze a reaction–diffusion equation of the form

$$\partial_t u - \operatorname{div} a(\nabla u) + f(u) + h(\nabla u) = g, \quad \text{in } \mathbb{R}^d \times (0, +\infty), \tag{1.1}$$

where a, f, h are suitable nonlinear functions and f has a polynomially controlled growth. Then, we introduce and solve an appropriate weighted weak formulation of the Cauchy problem for (1.1) with g and the initial datum uniformly locally bounded in L^2. By adapting the short trajectory method, we easily prove the existence of the global attractor in an L^2_{loc}-topology. In addition, using once more that approach, we estimate its ε-entropy from above by exploiting a minimal regularity property of the solutions which holds for $d \leqslant 3$.

To summarize, as we already mentioned, the main novelties with respect to the existing literature (and, in particular, to [26]) are the following:

(i) we can work in the usual "parabolic" functional setting, as a consequence we only need a handful of (relatively) simple estimates;
(ii) we consider weak solutions so that the phase space is rather large (namely, the initial data must be uniformly L^2_{loc} only);
(iii) we account for nonlinear diffusion effects;
(iv) we estimate the attractor's ε-entropy in a (relatively) easy way;
(v) regularity requirements on a, f, h, g are mild since the smoothness properties of solutions are weak.

Of course, our assumptions on the nonlinearities cannot be considered as optimal, but this is due to the generality of our aims. However, note that typical reaction terms like $f(u) = u^3 - \gamma u$, $\gamma > 0$, are included. Possible extensions to systems are also pointed out. Further generalizations may incorporate delay effects (see [13] and references therein, cf. also [12] for bounded domains) and/or the challenging issue of the (local) smoothness of the global attractor.

The plan of this paper goes as follows. The next Section 2 is devoted to introduce the functional setup which is, of course, a bit more complicated than the one with bounded domains. The notion of Kolmogorov's ε-entropy adapted to our framework is also introduced. Well-posedness and regularity issues are analyzed in Section 3. The existence of the global attractor is proven in Section 4 and an upper bound for its ε-entropy is established in Section 5.

2. Functional spaces

There are three classes of function spaces to be used in this paper. The standard Lebesgue and Sobolev spaces together with their weighted variants are briefly recalled in Section 2.1. These spaces are mainly used for formulating the existence theorem.

Throughout the paper $c_1, c_2, \dots$ denote universal constants whose meaning can change with the context, but which are independent of the data of the equation and also of the weight functions. We also occasionally simplify the notation by writing $A \approx B$ meaning that $c_1 A \leqslant B \leqslant c_2 A$.

In Section 2.2, we introduce the so-called *uniformly bounded* spaces and provide some equivalent descriptions of their norms. These spaces are aimed at describing the dynamics associated with the equation, and formulating the main results. As we mentioned in the Introduction, here we mostly follow [26,27], though in a slightly simplified setting (thanks to the fact that $\Omega = \mathbb{R}^d$ has no boundary). We remark that we perform the analysis by taking $\Omega = \mathbb{R}^d$ for simplicity; however, obvious technical adjustments would allow us to treat any suitably regular unbounded domain Ω.

Finally, in Section 2.3, we describe a class of spaces that can be thought of as a parabolic version of the uniformly bounded spaces. These spaces are the main technical novelty of the paper and are also the crucial tool for the application of the "method of trajectories" to the problem of the dimension of the attractor.

2.1. Weighted Sobolev spaces

For a domain $\mathcal{O} \subset \mathbb{R}^d$, we use $L^p(\mathcal{O})$, $W^{1,p}(\mathcal{O})$, $W^{1,p}_0(\mathcal{O})$ and $W^{-1,p'} = [W^{1,p}_0(\mathcal{O})]'$ to denote the standard Sobolev spaces. Observe that $W^{1,p}(\Omega) = W^{1,p}_0(\Omega)$ as $\Omega = \mathbb{R}^d$ throughout the paper. By $L^p_{\mathrm{loc}}(\Omega)$, $W^{1,p}_{\mathrm{loc}}(\Omega)$ we denote their locally integrable variants.

A prominent role will be played by the weight functions $e^{-|\cdot-\bar{x}|}$, $\bar{x} \in \mathbb{R}^d$. These give rise to weighted spaces $L^p_{\bar{x}}(\Omega)$, $W^{1,p}_{\bar{x}}(\Omega)$, $W^{-1,p'}_{\bar{x}}(\Omega) = [W^{1,p}_{\bar{x}}(\Omega)]'$, given via the respective norms

$$\|u\|^p_{L^p_{\bar{x}}(\Omega)} = \int_\Omega |u(x)|^p e^{-|x-\bar{x}|}\,\mathrm{d}x,$$

$$\|u\|^p_{W^{1,p}_{\bar{x}}(\Omega)} = \int_\Omega \left(|\nabla u(x)|^p + |u(x)|^p\right) e^{-|x-\bar{x}|}\,\mathrm{d}x,$$

$$\|u\|_{W^{-1,p'}_{\bar{x}}(\Omega)} = \sup_v \int_\Omega u(x)v(x)e^{-|x-\bar{x}|}\,\mathrm{d}x,$$

the last supremum being taken over $v \in W^{1,p}_{\bar{x}}(\Omega)$ with unit norm. These spaces share the usual good properties of Sobolev spaces (separability, reflexivity), note also that

$$L^p_{\bar{x}}(\Omega) \subset L^q_{\bar{x}}(\Omega), \quad p \geqslant q, \tag{2.1}$$

since Ω has finite measure with respect to the weight $e^{-|\cdot-\bar{x}|}$. It is also easy to see that the spaces $L^p_{\bar{x}}(\Omega)$, $L^p_{\bar{y}}(\Omega)$ in fact coincide and the equivalence constants only depend on $|\bar{x}-\bar{y}|$.

Finally, we remark that $L^2_{\bar{x}}(\Omega)$, $W^{1,2}_{\bar{x}}(\Omega)$ are Hilbert spaces using the obvious scalar product; however, it is worth noting that $\|\nabla\cdot\|_{L^2_{\bar{x}}(\Omega)}$ is not an equivalent norm on $W^{1,2}_{\bar{x}}(\Omega)$ (just consider a constant function). The notation $\langle\cdot,\cdot\rangle_{\bar{x}}$ will stand for the duality pairing between $W^{-1,2}_{\bar{x}}(\Omega)$ and $W^{1,2}_{\bar{x}}(\Omega)$.

2.2. Uniformly bounded spaces

First, we introduce the space $L^2_b(\Omega)$ of the *uniformly locally L^2-functions* as

$$L^2_b(\Omega) := \Big\{ u \in L^2_{\mathrm{loc}}(\Omega)\colon \sup_{x_0 \in \Omega} \|u\|_{L^2(C(x_0))} < +\infty \Big\}. \tag{2.2}$$

Here and below in the paper, $C(x_0)$ denotes the closed unit cube in $\mathbb{R}^d$ centered at x_0, namely $C(x_0) = \prod_{i=1,\dots,d} [x_{0,i} - 1/2, x_{0,i} + 1/2]$, where $x_{0,i}$ are the components of x_0. Clearly, $L^2_b(\Omega)$, endowed with the graph norm, is a Banach space. An equivalent norm is given by

$$\|u\|_{L^2_b(\Omega)} := \sup_k \|u\|_{L^2(C_k)}. \tag{2.3}$$

Here and in what follows, C_k are an enumeration of the unit cubes centered at $x_k \in (\mathbb{Z}/2)^d$. An advantage of this norm is that the supremum is taken over a *countable* family of cubes. Note also that, for later convenience, we allow a partial superposition of the cubes.

We will also need the weighted analogue of $L^2_b(\Omega)$. Given $\mu \geqslant 0$, an *admissible weight* of rate of growth μ is a (measurable and bounded) function $\phi : \mathbb{R}^N \to (0, +\infty)$ satisfying, for some $c \geqslant 1$,

$$c^{-1} e^{-\mu|x-y|} \leqslant \phi(x)/\phi(y) \leqslant c e^{\mu|x-y|}, \quad \forall x, y \in \mathbb{R}^d, \tag{2.4}$$

as well as the estimate

$$\big|\nabla\phi(x)\big| \leqslant \big|\phi(x)\big|. \tag{2.5}$$

A typical example is given by the exponential $\phi(x) = e^{m|x-\bar{x}|}$, where $\bar{x} \in \mathbb{R}^d$ and $m \in [-1, 0]$, which of course has rate of growth $\mu = |m|$. In fact, we can observe that $|m| \leqslant 1$ would be enough in order to have (2.4)–(2.5). However, since we also need global boundedness of ϕ in the sequel, we will only consider negative exponential weights.

It is easy to prove (see [26, Prop. 1.3]) that if ϕ_1 and ϕ_2 are admissible weights of growth rates μ_1 and μ_2, then $\max\{\phi_1, \phi_2\}$ and $\min\{\phi_1, \phi_2\}$ are still admissible weights both having growth rate $\max\{\mu_1, \mu_2\}$.

We now have the analogue of (2.2), i.e., the space of the functions which are uniformly locally L^2 with respect to the weight ϕ. This is defined as

$$L^2_{b,\phi}(\Omega) := \Big\{ u \in L^2_{\mathrm{loc}}(\Omega)\colon \sup_{x_0 \in \Omega} \phi^{1/2}(x_0) \|u\|_{L^2(C(x_0))} < +\infty \Big\}. \tag{2.6}$$

As before, we will take on $L^2_{b,\phi}(\Omega)$ the equivalent norm

$$\|u\|^2_{L^2_{b,\phi}(\Omega)} := \sup_k \phi(x_k) \|u\|^2_{L^2(C_k)}. \tag{2.7}$$

It is easy to check that $L^2_{b,\phi}(\Omega)$ is then a Banach space. Note that the constant function 1 is an admissible weight with growth rate 0 and $L^2_{b,1}(\Omega) = L^2_b(\Omega)$. Moreover, if the weight ϕ is of the form $\phi(x) = e^{-\mu|x-\bar{x}|}$ for some $\bar{x} \in \mathbb{R}^d$ and for $\mu \in [0, 1]$, then it is $L^2_b(\Omega) \subset L^2_{b,\phi}(\Omega)$ with continuous inclusion.

Given now an admissible weight ϕ with rate of growth *strictly* smaller than 1, we also define

$$\tilde{L}^2_{b,\phi}(\Omega) := \left\{ u \in L^2_{\mathrm{loc}}(\Omega)\colon\ u \in L^2_{\bar{x}}(\Omega)\ \forall \bar{x} \in \Omega,\ \sup_{\bar{x}\in\Omega} \phi(\bar{x}) \int_\Omega \left|u(x)\right|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x < +\infty \right\}. \tag{2.8}$$

It is not difficult to prove that $\tilde{L}^2_{b,\phi}(\Omega)$, endowed with the graph norm, is also a Banach space (note that it is not a Hilbert space). More precisely, we can prove

Theorem 2.1. *The spaces $L^2_{b,\phi}(\Omega)$ and $\tilde{L}^2_{b,\phi}(\Omega)$ coincide and, in particular, their norms are equivalent.*

Proof. Recall that $\{C_k\}_{k\in\mathbb{N}}$ are an enumeration of the unit cubes of Ω centered in the points of $x_k \in (\mathbb{Z}/2)^d$. It is then clear that, for fixed $\bar{x} \in \Omega$, we have

$$\phi(\bar{x}) \int_\Omega \left|u(x)\right|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x \leqslant \sum_{k\in\mathbb{N}} \int_{C_k} \phi(\bar{x}) \left|u(x)\right|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x. \tag{2.9}$$

Let us also notice that, for $x \in C_k$, there hold

$$e^{-|x-\bar{x}|} \leqslant c_1 e^{-|x_k-\bar{x}|} \quad \text{and} \quad \phi(\bar{x}) \leqslant c_2 \phi(x_k) e^{\mu|x_k-\bar{x}|}, \tag{2.10}$$

for suitable $c_1, c_2 > 0$ independent of $k, \bar{x}$. Hence,

$$\begin{aligned}\sum_{k\in\mathbb{N}} \int_{C_k} \phi(\bar{x}) \left|u(x)\right|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x &\leqslant c_3 \sum_{k\in\mathbb{N}} e^{-(1-\mu)|x_k-\bar{x}|} \phi(x_k) \int_{C_k} \left|u(x)\right|^2 \mathrm{d}x \\ &\leqslant c_3 \Big(\sup_{k\in\mathbb{N}} \phi(x_k) \|u\|^2_{L^2(C_k)} \Big) \sum_{k\in\mathbb{N}} e^{-(1-\mu)|x_k-\bar{x}|}. \end{aligned}\tag{2.11}$$

Assuming $\mu < 1$, the sum is bounded independently of $\bar{x}$. Passing to the supremum, this entails that

$$\|u\|_{\tilde{L}^2_{b,\phi}(\Omega)} \leqslant c\|u\|_{L^2_{b,\phi}(\Omega)}. \tag{2.12}$$

To prove the opposite inequality, note that, for $x \in C(\bar{x})$,

$$1 \leqslant c_4 e^{-|x-\bar{x}|}. \tag{2.13}$$

Hence,

$$\begin{aligned}\phi(\bar{x}) \|u\|^2_{L^2(C(\bar{x}))} &= \phi(\bar{x}) \int_{C(\bar{x})} \left|u(x)\right|^2 \mathrm{d}x \\ &\leqslant c_4 \phi(\bar{x}) \int_{C(\bar{x})} \left|u(x)\right|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x \\ &\leqslant c_4 \phi(\bar{x}) \int_\Omega \left|u(x)\right|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x. \end{aligned}\tag{2.14}$$

The proof is complete. □

For $p \in [1, \infty)$, we can define analogously as above the spaces $L^p_{b,\phi}(\Omega)$ and $\tilde{L}^p_{b,\phi}(\Omega)$, where

$$\|u\|^p_{L^p_{b,\phi}(\Omega)} := \sup_k \phi(x_k) \|u\|^p_{L^p(C_k)},$$
$$\|u\|^p_{\tilde{L}^p_{b,\phi}(\Omega)} := \sup_{\bar{x}} \phi(\bar{x}) \int_\Omega |u(x)|^p e^{-|x-\bar{x}|}\,\mathrm{d}x.$$

As above, one proves that $L^p_{b,\phi}(\Omega) = \tilde{L}^p_{b,\phi}(\Omega)$ provided that the growth rate of ϕ is smaller than 1; the equivalence relation can be succinctly written as

$$\|u\|^p_{L^p_{b,\phi}(\Omega)} \approx \sup_{\bar{x}} \phi(\bar{x}) \|u\|^p_{L^p_{\bar{x}}(\Omega)}. \tag{2.15}$$

Note also that the equivalence constants only depend on μ, c in (2.4) and not on the particular expression of the weight function; this fact will be used repeatedly in various a priori estimates.

As a next step, we extend the above construction to Sobolev spaces. First of all, given an admissible weight ϕ, $W^{1,2}_b(\Omega)$ and $W^{1,2}_{b,\phi}(\Omega)$ are defined as the spaces of $L^2_{\mathrm{loc}}(\Omega)$-functions which belong to $L^2_b(\Omega)$ and, respectively, $L^2_{b,\phi}(\Omega)$ together with their first (partial, distributional) derivatives. These are, of course, Banach spaces with the natural norms modeled on (2.3) and (2.7). We can also define $\tilde{W}^{1,2}_{b,\phi}(\Omega)$ as the space of $L^2_{\mathrm{loc}}(\Omega)$-functions such that

$$\sup_{\bar{x}\in\Omega} \phi(\bar{x}) \int_\Omega \left(|u(x)|^2 + |\nabla u(x)|^2\right) e^{-|x-\bar{x}|}\,\mathrm{d}x < +\infty. \tag{2.16}$$

In particular, we write $\tilde{W}^{1,2}_b(\Omega)$ in case $\phi \equiv 1$. The analogue of Theorem 2.1, whose proof is omitted for brevity since it does not present further difficulties, then reads

Theorem 2.2. *Given an admissible weight ϕ of growth rate $\mu \in [0, 1)$, the spaces $\tilde{W}^{1,2}_{b,\phi}(\Omega)$ and $W^{1,2}_{b,\phi}(\Omega)$ coincide and their norms are equivalent.*

Next, we come to *negative order* spaces. Firstly, we define

$$W^{-1,2}_{b,\phi}(\Omega) := \left\{\zeta \in \mathcal{D}'(\Omega):\ \zeta|_{C(\bar{x})} \in W^{-1,2}\big(C(\bar{x})\big)\ \forall \bar{x} \in \Omega,\ \sup_{\bar{x}\in\Omega} \phi(\bar{x}) \|\zeta\|^2_{W^{-1,2}(C(\bar{x}))} < +\infty\right\}. \tag{2.17}$$

Of course, the above, endowed with the graph norm, is a Banach space (and the supremum could be restricted to $\bar{x} \in (\mathbb{Z}/2)^d$, see the proof of the next theorem). As before, we can also define the counterpart $\tilde{W}^{-1,2}_{b,\phi}(\Omega)$. Let us take first $u \in \tilde{L}^2_{b,\phi}(\Omega)$ and set

$$\|u\|_{\tilde{W}^{-1,2}_{b,\phi}(\Omega)} := \sup_v \sup_{\bar{x}\in\Omega} \phi(\bar{x}) \int_\Omega u(x) v(x) e^{-|x-\bar{x}|}\,\mathrm{d}x, \tag{2.18}$$

where the first supremum is taken with respect to

$$\left\{v \in W^{1,2}_{b,\phi}(\Omega):\ \|v\|_{\tilde{W}^{1,2}_{b,\phi}(\Omega)} \leqslant 1\right\}. \tag{2.19}$$

The space $\tilde{W}^{-1,2}_{b,\phi}(\Omega)$ is then defined as the completion of $L^2_{b,\phi}(\Omega)$ with respect to the norm (2.18).

Theorem 2.3. *Given an admissible weight ϕ of growth rate $\mu \in [0,1)$, the spaces $\tilde{W}^{-1,2}_{b,\phi}(\Omega)$ and $W^{-1,2}_{b,\phi}(\Omega)$ coincide and their norms are equivalent.*

Proof. Let C_k and x_k be as in the proof of Theorem 2.1. Take $u \in L^2_{b,\phi}(\Omega)$. Then, it is clear that

$$\phi^{1/2}(x_k)\|u\|_{W^{-1,2}(C_k)} = \sup_v \phi^{1/2}(x_k)\int_{C_k} u(x)v(x)\,dx, \tag{2.20}$$

where the supremum is referred to the v's in $W^{1,2}_0(C_k)$ with $\|v\|_{W^{1,2}_0(C_k)} \leqslant 1$. Let us take any such v and extend it by zero outside C_k. Then,

$$\begin{aligned}\phi^{1/2}(x_k)\int_{C_k} u(x)v(x)\,dx &= \phi^{1/2}(x_k)\int_{\Omega} u(x)v(x)\,dx \\ &= \phi(x_k)\int_{\Omega} u(x)\underbrace{\phi^{-1/2}(x_k)v(x)e^{|x-x_k|}}_{\tilde{v}(x)}e^{-|x-x_k|}\,dx. \end{aligned}\tag{2.21}$$

One easily verifies that $\tilde{v}$ (which is in fact only supported in C_k) belongs to $\tilde{W}^{1,2}_{b,\phi}(\Omega)$ and has the norm smaller than some constant c_1. Taking the suprema with respect to v and k, we eventually get that

$$\|u\|_{W^{-1,2}_{b,\phi}(\Omega)} \leqslant c\|u\|_{\tilde{W}^{-1,2}_{b,\phi}(\Omega)}. \tag{2.22}$$

The proof of the opposite inequality is a little bit harder. Let $u \in L^2_{b,\phi}(\Omega)$, $v \in W^{1,2}_{b,\phi}(\Omega)$ and $\bar{x} \in \Omega$. Let also $\{\psi_k\}_{k\in\mathbb{N}}$ be a smooth partition of unity associated to the cubes C_k. Then,

$$\begin{aligned}\phi(\bar{x})\int_{\Omega} u(x)v(x)e^{-|x-\bar{x}|}\,dx &= \sum_{k\in\mathbb{N}}\phi(\bar{x})\int_{\Omega} u(x)(v\psi_k)(x)e^{-|x-\bar{x}|}\,dx \\ &\leqslant \sum_{k\in\mathbb{N}}\phi^{1/2}(x_k)\int_{C_k} u(x)(v\psi_k)(x)\frac{\phi(\bar{x})}{\phi^{1/2}(x_k)}e^{-|x-\bar{x}|}\,dx \\ &\leqslant \sum_{k\in\mathbb{N}}\phi^{1/2}(x_k)\|u\|_{W^{-1,2}(C_k)}\|V_k\|_{W^{1,2}_0(C_k)} \\ &\leqslant \|u\|_{W^{-1,2}_{b,\phi}(\Omega)}\sum_{k\in\mathbb{N}}\|V_k\|_{W^{1,2}_0(C_k)}, \end{aligned}\tag{2.23}$$

where we have set

$$V_k(x) := v(x)\psi_k(x)\phi(\bar{x})\phi^{-1/2}(x_k)e^{-|x-\bar{x}|}. \tag{2.24}$$

Then, a direct computation (notice that the functions ψ_k can be chosen uniformly bounded together with their first derivatives) shows that

$$\|V_k\|_{W^{1,2}_0(C_k)} \leqslant c\phi(\bar{x})\phi^{-1/2}(x_k)e^{-|x_k-\bar{x}|}\|v\|_{W^{1,2}(C_k)}, \tag{2.25}$$

where c is independent of $\bar{x}$, k. Thus, coming back to (2.23) and using Theorem 2.2 and (2.10) once more, we arrive at

$$\begin{aligned}\phi(\bar{x})\int_\Omega u(x)v(x)e^{-|x-\bar{x}|}\,dx &\leqslant c\|u\|_{W^{-1,2}_{b,\phi}(\Omega)}\sum_{k\in\mathbb{N}}\bigl(\phi(\bar{x})\phi^{-1}(x_k)e^{-|x_k-\bar{x}|}\phi^{1/2}(x_k)\|v\|_{W^{1,2}(C_k)}\bigr)\\ &\leqslant c\|u\|_{W^{-1,2}_{b,\phi}(\Omega)}\|v\|_{W^{1,2}_{b,\phi}(\Omega)}\sum_{k\in\mathbb{N}}e^{-(1-\mu)|x_k-\bar{x}|}\\ &\leqslant c\|u\|_{W^{-1,2}_{b,\phi}(\Omega)}\|v\|_{\tilde{W}^{1,2}_{b,\phi}(\Omega)}.\end{aligned}\tag{2.26}$$

Thus, dividing by $\|v\|_{\tilde{W}^{1,2}_{b,\phi}(\Omega)}$ and taking the supremum first with respect to $\bar{x}$ and then with respect to v, we obtain the opposite inequality of (2.22).

To conclude the proof, we observe that, a priori, the equivalence of the norms of $W^{-1,2}_{b,\phi}(\Omega)$ and $\tilde{W}^{-1,2}_{b,\phi}(\Omega)$ has been proved just for the functions of $L^2_{b,\phi}(\Omega)$. However, it can be easily extended to the whole spaces by means of a standard density argument. □

It is worth while observing that the above defined spaces share some usual properties of Lebesgue spaces, as for example

$$\|uv\|_{L^r_{b,\phi}(\Omega)} \leqslant \|u\|_{L^p_{b,\phi}(\Omega)}\|v\|_{L^q_{b,\phi}(\Omega)}, \quad \frac{1}{r}=\frac{1}{p}+\frac{1}{q}.$$

On the other hand, the Sobolev embedding $W^{1,2}_{b,\phi}(\Omega)\subset L^p_{b,\phi}(\Omega)$, $p=2d/(d-2)$ does not hold (unless $\phi\equiv 1$) due to the incompatibility of the powers of $\phi(x_k)$. Also, the $L^2_{b,\phi}(\Omega)$ norm of ∇u is not an equivalent norm in $W^{1,2}_{b,\phi}(\Omega)$.

Finally, we will need seminorms that correspond to restrictions to some (bounded) subdomain $\mathcal{O}\subset\Omega$. For arbitrary $\mathcal{O}\subset\Omega$, we set

$$\mathbb{I}(\mathcal{O}) := \{k\in\mathbb{N};\ C_k\cap\mathcal{O}\neq\emptyset\},$$
$$\|u\|^2_{L^2_{b,\phi}(\mathcal{O})} := \sup_{k\in\mathbb{I}(\mathcal{O})}\phi(x_k)\|u\|^2_{L^2(C_k)}.$$

2.3. Parabolic uniformly bounded spaces

As an auxiliary tool, we will work with a sort of "parabolic version" of uniformly local spaces – a main technical novelty of the present paper. This setup seems rather natural for the study of dynamics of parabolic-like evolutionary problems in unbounded domains.

Given an admissible weight function ϕ, we define spaces $L^2_{b,\phi}(0,\ell;L^2(\Omega))$, $L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))$ and $L^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))$, where for any function $u(x,t):\Omega\times(0,\ell)\to\mathbb{R}$, we set

$$\|u\|_{L^2_{b,\phi}(0,\ell;L^2(\Omega))} = \sup_{k\in\mathbb{N}}\phi^{1/2}(x_k)\|u\|_{L^2(0,\ell;L^2(C_k))},$$
$$\|u\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))} = \sup_{k\in\mathbb{N}}\phi^{1/2}(x_k)\|u\|_{L^2(0,\ell;W^{1,2}(C_k))},$$
$$\|u\|_{L^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))} = \sup_{k\in\mathbb{N}}\phi^{1/2}(x_k)\|u\|_{L^2(0,\ell;W^{-1,2}(C_k))}.$$

We also introduce the space $L^p_{b,\phi}(0,\ell;L^p(\Omega))$ as

$$\|u\|_{L^p_{b,\phi}(0,\ell;L^p(\Omega))}=\sup_{k\in\mathbb{N}}\phi^{1/p}(x_k)\|u\|_{L^p(0,\ell;L^p(C_k))}.$$

As customary, we omit the symbol ϕ if $\phi\equiv 1$. We will also need localized seminorm of the space $L^2_{b,\phi}(0,\ell;L^2(\Omega))$ to some domain $\mathcal{O}\subset\Omega$, namely

$$\|u\|_{L^2_{b,\phi}(0,\ell;L^2(\mathcal{O}))}=\sup_{k\in\mathbb{I}(\mathcal{O})}\phi^{1/2}(x_k)\|u\|_{L^2(0,\ell;L^2(C_k))}.$$

In analogy with Theorems 2.1–2.3, one then proves:

Theorem 2.4. *Let ϕ be an admissible weight function of growth rate $\mu\in[0,1)$. Then the function spaces $L^2_{b,\phi}(0,\ell;L^2(\Omega))$, $L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))$, $L^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))$ and $L^p_{b,\phi}(0,\ell;L^p(\Omega))$ coincide with the spaces $\tilde{L}^2_{b,\phi}(0,\ell;L^2(\Omega))$, $\tilde{L}^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))$, $\tilde{L}^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))$, and $\tilde{L}^p_{b,\phi}(0,\ell;L^p(\Omega))$, whose (equivalent) norms are given by*

$$\|u\|^2_{\tilde{L}^2_{b,\phi}(0,\ell;L^2(\Omega))}=\sup_{\bar{x}\in\Omega}\phi(\bar{x})\int_{\Omega\times(0,\ell)}\left|u(x,t)\right|^2e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t, \tag{2.27}$$

$$\|u\|^2_{\tilde{L}^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))}=\sup_{\bar{x}\in\Omega}\phi(\bar{x})\int_{\Omega\times(0,\ell)}\left(\left|u(x,t)\right|^2+\left|\nabla u(x,t)\right|^2\right)e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t, \tag{2.28}$$

$$\|u\|_{\tilde{L}^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))}=\sup_{v}\sup_{\bar{x}\in\Omega}\phi(\bar{x})\int_{\Omega\times(0,\ell)}u(x,t)v(x,t)e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t, \tag{2.29}$$

$$\|u\|^p_{\tilde{L}^p_{b,\phi}(0,\ell;L^p(\Omega))}=\sup_{\bar{x}\in\Omega}\phi(\bar{x})\int_{\Omega\times(0,\ell)}\left|u(x,t)\right|^pe^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t, \tag{2.30}$$

respectively. The supremum in (2.29) is taken over all v such that the norm in $\tilde{L}^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))$ is less than or equal to 1.

Proof. Omitted as being completely analogous to the three preceding theorems. The only difference is an extra integration over $t\in(0,\ell)$. Note that the already proven equivalences can be simply written as

$$\begin{aligned}\|u\|^p_{L^p_{b,\phi}(0,\ell;L^p(\Omega))}&\approx\sup_{\bar{x}}\phi(\bar{x})\|u\|^p_{L^p(0,\ell;L^p_{\bar{x}}(\Omega))},\\ \|u\|^2_{L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))}&\approx\sup_{\bar{x}}\phi(\bar{x})\|u\|^2_{L^2(0,\ell;W^{1,2}_{\bar{x}}(\Omega))},\quad\text{etc.}\qquad\square\end{aligned} \tag{2.31}$$

Remark 2.5. Note that in the above definitions, one *first* integrates over $t\in(0,\ell)$ and *then* takes the weighted supremum. It is thus clear that, e.g.,

$$L^2\big(0,\ell;L^2_{b,\phi}(\Omega)\big)\subset L^2_{b,\phi}\big(0,\ell;L^2(\Omega)\big)\subset L^2_{\mathrm{loc}}(Q),$$

where $Q=[0,\ell]\times\Omega$ and both inclusions are indeed strict.

2.4. Some auxiliary results

Given a precompact set K in a metric space M, we define Kolmogorov's ε-entropy as

$$H_\varepsilon(K,M) := \ln N_\varepsilon(K,M),$$

where $N_\varepsilon(K,M)$ is the smallest number of ε-balls that cover K. Also, the symbol $B_r(u;M)$ denotes a ball centered in u, of radius $r>0$, measured in the metric of M.

The following explicit version of the Aubin–Lions Lemma will be instrumental in the proof of the main theorem.

Lemma 2.6. *Set $Q=[0,\ell]\times\Omega$ and*

$$\|\chi\|_{W_{b,\phi}(Q)} := \|\chi\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))} + \|\partial_t\chi\|_{L^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))}. \tag{2.32}$$

Let $\mathcal{O}\subset\Omega$ be a "reasonable" domain in the sense that

$$\#\mathbb{I}(\mathcal{O}) \leqslant c_1 \operatorname{vol}(\mathcal{O}). \tag{2.33}$$

Let $r>0$, $\theta\in(0,1)$ be given. Then

$$H_{\theta r}\big(B_r\big(\chi;W_{b,\phi}(Q)\big), L^2_{b,\phi}\big(0,\ell;L^2(\mathcal{O})\big)\big) \leqslant c_0 \operatorname{vol}(\mathcal{O}),$$

where the constant c_0 only depends on c_1, ℓ and θ, but is independent of χ, r, $\mathcal{O}$ and the weight function ϕ as long as (2.4) *and* (2.5) *are satisfied.*

Proof. Observe that balls of radii $R\geqslant 1$ are "reasonable" class of domains and we will not work with any other $\mathcal{O}$.

STEP 1. Assume $\phi\equiv 1$. Then $W_{b,\phi}(Q)$ estimates from above each seminorm

$$\|\chi\|_{L^2(0,\ell;W^{1,2}(C_k))} + \|\partial_t\chi\|_{L^2(0,\ell;W^{-1,2}(C_k))}, \quad k\in\mathbb{I}(\mathcal{O}).$$

By the usual version of Aubin–Lions Lemma (see, e.g., [21]), we then have

$$H_{\theta r}\big(B_r\big(\chi;W_{b,\phi}(Q)\big), L^2\big(0,\ell;L^2(C_k)\big)\big) \leqslant c_1,$$

where c_1 is independent of k. The desired covering arises as a product of those and, in view of (2.33), the final estimate follows.

STEP 2. The case with general ϕ is reduced to the previous step using the operator

$$F:\chi\mapsto\phi^{1/2}\chi.$$

The proof will be finished once we show that

$$\|\chi\|_{N_{b,\phi}} \approx \|F\chi\|_{N_{b,1}}$$

and the equivalence constants can be taken independently of choosing $N_{b,\phi}$ as any of the spaces $L^2_{b,\phi}(0,\ell;L^2(\mathcal{O}))$, $L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))$ or $L^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))$.

(i) The case $N_{b,\phi} = L^2_{b,\phi}(0,\ell; L^2(\mathcal{O}))$ clearly follows from the fact that

$$\left|F\chi(x,t)\right|^2 = \phi(x)\left|\chi(x,t)\right|^2 \approx \phi(x_k)\left|\chi(x,t)\right|^2,$$

if $x \in C_k$.

(ii) Regarding the space $L^2_{b,\phi}(0,\ell; W^{1,2}(\Omega))$, one obviously has (cf. (2.5))

$$|\nabla F\chi|^2 \leqslant c_1\phi\big(|\nabla\chi|^2 + |\chi|^2\big) \leqslant c_2\phi(x_k)\big(|\nabla\chi|^2 + |\chi|^2\big), \tag{2.34}$$

for $x \in C_k$. The opposite inequality is more delicate. It is now crucial that (2.5) holds with 1, hence

$$|\nabla F\chi| \geqslant \left|\phi^{1/2}\nabla\chi\right| - \frac{1}{2}\phi^{-1/2}|\nabla\phi||\chi| \geqslant \phi^{1/2}\left(|\nabla\chi| - \frac{1}{2}|\chi|\right).$$

It then follows that

$$|\nabla F\chi|^2 + |F\chi|^2 \approx \big(|\nabla F\chi| + |F\chi|\big)^2 \geqslant c_3\phi(x_k)\big(|\nabla\chi|^2 + |\chi|^2\big)$$

and the equivalence is concluded as in (i).

(iii) We first have to remark that in $L^2_{b,\phi}(0,\ell; W^{-1,2}(\Omega))$ the operator F is defined by duality, i.e.,

$$\langle F\chi, v\rangle := \langle\chi, Fv\rangle.$$

But then

$$\begin{aligned}
\|F\chi\|_{L^2_{b,1}(0,\ell;W^{-1,2}(\Omega))} &= \sup_k \|F\chi\|_{L^2(0,\ell;W^{-1,2}(C_k))} \\
&= \sup_k \sup_v \int_{C_k\times(0,\ell)} \chi\phi^{1/2}v\,\mathrm{d}x\,\mathrm{d}t \\
&\approx \sup_k \sup_v \phi^{1/2}(x_k) \int_{\Omega\times(0,\ell)} \chi v\,\mathrm{d}x\,\mathrm{d}t \\
&= \|\chi\|_{L^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))}.
\end{aligned}$$

Here the supremum is taken over all $v \in L^2(0,\ell; W^{1,2}_0(C_k))$ with unit norm; in the second step we have used the equivalence

$$\left\|\phi^{1/2}v\right\|_{W^{1,2}(C_k)} \approx \phi(x_k)^{1/2}\|v\|_{W^{1,2}(C_k)} \tag{2.35}$$

established in part (ii). $\square$

3. Well-posedness

Here we give a rigorous mathematical formulation of Eq. (1.1) within the spaces of uniformly locally L^2-functions. We first specify our basic assumptions on the data, starting with the nonlinear diffusion term:

$$a \in C^0(\mathbb{R}^d; \mathbb{R}^d), \qquad a(0) = 0, \qquad \big(a(\xi) - a(\eta)\big) \cdot (\xi - \eta) \geqslant \kappa |\xi - \eta|^2, \quad \forall \xi, \eta \in \mathbb{R}^d, \tag{3.1}$$

$$\big|a(\xi) - a(\eta)\big| \leqslant c\kappa |\xi - \eta|, \quad \forall \xi, \eta \in \mathbb{R}^d, \tag{3.2}$$

$$\xi \mapsto a(\xi) \cdot \xi \text{ is a convex function on } \mathbb{R}^d, \tag{3.3}$$

where $\kappa > 0$ and $c \geqslant 1$ are suitable constants. We now introduce the family of nonlinear elliptic operators $\{A_{\bar{x}}\}_{\bar{x} \in \mathbb{R}^d}$ as

$$A_{\bar{x}} : W^{1,2}_{\bar{x}}(\Omega) \to W^{-1,2}_{\bar{x}}(\Omega),$$
$$\langle A_{\bar{x}} v, z \rangle_{\bar{x}} := \int_\Omega a\big(\nabla v(x)\big) \cdot \left(\nabla z(x) - z(x) \frac{x - \bar{x}}{|x - \bar{x}|}\right) e^{-|x - \bar{x}|}\, dx. \tag{3.4}$$

In particular, if $v \in W^{1,2}_{b,\phi}(\Omega)$ for an admissible weight ϕ of growth rate $\mu < 1$, then $A_{\bar{x}} v$ is an element of $W^{-1,2}_{\bar{x}}(\Omega)$ for all $\bar{x} \in \mathbb{R}^d$. The nonlinear function f is assumed to satisfy

$$f \in C^0(\mathbb{R}; \mathbb{R}), \qquad f(0) = 0, \tag{3.5}$$

$$\big|f(r) - f(s)\big| \leqslant c_2\big(1 + |r| + |s|\big)^{p-2} |r - s|, \quad \forall r, s \in \mathbb{R}, \tag{3.6}$$

$$\big(f(r) - f(s)\big)(r - s) \geqslant -C|r - s|^2, \quad \forall r, s \in \mathbb{R}, \tag{3.7}$$

$$c_4 |r|^p - c_5 \leqslant f(r) r \leqslant c_6\big(|r|^p + 1\big), \quad \forall r \in \mathbb{R}, \tag{3.8}$$

for some C, $c_i > 0$ and some $p \in (2, \infty)$. Hence, we are requiring that f grows superlinearly at infinity, which holds in most applications. As far as h is concerned, we let

$$h : \Omega \times \mathbb{R}^d \to \mathbb{R}, \quad \xi \mapsto h(x, \xi) \text{ is globally Lipschitz for a.e. } x \in \Omega, \tag{3.9}$$

$$x \mapsto h(x, \xi) \text{ is measurable and essentially bounded for all } \xi \in \mathbb{R}^d. \tag{3.10}$$

Remark 3.1. With minor modifications in the proofs, one could admit also a (Lipschitz) dependence on u in the convective term h. We limit ourselves to the slightly more restrictive setting (3.9)–(3.10) just for the sake of notational simplicity.

Finally, we take

$$g \in L^2_b(\Omega). \tag{3.11}$$

We are now able to state our result on well-posedness and dissipativity of the reaction–diffusion system in the space $L^2_b(\Omega)$. Notice that, since we are only considering *negative* exponential weights, $L^2_b(\Omega)$ is continuously included into $L^2_{b,\phi}(\Omega)$ for any such weight. In particular, estimate (3.16) below makes sense.

Theorem 3.2. *Let assumptions* (3.1)–(3.3) *and* (3.5)–(3.11) *hold. Let also*

$$u_0 \in L^2_b(\Omega). \tag{3.12}$$

Then, there exists a unique *function u such that, for any $\bar{x} \in \Omega$, one has*

$$\begin{aligned} &u \in C^0\big([0,T]; L^2_{\bar{x}}(\Omega)\big) \cap L^2\big(0,T; W^{1,2}_{\bar{x}}(\Omega)\big) \cap L^p\big(0,T; L^p_{\bar{x}}(\Omega)\big), \\ &u_t \in L^2\big(0,T; W^{-1,2}_{\bar{x}}(\Omega)\big) + L^{p'}\big(0,T; L^{p'}_{\bar{x}}(\Omega)\big), \end{aligned} \tag{3.13}$$

and for all $\bar{x} \in \Omega$ there holds

$$u_t + A_{\bar{x}} u + f(u) + h(\cdot, \nabla u) = g, \quad \text{in } L^2\big(0,T; W^{-1,2}_{\bar{x}}(\Omega)\big) + L^{p'}\big(0,T; L^{p'}_{\bar{x}}(\Omega)\big). \tag{3.14}$$

Moreover, we have

$$u|_{t=0} = u_0, \quad \text{in } L^2_{\bar{x}}(\Omega). \tag{3.15}$$

Finally, for every admissible weight function ϕ with growth rate $\mu < 1$ and almost all $t \geqslant 0$, there holds the dissipative estimate

$$\big\|u(t)\big\|^2_{L^2_{b,\phi}(\Omega)} + c_1 \|u\|^2_{L^2_{b,\phi}(t,t+1;W^{1,2}(\Omega))} + c_2 \|u\|^p_{L^p_{b,\phi}(t,t+1;L^p(\Omega))} \leqslant \|u_0\|^2_{L^2_{b,\phi}(\Omega)} e^{-\sigma t} + c_3, \tag{3.16}$$

where σ and c_i are positive constants depending on the parameters of the system, but independent of the initial datum u_0.

A function u under the conditions of Theorem 3.2 will be simply called a "solution" in the sequel. Of course, due to arbitrariness of T any solution can be thought to be defined for almost any $t \in (0, \infty)$.

Remark 3.3. Eq. (3.14) can be also written in an expanded way as

$$\begin{aligned} &\int_\Omega u_t(x,t) v(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x + \int_\Omega a\big(\nabla u(x,t)\big) \cdot \left(\nabla v(x,t) - v(x,t) \frac{x-\bar{x}}{|x-\bar{x}|}\right) e^{-|x-\bar{x}|}\,\mathrm{d}x \\ &\quad + \int_\Omega \big(f\big(u(x,t)\big) + h\big(x, \nabla u(x,t)\big) - g(x)\big) v(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x = 0, \end{aligned} \tag{3.17}$$

the above being intended to hold for any $\bar{x} \in \Omega$, almost any $t \in (0,T)$ and any test function $v \in L^2(0,T; W^{1,2}_{\bar{x}}(\Omega)) \cap L^p(0,T; L^p_{\bar{x}}(\Omega))$. In particular, by (3.13), one can take $v = u$.

Proof of Theorem 3.2. The proof is carried out by suitably approximating (3.14) through a family of problems defined on bounded domains and then passing to the limit via monotonicity and compactness methods.

As a first step, we then define $\Omega_n := B_n(0, \mathbb{R}^d)$, $n \in \mathbb{N}$, and for any n consider a cutoff function $\psi_n \in C^\infty(\Omega; [0,1])$ such that $\psi \equiv 1$ in $\bar{\Omega}_{n-1}$ and $\operatorname{supp}(\psi) \subset \Omega_n$. Then, we set $u_{0,n} := u_0 \psi_n$ and $g_n := g\psi_n$. Thanks to (3.11) and (3.12), applying Lebesgue's Theorem one can easily check that, for every $\bar{x} \in \Omega$,

$$u_{0,n} \to u_0 \quad \text{and} \quad g_n \to g, \quad \text{strongly in } L^2_{\bar{x}}(\Omega). \tag{3.18}$$

We also set $X_n := L^2(\Omega_n)$, $V_n := W_0^{1,2}(\Omega_n)$ and define the elliptic operator

$$A_n : V_n \to V_n', \qquad \langle A_n v, z\rangle := \int_\Omega \nabla a\big(v(x)\big) \cdot \nabla z(x)\,\mathrm{d}x, \tag{3.19}$$

where $v, z \in V_n$. Then, we can introduce our approximate problem

$$u_{n,t} + A_n u_n + f(u_n) + h(\cdot, \nabla u_n) = g_n, \quad \text{in } L^2(0,T;V_n) + L^{p'}\big(0,T;L^{p'}(\Omega_n)\big), \tag{3.20}$$

$$u_n|_{t=0} = u_{0,n}, \quad \text{a.e. in } \Omega_n. \tag{3.21}$$

We have the following

Lemma 3.4. *For all $n \in \mathbb{N}$, there exists one and only one solution u_n to* (3.20)–(3.21) *such that*

$$\begin{aligned} &u_{n,t} \in L^2\big(0,T;V_n'\big) + L^{p'}\big(0,T;L^{p'}(\Omega_n)\big),\\ &u_n \in C^0\big([0,T];X_n\big) \cap L^2(0,T;V_n) \cap L^p\big(0,T;L^p(\Omega_n)\big). \end{aligned} \tag{3.22}$$

The proof of the lemma is more or less standard and mainly relies on the basic tools of the theory of maximal monotone operators. We will not give it since most of the difficulties will be the same we will face in the passage to the limit $n \nearrow \infty$ we now describe.

Assume u_n be extended to 0 outside Ω_n and test (3.20) by $u_n e^{-|\cdot-\bar{x}|}$, for arbitrary $\bar{x} \in \Omega$. Then, we readily obtain the basic estimate

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|u_n(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x + \int_\Omega a\big(\nabla u_n(x,t)\big)\cdot\left(\nabla u_n(x,t) - u_n(x,t)\frac{x-\bar{x}}{|x-\bar{x}|}\right)e^{-|x-\bar{x}|}\,\mathrm{d}x\\ &\quad + \int_\Omega \big(f\big(u_n(x,t)\big) + h\big(x,\nabla u_n(x,t)\big)\big)u_n(x,t)e^{-|x-\bar{x}|}\,\mathrm{d}x = \int_\Omega g_n(x)u_n(x,t)e^{-|x-\bar{x}|}\,\mathrm{d}x. \end{aligned} \tag{3.23}$$

Using hypotheses (3.1)–(3.2) and (3.6), it is then not difficult to deduce from (3.23) a priori estimates in weighted spaces which entail

$$u_n \to u \quad \text{weakly star in } L^\infty\big(0,T;L^2_{\bar{x}}(\Omega)\big) \cap L^p\big(0,T;L^p_{\bar{x}}(\Omega)\big) \cap L^2\big(0,T;W^{1,2}_{\bar{x}}(\Omega)\big). \tag{3.24}$$

Note that, here and below, all convergence relations are intended up to the extraction of subsequences, not relabeled (see also Remark 3.5 below for more details). Next, writing (3.20) in the form corresponding to (3.17), namely

$$\begin{aligned} &\int_\Omega u_{n,t}(x,t)v(x,t)e^{-|x-\bar{x}|}\,\mathrm{d}x + \int_\Omega a\big(\nabla u_n(x,t)\big)\cdot\left(\nabla v(x,t) - v(x,t)\frac{x-\bar{x}}{|x-\bar{x}|}\right)e^{-|x-\bar{x}|}\,\mathrm{d}x\\ &\quad + \int_\Omega \big(f\big(u_n(x,t)\big) + h\big(x,\nabla u_n(x,t)\big) - g_n(x)\big)v(x,t)e^{-|x-\bar{x}|}\,\mathrm{d}x = 0, \end{aligned} \tag{3.25}$$

and letting v vary in $L^2(0,T;W^{1,2}_{\bar{x}}(\Omega))\cap L^p(0,T;L^p_{\bar{x}}(\Omega))$, passing to the supremum with respect to v of unit norm, it is not difficult to obtain

$$u_{n,t}\to u_t \quad \text{weakly in } L^2\big(0,T;W^{-1,2}_{\bar{x}}(\Omega)\big)+L^{p'}\big(0,T;L^{p'}_{\bar{x}}(\Omega)\big). \tag{3.26}$$

At this point, if one considers the restrictions to a fixed domain Ω_m, then (3.24) implies in particular

$$u_n\to u \quad \text{weakly star in } L^\infty(0,T;X_m)\cap L^p\big(0,T;L^p(\Omega_m)\big)\cap L^2\big(0,T;W^{1,2}(\Omega_m)\big). \tag{3.27}$$

On the other hand, if we write (3.20) for $n>m$ and test it by a generic $v\in L^2(0,T;V_m)\cap L^p(0,T;L^p(\Omega_m))$ (extended by 0 outside Ω_m), then, using (3.27) and applying duality arguments, we readily infer

$$u_{n,t}\to u_t \quad \text{weakly in } L^2\big(0,T;V_m'\big)+L^{p'}\big(0,T;L^{p'}(\Omega_m)\big). \tag{3.28}$$

In particular, by the Aubin–Lions Lemma, we get from (3.27)–(3.28) that

$$u_n\to u \quad \text{strongly in } L^2(0,T;X_m). \tag{3.29}$$

More precisely, by arbitrariness of m, we have

$$u_n\to u \quad \text{a.e. in } \Omega\times(0,T). \tag{3.30}$$

Thus, recalling (3.24) and applying Lebesgue's Theorem with respect to the measure $\mathrm{d}_{\bar{x}}x=e^{-|x-\bar{x}|}\,\mathrm{d}x$ (notice that $\Omega=\mathbb{R}^d$ has *finite* $\mathrm{d}_{\bar{x}}x$-measure), we readily obtain

$$u_n\to u \quad \text{strongly in } L^q\big(0,T;L^q_{\bar{x}}(\Omega)\big),\ \forall q\in[1,p) \tag{3.31}$$

and, thanks to (3.6),

$$f(u_n)\to f(u) \quad \text{strongly in } L^q\big(0,T;L^q_{\bar{x}}(\Omega)\big),\ \forall q\in\big[1,p'\big). \tag{3.32}$$

Thus, we are now ready to pass to the limit in Eq. (3.20). To do this, we first observe that, by (3.24) and assumptions (3.2) and (3.9), if $\bar{x}\in\Omega$ is fixed, there exist $\boldsymbol{\alpha}\in L^2(0,T;L^2_{\bar{x}}(\Omega)^d)$ and $\boldsymbol{h}\in L^2(0,T;L^2_{\bar{x}}(\Omega))$ such that

$$a(\nabla u_n)\to\boldsymbol{\alpha} \quad \text{weakly in } L^2\big(0,T;L^2_{\bar{x}}(\Omega)^d\big), \tag{3.33}$$

$$h(\cdot,\nabla u_n)\to\tilde{h} \quad \text{weakly in } L^2\big(0,T;L^2_{\bar{x}}(\Omega)\big). \tag{3.34}$$

Notice that, a priori, $\boldsymbol{\alpha}$ and $\tilde{h}$ might depend on the choice of $\bar{x}$. Let us now come back to (3.25). It is clear that we can take its limit, which assumes the form

$$\int_\Omega u_t(x,t)v(x,t)e^{-|x-\bar{x}|}\,\mathrm{d}x+\int_\Omega\boldsymbol{\alpha}(x,t)\cdot\left(\nabla v(x,t)-v(x,t)\frac{x-\bar{x}}{|x-\bar{x}|}\right)e^{-|x-\bar{x}|}\,\mathrm{d}x$$
$$+\int_\Omega\big(f\big(u(x,t)\big)+\tilde{h}(x,t)-g(x)\big)v(x,t)e^{-|x-\bar{x}|}\,\mathrm{d}x=0. \tag{3.35}$$

Now, let us choose $v = u_n$ in (3.25), rearrange some terms, integrate over $(0, T)$, and take the supremum limit. This procedure gives

$$\begin{aligned}
&\limsup_{n\nearrow\infty} \int_0^T \int_\Omega a\big(\nabla u_n(x,t)\big) \cdot \nabla u_n(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant -\frac{1}{2} \liminf_{n\nearrow\infty} \int_\Omega \big|u_n(x,T)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x + \frac{1}{2} \limsup_{n\nearrow\infty} \int_\Omega \big|u_{0,n}(x)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x \\
&\qquad - \liminf_{n\nearrow\infty} \int_0^T \int_\Omega \big(f\big(u_n(x,t)\big) + Cu_n(x,t)\big) u_n(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \limsup_{n\nearrow\infty} \int_0^T \int_\Omega C\big|u_n(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t + \limsup_{n\nearrow\infty} \int_0^T \int_\Omega g_n(x) u_n(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \liminf_{n\nearrow\infty} \int_0^T \int_\Omega h\big(x, \nabla u_n(x,t)\big) u_n(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \limsup_{n\nearrow\infty} \int_0^T \int_\Omega a\big(\nabla u_n(x,t)\big) \cdot \frac{x-\bar{x}}{|x-\bar{x}|} u_n(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.36)
\end{aligned}$$

At this point, we aim to compute the limits on the right-hand side. First, let us observe that the first two terms are treated by means of (3.18), (3.24), and semicontinuity of norms with respect to weak star convergence. Next, recalling (3.5) and (3.7), by (3.30) and Fatou's Lemma we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \big(f\big(u(x,t)\big) + Cu(x,t)\big) u(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \liminf_{n\nearrow\infty} \int_0^T \int_\Omega \big(f\big(u_n(x,t)\big) + Cu_n(x,t)\big) u_n(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.37)
\end{aligned}$$

The subsequent three terms are treated thanks to (3.18), (3.31) (where we can take $q = 2$) and (3.34). Finally, using (3.33) and again (3.31), we arrive at

$$\begin{aligned}
&\lim_{n\nearrow\infty} \int_0^T \int_\Omega a\big(\nabla u_n(x,t)\big) \cdot \frac{x-\bar{x}}{|x-\bar{x}|} u_n(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \boldsymbol{\alpha}(x,t) \cdot \frac{x-\bar{x}}{|x-\bar{x}|} u(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.38)
\end{aligned}$$

Thus, comparing (3.36) with (3.35) (written for $v = u$ and integrated in time), we finally deduce that

$$\limsup_{n\nearrow\infty} \int_0^T \int_\Omega a\big(\nabla u_n(x,t)\big)\cdot \nabla u_n(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^T \int_\Omega \boldsymbol{\alpha}(x,t)\cdot \nabla u(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.39}$$

Noting now that, by assumption (3.1), a induces a maximal monotone operator on the Hilbert space $L^2(0,T;L^2_{\bar{x}}(\Omega)^d)$, the usual monotonicity argument (cf., e.g., [6, Prop. 1.1, p. 42]) permits to say that

$$\boldsymbol{\alpha}(x,t) = a\big(\nabla u(x,t)\big) \quad \mathrm{d}_{\bar{x}}x\text{-a.e. in } \Omega \text{ and a.e. in } (0,T), \tag{3.40}$$

whence the same holds almost everywhere with respect to Lebesgue's measure in $\Omega\times(0,T)$. In particular, $\boldsymbol{\alpha}$ is independent of the choice of $\bar{x}$. Thus, substituting in (3.35), we get exactly (3.17). Finally, we notice that, as a consequence of (3.39)–(3.40) and lower semicontinuity,

$$\int_0^T \int_\Omega a\big(\nabla u_n(x,t)\big)\cdot \nabla u_n(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega a\big(\nabla u(x,t)\big)\cdot \nabla u(x,t) e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.41}$$

Thus, using (3.3) and, e.g., [10, Thm. 2.11], we obtain

$$\nabla u_n(x,t) \to \nabla u(x,t) \quad \text{a.e. in } \Omega\times(0,T), \tag{3.42}$$

whence, by (3.9), (3.33) and Lebesgue's Theorem,

$$\nabla u_n \to \nabla u \quad \text{and} \quad h(\cdot,\nabla u_n) \to h(\cdot,\nabla u) \quad \text{strongly in } L^q\big(0,T;L^q_{\bar{x}}(\Omega)\big) \tag{3.43}$$

for all $q\in[1,2)$. In particular, $\tilde{h} = h(\cdot,\nabla u)$ (cf. (3.34)), which concludes the proof of existence.

Remark 3.5. It is worth observing that relations (3.24)–(3.26) and (3.31)–(3.32) hold for *any* $\bar{x}\in\mathbb{R}^d$ and the limits are independent of $\bar{x}$. This follows already from the fact that the spaces $L^p_{\bar{x}}(\Omega)$ coincide for different values of $\bar{x}$. However, it is still necessary to consider the weak formulation for all $\bar{x}$ simultaneously to make sure that the a priori estimates are also uniform with respect to $\bar{x}$. In virtue of the equivalence relations (2.15) and (2.31) this then leads to the estimates in the uniformly bounded spaces.

Remark 3.6. In the case when h is a *linear* convection term (namely, $h(x,\xi) = \boldsymbol{v}(x)\cdot\xi$ for some measurable and bounded function $\boldsymbol{v}$), then assumption (3.3) can be avoided. Actually, the only role of (3.3) is that of guaranteeing the strong convergence (3.43) of gradients, which is not required for taking the limit in case h is linear.

Remark 3.7. It is not difficult to realize that Theorem 3.2 can be extended to systems of m equations provided that the nonlinear function a is replaced by $\mathbf{a}\in C^0(\mathbb{R}^{m\times d};\mathbb{R}^{m\times d})$ satisfying suitable reformulations of (3.1) and (3.2) and h is replaced by a linear function of the form $\mathbf{h}(x,\mathbf{M}) = \boldsymbol{v}(x)\cdot\mathbf{M}$, where $\mathbf{M}\in\mathbb{R}^{m\times d}$ (see Remark 3.6). Another possibility is to preserve a nonlinear convective term $\mathbf{h}:\Omega\times\mathbb{R}^{m\times d}\to\mathbb{R}^m$ satisfying suitable generalizations of (3.9) and (3.10) and taking the vector Laplacian $-\Delta$ as diffusion operator.

We now move to dissipativity. To prove it, let us go back to (3.35), take $v = u$ and use (3.40), (3.1)–(3.2), (3.9) and (3.6). Then, we deduce, for some $\sigma > 0$ independent of $\bar{x}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_{\bar{x}}(\Omega)} + \sigma\big(\|\nabla u\|^2_{L^2_{\bar{x}}(\Omega)} + \|u\|^p_{L^p_{\bar{x}}(\Omega)}\big) \leqslant c\int_\Omega \big(1 + g^2(x)\big)e^{-|x-\bar{x}|}\,\mathrm{d}x \leqslant c_1, \tag{3.44}$$

where c_1 only depends on $\|g\|_{L^2_b(\Omega)}$ and on the Lipschitz constant of h (and is independent of $\bar{x}$). By a standard application of Gronwall's Lemma, we further deduce

$$\big\|u(t)\big\|^2_{L^2_{\bar{x}}(\Omega)} + \sigma\int_t^{t+1}\big(\|\nabla u\|^2_{W^{1,2}_{\bar{x}}(\Omega)} + \|u\|^p_{L^p_{\bar{x}}(\Omega)}\big)\,\mathrm{d}s \leqslant \|u_0\|^2_{L^2_{\bar{x}}(\Omega)}e^{-\sigma t} + c_2.$$

Next, we multiply with $\phi(\bar{x})$, and take the supremum over $\bar{x} \in \mathbb{R}^d$. Using the fact that ϕ is uniformly bounded and also the equivalence relations (2.15), (2.31), we finally conclude (3.16).

Notice that the above is a dissipative estimate in any of the spaces $L^2_{b,\phi}(\Omega)$ where ϕ is an admissible weight of growth rate strictly lower than 1.

Finally, let us prove uniqueness, which is standard. Indeed, it is sufficient to write (3.14) for a couple of solutions u_1 and u_2, take the difference, test it by $u_1 - u_2$ (in the appropriate functional sense) and integrate with respect to the measure $\mathrm{d}_{\bar{x}}x \otimes \mathrm{d}t$. Then the thesis follows as before by using Gronwall's Lemma and taking the supremum with respect to $\bar{x}$. We omit the details since we shall prove more refined contractive estimates in the next section (Theorem 4.2). The proof of Theorem 3.2 is complete. □

In order to prepare the long time analysis, we need a further regularity result.

Theorem 3.8. *Let $d \leqslant 3$ and consider a solution u. Then, for any $q \in (1,\infty)$ and any $\tau > 0$, u enjoys the additional regularity*

$$u \in L^\infty\big(\tau,\infty; L^q_b(\Omega)\big). \tag{3.45}$$

More precisely, for any $q \in (1,\infty)$ there exists a computable nonnegative-valued function $\mathcal{Q}$, depending on q and increasingly monotone in each of its arguments, such that

$$\big\|u(t)\big\|_{L^q_b(\Omega)} \leqslant \mathcal{Q}\big(\tau^{-1}, \|u_0\|_{L^2_b(\Omega)}\big), \quad \forall t \geqslant \tau > 0. \tag{3.46}$$

Proof. The proof is performed by means of (finitely many) iterative estimates. As a first step, we notice that, due to the dissipative estimate (3.16), for any $t \geqslant 0$, any $\tau \in (0,1)$ and any $\bar{x} \in \Omega$ there exists $t_0 \in [t, t+\tau]$ (possibly depending also on $\bar{x}$) such that

$$\big\|u(t_0)\big\|_{L^p_{\bar{x}}(\Omega)} + \|u\|_{L^2_b(t,t+1;W^{1,2}(\Omega))} \leqslant \mathcal{Q}\big(\tau^{-1}, \|u_0\|_{L^2_b(\Omega)}\big), \tag{3.47}$$

where $\mathcal{Q}$ is as in the statement.

Then, we can test the equation by $v = |u|^\alpha u$, where $\alpha = p - 2 > 0$ due to our assumptions. Such a test function is indeed admissible at least on the level of approximations, thanks to uniqueness. Thus, using (3.8) and (3.9), and observing that $a(\nabla u)\cdot\nabla v \geqslant 0$ by (3.1), one deduces after obvious manipulations

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^p_{L^p_{\bar{x}}(\Omega)} + c_1\|u\|^{p+\alpha}_{L^{p+\alpha}_{\bar{x}}(\Omega)} \leqslant c_2 + c_3\|u\|^p_{L^p_{\bar{x}}(\Omega)} + c_4\int_\Omega \big(1+|\nabla u|+|g|\big)|u|^{\alpha+1}e^{-|x-\bar{x}|}\,\mathrm{d}x. \tag{3.48}$$

The last integrand in (3.48) is then simply estimated as

$$\big(|\nabla u|+|g|\big)|u|^{a+1} \leqslant \varepsilon|u|^{2\alpha+2} + \varepsilon^{-1}\big(|\nabla u|^2+|g|^2\big).$$

Then, choosing ε small enough and remarking that $2a+2=p+a$, we further deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^p_{L^p_{\bar{x}}(\Omega)} + \frac{c_1}{2}\|u\|^{p+\alpha}_{L^{p+\alpha}_{\bar{x}}(\Omega)} \leqslant c_5 + c_6\|u\|^p_{L^p_{\bar{x}}(\Omega)} + c_7\|\nabla u\|^2_{L^2_{\bar{x}}(\Omega)}. \tag{3.49}$$

Then, we can integrate (3.49) over (t_0, t_0+2). Recalling (3.47) and using Gronwall's Lemma, we can then pass to the supremum with respect to $\bar{x}$ first on the right-hand side and then on the left-hand side. Noting that for any $\bar{x}$ it is $t_0(\bar{x}) \leqslant t+\tau$, by arbitrariness of t in $\mathbb{R}^+$ we deduce

$$\|u\|_{L^\infty(\tau,\infty;L^p_b(\Omega))} \leqslant \mathcal{Q}\big(\tau^{-1}, \|u_0\|_{L^2_b(\Omega)}\big). \tag{3.50}$$

Moreover, we also obtain that, for each $t \geqslant \tau$ and any $\bar{x}\in\Omega$, there exists $t_1 \in [t, t+\tau]$ such that

$$\big\|u(t_1)\big\|_{L^{p+\alpha}_{\bar{x}}(\Omega)} + \|u\|_{L^{p+\alpha}_b(t,t+1;L^{p+\alpha}(\Omega))} \leqslant \mathcal{Q}\big(\tau^{-1}, \|u_0\|_{L^2_b(\Omega)}\big). \tag{3.51}$$

We can now proceed by an induction argument. More precisely, we will just need a finite number of steps. Actually, since $\alpha = p-2 > 1$, we will stop after n iterations when $n\in\mathbb{N}$ is such that $p+(n-1)\alpha = n\alpha+2 \geqslant q$.

So, we can assume that, given $k \leqslant n$, for each $t \geqslant \tau$ and any $\bar{x}\in\Omega$, there exists $t_{k-1}\in[t,t+\tau]$ such that

$$\big\|u(t_{k-1})\big\|_{L^{k\alpha+2}_{\bar{x}}(\Omega)} + \|u\|_{L^{k\alpha+2}_b(t,t+1;L^{k\alpha+2}(\Omega))} \leqslant \mathcal{Q}\big(\tau^{-1}, \|u_0\|_{L^2_b(\Omega)}\big), \tag{3.52}$$

and prove now the same relation with $k-1$ replaced by k.

To do this, we test the equation by $v=|u|^{k\alpha}u$, where $\alpha=p-2>0$ as before. Then, we obtain the analogue of (3.48), where, however, we need to use (3.1) a bit more precisely. Namely, we get

$$c_{1,k}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^{k\alpha+2}_{L^{k\alpha+2}_{\bar{x}}(\Omega)} + c_{2,k}\int_\Omega |\nabla u|^2|u|^{k\alpha}e^{-|x-\bar{x}|}\,\mathrm{d}x + c_{3,k}\|u\|^{p+k\alpha}_{L^{p+k\alpha}_{\bar{x}}(\Omega)} \leqslant c_{4,k} + c_{5,k}\|u\|^{k\alpha+2}_{L^{k\alpha+2}_{\bar{x}}(\Omega)} + c_{6,k}\int_\Omega \big(1+|\nabla u|+|g|\big)|u|^{k\alpha+1}e^{-|x-\bar{x}|}\,\mathrm{d}x. \tag{3.53}$$

All constants c or $c_{i,k}$ here and below will be allowed to depend on k. However, since a finite number of induction steps will suffice, we will not need to compute them explicitly. To estimate the terms on the right-hand side, we then observe that

$$c_{6,k}\int_\Omega \big(1+|\nabla u|\big)|u|^{k\alpha+1}e^{-|x-\bar{x}|}\,\mathrm{d}x \leqslant \epsilon\int_\Omega |\nabla u|^2|u|^{k\alpha}e^{-|x-\bar{x}|}\,\mathrm{d}x + c_\epsilon + c_\epsilon\|u\|^{k\alpha+2}_{L^{k\alpha+2}_{\bar{x}}(\Omega)}. \tag{3.54}$$

As for the g-term, we need however to be much more accurate than before. Firstly, we notice that, for positive λ_i, $i = 1, 2, 3$, such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ (and that will be chosen below), we have

$$\begin{aligned} c_{6,k} \int_\Omega |g||u|^{k\alpha+1} e^{-|x-\bar{x}|}\, dx &= c_{6,k} \int_\Omega \big(|g| e^{-\lambda_1|x-\bar{x}|}\big)\big(|u|^{\frac{3k\alpha}{4}} e^{-\lambda_2|x-\bar{x}|}\big)\big(|u|^{\frac{k\alpha+4}{4}} e^{-\lambda_3|x-\bar{x}|}\big)\, dx \\ &\leqslant c \big\| |g| e^{-\lambda_1|\cdot-\bar{x}|}\big\|_{L^2(\Omega)} \times \big\| |u|^{\frac{3k\alpha}{4}} e^{-\lambda_2|\cdot-\bar{x}|}\big\|_{L^{q_2}(\Omega)} \\ &\quad \times \big\| |u|^{\frac{k\alpha+4}{4}} e^{-\lambda_3|\cdot-\bar{x}|}\big\|_{L^{q_3}(\Omega)} \\ &=: \mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3, \quad \text{where } \frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{2}. \end{aligned} \tag{3.55}$$

Let us now estimate the quantities $\mathcal{I}_i$. Actually, taking

$$q_2 = \frac{4(k\alpha+2)}{k\alpha}, \qquad q_3 = \frac{4(k\alpha+2)}{k\alpha+4}, \tag{3.56}$$

it is not difficult to obtain

$$\mathcal{I}_2 = \big\| |u|^{\frac{k\alpha+2}{2}} e^{-\frac{2(k\alpha+2)\lambda_2|\cdot-\bar{x}|}{3k\alpha}} \big\|_{L^6(\Omega)}^{\frac{3k\alpha}{2(k\alpha+2)}}, \tag{3.57}$$

whence, by continuity of the embedding $H^1(\Omega) \subset L^6(\Omega)$, it is straightforward to arrive at

$$\mathcal{I}_2 \leqslant c \left| \int_\Omega |u|^{k\alpha} |\nabla u|^2 e^{-\frac{4(k\alpha+2)\lambda_2|x-\bar{x}|}{3k\alpha}} \right|^{\frac{3k\alpha}{4(k\alpha+2)}} + c \left| \int_\Omega |u|^{k\alpha+2} e^{-\frac{4(k\alpha+2)\lambda_2|x-\bar{x}|}{3k\alpha}} \right|^{\frac{3k\alpha}{4(k\alpha+2)}}. \tag{3.58}$$

Computing $\mathcal{I}_3$ directly, we similarly obtain

$$\mathcal{I}_3 \leqslant \left| \int_\Omega |u|^{k\alpha+2} e^{-\frac{4(k\alpha+2)\lambda_3|x-\bar{x}|}{k\alpha+4}} \right|^{\frac{k\alpha+4}{4(k\alpha+2)}}. \tag{3.59}$$

At this point, in order to get the same weight functions as on the left-hand side, we choose

$$\lambda_2 = \frac{3k\alpha}{4(k\alpha+2)}, \qquad \lambda_3 = \frac{k\alpha+4}{4(k\alpha+2)}, \quad \text{so that} \quad \lambda_1 = \frac{1}{k\alpha+2} \tag{3.60}$$

and consequently we obtain

$$\mathcal{I}_1 \leqslant c \left| \int_\Omega g^2(x) e^{-\frac{2|x-\bar{x}|}{k\alpha+2}}\, dx \right|^{1/2} \leqslant c \sup_{\bar{x}\in\Omega} \left\{ \left| \int_\Omega g^2(x) e^{-\frac{2|x-\bar{x}|}{k\alpha+2}}\, dx \right|^{1/2} \right\} \leqslant c, \tag{3.61}$$

where the last inequality follows from the fact that we have obtained a norm of g that is equivalent to the usual norm of $L_b^2(\Omega)$ (this fact can be shown proceeding similarly with the proof of Theorem 2.1 in the case $\phi \equiv 1$). Notice that, the larger is k, the slower is the decay of the exponential weight (however, we will not need to take $k \to \infty$ here).

Thus, using also the Young inequality, (3.55) gives

$$\mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3 \leqslant \epsilon\left(\int_\Omega |u|^{k\alpha}|\nabla u|^2 e^{-|x-\bar{x}|} + \int_\Omega |u|^{k\alpha+2} e^{-|x-\bar{x}|}\right) \\ + c_\epsilon \left|\int_\Omega |u|^{k\alpha+2} e^{-|x-\bar{x}|}\right|^{\frac{k\alpha+4}{k\alpha+8}}, \tag{3.62}$$

where of course the latter exponent is (strictly) lower than 1.

Thus, integrating (3.53) over $(t_{k-1}, t_{k-1}+2)$, taking ϵ small enough, using Gronwall's Lemma, and taking as before the supremum with respect to $\bar{x}$ first on the right-hand side and then on the left-hand side, it is almost immediate to obtain (3.52) with $k-1$ replaced by k. This concludes the proof. □

Remark 3.9. Note that should we assume $g \in L^\infty(\Omega)$, Theorem 3.8 can be proved in a simpler way and the restriction $d \leqslant 3$ can be removed.

Remark 3.10. It is not difficult to realize that Theorem 3.8 can be extended to systems of m equations provided that the diffusion operator is the vector Laplacian $-\Delta$ and the nonlinear convective term h is replaced by $\mathbf{h} : \Omega \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ satisfying suitable generalizations of (3.9) and (3.10) (see also Remark 3.7).

4. Global attractor

Thanks to Theorem 3.2 we can introduce the solution operator

$$S(t) : L^2_b(\Omega) \to L^2_b(\Omega), \quad u_0 \mapsto u(\cdot, t).$$

Before showing that $S(\cdot)$ is a continuous semigroup, we prove a simple

Corollary 4.1. *The semiflow $S(\cdot)$ admits an absorbing set of the form*

$$\mathcal{B} := B_K\big(0; L^2_b(\Omega)\big), \tag{4.1}$$

with a sufficiently large $K > 0$. Moreover, $\mathcal{B}$ can be chosen to be positively invariant and bounded in the space $L^q_b(\Omega)$ for q arbitrarily large.

Proof. The existence of an absorbing set $\mathcal{B}_0$ satisfying (4.1) is an immediate consequence of (3.16). Setting

$$\mathcal{B} := \bigcup_{t \geqslant 1} S(t)\mathcal{B}_0, \tag{4.2}$$

we immediately obtain the positive invariance, as well as the L^q_b-boundedness, thanks to (3.45). □

Notice that, however, we cannot expect that the dynamics be compact in $L^2_b(\Omega)$. The standard way out of this impasse is the local topology $L^2_{\mathrm{loc}}(\Omega)$. Indeed, thanks to Corollary 4.1, we can restrict our analysis to those trajectories taking values, for all nonnegative times in the L^q_b-bounded set $\mathcal{B}$. Then, one easily verifies that

$$u_n \to u_0 \quad \text{in } L^2_{\mathrm{loc}}(\Omega) \quad \Longleftrightarrow \quad u_n \to u_0 \quad \text{in } L^2_{\bar{x}}(\Omega), \tag{4.3}$$

for any $u_n, u_0 \in \mathcal{B}$. Namely, whatever is $\bar{x} \in \Omega$, the norm of $L^2_{\bar{x}}(\Omega)$ induces to $\mathcal{B}$ exactly the $L^2_{\mathrm{loc}}(\Omega)$-topology (in particular, one could directly choose $\bar{x} = 0$ at this stage). Thus, recalling also that the solutions are continuous as functions with values in $L^2_{\bar{x}}(\Omega)$, this space seems to be most convenient for the construction of the global attractor. More precisely, we are going to establish the existence of the $(L^2_b(\Omega), L^2_{\mathrm{loc}}(\Omega))$-attractor, following the terminology of [5].

We recall that one possible strategy to show the compactness of the dynamics in $L^2_{\mathrm{loc}}(\Omega)$ is to derive higher regularity estimates, as for example in $W^{1,2}_b(\Omega)$. However, as we mentioned in the Introduction, here we adopt a more elementary approach, which circumvents more advanced regularity techniques, resting only on the natural parabolic compactness of solutions. This is easy to obtain while we look at the dynamics from the perspective of "trajectories" with some finite fixed length ℓ.

We then introduce the set of the *short trajectories* taking values in $\mathcal{B}$:

$$\mathcal{X} := \{\chi \in L^2(0, \ell; L^2_{\bar{x}}(\Omega));\ \chi \text{ is a solution of (1.1)},\ \chi(0) \in \mathcal{B}\}.$$

Further, we define the semigroup

$$L(t) : \mathcal{X} \to \mathcal{X}, \qquad [L(t)\chi](s) := S(t)\chi(s), \quad s \in (0, \ell),$$

and the mapping

$$e : \mathcal{X} \to L^2_b(\Omega), \quad \chi \mapsto \chi(\ell).$$

The solutions are understood in the sense of Theorem 3.2, hence elements of $\mathcal{X}$ possess additional regularity. In particular, for any $\chi \in \mathcal{X}$, one has

$$\chi \in L^\infty(0, \ell; L^2_b(\Omega)) \cap L^2_{b,\phi}(0, \ell; W^{1,2}(\Omega)) \cap L^p_{b,\phi}(0, \ell; L^p(\Omega)); \tag{4.4}$$

$$\chi_t \in L^2(0, \ell; W^{-1,2}_{\bar{x}}(\Omega)) + L^{p'}(0, \ell; L^{p'}_{\bar{x}}(\Omega)). \tag{4.5}$$

Also, thanks to Corollary 4.1, we can assume that

$$\chi \in L^\infty(0, \ell; L^q_b(\Omega)). \tag{4.6}$$

All the above estimates are independent of χ and $\bar{x}$. Consequently, $\chi \in C([0, \ell]; L^2_{\bar{x}}(\Omega))$ in the sense of representative, and it thus makes sense to talk about point values of elements of $\mathcal{X}$. Continuity properties of the above introduced operators are summarized in the following

Theorem 4.2.

1. *$S(t) : L^2_{\bar{x}}(\Omega) \to L^2_{\bar{x}}(\Omega)$ are Lipschitz continuous uniformly w.r.t. $t \in [0, T]$;*
2. *$L(t) : L^2(0, \ell; L^2_{\bar{x}}(\Omega)) \to L^2(0, \ell; L^2_{\bar{x}}(\Omega))$ are Lipschitz continuous uniformly w.r.t. $t \in [0, T]$;*
3. *$e : L^2(0, \ell; L^2_{\bar{x}}(\Omega)) \to L^2_{\bar{x}}(\Omega)$ is Lipschitz continuous.*

Proof. Let u_1, u_2 be weak solutions. Subtract the equations and test by $w := u_1 - u_2$. We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |w(x,t)|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x$$
$$+ \int_\Omega \big(a(\nabla u_1(x,t)) - a(\nabla u_2(x,t))\big) \cdot \left(\nabla w(x,t) - w(x,t)\frac{x - \bar{x}}{|x - \bar{x}|}\right) e^{-|x-\bar{x}|}\,\mathrm{d}x$$

$$
\begin{aligned}
&+\int_\Omega \big(f\big(u_1(x,t)\big)+h\big(x,\nabla u_1(x,t)\big)-f\big(u_2(x,t)\big)-h\big(x,\nabla u_2(x,t)\big)\big)w(x,t)e^{-|x-\bar{x}|}\,\mathrm{d}x \\
&=0. \qquad (4.7)
\end{aligned}
$$

Invoking (3.1)–(3.2), (3.7) and (3.9) and using Young's inequality, one deduces

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \big|w(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x+\kappa\int_\Omega \big|\nabla w(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x \leqslant c_1\int_\Omega \big|w(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x.
$$

Integration over $t \in (t_1,t_2)$ yields

$$
\begin{aligned}
&\int_\Omega \big|w(x,t_2)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x+\kappa\int_{\Omega\times(t_1,t_2)} \big|\nabla w(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant \int_\Omega \big|w(x,t_1)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x+c_1\int_{\Omega\times(t_1,t_2)} \big|w(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

and Gronwall's Lemma applied to

$$
Y(t):=\int_\Omega \big|w(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x+\kappa\int_{\Omega\times(t_1,t)} \big|\nabla w(x,s)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}s
$$

implies the basic estimate

$$
\begin{aligned}
&\sup_{t\in[t_1,t_2]}\int_\Omega \big|w(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x+\kappa\int_{\Omega\times(t_1,t_2)} \big|\nabla w(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant c_2\int_\Omega \big|w(x,t_1)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x. \qquad (4.8)
\end{aligned}
$$

Part 1 of the theorem follows immediately. One also has

$$
\big\|w(t+s)\big\|^2_{L^2_{\bar{x}}(\Omega)} \leqslant c_3\big\|w(s)\big\|^2_{L^2_{\bar{x}}(\Omega)}, \qquad \big\|w(\ell)\big\|^2_{L^2_{\bar{x}}(\Omega)} \leqslant c_4\big\|w(s)\big\|^2_{L^2_{\bar{x}}(\Omega)},
$$

for any $s \in (0,\ell)$, $t \in (0,T)$, where the constants c_3, c_4 only depend on T. Then, integrating the above relations over s yields parts 2 and 3 of the theorem, respectively. □

Remark 4.3. We can establish even stronger continuity of $S(t)$. From the above theorem, one has

$$
\big\|w(t_2)\big\|^2_{L^2_{\bar{x}}(\Omega)} \leqslant c_4\big\|w(t_1)\big\|^2_{L^2_{\bar{x}}(\Omega)};
$$

multiplying by $\phi(\bar{x})$ and taking suprema over $\bar{x}$, together with Theorem 2.1, yields the continuity of $S(t)$ with respect to the $L^2_{b,\phi}(\Omega)$-norm.

The existence of a global attractor is now proved in a straightforward manner. Recall that, following [5], a set $\mathcal{A}$ is called (X,Y)-attractor for the dynamical system $(S(t),X)$, provided that $\mathcal{A}$ is fully invariant, compact in the topology Y, and attracts bounded subsets of X uniformly in the topology of Y.

Theorem 4.4. *The dynamical system $(S(t), L^2_b(\Omega))$ has an $(L^2_b(\Omega), L^2_{\mathrm{loc}}(\Omega))$-attractor.*

Proof. 1. We first establish the attractor for $(L(t), \mathcal{X})$. Recalling Theorem 2.4 above, it follows from (4.4), (4.5) that $\mathcal{X}$ is bounded in each of the seminorms

$$\chi \in L^2\big(0, \ell; W^{1,2}(C_k)\big) \cap L^p\big(0, \ell; L^p(C_k)\big),$$
$$\chi_t \in L^2\big(0, \ell; W^{-1,2}(C_k)\big) + L^{p'}\big(0, \ell; L^{p'}(C_k)\big).$$

By the Aubin–Lions Lemma, we then have compactness in $L^2(0, \ell; L^2(C_k))$ for any k; invoking the boundedness of $\mathcal{X}$ in $L^\infty(0, \ell; L^2_b(\Omega))$, we have indeed the compactness in $L^2(0, \ell; L^2_{\bar{x}}(\Omega))$. Recalling the continuity of $L(t)$, we deduce the existence of $\mathcal{A}_\ell$, the global attractor for $(L(t), \mathcal{X})$, by standard arguments.

2. Set

$$\mathcal{A} := e(\mathcal{A}_\ell). \tag{4.9}$$

From the continuity of e and the equivalence (4.3) one immediately obtains that this is the desired $(L^2_b(\Omega), L^2_{\mathrm{loc}}(\Omega))$-attractor for $(S(t), L^2_b(\Omega))$. $\square$

Remark 4.5. The *existence* of the global attractor can be proven solely in virtue of the regularity established in Theorem 3.2 above. Of course, in this case we can no longer choose $\mathcal{B}$ bounded in $L^q_b(\Omega)$. Moreover, extensions to systems are possible on account of Remarks 3.7 and 3.10.

5. Entropy estimates

The aim of the last section is to study finite-dimensionality of the attractor. As is well known, for reaction–diffusion equations in the case of a *bounded* domain Ω, the attractor $\mathcal{A}_\Omega$ satisfies (see, e.g., [22, Chap. VI, Sec. 2.1])

$$H_\varepsilon\big(\mathcal{A}_\Omega, L^2(\Omega)\big) \leqslant c_0 \operatorname{vol}(\Omega) \ln \frac{1}{\varepsilon}, \quad \varepsilon \in (0, \varepsilon_0). \tag{5.1}$$

Here the constant c_0 only depends on the structural properties of the equation, but not on the size of Ω. In particular, we have finite fractal dimension of $\mathcal{A}_\Omega$. Such an estimate being meaningless if Ω has infinite volume, we will follow [26,27] to estimate the entropy of $\mathcal{A}$ in the seminorm $L^2_b(\mathcal{O})$, where $\mathcal{O}$ is a suitable bounded subdomain of Ω. Also, note that the left-hand side of (5.1) is larger if $L^2(\Omega)$ is replaced by $L^2_b(\Omega)$ (cf. also (5.2) below).

Our main result is the following theorem.

Theorem 5.1. *Let $d \leqslant 3$ and set*

$$\Omega_{x_0,R} := \Omega \cap B_R\big(x_0, \mathbb{R}^d\big) = B_R\big(x_0, \mathbb{R}^d\big).$$

Then, there exist c_0, c_1 and $\varepsilon_0 > 0$, such that, for any $x_0 \in \Omega$, $R \geqslant 1$ and $\varepsilon \in (0, \varepsilon_0)$ one has

$$H_\varepsilon\big(\mathcal{A}, L^2_b(\Omega_{x_0,R})\big) \leqslant c_0 \left(R + c_1 \ln \frac{1}{\varepsilon}\right)^d \ln \frac{1}{\varepsilon}. \tag{5.2}$$

The rest of this section is devoted to the proof of this result. Remark that (5.2) is completely analogous to (5.1), but for the "extra term" $c_1 \ln \frac{1}{\varepsilon}$. Heuristically, the finer description of $\mathcal{A}$ one seeks, the larger portion of Ω influences the dynamics. Moreover, the optimality of this estimate is suggested by the results of [27] where a similar bound is proved to be sharp, albeit in a different regularity setting.

Given $x_0 \in \Omega$ and $R \geqslant 1$, we set

$$\psi_{x_0,R} := \begin{cases} 1; & |x-x_0| \leqslant R+\sqrt{d}, \\ \exp((R+\sqrt{d}-|x-x_0|)/2); & \text{otherwise}. \end{cases} \tag{5.3}$$

Clearly, one has

$$H_\varepsilon\big(\mathcal{A}, L^2_b(\Omega_{x_0,R})\big) \leqslant H_\varepsilon\big(\mathcal{A}, L^2_{b,\psi_{x_0,R}}(\Omega)\big),$$

hence it is enough to estimate the right-hand side. As usual, one arrives at such a result through suitable iterative coverings obtained by combining the "smoothing property" of solution operators with compact embeddings in the appropriate function spaces. As in the previous section, we will rely on the natural parabolic estimates. Let us start by an improved continuity result for the evolution operators.

Theorem 5.2. *Let ϕ be an admissible weight function of growth rate $\mu < 1$. Then,*

1. $L(t) : L^2_{b,\phi}(0,\ell; L^2(\Omega)) \to L^2_{b,\phi}(0,\ell; L^2(\Omega))$ *are Lipschitz continuous uniformly w.r.t.* $t \in [0,T]$;
2. $e : L^2_{b,\phi}(0,\ell; L^2(\Omega)) \to L^2_{b,\phi}(\Omega)$ *is Lipschitz continuous.*

Proof. It follows from (4.8) that

$$\int_\Omega \big|w(x,t+s)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x \leqslant c_4 \int_\Omega \big|w(x,s)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x, \quad t \in [0,T],$$

where c_4 only depends on T. Hence, integrating in $\mathrm{d}s$ over $(0,\ell)$, one has

$$\int_{\Omega\times(t,t+\ell)} \big|w(x,s)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}s \leqslant c_4 \int_{\Omega\times(0,\ell)} \big|w(x,s)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}s.$$

Applying $\sup_{\bar{x}\in\Omega} \phi(\bar{x})$ and using Theorem 2.4 again, one obtains – in terms of trajectories –

$$\big\|L(t)\chi_1 - L(t)\chi_2\big\|_{L^2_{b,\phi}(0,\ell;L^2(\Omega))} \leqslant c_5 \|\chi_1-\chi_2\|_{L^2_{b,\phi}(0,\ell;L^2(\Omega))}. \tag{5.4}$$

This proves part 1. Analogously, one deduces

$$\int_\Omega \big|w(x,\ell)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x \leqslant c_6 \int_{\Omega\times(0,\ell)} \big|w(x,t)\big|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t,$$

and thus

$$\big\|e(\chi_1)-e(\chi_2)\big\|_{L^2_{b,\phi}(\Omega)} \leqslant c_5 \|\chi_1-\chi_2\|_{L^2_{b,\phi}(0,\ell;L^2(\Omega))}, \tag{5.5}$$

i.e., part 2. $\square$

A key step towards our entropy estimate is the following "smoothing property" of the operator $L(t)$ – a sort of typical result in the spirit of the method of trajectories.

Theorem 5.3. *Let ϕ be an admissible weight function of growth rate $\mu < 1$. Then for any $\chi_1, \chi_2 \in \mathcal{X}$, one has*

$$\left\|L(\ell)\chi_1 - L(\ell)\chi_2\right\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))} \leqslant K_1\|\chi_1 - \chi_2\|_{L^2_{b,\phi}(0,\ell;L^2(\Omega))}, \tag{5.6}$$

$$\left\|\partial_t\big(L(\ell)\chi_1 - L(\ell)\chi_2\big)\right\|_{L^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))} \leqslant K_2\|\chi_1 - \chi_2\|_{L^2_{b,\phi}(0,\ell;L^2(\Omega))}, \tag{5.7}$$

where K_1, K_2 only depend on the constants μ and c characterizing the growth of ϕ in (2.4)*.*

Proof. Let u, v be the weak solutions such that $u|_{[0,\ell]} = \chi_1$, $v|_{[0,\ell]} = \chi_2$, and set $w := u - v$.

STEP 1. One deduces from (4.8) that

$$\int\limits_{\Omega\times(\ell,2\ell)} \big(\left|\nabla w(x,t)\right|^2 + \left|w(x,t)\right|^2\big)e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \leqslant c_1 \int\limits_{\Omega} \left|w(x,s)\right|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x, \quad \forall s\in(0,\ell).$$

Integrating over s and applying $\sup_{\bar{x}}\phi(\bar{x})$, in view of Theorems 2.1, 2.4, yields (5.6).

STEP 2. We will show that

$$\|\partial_t w\|_{L^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))} \leqslant c_2\|w\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))},$$

which combined with (5.6) implies (5.7). Using Eq. (3.14) and Theorem 2.4, we can write

$$\begin{aligned}
\|\partial_t w\|_{L^2_{b,\phi}(0,\ell;W^{-1,2}(\Omega))} &\leqslant c_3 \sup_z \sup_{\bar{x}} \phi(\bar{x}) \int\limits_{\Omega\times(0,\ell)} \partial_t w z e^{-|x-\bar{x}|}\,\mathrm{d}x \\
&= \sup_z \sup_{\bar{x}} \phi(\bar{x}) \int\limits_{\Omega\times(0,\ell)} \bigg[\big(a(\nabla u) - a(\nabla v)\big)\cdot\bigg(\nabla z - z\frac{x-\bar{x}}{|x-\bar{x}|}\bigg) + \big(h(x,\nabla u) - h(x,\nabla v)\big)z \\
&\quad + \big(f(u) - f(v)\big)z\bigg]e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.8}$$

The first supremum is taken over $z \in L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))$ with unit norm. Recalling (3.2), the first two terms on the right-hand side get estimated as

$$\begin{aligned}
&c_4\phi(\bar{x}) \int\limits_{\Omega\times(0,\ell)} |\nabla w|\big(|\nabla z| + |z|\big)e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant c_4\Bigg(\phi(\bar{x}) \int\limits_{\Omega\times(0,\ell)} |\nabla w|^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t\Bigg)^{1/2}\Bigg(\phi(\bar{x}) \int\limits_{\Omega\times(0,\ell)} \big(|\nabla z| + |z|\big)^2 e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t\Bigg)^{1/2} \\
&\quad\leqslant c_5\|w\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))}\|z\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))} = c_5\|w\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))}.
\end{aligned}$$

Invoking (3.6), the last term in the right-hand side of (5.8) is estimated as

$$c_6\phi(\bar{x})\int\limits_{\Omega\times(0,\ell)}\big(1+|u|+|v|\big)^{p-2}|w||z|e^{-|x-\bar{x}|}\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant c_7\sup_k\phi(x_k)\int\limits_0^\ell\big\|\big(1+|u|+|v|\big)^{p-2}wz\big\|_{L^1(C_k)}\,\mathrm{d}t. \tag{5.9}$$

We have used (2.31) with $p=1$. Furthermore, thanks to the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ and the additional regularity (4.6) (where we take $q=3(p-2)/2$), we can estimate by Hölder's inequality

$$\big\|\big(1+|u|+|v|\big)^{p-2}wz\big\|_1\leqslant\big\|\big(1+|u|+|v|\big)\big\|^{p-2}_{\frac{3(p-2)}{2}}\|w\|_6\|z\|_6\leqslant c_8\|w\|_{W^{1,2}(C_k)}\|z\|_{W^{1,2}(C_k)}.$$

Hence, (5.9) can further be estimated as

$$c_9\left(\sup_k\phi(x_k)\int\limits_0^\ell\|w\|^2_{W^{1,2}(C_k)}\,\mathrm{d}t\right)^{1/2}\left(\sup_k\phi(x_k)\int\limits_0^\ell\|z\|^2_{W^{1,2}(C_k)}\,\mathrm{d}t\right)^{1/2}\leqslant c_{10}\|w\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\Omega))}.$$

This finishes the proof. □

What we have just shown is the Lipschitz continuity of $L(\ell)$ from $L^2_{b,\phi}(0,\ell;L^2(\Omega))$ into $W_{b,\phi}(Q)$, where the latter space was defined in (2.32). However – and this is the peculiar feature of the analysis in unbounded domains – the space $W_{b,\phi}(Q)$ is NOT compactly embedded into $L^2_{b,\phi}(0,\ell;L^2(\Omega))$. The compactness can only be employed using seminorms related to restrictions to bounded sets $\mathcal{O}\subset\Omega$ (cf. Lemma 2.6) which also exhibit the correct dependence on the volume of the domain of restriction $\mathcal{O}$.

The last ingredient is to employ the boundedness of $\mathcal{X}$ in $L^2_b(\Omega)$ together with the decay of the weight $\psi_{x_0,R}$ to localize the entropy of an attractor to a bounded domain up to some error. This estimate is actually the origin of the "extra term" in (5.2).

Lemma 5.4. *Let $\varepsilon_0>0$ be given. Then there exists c_1 such that, for any $x_0\in\mathbb{R}^d$, $R\geqslant1$ and $\varepsilon\in(0,\varepsilon_0)$, having set*

$$R(\varepsilon):=R+c_1\left(1+\ln\frac{1}{\varepsilon}\right),$$

for arbitrary $\chi_1,\chi_2\in\mathcal{X}$, one has

$$\|\chi_1-\chi_2\|_{L^2_{b,\psi_{x_0,R}}(0,\ell;L^2(\Omega))}\leqslant\max\big\{\|\chi_1-\chi_2\|_{L^2_{b,\psi_{x_0,R}}(0,\ell;L^2(\Omega_{x_0,R(\varepsilon)}))},\varepsilon\big\}.$$

Proof. Recall that

$$\|\chi_1-\chi_2\|^2_{L^2_{b,\psi_{x_0,R}}(0,\ell;L^2(\Omega))}$$

$$=\max\Big\{\sup_{x_k\notin\Omega_{x_0,R(\varepsilon)}}\psi_{x_0,R}(x_k)\|\chi_1-\chi_2\|^2_{L^2(0,\ell;L^2(C_k))},\|\chi_1-\chi_2\|_{L^2_{b,\psi_{x_0,R}}(0,\ell;L^2(\Omega_{x_0,R(\varepsilon)}))}\Big\}.$$

However, thanks to the decay of $\psi_{x_0,R}$ and the boundedness of $\mathcal{X}$, the first term is automatically smaller than ε^2 due to proper choice of constant c_1. $\square$

We are now ready to prove the main result.

Proof of Theorem 5.1. In what follows, $\psi_{x_0,R}$ is the weight function defined in (5.3). Remark that it has growth rate $\mu = 1/2$, and satisfies (2.4) with $c = 1$ for any $x_0 \in \mathbb{R}^d$, $R \geqslant 1$.

1. First (and the key) step of the proof is the recurrent estimate

$$H_{\alpha/2}\big(\mathcal{A}_\ell, L^2_{b,\psi_{x_0,R}}\big(0,\ell;L^2(\Omega)\big)\big) \leqslant H_\alpha\big(\mathcal{A}_\ell, L^2_{b,\psi_{x_0,R}}\big(0,\ell;L^2(\Omega)\big)\big) + c_0\left(R + c_1 \ln\frac{1}{\alpha}\right)^d. \tag{5.10}$$

Indeed, let

$$\mathcal{A}_\ell \subset \bigcup_m B_\alpha\big(\chi_m; L^2_{b,\psi_{x_0,R}}\big(0,\ell;L^2(\Omega)\big)\big).$$

Thanks to Theorem 5.3 and invariance of $\mathcal{A}_\ell$, we then deduce that, for some $\tilde{\chi}_m \in \mathcal{X}$ and some $\kappa > 0$

$$\mathcal{A}_\ell \subset \bigcup_m B_{\kappa\alpha}\big(\tilde{\chi}_m; W_{b,\psi}(Q)\big).$$

By Lemma 2.6, each of the latter balls can be covered so that

$$H_{\alpha/2}\big(B_{\kappa\alpha}\big(\tilde{\chi}_m; W_{b,\psi}(Q)\big), X_{b,\psi}(\Omega_{x_0,R(\alpha/2)})\big) \leqslant c_0\left(R + c_1 \ln\frac{1}{\alpha}\right)^d.$$

Finally, by Lemma 5.4, covering of $\mathcal{A}_\ell$ by $\alpha/2$-balls in the $X_{b,\psi}(\Omega_{x_0,R(\alpha/2)})$ seminorm is also covering by $\alpha/2$-balls in the norm $L^2_{b,\psi_{x_0,R}}(0,\ell;L^2(\Omega))$.

2. Choose $\varepsilon_0 > 0$ such that $H_{\varepsilon_0}(\mathcal{A}_\ell, L^2_{b,\psi_{x_0,R}}(0,\ell;L^2(\Omega))) = 0$. Given $\varepsilon \in (0,\varepsilon_0)$, one picks $k \in \mathbb{N}$ such that

$$2^{-k}\varepsilon_0 \leqslant \varepsilon < 2^{-k+1}\varepsilon_0.$$

Note that this means $k \leqslant c \ln 1/\varepsilon$, at least provided ε is small enough. Then, using (5.10), one can estimate

$$\begin{aligned}
&H_\varepsilon\big(\mathcal{A}_\ell, L^2_{b,\psi_{x_0,R}}\big(0,\ell;L^2(\Omega)\big)\big)\\
&\quad\leqslant H_{2^{-k}\varepsilon_0}\big(\mathcal{A}_\ell, L^2_{b,\psi_{x_0,R}}\big(0,\ell;L^2(\Omega)\big)\big)\\
&\quad\leqslant \sum_{l=1}^k H_{2^{-l}\varepsilon_0}\big(\mathcal{A}_\ell, L^2_{b,\psi_{x_0,R}}\big(0,\ell;L^2(\Omega)\big)\big) - H_{2^{-l+1}\varepsilon_0}\big(\mathcal{A}_\ell, L^2_{b,\psi_{x_0,R}}\big(0,\ell;L^2(\Omega)\big)\big)\\
&\quad\leqslant \sum_{l=1}^k c_0\left(R + c_1 \ln\frac{2^{l-1}}{\varepsilon_0}\right)^d\\
&\quad\leqslant c_0\left(R + c_1 \ln\frac{1}{\varepsilon}\right)^d \ln\frac{1}{\varepsilon}.
\end{aligned}$$

3. Finally, in view of Lipschitz continuity of e (Theorem 5.2) and (4.9) we conclude

$$H_\varepsilon\big(\mathcal{A}, L^2_{b,\psi_{x_0,R}}(\Omega)\big) \leqslant H_{\varepsilon/\kappa}\big(\mathcal{A}_\ell, L^2_{b,\psi_{x_0,R}}\big(0,\ell; L^2(\Omega)\big)\big),$$

where κ is the Lipschitz constant of the mapping e, which is independent of the particular weight function $\psi_{x_0,R}$. This finishes the proof. $\square$

Remark 5.5. Recalling Remark 3.10, we point out that Theorem 5.1 can be extended to systems where the diffusion operator is the vector Laplacian $-\Delta$ and the nonlinear convective term h is replaced by $\mathbf{h} : \Omega \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ satisfying suitable generalizations of (3.9) and (3.10).

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