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Recovery of the m-function from spectral data for generalized Sturm–Liouville problems

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Abstract

The Sturm–Liouville problem $-y'' + qy = \lambda y$, $y(0)\cos\alpha = y'(0)\sin\alpha$, $(y'/y)(1) = h(\lambda)/g(\lambda)$ is studied, where h and g are real polynomials. Generalized norming constants ρ_n^k associated with eigenvalues Λ_n are defined and formulae are given for recovering the m-function from these constants. This leads to a uniqueness theorem for the associated inverse problem.

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1. Introduction

In this article we continue our treatment of the regular Sturm–Liouville equation

$$ly := -y'' + qy = \lambda y \text{ on } [0,1] \tag{1.1}$$

with $q \in AC[0,1]$, subject to the boundary conditions

$$y(0)\cos\alpha = y'(0)\sin\alpha, \quad \alpha \in [0,\pi) \tag{1.2}$$

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and

$$\frac{y'}{y}(1) = f(\lambda), \tag{1.3}$$

where

$$f(\lambda) = \frac{h(\lambda)}{g(\lambda)}$$

and g and h are polynomials with real coefficients and no common zeros. When $f(\lambda)=\infty$, (1.3) is interpreted as a Dirichlet condition $y(1)=0$.

This type of boundary condition has been discussed by numerous authors, e.g., [1,2,4–7,9,11–13,18–21]. For applications to science and engineering, see the references [7].

Our basic goal is to obtain formulae for the Titchmarsh–Weyl m-function in terms of certain spectral data for (1.1)–(1.3). Here we recall

Definition 1.1. Let $v(x,\lambda)$ be a non-identically zero solution of (1.1) satisfying (1.3). The Weyl m-function of (1.1)–(1.3) is defined by

$$m(\lambda) = \frac{v'(0,\lambda)\cos\alpha + v(0,\lambda)\sin\alpha}{v'(0,\lambda)\sin\alpha - v(0,\lambda)\cos\alpha}. \tag{1.4}$$

The poles of the m-function are precisely the eigenvalues of (1.1)–(1.3) and the order of each pole coincides with the multiplicity (which we define in Section 2) of the corresponding eigenvalue. The spectral data will consist of the eigenvalues λ_n and a generalization (see below) ρ_n^k of the "norming constants" which, for problems with λ-independent boundary conditions, take the form

$$\rho_n^0 = \int_0^1 \frac{v(x,\lambda_n)^2}{v(0,\lambda_n)^2}\,\mathrm{d}x$$

in the case $\alpha \neq 0$.

It is known (even for the case of $f(\lambda)$ and $g(\lambda)$ affine with real coefficients, cf. [3]) that the eigenvalues may be nonreal or even nonsemisimple. In fact one of our main contributions is to find (as far as we know for the first time) an appropriate generalization of the norming constants for such eigenvalues. We employ a bilinear form on $L^2 \oplus \mathbb{C}^M$, for suitable M, evaluated on appropriately defined "links" of the Jordan chains for an operator L corresponding to (1.1)–(1.3). This is equivalent to using a sesquilinear form σ on left and right Jordan chains for L.

It is worth noting that the above form σ is in general indefinite, in contrast with the work of several of the works cited above, e.g., [6,21], where f is a Nevanlinna function (i.e., $f \in N_0$). In our case f belongs to a generalized Nevanlinna class N_k and as a consequence L is self-adjoint in a Pontryagin, not Hilbert, space in general. Thus the nonreality and non-semi-simplicity noted above is limited to a finite number of eigenvalues and can be seen directly from our data λ_n and ρ_n^k. We note that m is also a generalized Nevanlinna function, a fact exploited in [10,11] and several references therein. The existence (and uniqueness) of σ was noted by Russakovskii [18,19]. The general case can be tackled via Bezoutians (cf. [19]), but to shorten the presentation we make an assumption (see Section 2) about the simplicity of the roots of g or of h since then there is a particularly simple formula for σ—see (2.2).

The basic constructions noted above are introduced in Section 2, and the singular part of the Laurent series of m at λ_n is developed in terms of the λ_n and ρ_n^k in Section 3. The full formulae for the recovery of m are given for the non-Dirichlet and Dirichlet cases in Sections 4 and 5, respectively, our main results being Theorems 4.4, 5.2 and 5.3.

An important by-product of our formulae is an "inverse uniqueness" result. From [7] we know that q, α and f depend uniquely on m (i.e., if $\tilde{q}$, $\tilde{\alpha}$ and $\tilde{f}$ also give rise to the same m, then $q=\tilde{q}$, $\alpha=\tilde{\alpha}$ and $f=\tilde{f}$). It will follow from our formulae that q, α and f also depend uniquely on λ_n and ρ_n^k (see Theorem 6.1), thus generalizing the classical result of Börg [8], Krein [15] and Marchenko [16] to boundary conditions of the form (1.3). In fact, combining the results of [7] with those here, we see that each member of the following list of data sets depends uniquely on each of the others: (q,α,f); m; (ϕ,α) where $\phi(x,\lambda)$ is the Prüfer angle (specified at $x=1$); and (λ_n,ρ_n^k). Pairs of spectra (λ_n,μ_n), where μ_n corresponds to (q,β,f), may be added to this list although obviously β must then be assumed known.

2. Preliminaries

If $\deg(g)\geqslant\deg(h)$ then we set $M=\deg(g)$ and assume that g is monic with real simple zeros, and if $\deg(g)<\deg(h)$ then we set $M=\deg(h)$ and assume that h is monic with real simple zeros.

We now define v as the solution of (1.1) satisfying the terminal conditions

$$v(1,\lambda)=g(\lambda),$$

$$v'(1,\lambda)=h(\lambda)$$

and write

$$D(\lambda)=v'(0,\lambda)\sin\alpha-v(0,\lambda)\cos\alpha,$$

where, as in the rest of this paper, $'=\partial/\partial x$ giving $v'(1,\lambda)=\frac{\partial v(x,\lambda)}{\partial x}|_{x=1}$ and $^{(j)}=\partial^j/\partial\lambda^j$ giving $v^{(j)}(x,\Lambda)=\frac{\partial^j v(x,\lambda)}{\partial\lambda^j}|_{\lambda=\Lambda}$.

The eigenvalues λ_n are the zeros of $D(\lambda)$, listed by increasing real part, according to multiplicity as zeros of $D(\lambda)$. For brevity (1.1)–(1.3) will frequently be denoted by the shorthand (α,f,q).

At various points in the analysis we shall need to distinguish the cases $\deg(h)\leqslant\deg(g)$ (respectively $\deg(h)>\deg(g)$). The abbreviation 'resp*' will be used consistently for this purpose. Our assumptions on g and h allow us to write $f(\lambda)$ (1.3) in the form

$$f(\lambda)(\text{resp}^*\ 1/f(\lambda))=b-\sum_{k=1}^{M}\frac{b_k}{\lambda-c_k} \tag{2.1}$$

in the case $\deg(h)\leqslant\deg(g)$ (resp* $\deg(h)>\deg(g)$), where the coefficients b,b_k,c_k for $k=1,\dots,M$ are real.

We define a sesquilinear form on $H=L^2[0,1]\oplus\mathbb{C}^M$ by

$$\langle Y,Z\rangle=\int_0^1 y\bar{z}+\sum_{k=1}^{M}\frac{y_k\bar{z}_k}{b_k}\left(\text{resp}^*\ \int_0^1 y\bar{z}-\sum_{k=1}^{M}\frac{y_k\bar{z}_k}{b_k}\right), \tag{2.2}$$

where y and z denote the L^2-components of Y and Z, respectively, and (y_i) and (z_i) the $\mathbb{C}^M$ components of Y and Z. For later use, we define the bilinear form $[X,Y]=\langle X,\bar{Y}\rangle$ for $X,Y\in H$.

The boundary value problem (1.1)–(1.3) can then be posed in the Pontryagin space $(H,\langle\cdot,\cdot\rangle)$ as an eigenvalue problem for the operator

$$LY=\begin{bmatrix} ly \\ c_1y_1-b_1y(1) \\ \cdots \\ c_My_M-b_My(1)\end{bmatrix}\left(\text{resp}^*\begin{bmatrix} ly \\ c_1y_1-b_1y'(1) \\ \cdots \\ c_My_M-b_My'(1)\end{bmatrix}\right)\quad\text{with}\quad Y=\begin{bmatrix} y \\ y_1 \\ \cdots \\ y_M\end{bmatrix} \tag{2.3}$$

and domain

$$D(L)=\left\{Y\in H: y,y'\in AC, ly\in L^2, y(0)\cos\alpha=y'(0)\sin\alpha, [y'-by](1)=\sum_{k=1}^{M}y_k\right\}$$

$$\left(\text{resp}^*\ D(L)=\left\{Y\in H: y,y'\in AC, ly\in L^2, y(0)\cos\alpha=y'(0)\sin\alpha, [y-by'](1)=\sum_{k=1}^{M}y_k\right\}\right).$$

Note that λ is an eigenvalue of (1.1)–(1.3) with eigenfunction y if and only if λ is an eigenvalue of L and a corresponding eigenvector of L is

$$Y=\begin{bmatrix} y(x) \\ \frac{b_1}{c_1-\lambda}y(1) \\ \vdots \\ \frac{b_M}{c_M-\lambda}y(1)\end{bmatrix}\quad\left(\text{resp}^*\begin{bmatrix} y(x) \\ \frac{b_1}{c_1-\lambda}y'(1) \\ \vdots \\ \frac{b_M}{c_M-\lambda}y'(1)\end{bmatrix}\right)$$

provided that $\lambda\neq c_j,\ j=1,\dots,M$. The operator L is self-adjoint and bounded below with compact resolvent in the Pontryagin space H from which we can deduce that the spectrum of L is discrete, has a finite number of nonreal eigenvalues, and has $+\infty$ as its only accumulation point. These results can be found in [18], although some are only stated, and the inner product is not specified explicitly. Complete proofs in the Hilbert space case can be found in [6], and they extend readily to the case here. See also Remarks 2.1.

For $\lambda\neq c_j,\ j=1,\dots,M$, let

$$V(x,\lambda)=\begin{bmatrix} v(x,\lambda) \\ \frac{b_1}{c_1-\lambda}v(1,\lambda) \\ \vdots \\ \frac{b_M}{c_M-\lambda}v(1,\lambda)\end{bmatrix}\quad\left(\text{resp}^*\begin{bmatrix} v(x,\lambda) \\ \frac{b_1}{c_1-\lambda}v'(1,\lambda) \\ \vdots \\ \frac{b_M}{c_M-\lambda}v'(1,\lambda)\end{bmatrix}\right)$$

and in the case of $\lambda = c_j$ let

$$V(x,c_j)=\begin{bmatrix} v(x,c_j)\\ 0\\ \vdots\\ 0\\ v'(1,c_j)\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix}=\begin{bmatrix} v(x,c_j)\\ 0\\ \vdots\\ 0\\ -b_j g^{(1)}(c_j)\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix}\left(\text{resp}^*\ \begin{bmatrix} v(x,c_j)\\ 0\\ \vdots\\ 0\\ v(1,c_j)\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix}=\begin{bmatrix} v(x,c_j)\\ 0\\ \vdots\\ 0\\ -b_j h^{(1)}(c_j)\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix}\right).$$

The $V(x,\lambda)$ will provide the key to defining the generalized norming constants, but first they need an appropriate normalization as defined below. Let

$$U(x,\lambda)=\begin{cases}\sin\alpha\,\dfrac{V(x,\lambda)}{v(0,\lambda)}, & \alpha\neq 0\\[2mm] \dfrac{V(x,\lambda)}{v'(0,\lambda)}, & \alpha=0\end{cases}\qquad \text{for all}\quad x\in[0,1],\lambda\in\mathbb{C}$$

($U(x,\lambda)$ has a pole when $v(0,\lambda)=0$ in the case $\alpha\neq 0$ and when $v'(0,\lambda)=0$ in the case $\alpha=0$, and is consequently not defined at such values of λ).

Let Λ_n denote the eigenvalues of (α,f,q) listed without repetition and let $v_n\geqslant 1$ denote the multiplicity of the eigenvalue $\Lambda_n, n=0,1,2,\ldots$. For large n, $\Lambda_n=\lambda_{n+v}$ where

$$v=\sum_{n=0}^{\infty}(v_n-1). \tag{2.4}$$

Here $0\leqslant v<\infty$, [18], finiteness again following from self-adjointness of L in H. For an eigenvalue Λ_n, let

$$U_n^0(x)=U(x,\Lambda_n).$$

If $v_n>1$ we denote the associated root functions, [17, pp. 16–21], by

$$U_n^j(x)=U^{(j)}(x,\Lambda_n),\quad j=1,\ldots,v_n-1.$$

It should be noted [7, Lemma 2.1], that the algebraic multiplicity of the eigenvalue Λ_n of (1.1)–(1.3) coincides with the order of Λ_n as a zero of D. Moreover, the eigenvalues are geometrically simple and the $U_n^j, 0\leqslant j\leqslant v_n-1$, form a Jordan basis of the algebraic eigenspace at Λ_n. We denote the $L^2[0,1]$ components of $U_n^{(j)}(x)$ and $V^{(j)}(x,\lambda)$ by $u_n^{(j)}(x)$ and $v^{(j)}(x,\lambda)$. It should be observed that $u_n^{(j)}(x)$ and $v^{(j)}(x,\lambda)$ are also the jth partial derivatives of $u(x,\lambda)$ and $v(x,\lambda)$ with respect to λ and evaluated at Λ_n and λ, respectively.

Note that $u(x,\lambda)$ is a solution of (1.1) and for $\alpha\neq 0$, $u(0,\lambda)=\sin\alpha$ and consequently $u^{(j)}(0,\lambda)=0$ for all $j\geqslant 1$. At $\lambda=\Lambda_n$, $u'(0,\lambda)\sin\alpha-u(0,\lambda)\cos\alpha$ has a zero of order v_n, so $u^{(j)\prime}(0,\Lambda_n)=0$ for

$1 \leqslant j \leqslant v_n - 1$ and $u^{(v_n)\prime}(0, \Lambda_n) \neq 0$. In the case $\alpha = 0$, $u'(0,\lambda) = 1$ giving $u^{(j)\prime}(0,\lambda) = 0$ for all $j \geqslant 1$, and $u(0,\lambda)$ has a zero of order v_n at Λ_n, giving $u^{(j)}(0,\Lambda_n) = 0$ for $0 \leqslant j \leqslant v_n - 1$ and $u^{(v_n)}(0,\Lambda_n) \neq 0$. In addition we observe that $\tilde{L}V(x,\lambda) = \lambda V(x,\lambda)$ and $\tilde{L}U(x,\lambda) = \lambda U(x,\lambda)$ for all $x \in [0,1]$ where $\tilde{L}$ is the extension of L to the domain

$$D(\tilde{L}) = \{Y \in H : y, y' \in AC, ly \in L^2\}.$$

We are now ready for the construction of our generalized norming constants associated with the eigenvalue Λ_n via the equations

$$\rho_n^j = \left[U_n^j, U_n^{v_n - 1}\right], \quad j = 0, \ldots, v_n - 1.$$

Together with the λ_n, these will form our data from which the m-function will be reconstructed.

Remark 2.1. Since L is self-adjoint in H, the nonreal eigenvalues appear as a finite number of conjugate pairs, even as to multiplicity. Moreover, if Λ_n is nonreal then the above constructions also appear in conjugate pairs, e.g., $v(x, \bar{\Lambda}_n) = \overline{v(x,\Lambda_n)}$, $U^j(x,\bar{\Lambda}_n) = \overline{U^j(x,\Lambda_n)}$ and similarly for the $\rho_n^j, j = 0, \ldots, v_n - 1$.

3. The Laurent expansion of *m*

The first lemma in this section will supply the necessary background to express the coefficients of the negative power terms in the Laurent expansion of $m(\lambda)$ at an eigenvalue Λ_n in terms of the λ derivatives of $u(0,\lambda)$ and $u'(0,\lambda)$ at Λ_n.

Lemma 3.1. *Let $Z(\lambda)$ be analytic in a neighbourhood of a and have a zero of order N at a. If $a_k = (1/k!)(\partial^k Z/\partial\lambda^k)(a)$ then the Laurent expansion of $1/Z$ about a has the form*

$$\frac{1}{Z}(\lambda) = E(\lambda) + \sum_{k=1}^{N} \frac{p_k}{(\lambda - a)^k},$$

where $E(\lambda)$ is analytic in a neighbourhood of a and

$$p_N = \frac{1}{a_N},$$

$$p_{N-j} = -\frac{1}{a_N} \sum_{k=0}^{j-1} p_{N-k} a_{N+j-k}, \quad j = 1, \ldots, N-1.$$

The proof is left to the reader.

The next theorem links the derivatives $u^{(v_n+j)}(0,\Lambda_n)$ to the norming constants $\rho_n^j, j = 0, \ldots, v_n - 1$.

Theorem 3.2. *For $j = 0, \ldots, v_n - 1$,*

$$v_n \rho_n^j \binom{v_n + j}{v_n} = \begin{cases} u^{(v_n+j)\prime}(0,\Lambda_n) \sin\alpha, & \alpha \neq 0, \\ -u^{(v_n+j)}(0,\Lambda_n), & \alpha = 0. \end{cases}$$

Proof. Integration by parts, along with the definitions of $V(x,\lambda)$ and $[\cdot,\cdot]$, gives

$$\begin{aligned}
&(\lambda-\mu)[V(x,\lambda),V(x,\mu)]\\
&\quad=[\lambda V(x,\lambda),V(x,\mu)]-[V(x,\lambda),\mu V(x,\mu)]\\
&\quad=[\tilde{L}V(x,\lambda),V(x,\mu)]-[V(x,\lambda),\tilde{L}V(x,\mu)]\\
&\quad=[v(x,\lambda)v'(x,\mu)-v'(x,\lambda)v(x,\mu)]_{x=0}^{x=1}\\
&\qquad+\sum_{k=1}^{M}\frac{v(1,\lambda)v(1,\mu)}{b_k}\left[\left(\frac{c_kb_k}{c_k-\lambda}-b_k\right)\frac{b_k}{c_k-\mu}-\left(\frac{c_kb_k}{c_k-\mu}-b_k\right)\frac{b_k}{c_k-\lambda}\right]\\
&\quad=[v(1,\lambda)v'(1,\mu)-v'(1,\lambda)v(1,\mu)]-[v(0,\lambda)v'(0,\mu)-v'(0,\lambda)v(0,\mu)]\\
&\qquad+(\lambda-\mu)v(1,\lambda)v(1,\mu)\sum_{k=1}^{M}\frac{b_k}{(c_k-\lambda)(c_k-\mu)}\\
&\quad=v(1,\lambda)v(1,\mu)[f(\mu)-f(\lambda)]-[v(0,\lambda)v'(0,\mu)-v'(0,\lambda)v(0,\mu)]\\
&\qquad+v(1,\lambda)v(1,\mu)[f(\lambda)-f(\mu)]\\
&\quad=v'(0,\lambda)v(0,\mu)-v(0,\lambda)v'(0,\mu)
\end{aligned}$$

$$\begin{aligned}
&(\text{resp}^*\ (\lambda-\mu)[V(x,\lambda),V(x,\mu)]\\
&\quad=[\lambda V(x,\lambda),V(x,\mu)]-[V(x,\lambda),\mu V(x,\mu)]\\
&\quad=[\tilde{L}V(x,\lambda),V(x,\mu)]-[V(x,\lambda),\tilde{L}V(x,\mu)]\\
&\quad=[v(x,\lambda)v'(x,\mu)-v'(x,\lambda)v(x,\mu)]_0^1\\
&\qquad+\sum_{k=1}^{M}v'(1,\lambda)v'(1,\mu)\left[\frac{b_k}{c_k-\mu}-\frac{b_k}{c_k-\lambda}\right]\\
&\quad=v'(0,\lambda)v(0,\mu)-v(0,\lambda)v'(0,\mu)).
\end{aligned}$$

Thus, independently of $\deg(g)$ and $\deg(h)$,

$$(\lambda-\mu)[V(x,\lambda),V(x,\mu)]=v'(0,\lambda)v(0,\mu)-v(0,\lambda)v'(0,\mu). \tag{3.1}$$

In the above we have assumed $\lambda,\mu\neq c_j$ for all $j=1,\dots,M$. However, by continuity, (3.1) still holds if λ or μ is equal to c_j for some j.

We consider next the case $\alpha\neq 0$. Dividing (3.1) by $v(0,\mu)v(0,\lambda)$ and multiplying by $\sin\alpha$ we have

$$\frac{(\lambda-\mu)}{\sin\alpha}[U(x,\lambda),U(x,\mu)]=u'(0,\lambda)-u'(0,\mu).$$

Taking the jth derivative, $j\geqslant 1$, of the above expression with respect to λ we see that

$$(\lambda-\mu)[U^{(j)}(x,\lambda),U(x,\mu)]+j[U^{(j-1)}(x,\lambda),U(x,\mu)]=\sin\alpha u^{(j)\prime}(0,\lambda) \tag{3.2}$$

and hence for all $j \geqslant 1$:

$$j[U_n^{j-1}(x), U_n^0(x)] = \sin\alpha u^{(j)\prime}(0, \Lambda_n). \tag{3.3}$$

In particular

$$[U_n^{v_n-1}(x), U_n^0(x)] = \frac{\sin\alpha}{v_n}\, u^{(v_n)\prime}(0, \Lambda_n). \tag{3.4}$$

Differentiating (3.2) $k \geqslant 1$ times with respect to μ we obtain

$$(\mu-\lambda)[U^{(j)}(x,\lambda), U^{(k)}(x,\mu)] + k[U^{(j)}(x,\lambda), U^{(k-1)}(x,\mu)] = j[U^{(j-1)}(x,\lambda), U^{(k)}(x,\mu)]$$

which evaluated at $\lambda = \Lambda_n = \mu$ results in the following recurrence formula:

$$k[U_n^j(x), U_n^{k-1}(x)] = j[U_n^{j-1}(x), U_n^k(x)].$$

Iterating the above relation we obtain

$$\binom{v_n+k-1}{k}[U_n^{v_n-1}, U_n^k] = [U_n^{v_n+k-1}, U_n^0]$$

which along with (3.3) and (3.4) gives

$$\begin{aligned}\sin\alpha u^{(j+v_n)\prime}(0,\Lambda_n) &= (v_n+j)[U_n^{j+v_n-1}, U_n^0]\\ &= (v_n+j)\frac{(v_n+j-1)!}{(v_n-1)!j!}[U_n^{v_n-1}, U_n^j]\\ &= v_n\,\frac{(v_n+j)!}{v_n!j!}\,\rho_n^j,\end{aligned}$$

as required.

Now assume $\alpha = 0$. Dividing (3.1) by $v'(0,\mu)v'(0,\lambda)$ we have

$$(\lambda-\mu)[U(x,\lambda), U(x,\mu)] = u(0,\mu) - u(0,\lambda).$$

Thus we may proceed as in the non-Dirichlet case but with u' replaced by u and $\sin\alpha$ by -1. □

Combining Lemma 3.1 and Theorem 3.2 we are able to express the coefficients of the singular part of the Laurent expansion of $m(\lambda)$ in terms of the norming constants as follows. Note that $\rho_n^0 \neq 0$ –cf. (3.4) and its analogue for $\alpha = 0$ (see above).

Corollary 3.3. *The Laurent expansion of $m(\lambda)$ about Λ_n takes the form*

$$m(\lambda) = E_n(\lambda) + \sum_{k=1}^{v_n}\frac{p_n^k}{(\lambda-\Lambda_n)^k},$$

where $E_n(\lambda)$ is analytic in a neighbourhood of Λ_n and

$$p_n^{v_n} = \frac{(v_n-1)!}{\rho_n^0},$$

$$p_n^{v_n-j} = -\sum_{k=0}^{j-1} p_n^{v_n-k}\,\frac{\rho_n^{j-k}}{(j-k)!\rho_n^0} \quad j = 1,\ldots,v_n-1.$$

Proof. We begin by considering the non-Dirichlet case (i.e., $\alpha \neq 0$). Simple manipulation using the definitions of $m(\lambda)$ and $u'(0,\lambda)$ shows that the Weyl m-function (1.4) can expressed as

$$m(\lambda) = \cot\alpha + \frac{1}{Z(\lambda)},$$

where

$$Z(\lambda) = \sin\alpha u'(0,\lambda) - \cos\alpha\sin\alpha.$$

Applying Lemma 3.1, we obtain

$$m(\lambda) = E_n(\lambda) + \sum_{k=1}^{v_n} \frac{p_n^k}{(\lambda - \Lambda_n)^k},$$

where $E_n(\lambda)$ is analytic in a neighbourhood of Λ_n and

$$a_n^k = \frac{1}{k!}\frac{d^k Z}{d\lambda^k}(\Lambda_n),$$

$$p_n^{v_n} = \frac{1}{a_n^{v_n}},$$

$$p_n^{v_n-j} = -\frac{1}{a_n^{v_n}}\sum_{k=0}^{j-1} p_n^{v_n-k} a_n^{v_n+j-k}, \quad j = 1,\ldots,M-1.$$

But then by Theorem 3.2

$$a_n^{v_n+j} = \frac{\sin\alpha}{(j+v_n)!} u^{(v_n+j)\prime}(0,\Lambda_n) = \frac{\rho_n^j}{(v-1)!j!}$$

giving

$$p_n^{v_n} = \frac{(v_n-1)!}{\rho_n^0},$$

$$p_n^{v_n-j} = -\sum_{k=0}^{j-1} \frac{p_n^{v_n-k}\rho_n^{j-k}}{(j-k)!\rho_n^0},$$

concluding the non-Dirichlet case.

For the Dirichlet case $\alpha = 0$, the definitions of $m(\lambda)$ and $u(0,\lambda)$ give

$$m(\lambda) = -\frac{1}{u(0,\lambda)}.$$

Applying Lemma 3.1 with $Z(\lambda) = -u(0,\lambda)$ we obtain

$$a_n^{v_n+j} = \frac{-1}{(v_n+j)!} u^{(v_n+j)}(0,\Lambda_n) = \frac{\rho_n^j}{(v_n-1)!j!},$$

and we now proceed as in the non-Dirichlet case. □

4. Representation of the *m*-function

Our task now is to represent the analytic part of m, i.e., E_n of Corollary 3.3, in terms of the spectral data. From this point on, asymptotics will be needed for several quantities related to (1.1)–(1.3). For the m-function, the basic asymptotics as $\lambda \to -\infty$ are as follows.

Theorem 4.1 (Binding et al. [7, Lemma 4.1]). *For $\lambda \to -\infty$ we have*

$$m(\lambda) = \begin{cases} \cot\alpha + O\left(\dfrac{1}{\sqrt{|\lambda|}}\right), & \alpha \neq 0, \\ \sqrt{|\lambda|} + O(1), & \alpha = 0. \end{cases}$$

Here and below we take positive square roots. We recall that for large n, each eigenvalue is simple. Thus asymptotic expressions for the norming constants relate only to simple eigenvalues, so only ρ_n^0 will be required, and we now give the corresponding asymptotics to the order we need.

Theorem 4.2. *For $q \in W^{1,1}(0,1)$ (i.e., q in AC), the norming constants for (α, f, q) obey the following asymptotics as $n \to \infty$:*

$$\rho_n^0 = \begin{cases} \dfrac{\sin^2\alpha}{2} + O\left(\dfrac{1}{n^2}\right), & \alpha \neq 0, \\ \dfrac{1}{2\Lambda_n} + O\left(\dfrac{1}{n^4}\right), & \alpha = 0. \end{cases} \tag{4.1}$$

Remark. Note that for $q \in L^1$ we obtain the above theorem but with $O(1/n^2)$ replaced by $O(1/n)$ which is sufficient for us in the case $\alpha \neq 0$, but is inadequate for $\alpha = 0$ where we need $O(1/n^{1+\varepsilon})$ for some $\varepsilon > 0$. Weaker conditions than $q \in AC$ which suffice for us are $q \in H^s$ for some $s > 0$ [14], or q Hölder continuous for some non-zero Hölder exponent [23].

Proof. As $v_n = 1$ for large n

$$\rho_n^0 = [U_n^0, U_n^0] = \int_0^1 u_n^2 + \sum_{k=1}^{M} \left(\frac{b_k}{c_k - \Lambda_n}\right)^2 \frac{u_n^2(1)}{b_k} = \int_0^1 u_n^2 + f^{(1)}(\Lambda_n) u_n^2(1),$$

$$\left(\text{resp}^* \;\; \rho_n^0 = [U_n^0, U_n^0] = \int_0^1 u_n^2 - \sum_{k=1}^{M} \left(\frac{b_k}{c_k - \Lambda_n}\right)^2 \frac{(u_n')^2(1)}{b_k} = \int_0^1 u_n^2 + \frac{f^{(1)}}{f}(\Lambda_n) u_n^2(1)\right)$$

(recall $f^{(1)}$ means $\partial f/\partial\lambda$). But from [7, Appendix] we have as $n \to \infty$

$$u_n^2(x,\lambda) = \begin{cases} \dfrac{2\cos\sqrt{\Lambda_n}x}{\sqrt{\Lambda_n}}[\cos\alpha\sin\alpha\sin\sqrt{\Lambda_n}x + \sin^2\alpha\displaystyle\int_0^x q(t)\cos\sqrt{\Lambda_n}t\sin\sqrt{\Lambda_n}(x-t)\,\mathrm{d}t] \\ \quad + \sin^2\alpha\cos^2\sqrt{\Lambda_n}x + \mathrm{O}\left(\dfrac{e^{|\Im\sqrt{\Lambda_n}x|}}{\Lambda_n}\right), & \alpha \neq 0, \\ \dfrac{\sin^2\sqrt{\Lambda_n}x}{\Lambda_n} + \dfrac{2\sin\sqrt{\Lambda_n}x}{\Lambda_n^{3/2}}\displaystyle\int_0^x q(t)\sin\sqrt{\Lambda_n}t\sin\sqrt{\Lambda_n}(x-t)\,\mathrm{d}t \\ \quad + \mathrm{O}\left(\dfrac{e^{|\Im\sqrt{\Lambda_n}x|}}{\Lambda_n^2}\right), & \alpha = 0. \end{cases}$$

Recalling that $M = \max\{\deg(g),\deg(h)\}$ and $v = \sum_{j=0}^{\infty}(v_j - 1)$, we obtain from [7, Theorem 2.2]

$$\sqrt{\Lambda_n} = \begin{cases} (v+n-M)\pi + \mathrm{O}(1/n), & \alpha \neq 0, \\ (v+n+\frac{1}{2}-M)\pi + \mathrm{O}(1/n), & \alpha = 0 \end{cases} \tag{4.2}$$

(resp*

$$\sqrt{\Lambda_n} = \begin{cases} (v+n+\frac{1}{2}-M)\pi + \mathrm{O}(1/n), & \alpha \neq 0, \\ (v+n+1-M)\pi + \mathrm{O}(1/n), & \alpha = 0, \end{cases} \tag{4.3}$$

which gives

$$u_n^2(1) = \begin{cases} \sin^2\alpha + \mathrm{O}(1/n), & \alpha \neq 0, \\ \dfrac{1}{\Lambda_n} + \mathrm{O}(1/n^3), & \alpha = 0 \end{cases}$$

(resp*

$$u_n^2(1) = \begin{cases} \mathrm{O}(1/n^2), & \alpha \neq 0, \\ \mathrm{O}(1/n^4), & \alpha = 0 \end{cases}$$

and

$$u_n^2(1) f^{(1)}(\Lambda_n)\left(\text{resp}^*\ u_n^2(1)\,\frac{f^{(1)}}{f}(\Lambda_n)\right) = \begin{cases} \mathrm{O}(1/n^2), & \alpha \neq 0, \\ \mathrm{O}(1/n^4), & \alpha = 0. \end{cases}$$

From the above expressions for $\sqrt{\Lambda_n}$ and $u_n^2(x)$ we get

$$\int_0^1 \cos^2\sqrt{\Lambda_n}t\,\mathrm{d}t = \frac{1}{2} + \frac{\sin 2\sqrt{\Lambda_n}}{2\sqrt{\Lambda_n}} = \frac{1}{2} + \mathrm{O}\left(\frac{1}{n^2}\right),$$

$$\int_0^1 \sin^2\sqrt{\Lambda_n}t\,\mathrm{d}t = \frac{1}{2} + \mathrm{O}\left(\frac{1}{n^2}\right)$$

and

$$\frac{1}{\sqrt{\Lambda_n}}\int_0^1 2\cos\sqrt{\Lambda_n}x\sin\sqrt{\Lambda_n}x\,\mathrm{d}x = \frac{1}{\sqrt{\Lambda_n}}\int_0^1 \sin 2\sqrt{\Lambda_n}x\,\mathrm{d}x$$
$$= \frac{1-\cos 2\sqrt{\Lambda_n}}{2\Lambda_n}$$
$$= \mathrm{O}(1/n^2).$$

Further as $n \to \infty$

$$\int_0^1 \frac{2\cos\sqrt{\Lambda_n}x}{\sqrt{\Lambda_n}}\int_0^x q(t)\cos\sqrt{\Lambda_n}t\sin\sqrt{\Lambda_n}(x-t)\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^1 2q(t)\cos\sqrt{\Lambda_n}t\int_t^1 \frac{\cos\sqrt{\Lambda_n}x\sin\sqrt{\Lambda_n}(x-t)}{\sqrt{\Lambda_n}}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^1 \frac{q(t)\cos\sqrt{\Lambda_n}t}{\sqrt{\Lambda_n}}\int_t^1 \left[\sin\sqrt{\Lambda_n}(2x-t) - \sin\sqrt{\Lambda_n}t\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\frac{1}{2\sqrt{\Lambda_n}}\int_0^1 (1-t)q(t)\sin 2\sqrt{\Lambda_n}t\,\mathrm{d}t + \mathrm{O}(1/\Lambda_n)$$
$$= -\frac{1}{4\Lambda_n}\left[q(0) + \int_0^1 \frac{\mathrm{d}[(1-t)q(t)]}{\mathrm{d}t}\cos 2\sqrt{\Lambda_n}t\,\mathrm{d}t\right] + \mathrm{O}(1/\Lambda_n)$$
$$= \mathrm{O}(1/\Lambda_n).$$

Similarly

$$\int_0^1 \frac{2\sin\sqrt{\Lambda_n}x}{\sqrt{\Lambda_n}}\int_0^x q(t)\sin\sqrt{\Lambda_n}t\sin\sqrt{\Lambda_n}(x-t)\,\mathrm{d}t\,\mathrm{d}x = \mathrm{O}(1/\Lambda_n).$$

The result now follows by combining the above asymptotics. □

In the remainder of this paper we assume that 0 is not an eigenvalue of (α, f, q). This does not result in a loss of generality, since if 0 is an eigenvalue we can replace the problem $(\alpha, f(\lambda), q(x))$ by $(\alpha, f(\lambda+\zeta), q(x)-\zeta)$, where $\zeta \in \mathbb{R}$ is not an eigenvalue of (α, f, q). Then $(\alpha, f(\lambda+\zeta), q(x)-\zeta)$ does not have 0 as an eigenvalue. The spectrum of $(\alpha, f(\lambda+\zeta), q(x)-\zeta)$ is obtained from that of $(\alpha, f(\lambda), q(x))$ by translation through ζ. The norming constants are the same for both problems.

Lemma 4.3. *If λ is not an eigenvalue of* (1.1)–(1.3), *then*

$$m(\lambda) = m(0) + \sum_{n=0}^{\infty}\sum_{j=1}^{v_n} p_n^j\left[\frac{1}{(\lambda-\Lambda_n)^j} - \frac{1}{(-\Lambda_n)^j}\right] \tag{4.4}$$

with p_n^j as given in Corollary 3.3.

Proof. Let $\Gamma_n = \{\xi^2 \mid \xi \in \gamma_n\}$ where γ_n is the contour from $-\mathrm{i}\zeta_n$ to $\mathrm{i}\zeta_n$ as indicated in Fig. 1 and

$\zeta_n = (n + \frac{1}{4})\pi.$

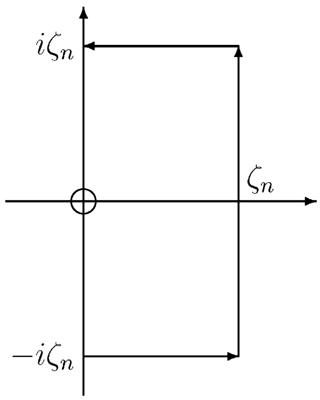

Fig. 1. γ_n in the ξ-plane.

We observe that for large n

$$m(\mu) = O(n) \quad \text{uniformly for} \quad \mu \in \Gamma_n$$

and that length$(\Gamma_n) \leqslant n^2 K$ for some positive constant K, independent of n.

Letting

$$H(\mu) = \frac{m(\mu)}{\mu(\mu - \lambda)},$$

we see that $H(\mu) = O(n^{-3})$ on Γ_n and so

$$\lim_{n\to\infty} \int_{\Gamma_n} H(\mu)\,d\mu = 0.$$

Thus, assuming $\lambda \neq 0$ and that λ is not an eigenvalue, we have

$$0 = \frac{m(\lambda)}{\lambda} - \frac{m(0)}{\lambda} + \sum_{n=0}^{\infty} \sum_{j=1}^{v_n} \frac{p_n^j}{(j-1)!} \frac{\partial^{j-1}}{\partial \mu^{j-1}} \left(\frac{1}{\mu(\mu - \lambda)} \right) \bigg|_{\mu = \Lambda_n}$$

giving

$$m(\lambda) = m(0) + \sum_{n=0}^{\infty} \sum_{j=1}^{v_n} \frac{p_n^j}{(j-1)!} \frac{\partial^{j-1}}{\partial \mu^{j-1}} \left(\frac{1}{\mu} + \frac{1}{\lambda - \mu} \right) \bigg|_{\mu = \Lambda_n}$$

from which the statement of the lemma follows. □

We are now able to give an expansion of m directly in terms of the spectral data Λ_n, v_n and ρ_n^j, $j = 0, \dots, v_n - 1$, (and α which can be obtained from (4.1)).

Theorem 4.4. *For $\alpha \neq 0$ and $\lambda \neq \Lambda_n$, $n = 0, 1, 2, \dots$,*

$$m(\lambda) = \cot \alpha + \sum_{k=0}^{\infty} \sum_{j=1}^{v_k} \frac{p_k^j}{(\lambda - \Lambda_k)^j},$$

where p_k^j satisfy Corollary 3.3.

Proof. For large k, $v_k = 1$ and so $p_k^j = p_k^1 = 1/\rho_k^0 = \mathrm{O}(1)$ by Corollary 3.3 and (4.1). Thus we may let λ tend to $-\infty$ in (4.4) and using Theorem 4.1 we obtain

$$\cot\alpha = m(0) - \sum_{n=0}^{\infty}\sum_{j=1}^{v_n} \frac{p_n^j}{(-\Lambda_n)^j}.$$

The result now follows by using the above expression to eliminate $m(0)$ from (4.4). □

5. The Dirichlet case

In this section we establish analogues of Corollary 4.4 for the case $\alpha = 0$. The primary difficulty is that eliminating $m(0)$ from (4.4) requires significantly more care.

We start with an improved version of Theorem 4.1 for this case.

Lemma 5.1. *In the Dirichlet case* $\alpha = 0$,

$$m(-\sigma^2) = \sigma + \mathrm{O}(1/\sigma), \quad \text{as} \quad \sigma \to \infty.$$

In particular $m(\lambda) + \mathrm{i}\sqrt{\lambda} \to 0$ *as* $\lambda \to -\infty$.

Proof. Referring to [7, Appendix] with $\sqrt{\lambda} = \mathrm{i}\sigma$ we get

$$v(0, -\sigma^2) = \frac{(-1)^M \sigma^{2M} \mathrm{e}^{\sigma}}{4}[2 + \mathrm{O}(\sigma^{-1})],$$

$$v'(0, -\sigma^2) = \frac{(-1)^{M+1} \sigma^{2M+1} \mathrm{e}^{\sigma}}{4}[2 + \mathrm{O}(\sigma^{-1})]$$

(resp*

$$v(0, -\sigma^2) = \frac{(-1)^{M-1} \sigma^{2M-1} \mathrm{e}^{\sigma}}{4}[2 + \mathrm{O}(\sigma^{-1})]$$

$$v'(0, -\sigma^2) = \frac{(-1)^{M} \sigma^{2M} \mathrm{e}^{\sigma}}{4}[2 + \mathrm{O}(\sigma^{-1})]).$$

Thus

$$m(-\sigma^2) - \sigma = -\frac{v'(0, -\sigma^2) + \sigma v(0, -\sigma^2)}{v(0, -\sigma^2)} = \mathrm{O}\left(\frac{1}{\sigma}\right). \qquad \square$$

The following theorems express $m(\lambda)$ in terms of the eigenvalues and norming constants. The analysis will depend on the relative magnitudes of $\deg(h)$ and $\deg(g)$.

Theorem 5.2. *For* $\alpha = 0$ *and* $M = \deg(h) > \deg(g)$

$$m(\lambda) = 1 + 2(v - M) + \sum_{n=0}^{\infty}\left[2 + \sum_{j=1}^{v_n} \frac{p_n^j}{(\lambda - \Lambda_n)^j}\right],$$

where $v - M$ *can be obtained from* (4.3).

Proof. First, (4.3) follows from $\Lambda_n = (n+1+v-M)^2\pi^2 + O(1)$. Noting that $\sqrt{\lambda}\cot\sqrt{\lambda} = -i\sqrt{\lambda}[1 + O(e^{-|\Im\sqrt{\lambda}|})]$ as $\lambda \to -\infty$ we see by Lemma 5.1 that

$$\lim_{\lambda\to-\infty} [m(\lambda) - \sqrt{\lambda}\cot\sqrt{\lambda}] = 0.$$

Now from (4.4) we have

$$m(\lambda) = m(0) + S(\lambda) + \sum_{n=0}^{\infty} p_n^1 \left(\frac{1}{\lambda - \Lambda_n} + \frac{1}{\Lambda_n} \right), \tag{5.1}$$

where

$$S(\lambda) = \sum_{n=0}^{\infty} \sum_{j=2}^{v_n} p_n^j \left(\frac{1}{(\lambda - \Lambda_n)^j} - \frac{1}{(-\Lambda_n)^j} \right) \tag{5.2}$$

which is in fact a finite summation, as all but finitely many eigenvalues are simple. Let

$$S_\infty = \lim_{\lambda\to-\infty} S(\lambda) = -\sum_{n=0}^{\infty} \sum_{j=2}^{v_n} \frac{p_n^j}{(-\Lambda_n)^j}. \tag{5.3}$$

With the aid of the Mittag–Leffler expansion, [22, p. 113]

$$\sqrt{\lambda}\cot\sqrt{\lambda} = -1 - \sum_{k=0}^{\infty} \frac{2\lambda}{n^2\pi^2 - \lambda}$$

we obtain

$$\begin{aligned} m(\lambda) &- \sqrt{\lambda}\cot\sqrt{\lambda} \\ &= m(0) + S(\lambda) + 1 + \sum_{n=0}^{M-v-2} p_n^1 \left(\frac{1}{\lambda - \Lambda_n} + \frac{1}{\Lambda_n} \right) \\ &\quad + \sum_{n=0}^{\infty} \left[\frac{p_{n+M-v-1}^1}{\Lambda_{n+M-v-1}} \frac{\lambda}{\lambda - \Lambda_{n+M-v-1}} - 2\frac{\lambda}{\lambda - n^2\pi^2} \right] \\ &= m(0) + S(\lambda) + 1 + \sum_{n=0}^{\infty} \left[\frac{p_{n+M-v-1}^1}{\Lambda_{n+M-v-1}} - 2 \right] \frac{\lambda}{\lambda - \Lambda_{n+M-v-1}} \\ &\quad + \sum_{n=0}^{M-v-2} p_n^1 \left(\frac{1}{\lambda - \Lambda_n} + \frac{1}{\Lambda_n} \right) + \sum_{n=0}^{\infty} \left[\frac{2\lambda}{\lambda - \Lambda_{n+M-v-1}} - \frac{2\lambda}{\lambda - n^2\pi^2} \right]. \end{aligned} \tag{5.4}$$

Now Corollary 3.3 and (4.1) yield

$$p_n^1/\Lambda_n = 2 + O(n^{-2}) \tag{5.5}$$

so

$$\lim_{\lambda\to-\infty} \sum_{n=0}^{\infty} \left[\frac{p_{n+M-v-1}^1}{\Lambda_{n+M-v-1}} - 2 \right] \frac{\lambda}{\lambda - \Lambda_{n+M-v-1}} = \sum_{n=0}^{\infty} \left[\frac{p_{n+M-v-1}^1}{\Lambda_{n+M-v-1}} - 2 \right].$$

Noting that $\Lambda_{n+M-v-1} - n^2\pi^2 = \mathrm{O}(1)$, we see that there exists a constant K such that, for λ large and negative,

$$\sum_{n=0}^{\infty}\left|\frac{2\lambda}{\lambda-\Lambda_{n+M-v-1}} - \frac{2\lambda}{\lambda-n^2\pi^2}\right|$$

$$\leqslant K\sum_{n=0}^{\infty}\left[\frac{|\lambda|}{|\lambda|(n^2\pi^2-\lambda)}\right]$$

$$= -\frac{K(\sqrt{\lambda}\cot\sqrt{\lambda}+1)}{2|\lambda|}$$

$$\to 0 \quad \text{as} \quad \lambda \to -\infty.$$

Assembling these pieces, we see that, as $\lambda \to -\infty$,

$$m(\lambda) - \sqrt{\lambda}\cot\sqrt{\lambda} \to m(0) + S_\infty + 1 + \sum_{n=0}^{M-v-2} p_n^1 \frac{1}{\Lambda_n} + \sum_{n=0}^{\infty}\left[\frac{p^1_{n+M-v-1}}{\Lambda_{n+M-v-1}} - 2\right]$$

from which we obtain

$$m(0) = \sum_{n=0}^{\infty}\sum_{j=2}^{v_n}\frac{p_n^j}{(-\Lambda_n)^j} - 1 - 2(M-v-1) - \sum_{n=0}^{\infty}\left[\frac{p_n^1}{\Lambda_n} - 2\right].$$

Substituting these expressions back into (5.1) we get

$$m(\lambda) = 1 - 2M + 2v + \sum_{n=0}^{\infty}\sum_{j=2}^{v_n}\frac{p_n^j}{(-\Lambda_n)^j} - \sum_{n=0}^{\infty}\left[\frac{p_n^1}{\Lambda_n} - 2\right]$$

$$+ \sum_{n=0}^{\infty}\sum_{j=1}^{v_n} p_n^j\left[\frac{1}{(\lambda-\Lambda_n)^j} - \frac{1}{(-\Lambda_n)^j}\right]$$

$$= 1 - 2M + 2v + \sum_{n=0}^{\infty}\sum_{j=2}^{v_n} p_n^j\frac{1}{(\lambda-\Lambda_n)^j} + \sum_{n=0}^{\infty}\left[2 + \frac{p_n^1}{\lambda-\Lambda_n}\right],$$

proving the theorem. □

We now discuss the remaining case. Note that $v - M$ can now be obtained from (4.2).

Theorem 5.3. *For* $\alpha = 0$ *and* $\deg(h) \leqslant \deg(g) = M$ *we have*

$$m(\lambda) = 2(v-M) + \sum_{n=0}^{\infty}\left[2 + \sum_{j=1}^{v_n}\frac{p_n^j}{(\lambda-\Lambda_n)^j}\right].$$

Proof. Here $\Lambda_n = (n + v + \frac{1}{2} - M)^2\pi^2 + O(1)$ and we note that $\sqrt{\lambda}\tan\sqrt{\lambda} = i\sqrt{\lambda}[1 + O(e^{-2|\Im\sqrt{\lambda}|})]$ as $\lambda \to -\infty$. From [22, p. 113] we have

$$\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}} = \sum_{k=0}^{\infty} \frac{2}{(k + \frac{1}{2})^2\pi^2 - \lambda}.$$

Thus by Lemma 5.1

$$\lim_{\lambda\to-\infty} m(\lambda) + \sqrt{\lambda}\tan\sqrt{\lambda} = 0.$$

Using (5.1)–(5.3) and the above Mittag–Leffler expansion, we have

$$m(\lambda) + \sqrt{\lambda}\tan\sqrt{\lambda} = m(0) + S(\lambda) + \sum_{n=0}^{\infty}\left[\frac{p^1_{n+M-v}}{\Lambda_{n+M-v}} - 2\right]\frac{\lambda}{\lambda - \Lambda_{n+M-v}}$$
$$+ \sum_{n=0}^{M-v-1} p^1_n\left(\frac{1}{\lambda - \Lambda_n} + \frac{1}{\Lambda_n}\right) + \sum_{n=0}^{\infty}\left[\frac{2\lambda}{\lambda - \Lambda_{n+M-v}} - \frac{2\lambda}{\lambda - (n + \frac{1}{2})^2\pi^2}\right],$$

cf. (5.4). By (5.5) we have

$$\lim_{\lambda\to-\infty}\sum_{n=0}^{\infty}\left[\frac{p^1_{n+M-v}}{\Lambda_{n+M-v}} - 2\right]\frac{\lambda}{\lambda - \Lambda_{n+M-v}} = \sum_{n=0}^{\infty}\left[\frac{p^1_{n+M-v}}{\Lambda_{n+M-v}} - 2\right].$$

Noting that $\Lambda_{n+M-v} - (n + \frac{1}{2})^2\pi^2 = O(1)$, we see that there exists a constant K such that, for λ large and negative,

$$\sum_{n=0}^{\infty}\left|\frac{2\lambda}{\lambda - \Lambda_{n+M-v}} - \frac{2\lambda}{\lambda - (n + \frac{1}{2})^2\pi^2}\right| \leqslant K\sum_{n=0}^{\infty}\left[\frac{|\lambda|}{|\lambda|((n + \frac{1}{2})^2\pi^2 - \lambda)}\right]$$
$$= \frac{K\tan\sqrt{\lambda}}{2\sqrt{\lambda}}$$
$$\to 0 \quad \text{as} \quad \lambda \to -\infty.$$

As $\lambda \to -\infty$ we thus have

$$m(\lambda) + \sqrt{\lambda}\tan\sqrt{\lambda} \to m(0) + S_\infty + \sum_{n=0}^{M-v-1} p^1_n\frac{1}{\Lambda_n} + \sum_{n=0}^{\infty}\left[\frac{p^1_{n+M-v}}{\Lambda_{n+M-v}} - 2\right]$$

from which we obtain

$$0 = m(0) + 2(M - v) + S_\infty + \sum_{n=0}^{\infty}\left[\frac{p^1_n}{\Lambda_n} - 2\right].$$

Substituting this back into the expression for $m(\lambda)$ we get the required result

$$m(\lambda) = -2(M - v) + \sum_{n=0}^{\infty}\sum_{j=2}^{v_n}\frac{p^j_n}{(\lambda - \Lambda_n)^j} + \sum_{n=0}^{\infty}\left[2 + \frac{p^1_n}{\lambda - \Lambda_n}\right]$$

as for the proof of Theorem 5.2. □

6. Uniqueness

We conclude the present work by proving that the eigenvalues, their multiplicities and corresponding norming constants of (1.1)–(1.3) uniquely determine g, h and q.

Theorem 6.1. *The eigenvalues λ_n and their generalized norming constants ρ_n^k uniquely determine q, α and f.*

Proof. By Theorems 4.4, 5.2 and 5.3, the λ_n and ρ_n^k uniquely determine m. The theorem now follows from [7]. □

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