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Optimal Existence Theorems for Nonhomogeneous Differential Inclusions[1]

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In this paper we address the question of solvability of the differential inclusions

of the gradients by appropriate selection of the elements of the sequence. In this paper we identify an optimal setting of this method. In particular we show that the existence result holds for general upper semicontinuous functions H without extra requirements like quasiconvexity of H with respect to Du, which was assumed in previous works, where the idea to apply the Baire category lemma to the sets of approximate solutions was developed. We also apply our result to identify the minimal sets, where the function H should vanish to guarantee solvability of the inclusions. © 2001 Academic Press

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1. INTRODUCTION

In this paper we are interested in identifying an optimal principle which guarantees solvability of the problems

$$H(\cdot, u(\cdot), Du(\cdot)) = 0, \qquad u|_{\partial\Omega} = f, \qquad u \in W^{1,\infty}(\Omega; \mathbf{R}^m), \tag{1.1}$$

where $H \geqslant 0$ is defined in a subset of $\Omega \times \mathbf{R}^m \times \mathbf{R}^{m \times n}$ and $(x, f(x), Df(x))$ belongs to this subset for a.e. $x \in \Omega$. Here and everywhere in the paper we assume that Ω is a Lipschitz bounded domain in $\mathbf{R}^n$.

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Consider first the homogeneous case $H = H(Du)$, $f = l_A$, where l_A is an affine function with the gradient equal to A. Assume that $U \subset \mathbf{R}^{m \times n}$ is a domain of definition of a continuous nonnegative function H and assume that the set $K := \{v \in U : H(v) = 0\}$ is compact.

If we can solve the problem (1.1) with $f = l_A$, $A \in U$, then there exists a sequence of functions $\phi_k \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with the properties $D\phi_k \in U$ a.e., $\operatorname{dist}(D\phi_k(\cdot), K) \to 0$ in L^1 as $k \to \infty$. This motivates

Definition 1.1. Let U, K be bounded subsets of $\mathbf{R}^{m \times n}$.

We say that U can be reduced to K if for every $A \in U$ there is a sequence of piece-wise affine functions $\phi_k \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with the properties:

(1) $D\phi_k \in U$ a.e. in Ω, $k \in \mathbf{N}$,

(2) $\|\operatorname{dist}(D\phi_k, K)\|_{L^1(\Omega)} \to 0$ for $k \to \infty$.

Here and in the following we say that ϕ is piece-wise affine if it is Lipschitz and there exist at most countably many disjoint open sets $\Omega_j \subset \Omega$, whose union has full measure, such that $\phi|_{\Omega_j}$ is affine.

It turns out that the conditions that arise in the definition already imply solvability of the differential inclusion.

Theorem 1.2. *Assume that U is a bounded subset in $\mathbf{R}^{m \times n}$, and assume that K is a compact subset in $\mathbf{R}^{m \times n}$ to which U can be reduced.*

Then for each piece-wise affine function $f \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ with $Df \in (U \cup K)$ a.e. in Ω the problem

$$Du \in K \text{ a.e. in } \Omega, \qquad u \in W^{1,\infty}(\Omega; \mathbf{R}^m), \qquad u|_{\partial\Omega} = f|_{\partial\Omega}$$

has a solution. Moreover, each ε-neighborhood of f in the $L^\infty(\Omega; \mathbf{R}^m)$-norm contains a solution of this problem.

Before we state the main result in the nonhomogeneous case we recall the definitions of standard distance functions. For a point $A \in \mathbf{R}^{m \times n}$ and a set $S \subset \mathbf{R}^{m \times n}$ we define

$$\operatorname{dist}(A, S) := \inf_{v \in S} |A - v|.$$

For two sets S_1, S_2 we define

$$\operatorname{dist}(S_1, S_2) := \sup_{A \in S_1} \operatorname{dist}(A, S_2).$$

The Hausdorff distance between the sets S_1 and S_2 is

$$\operatorname{dist}_H(S_1, S_2) := \operatorname{dist}(S_1, S_2) + \operatorname{dist}(S_2, S_1).$$

We will use some other standard notions and notations the complete list of which is located at the end of this section.

The main result of this paper is Theorem 1.3, which generalizes Theorem 1.2 to the nonhomogeneous case. We state it under rather general assumptions on the function d in view of further applications. More specific cases will be considered as corollaries.

THEOREM 1.3. *Let* $U: \Omega \times \mathbf{R}^m \to 2^{\mathbf{R}^{m\times n}}$, $K: \Omega \times \mathbf{R}^m \to 2^{\mathbf{R}^{m\times n}}$ *be multi-valued functions with equi-bounded values. Let also*

$$d: \{(x, u, v) \in \Omega \times \mathbf{R}^m \times \mathbf{R}^{m\times n} : v \in (U(x, u) \cup K(x, u))\} \to [0, M]$$

be an upper semicontinuous function such that for each $(x, u) \in \Omega \times \mathbf{R}^m$ *the set* $K(x, u)$ *is compact,* $K(x, u) = \{v \in (U(x, u) \cup K(x, u)) : d(x, u, v) = 0\}$, *and* $d(x, u, v_k) \to 0$ *if and only if* $\operatorname{dist}(v_k, K(x, u)) \to 0$, $k \to \infty$.

Assume that for each $(x_0, u_0) \in \Omega \times \mathbf{R}^m$, *each* $v_0 \in U(x_0, u_0)$, *and each* $\varepsilon > 0$ *there exists a piece-wise affine function* $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ *such that*

$$\int_\Omega d(x_0, u_0, v_0 + D\phi(y))\, dy \leqslant \varepsilon \qquad \text{and} \qquad v_0 + D\phi(\cdot) \in U(x, u) \text{ a.e. in } \Omega$$

for all (x, u) *sufficiently close to* (x_0, u_0).

Then for each piece-wise affine function $f \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ *with* $Df(\cdot) \in U(\cdot, f(\cdot))$ *a.e. in* Ω *and each* $\eta > 0$ *the problem*

$$Du(\cdot) \in K(\cdot, u(\cdot)) \text{ a.e. in } \Omega,$$

$$u \in W^{1,\infty}(\Omega; \mathbf{R}^m), \quad u|_{\partial\Omega} = f|_{\partial\Omega}, \quad \|u - f\|_{L^\infty} \leqslant \eta,$$

has a solution.

It is helpful to state explicitly a version of Theorem 1.3 in the case when d is the standard distance function.

COROLLARY 1.4. *Let* $U: \Omega \times \mathbf{R}^m \to 2^{\mathbf{R}^{m\times n}}$, $K: \Omega \times \mathbf{R}^m \to 2^{\mathbf{R}^{m\times n}}$ *be multi-valued functions with equi-bounded values, where the sets* $K(x, u)$ *are also compact and the mapping* $(x, u) \to K(x, u)$ *is lower semicontinuous.*

Assume that for each $(x_0, u_0) \in \Omega \times \mathbf{R}^m$, *each* $v_0 \in U(x_0, u_0)$, *and each* $\varepsilon > 0$ *there exists a piece-wise affine function* $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ *such that*

$$\int_\Omega \operatorname{dist}(v_0 + D\phi(y), K(x_0, u_0))\, dy \leqslant \varepsilon \qquad \text{and} \qquad v_0 + D\phi \in U(x, u) \text{ a.e. in } \Omega$$

for all (x, u) *sufficiently close to* (x_0, u_0).

Then for each piece-wise affine function $f\colon \Omega \to \mathbf{R}^m$ *with* $Df(x) \in U(x, f(x))$ *a.e. in* Ω *and each* $\eta > 0$ *we can find a function* $f_\eta \in f + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ *such that* $\|f - f_\eta\|_{L^\infty} \leqslant \eta$ *and* $Df_\eta(x) \in K(x, f_\eta(x))$ *a.e. in* Ω.

Note that one of the cases of Corollary 1.4 was stated by M. Gromov [G, p. 218] as a further development of the method of convex integration. The main difference is that in [G] a more special approximation of the sets K in Corollary 1.4 is required. Such approximations are not always possible in applications, see e.g. [MSv2]. Those applications are motivated by mathematical models of solid-solid phase transitions, see [BJ1], [BJ2], [BJFK] for discussions.

A general existence theory for the vectorial case $m > 1$ was developed by Dacorogna and Marcellini in [DM1–DM4]. Theorem 1.3 of this paper shows that the result of that theory can be refined since requirements on quasiconvexity of d with respect to Du can be dropped. We state Theorem 1.3 in the case of one equation $d = 0$, but one can also cover the cases of systems of equations $L_1 = 0, ..., L_q = 0$ from [DM1–DM4] by taking $d := L_1 + \cdots + L_q$ (again quasiconvexity of L_i, $i = 1, ..., q$, with respect to Du is not required).

Theorem 1.3 gives also some improvements in the scalar case $m = 1$ since we do not require the level sets $U(x, u) := \{v \in \mathbf{R}^n : L(x, u, v) < 0\}$ to form a continuous multi-valued mapping contrary to the case considered in [BF], [DeBP]. In the later case optimal results were obtained in the paper [BF], and assumptions on regularity of boundary data were weakened in [DeBP]. Of course in the scalar case there exists the well-known theory of viscosity solutions, see e.g., [Ba], [BCD], [CrEL], [K], [L], [Su]. However those solutions can not cover all the existence results available in the Sobolev class, as it was recently shown in [CDGG].

We will also discuss a problem which is inverse to the one considered in Corollary 1.4, i.e., given a multi-valued mapping U we identify the smallest subsets $K \subset U$ for which the differential inclusions are solvable (in optimal control this problem is known as bang-bang principle). In the case of convex sets U this problem can be completely solved, as Theorem 1.6 below shows.

We say that a set $E \subset \mathbf{R}^{m \times n}$ contains no rank-one connections if $\operatorname{rank}(A - B) > 1$ for all $A, B \in E$ with $A \neq B$.

Definition 1.5. For a compact convex subset U of $\mathbf{R}^{m \times n}$ we define the set of gradient extremum points $\operatorname{gr\,extr} U$ as the union of the set of all extremum points of U and of all those faces of ∂U which do not contain rank-one connections.

Theorem 1.6. *Let* $U : \mathbf{R}^n \times \mathbf{R}^m \to 2^{\mathbf{R}^{m \times n}}$ *be a continuous multi-valued mapping, which is compact and convex. Let* $f \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ *be a piece-wise* *affine function which satisfies the inclusion* $Df(\cdot) \in \operatorname{int} U(\cdot, f(\cdot))$ *a.e. in* Ω. *Then for each* $\varepsilon > 0$ *there exists* $u \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ *such that*

$$u|_{\partial\Omega} = f, \qquad \|u - f\|_{L^\infty} \leqslant \varepsilon, \qquad \textit{and}$$
$$Du(\cdot) \in \overline{\text{gr extr } U(\cdot, u(\cdot))} \text{ a.e. in } \Omega.$$

In Section 4 we will also show that the choice of the multi-valued mapping $(x, u) \to \text{gr extr } U(x, u)$ is optimal to solve the differential inclusion, cf. Theorem 4.4.

To identify an optimal result in the general case of nonconvex sets U one needs to introduce a definition of well-behaved gradient extremum points in this case, see Section 4 for a discussion. However Theorem 1.3 allows us to indicate a rather general sufficient condition, which is Theorem 4.5. In its turn Theorem 4.5 allows us to prove an attainment result in the non-homogeneous version of the 2D two-well problem, which was previously studied in [Sv], [MSv1], [DM2], [DM4] in the homogeneous case.

Consider matrices $A, B \in \mathbf{R}^{2\times 2}$ and let $\lambda_1(BA^{-1}) \leqslant \lambda_2(BA^{-1})$ denote the singular values of BA^{-1}, i.e. the eigenvalues of $[(BA^{-1})^t (BA^{-1})]^{1/2}$. Suppose that

$$\det B > \det A > 0, \qquad 0 < \lambda_1(BA^{-1}) < 1 < \lambda_2(BA^{-1}). \tag{1.2}$$

Then one easily checks that there are exactly two matrices B_1, B_2 in the set $SO(2)\, B$ which satisfy $\operatorname{rank}(B_i - A) = 1$, $i = 1, 2$. Let $K := SO(2)\, A \cup SO(2)\, B$ and let U be the set of all those $v \in \mathbf{R}^{2\times 2}$ for which there exists a sequence $\phi_j \in l_v + W_0^{1,\infty}(\Omega; \mathbf{R}^2)$ with the property $\operatorname{dist}(D\phi_j, K) \to 0$ in L^1. This set was explicitly computed in [Sv].

To indicate the dependence of U on A and B, we write $U_{A, B}$. If A and B are functions we use the notation $U(x, u) = U_{A(x, u), B(x, u)}$.

THEOREM 1.7. *Suppose that* $A, B\colon \Omega \times \mathbf{R}^2 \to \mathbf{R}^{2\times 2}$ *are continuous functions which satisfy* (1.2). *Then for each piece-wise affine function* $f \in W^{1,\infty}(\Omega; \mathbf{R}^2)$ *with*

$$Df(x) \in \{\operatorname{int} U(x, f(x)) \cup K(x, f(x))\}$$

and each $\varepsilon > 0$ *we can find a function* $u \in f + W_0^{1,\infty}(\Omega; \mathbf{R}^2)$ *such that*

$$Du(x) \in K(x, u(x)) \text{ a.e. in } \Omega, \qquad \|u - f\|_{L^\infty(\Omega; \mathbf{R}^2)} \leqslant \varepsilon.$$

The paper will be organized as follows.

In Section 2 we prove general reduction principles, which are Theorems 1.2, 1.3. The first theorem was proved in [S1], however we include its proof for convenience of a reader. The basic technical ingredient is Lemma 2.1, which is closely related to ideas of Nash [Na], Kuiper [Ku] and Gromov [G]. This lemma shows how to construct a sequence u_j of perturbations of a given function to assure strong convergence of Du_j. We follow the construction from [S1]. Another realization of the same idea can be found in [MSv1], [MSv2].

In Section 3 we prove solvability of Hamilton–Jacobi equations

$$L(\cdot, f(\cdot)+\phi(\cdot), Df(\cdot)+D\phi(\cdot))=0,$$

where $L\colon \bar{\Omega}\times\mathbf{R}^m\times\mathbf{R}^{m\times n}\to\mathbf{R}$ is a countinuous function such that $\liminf_{|v|\to\infty} L(x,u,v)>0$ and f is a piece-wise affine function with $L(\cdot, f(\cdot), Df(\cdot))\leqslant 0$ a.e. in Ω, ϕ is an unknown function, see Theorem 3.2. Of course this result is a straightforward corollary of the main result, which is Theorem 1.3. Note also that the result still holds for those L which are upper semicontinuous in x since, in fact, Theorem 1.3 can be generalized to this situation. However lower semicontinuity may prevent solvability of the problem. To show this we develop the arguments of the example by P. L. Lions [L, Ch. 7], which were communicated to us by M. Crandall.

In Section 4 we reduce Theorem 1.6 to Corollary 1.4. We show that the choice $K(x,u):=\operatorname{gr}\operatorname{extr} U(x,u)$ is optimal to resolve the differential inclusions in question for a convex-valued multifunction $(x,u)\to U(x,u)$. We discuss also which progress can be made in the case of general multi-valued functions. The main result in this direction is Theorem 4.5. As a consequence one obtains Theorem 1.7 stated above.

In Section 5 we compare our approach to the problem of solvability of equations and inclusions with the approach based on application of the Baire category lemma to the sets of approximate solutions. The latter approach was developed in particular by Italian School, see e.g., [DM1–DM4] and papers mentioned therein for the vectorial case and [C], [B], [BF], [DeBP] for the scalar case. We show that Theorem 1.3 allows to obtain sharper results than those in [DM1–DM4]. The main difference is that to apply the Baire category approach one needs to require openness of the sets of approximate solutions in the L^∞-norm, see Section 5. We compare the methods on example of convex sets, which is the best studied case in literature.

Notation

We use the following notation: for a subset U of $\mathbf{R}^n$ the sets int U, re int U, co U, and extr U are respectively the interior of U, the relative interior of U, the convex hull of U, and the set of extremum points of U (a point a belongs to extr U if it can not be represented as a convex combination of other points of U). The set $B(a,\varepsilon)$ denotes the ball of radius ε

which is centered at the point $a \in \mathbf{R}^n$. The boundary of the set U is denoted by ∂U. Note that if U is a convex and compact set then by the Hahn–Banach theorem for each $A \in \partial U$ we can find a hyperplane H such that $A \in (\partial U \cap H)$ and U lies on one side of this hyperplane. The sets $\partial U \cap H$ are also convex and compact.

For each point $A \in \partial U$ one defines faces (of ∂U) containing A inductively as follows. First there exists a hyperplane H such that $A \in (\partial U \cap H)$ and U lies on one side of H. The set $\partial U \cap H$ is a face containing A. If A is not an interior point (relative to H) of the set $\partial U \cap H$ then there exists a hyperplane H' in H such that $A \in (\partial U \cap H')$ and the set $\partial U \cap H$ lies on one side of H' in H. The set $\partial U \cap H'$ is also a face containing A. Proceeding inductively we come to the situation when either A is an interior point of the face or the face has dimension zero, i.e., it is the singleton $\{A\}$. In the latter case we also consider A as an interior point of the face.

It is not difficult to show that the face which contains A as an interior point is unique and that the dimension of this face is minimal among the dimensions of all the faces containing A. This face will be called the smallest face containing A and its dimension will be called the index ($\operatorname{ind} A$) of the point A. Note that if A is not an extremum point of U then $\operatorname{ind} A > 0$.

Weak and strong convergence of sequences are denoted by $\rightharpoonup$ and $\rightarrow$, respectively.

Recall that a multi-valued mapping $F: \Omega \times \mathbf{R}^m \rightarrow 2^{\mathbf{R}^{m \times n}}$ is called lower semicontinuous if for each $(x_0, u_0) \in \Omega \times \mathbf{R}^m$, each $v_0 \in F(x_0, u_0)$, and each sequence (x_k, u_k) converging to (x_0, u_0) one can find $v_k \in F(x_k, u_k)$ such that $v_k \rightarrow v_0$ as $k \rightarrow \infty$. If F has compact values then we call F continuous if it is continuous in the Hausdorff metric. F is called compact or convex if its values are compact or convex sets, respectively.

2. GENERAL REDUCTION PRINCIPLES

In this section we prove Theorems 1.2, 1.3 and then derive Corollary 1.4. Note that Theorem 1.2 is a homogeneous version of Theorem 1.3. However we include its proof for the convenience of the reader.

We recall the following version of the Vitali covering theorem. A family G of closed subsets of $\mathbf{R}^n$ is said to be a Vitali cover of a bounded set S if for each $x \in S$ there exists a positive number $r(x) > 0$, a sequence of balls $B(x_k, \varepsilon_k)$ with $\varepsilon_k \rightarrow 0$, and a sequence $C_k \in G$ such that $x \in C_k$, $C_k \subset B(x, \varepsilon_k)$, and $\{\text{meas } C_k / \text{meas } B(x, \varepsilon_k)\} > r(x)$ for all $k \in \mathbf{N}$.

The version of the Vitali covering theorem from [Sa, p. 109] says that each Vitali cover of S contains an at most countable subfamily of disjoint sets C_k such that $\text{meas}(S \setminus \bigcup_k C_k) = 0$.

We will frequently use the following construction which will be called shortly *the Vitali covering argument*. Let Ω be a Lipschitz bounded domain. Given an open set $\tilde{\Omega}$ and a function $f \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ we consider a decomposition of $\tilde{\Omega}$ into disjoint sets $x_i + \varepsilon_i \bar{\Omega}$, $i \in \mathbf{N}$, and a set of zero measure. Define $u(x) = \varepsilon_i f((x - x_i)/\varepsilon_i)$ for $x \in x_i + \varepsilon_i \bar{\Omega}$, $i \in \mathbf{N}$. Then $u \in W_0^{1,\infty}(\tilde{\Omega}; \mathbf{R}^m)$.

The basic two properties of this construction are that Du has the same distribution in $\tilde{\Omega}$ as Df in Ω, in particular for each subset K of $\mathbf{R}^{m \times n}$ we have

$$\frac{1}{\operatorname{meas} \tilde{\Omega}} \int_{\tilde{\Omega}} \operatorname{dist}(Du(x), K)\, dx = \frac{1}{\operatorname{meas} \Omega} \int_{\Omega} \operatorname{dist}(Df(x), K)\, dx,$$

and we can make L^∞-norm of u arbitrary small by taking ε_i, $i \in \mathbf{N}$, sufficiently small.

The first basic technical ingredient of our approach is the following lemma.

LEMMA 2.1 (controlled L^∞ convergence implies $W^{1,1}$ convergence). *Let u_j be a sequence of piece-wise affine functions such that*

$$u_{j+1} = u_j + \phi_j, \qquad \phi_j \in W_0^{1,\infty}(\Omega; \mathbf{R}^m),$$

and $\|u_j\|_{W^{1,\infty}(\Omega; \mathbf{R}^m)} \leqslant \text{const} < \infty$.

Let $\Omega_j \subset\subset \operatorname{int} \Omega$ be a sequence of subsets of Ω such that $\operatorname{meas}(\Omega \backslash \Omega_j) \to 0$ *as $j \to \infty$. Suppose that $\Omega_j := \bigcup_{i=1}^{i(j)} \Omega_j^i$ is a union of disjoint tetrahedra Ω_j^i on which u_j is affine and suppose*

$$\operatorname{diam} \Omega_j^i \leqslant c(\textit{in-radius of } \Omega_j^i), \qquad i \in \{1, ..., i(j)\},$$

with $c > 0$ independent of $j \in \mathbf{N}$. Let

$$d_j := \min_{1 \leqslant i \leqslant i(j)} \textit{in-radius of } \Omega_j^i, \qquad D_j := \max_{1 \leqslant i \leqslant i(j)} \operatorname{diam} \Omega_j^i$$

and suppose that $D_j \to 0$ as $j \to \infty$.

Then the estimates

$$\|\phi_j\|_{L^\infty} \leqslant \frac{d_j}{2^{j+1}}, \qquad \|\phi_{j+1}\|_{L^\infty} \leqslant \frac{\|\phi_j\|_{L^\infty}}{2}, \qquad j \in \mathbf{N}, \tag{2.1}$$

imply that u_j converges in $W^{1,1}(\Omega; \mathbf{R}^m) \cap L^\infty(\Omega; \mathbf{R}^m)$.

Proof. The inequalities (2.1) imply the inequalities

$$\sum_{i=j+1}^{\infty} \|\phi_i\|_{L^\infty} \leqslant \text{const}/2^j, \qquad \|u_j - u_0\|_{L^\infty} \leqslant \sum_{i=j}^{\infty} \|\phi_i\|_{L^\infty} \leqslant 2\, \|\phi_j\|_{L^\infty}. \tag{2.2}$$

Thus the sequence u_j converges in L^∞-norm. Hence there exists $u_0 \in u_1 + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that $u_j \rightharpoonup^* u_0$ in $W^{1,\infty}(\Omega; \mathbf{R}^m)$ as $j \to \infty$.

For each $j \in \mathbf{N}$ we can extend the triangulation $\Omega_j = \bigcup_{i=1}^{i(j)} \Omega_j^i$ to a triangulation of the whole domain Ω, i.e., $\Omega = \bigcup_{i=1}^{\infty} \Omega_j^i$.

Consider piece-wise affine approximations $u_0^j \colon \Omega_j \to \mathbf{R}^m$ of u_0 associated with the triangulations $\Omega = \bigcup_{i=1}^{\infty} \Omega_j^i$, i.e. u_0^j are affine in each set Ω_j^i, $i \in \mathbf{N}$, and equal to u_0 in vertices of these sets. It is not difficult to show that

$$\|u_0^j - u_0\|_{W^{1,1}(\Omega; \mathbf{R}^m)} \to 0, \qquad j \to \infty. \tag{2.3}$$

In view of (2.3) and the convergence

$$\|u_j\|_{W^{1,1}(\Omega \setminus \Omega_j; \mathbf{R}^m)} + \|u_0^j\|_{W^{1,1}(\Omega \setminus \Omega_j; \mathbf{R}^m)} \to 0, \qquad j \to \infty$$

it suffices to prove that $\|u_0^j - u_j\|_{W^{1,1}(\Omega_j; \mathbf{R}^m)} \to 0$. This convergence follows from (2.1). In fact, since both functions u_0^j and u_j are affine in Ω_j^i for each $i \in \{1, ..., i(j)\}$, maximum of the function $|u_0^j - u_j|$ in Ω_j^i is achieved in vertices, where $u_0^j = u_0$. Then the first inequality in (2.1) together with the second one in (2.2) imply the inequality

$$|Du_j - Du_0^j| \leqslant 1/2^j$$

in each set Ω_j^i, $i \in \{1, ..., i(j)\}$, and the convergence (2.3) follows. This proves the claim of the lemma. Q.E.D.

Proof of Theorem 1.2. Let f be a piece-wise affine function such that $Df \in (U \cup K)$ a.e. in Ω. We will construct a sequence of piece-wise affine functions $u_j \colon \Omega \to \mathbf{R}^m$ having the following properties:

$$Du_j \in (U \cup K) \text{ a.e. in } \Omega, \qquad \|\mathrm{dist}(Du_j; K)\|_{L^1} \to 0, \tag{2.4}$$

$$u_j|_{\partial\Omega} = f|_{\partial\Omega}, \tag{2.5}$$

$$u_j \to u_0 \text{ in } W^{1,1}(\Omega; \mathbf{R}^m) \cap L^\infty(\Omega; \mathbf{R}^m). \tag{2.6}$$

We take $u_1 = f$. Assume that u_j is already defined. We will show how to define u_{j+1}. Let $\Omega_j \subset\subset \Omega$ be such that

$$\mathrm{meas}(\Omega \setminus \Omega_j) \leqslant \frac{\mathrm{meas}\,\Omega}{2^j}, \tag{2.7}$$

and let $\Omega_j = \bigcup_{i=1}^{i(j)} \Omega_j^i$, where Ω_j^i are disjoint tetrahedra such that Du_j is constant in Ω_j^i for each $i \in \{1, ..., i(j)\}$, i.e., $Du_j = A_j^i$ in Ω_j^i, $i \in \{1, ..., i(j)\}$. We may assume also that

$$\mathrm{diam}\,\Omega_j^i \leqslant c(\text{in-radius of } \Omega_j^i), \qquad i \in \{1, ..., i(j)\},$$

with some $c > 0$ independent of $j \in \mathbf{N}$.

We assume that d_j is the minimum of the set of diameters of balls inscribed in the sets Ω_j^i, $i \in \{1, ..., i(j)\}$, D_j is the maximum of the set of diameters of the sets Ω_j^i, $i \in \{1, ..., i(j)\}$. We may assume also $D_j \in]0, 1/j]$.

Fix $i \in \{1, ..., i(j)\}$. By the assumptions of the theorem and by the Vitaly covering argument we can find a piece-wise affine function $\phi_j^i \in W_0^{1,\infty}(\Omega_j^i; \mathbf{R}^m)$ such that $\phi_j^i \neq 0$ if the inclusion $Du_j(x) \in K$ a.e. in Ω_j^i does not hold and

$$\|\operatorname{dist}(A_j^i + D\phi_j^i, K)\|_{L^1(\Omega_j^i)} < \frac{1}{2^j} \operatorname{meas} \Omega_j^i, \quad A_j^i + D\phi_j^i \in U, \tag{2.8}$$

$$\|\phi_j^i\|_{L^\infty(\Omega_j^i)} \leqslant \frac{d_j}{2^{j+1}}, \quad \|\phi_j^i\|_{L^\infty(\Omega_j^i)} \leqslant \frac{\|u_j - u_{j-1}\|_{L^\infty(\Omega)}}{2}. \tag{2.9}$$

Define $\phi_j = \phi_j^i$ in Ω_j^i, $\phi_j = 0$ otherwise.

Define also $u_{j+1} := u_j + \phi_j$ in Ω_j, $u_{j+1} = u_j$ otherwise. Then (2.8) implies (2.4). By Lemma 2.1 the inequalities (2.9) show that the limit in (2.6) exists. Finally (2.4), (2.5) give

$$Du_0 \in K \text{ a.e. in } \Omega, \qquad u_0|_{\partial\Omega} = f|_{\partial\Omega}, \qquad u_0 \in W_0^{1,\infty}(\Omega; \mathbf{R}^m). \tag{2.10}$$

This completes the proof. Q.E.D.

Proof of Theorem 1.3. The argument follows the lines of the proof of the previous theorem. Fix $\eta > 0$.

The sequence u_j will be constructed in a way to meet the requirements of Lemma 2.1, i.e., $u_{j+1} = u_j + \phi_j$, where $\phi_j \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ are piece-wise affine functions such that (2.1) holds with Ω_j such that $\operatorname{meas}(\Omega \backslash \Omega_j) \leqslant 1/2^j$. Note that to choose ϕ_j satisfying the requirement (2.1) we need only know the function ϕ_{j-1}. We will use this flexibility to take ϕ_j with

$$\|\phi_j\|_{L^\infty} \leqslant \eta/2^j. \tag{2.11}$$

Moreover the sequence ϕ_j will satisfy one more requirement. We show how to achieve this knowing the function ϕ_{j-1}.

Let x_0 be a point such that the restriction of Du_j to its neighborhood is a constant function. Let its value be A.

By assumptions we can find a set $V \subset U(x_0, u_j(x_0))$ such that there is a piece-wise affine function $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with $A + D\phi \in V$ a.e.,

$$\int_\Omega d(x_0, u_j(x_0), A + D\phi(x))\, dx \leqslant \frac{1}{j} \operatorname{meas} \Omega. \tag{2.12}$$

Moreover there exists $\delta > 0$ such that $Du_j = A$ in $B(x_0, \delta)$ and

$$V \subset \bigcap_{|x - x_0| \leqslant \delta,\ |u| \leqslant \delta} U(x, u_j(x) + u).$$

We will show that $\delta > 0$ can be taken so small that

$$\int_{\tilde{\Omega}} d(x, u_j(x) + u, A + D\tilde{\phi}(x))\, dx \leqslant \frac{3}{j} \operatorname{meas} \tilde{\Omega} \tag{2.13}$$

for each open set $\tilde{\Omega} \subset B(x_0, \delta)$, each u with $|u| \leqslant \delta$ and each function $\tilde{\phi} \in W_0^{1,\infty}(\tilde{\Omega}; \mathbf{R}^m)$, which is obtained by the Vitaly covering argument applied to ϕ, with $\|\tilde{\phi}\|_{L^\infty(\tilde{\Omega}; \mathbf{R}^m)} \leqslant \delta$. To prove (2.13) recall that $d \leqslant M$ everywhere and there is a finite set $\{A_1, ..., A_l\}$ of elements of $\mathbf{R}^{m \times n}$ with

$$\operatorname{meas}\{x \in \Omega : D\phi(x) \neq A_1, ..., D\phi(x) \neq A_l\} \leqslant \frac{1}{jM} \operatorname{meas} \Omega. \tag{2.14}$$

If δ is sufficiently small then upper semicontinuity of d implies

$$d(x, u_j(x) + u, A + A_i) - d(x_0, u_j(x_0), A + A_i) \leqslant 1/j, \qquad i \in \{1, ..., l\},$$

for each $x \in B(x_0, \delta)$ and $|u| \leqslant \delta$. Then for each $\tilde{\phi}$ under consideration we have

$$d(x, u_j(x) + u, A + D\tilde{\phi}(x)) - d(x_0, u_j(x_0), A + D\tilde{\phi}(x)) \leqslant 1/j \tag{2.15}$$

in the set $\tilde{\Omega}_1 := \{x \in \tilde{\Omega} : D\tilde{\phi} \in \{A_1, ..., A_l\}\}$. In view of (2.14) we have also

$$\int_{\tilde{\Omega} \setminus \tilde{\Omega}_1} d(x, u_j(x) + u, A + D\tilde{\phi}(x))\, dx \leqslant \frac{1}{j} \operatorname{meas} \tilde{\Omega}.$$

The latter inequality together with the inequalities (2.12) and (2.15) implies (2.13). Applying the Vitaly covering argument once more we can make the L^∞-norm of the function $\tilde{\phi}$ arbitrary small and we can assume that $\tilde{\Omega} \subset B(x_0, \delta)$ is a tetrahedron containing x_0.

Applying the Vitaly covering arguments together with (2.13) we obtain that for each $j \in \mathbf{N}$ there exists a subset $\Omega_j := \bigcup_{i=1}^{i(j)} \Omega_j^i$ of Ω such that $\operatorname{meas}(\Omega \setminus \Omega_j) \leqslant 1/2^j$, Ω_j^i, $i \in \{1, ..., i(j)\}$, are disjoint tetrahedra, and $Du_j = A_j^i$ in each tetrahedron Ω_j^i, $i \in \{1, ..., i(j)\}$. In addition we may assume

$$\operatorname{diam} \Omega_j^i \leqslant c(\text{in-radius of } \Omega_j^i), \qquad i \in \{1, ..., i(j)\},$$

with $c>0$ independent of $j \in \mathbf{N}$. Moreover there exist $\delta_j > 0$ and sets U_j^i, $i \in \{1, ..., i(j)\}$, such that

$$U_j^i \subset \bigcap_{x \in \Omega_j^i,\ |u| \leqslant \delta_j} U(x, u_j(x) + u), \tag{2.16}$$

and there exist piece-wise affine functions $\phi_j^i \in W_0^{1,\infty}(\Omega_j^i; \mathbf{R}^m)$ with $(A_j^i + D\phi_j^i) \in U_j^i$ a.e. and

$$\int_{\Omega_j^i} d(x, u_j(x) + u, A_j^i + D\phi_j^i(x))\, dx \leqslant \frac{3}{j} \operatorname{meas} \Omega_j^i,$$
$$\text{for all} \quad |u| \leqslant \delta_j, \qquad 1 \leqslant i \leqslant i(j). \tag{2.17}$$

Moreover we can select ϕ_j^i in such a way that

$$\|\phi_j^i\|_{L^\infty(\Omega_j^i)} \leqslant \delta_j/2, \qquad \|\phi_j^i\|_{L^\infty(\Omega_j^i)} \leqslant \|\phi_{j-1}\|_{L^\infty(\Omega)}/2, \qquad i \in \{1, ..., i(j)\}. \tag{2.18}$$

The function ϕ_j is then defined as ϕ_j^i in Ω_j^i, $i \in \{1, ..., i(j)\}$, $\phi_j = 0$ otherwise.

Remember that in addition to (2.18) we can assume that ϕ_j satisfies (2.11) and (2.1). By Lemma 2.1 the latter assumption implies convergence $u_j \to u_0$ in $L^\infty(\Omega; \mathbf{R}^m) \cap W^{1,1}(\Omega; \mathbf{R}^m)$. It turns out that (2.16)–(2.18) imply the identity $d(x, u_0(x), Du_0(x)) = 0$ a.e. in Ω. In fact by (2.16)–(2.18) we have

$$\int_\Omega d(x, u_0(x), Du_{j+1}(x))\, dx \leqslant \frac{3}{j} \operatorname{meas} \Omega + \frac{M}{2^j}.$$

We can take a subsequence u_j (not relabeled) such that Du_j converges to Du_0 a.e. in Ω, and $d(x, u_0(x), Du_j(x)) \to 0$ a.e. in Ω.

Since for each $(x, u) \in \Omega \times \mathbf{R}^m$ the set $K(x, u) := \{v \in U(x, u) : d(x, u, v) = 0\}$ is compact and the convergence $d(x, u, v_k) \to 0$ holds with $v_k \in U(x, u)$ if and only if $\operatorname{dist}(v_k, K(x, u)) \to 0$ we obtain that $Du_0(x) \in K(x, u_0(x))$ for a.e. $x \in \Omega$.

The proof is complete. Q.E.D.

Proof of Corollary 1.4. This is an easy consequence of Theorem 1.3. In fact it is enough to check that the function

$$d\colon \{(x, u, v) \in \Omega \times \mathbf{R}^m \times \mathbf{R}^{m \times n} : v \in U\} \to \mathbf{R},$$

defined by $d(x, u, v) := \operatorname{dist}(v, K(x, u))$, $v \in U(x, u)\ UK(x, u)$, is upper semicontinuous. The latter property follows from lower semicontinuity of the multi-valued mapping $(x, u) \to K(x, u)$. The verification of other requirements of Theorem 1.3 is straightforward. The proof is complete. Q.E.D.

3. SOBOLEV SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

In this section we show how Theorem 3.2 below can be derived from general principles discussed in the previous section. We discuss also how measurable dependence of L on x influences the result. It turns out that Theorem 3.2 still holds if L is upper semicontinuous with respect to x, but it might be false if L is only lower semicontinuous in a subset of nonzero measure.

In the proof of Theorem 3.2 we will use Lemma 3.1, which is a vector-valued version of Lemma 2.3 from [S2]. To prove Lemma 3.1 we make use of special functions w_s (see (3.3)) proposed in [Ma], [Gu].

LEMMA 3.1. *Assume that* $c \in \mathbf{R}^m$ *and assume that* $b \in \mathbf{R}^n$. *Let* $b_1 = t_1 b$, $b_2 = t_2 b$, *where* $t_2 < 0 < t_1$, *and let* $b_1, ..., b_q$ *be extreme points of a compact convex set with* $0 \in \operatorname{int} \operatorname{co}\{b_1, ..., b_q\}$. *Define* $B_i := c \otimes b_i$, $i \in \{1, ..., q\}$.

Then for each $\varepsilon > 0$ *there exists a piece-wise affine function* $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ *such that*

$$\operatorname{meas}\{x \in \Omega : D\phi(x) = B_1 \text{ or } D\phi(x) = B_2\} \geqslant \operatorname{meas} \Omega - \varepsilon, \tag{3.1}$$

$$D\phi \in \{B_1, ..., B_q\} \text{ a.e. in } \Omega. \tag{3.2}$$

Proof. It is enough to prove the lemma in the scalar case $m = 1$ (with $c = 1$). In fact, if (3.1), (3.2) hold for a function $\psi \in W_0^{1,\infty}(\Omega)$ then we can define a function $\phi: \Omega \to \mathbf{R}^m$ by the rule $\phi_i = c_i \psi$, $i \in \{1, ..., m\}$. Then $D\phi = c \otimes D\psi$ and the result holds in the general vector-valued case.

To prove the lemma in the scalar case consider first extremum points $v_1, ..., v_q$ of a compact subset in $\mathbf{R}^n$ with $0 \in \operatorname{int} \operatorname{co}\{v_1, ..., v_q\}$. Consider the function

$$w_s(\cdot) := \max_{v \in \{v_1, ..., v_q\}} \langle v, \cdot \rangle - s, \qquad s > 0. \tag{3.3}$$

It is clear that $w_s(\cdot)$ is a Lipschitz function such that $Dw_s \in \{v_1, ..., v_q\}$ a.e. and $w_s(\cdot) = 0$ in ∂P_s, where P_s are polyhedra with the property $P_s = sP_1$.

We can decompose Ω into domains $\Omega_i := x_i + s_i P_1$, $i \in \mathbf{N}$, and a set N of null measure, i.e., $\Omega := \bigcup_{i \in \mathbf{N}} (x_i + s_i P_1) \cup N$. Define $u(x) := w_{s_i}(x - x_i)$ for $x \in x_i + s_i P_1$, $i \in \mathbf{N}$, $u = 0$ otherwise. Then $u \in W_0^{1,\infty}(\Omega)$, $Du \in \{v_1, ..., v_q\}$ a.e. in Ω.

We can take $v_1 = b_1$, $v_2 = b_2$ and $v_i \in B(b_1, \varepsilon) \cap \operatorname{int} \operatorname{co}\{b_1, ..., b_q\}$, $i \in \{3, ..., q\}$. Then we can perturb the function u in each set $\Omega_i := \{x \in \Omega : Du(x) = v_i\}$, $i \in \{3, ..., q\}$, in such a way that the perturbation ϕ_ε has the property $D\phi_\varepsilon \in \{b_1, ..., b_q\}$. We can do this since $v_i \in \operatorname{int} \operatorname{co}\{b_1, ..., b_q\}$ and the construction in (3.3) can be applied to find a piece-wise affine function $f_i \in W_0^{1,\infty}(\Omega_i)$ such that $Df_i \in \{b_1 - v_i, ..., b_q - v_i\}$. Then, the function $l_{v_i} + f_i$ presents the perturbation in question.

Note that

$$\text{meas}\{x \in \Omega : D\phi_\varepsilon \notin \{b_1, b_2\}\} \to 0, \qquad \varepsilon \to 0,$$

since

$$\text{meas}\{x \in \Omega_i : Df_i(x) \neq b_1 - v_i\} \to 0, \qquad \varepsilon \to 0, \qquad \forall i \in \{3, ..., q\}.$$

This proves the claim of the lemma. Q.E.D.

THEOREM 3.2. *Let* $L: \bar{\Omega} \times \mathbf{R}^m \times \mathbf{R}^{m \times n} \to \mathbf{R}$ *be a continuous function such that* $\liminf_{|v| \to \infty} L(x, u, v) > 0$ *uniformly on compact sets in the* x *and* u *variables, and let* $f \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ *be a piece-wise affine function such that* $L(\cdot, f(\cdot), Df(\cdot)) \leqslant 0$ *a.e. in* Ω.

Then for each $\varepsilon > 0$ *one can find a function* $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ *such that* $\|\phi\|_{L^\infty} \leqslant \varepsilon$ *and*

$$L(x, f(x) + \phi(x), Df(x) + D\phi(x)) = 0 \text{ a.e. in } \Omega.$$

Proof. We assume

$$U(x, u) := \{v \in \mathbf{R}^{m \times n} : L(x, u, v) < 0\}, \qquad K(x, u) := \partial U(x, u). \quad (3.4)$$

We define $d := -L$.

To prove the assertion it is enough to verify the assumptions of Theorem 1.3. Let $v_0 \in U(x_0, u_0)$ and let $\varepsilon > 0$. It suffices to show that there exists a set $U_\varepsilon \ni v_0$ reducible to the set

$$K_\varepsilon := \{v \in U(x_0, u_0) : \text{dist}(v, K(x_0, u_0)) \leqslant \varepsilon\}$$

and such that $U_\varepsilon \subset U(x, u)$ for all (x, u) sufficiently close to (x_0, u_0).

Note that

$$\inf\{d(x_0, u_0, v) : v \in U(x_0, u_0) \backslash K_\varepsilon\} > \nu > 0.$$

Since $v_0 \in U(x_0, u_0)$ we infer $d(x_0, u_0, v_0) = \eta > 0$. It is clear that K_ε contains the boundary of the set

$$U_\varepsilon := \bigcap_{|x - x_0| \leqslant \delta,\ |u - u_0| \leqslant \delta} \{v \in U(x, u) : d(x, u, v) \geqslant \min\{\eta/2, \nu/2\}\}$$

and that $v_0 \in U_\varepsilon$ if $\delta = \delta(\varepsilon) > 0$ is sufficiently small. We can apply Lemma 3.1 to show that the set U_ε can be reduced to its boundary ∂U_ε. To do this consider a rank-one matrix A and consider $t_1 < 0$, $t_2 > 0$ such that $v_0 + t_1 A$, $v_0 + t_2 A \in \partial U_\varepsilon$, $v_0 + tA \in U_\varepsilon$ for $t \in]t_1, t_2[$. We can use Lemma 3.1 to assert that there exists a piece-wise affine function $\phi_\varepsilon \in l_{v_0} + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that

$$D\phi_\varepsilon \in U_\varepsilon \text{ a.e.,}$$

$$\text{meas}\{x \in \Omega : D\phi_\varepsilon(x) \in \{v_0 + t_1 A, v_0 + t_2 A\}\} \geqslant \text{meas } \Omega - \varepsilon.$$

Then $\int_\Omega d(x_0, u_0, D\phi_\varepsilon(y))\, dy \to 0$ as $\varepsilon \to 0$.

By construction

$$U_\varepsilon \subset \bigcap_{|x - x_0| \leqslant \delta,\ |u - u_0| \leqslant \delta} U(x, u)$$

if $\delta = \delta(\varepsilon) > 0$ is sufficiently small. The proof is complete. ∎

Note that Theorem 3.2 can be extended to the case of upper semicontinuous dependence of L on x. This follows from the possibility to replace the requirement of Theorem 1.3 on upper semicontinuity of the function

$$d: \{(x, u, v) \in \Omega \times \mathbf{R}^m \times \mathbf{R}^{m \times n} : v \in (U(x, u) \cup K(x, u))\} \to [0, M]$$

by a weaker assumption on the validity of this requirement with a sequence of subsets Ω_k of Ω instead of Ω itself, where $\text{meas}(\Omega \backslash \Omega_k) \leqslant 1/k$. In this case the proof follows the lines of the proof given in Section 2 with the only change that some estimates hold in the integral sense.

Note that the existence result is well-known in the scalar case $m = 1$ for Hamilton–Jacobi equations of the eikonal type $H(Du(\cdot)) = f(\cdot)$, see [L, Chap. 7]. Moreover for this type of equations a theory of well-posed solutions similar to the theory of viscosity solutions was developed recently in [NJ].

It is also obvious that instead of requiring upper semicontinuity in x in the whole domain Ω we can take an open subset Ω_0 of full measure. However if we admit that L is no longer upper semicontinuous in a subset Ω' of Ω with nonzero measure then the existence result may fail.

Consider the problem $|Du| = f$, $u \in W^{1,\infty}(\Omega)$, where $\Omega = [0, 1] \times [0, 1]$ and $u: \Omega \to \mathbf{R}$. It was remarked in [L, Remark 7.5], [Cr] that one can find an open, dense, and connected subset $\tilde{\Omega}$ of Ω with $\text{meas}\{\Omega \backslash \tilde{\Omega}\} > 0$. Then taking $f = 0$ in $\tilde{\Omega}$, $f = 1$ otherwise, we infer that each solution u of the problem satisfies $Du = 0$ in $\tilde{\Omega}$. Connectedness of $\tilde{\Omega}$ implies that u is constant in $\tilde{\Omega}$. Then density implies that u is constant everywhere in Ω, i.e., $Du = 0$ a.e. in Ω.

In this example f is forced to be equal to zero in a large set. It turns out that this example can be modified to include the case with $f \in \{1, 3\}$. In fact let G be an open dense subset of $[0, 1]$ with $(1-\varepsilon) < \text{meas}\, G < 1$, $\varepsilon > 0$ is given. Consider the set $\tilde{\Omega} := G \times G$. Assume $f = 1$ in $\tilde{\Omega}$, $f = 3$ in $\Omega \setminus \tilde{\Omega}$.

Assume that $u \in W^{1,\infty}(\Omega)$ and $|Du| \leqslant f$ in $\tilde{\Omega}$, i.e. $|Du| \leqslant 1$ in $\tilde{\Omega}$. Our claim is that $|Du| \leqslant 2$ a.e. in Ω. To see this notice that if $A_1 = (x_1, y_1) \in \tilde{\Omega}$ and $A_2 = (x_2, y_2) \in \tilde{\Omega}$ then the point $A = (x_1, y_2)$ also belongs to $\tilde{\Omega}$. Since

$$|A_1 - A_2| \geqslant \max\{|A - A_1|, |A - A_2|\}$$

and

$$|u(A_1) - u(A)| \leqslant |A_1 - A|, \qquad |u(A_2) - u(A)| \leqslant |A_2 - A|$$

we obtain that $|u(A_1) - u(A_2)| \leqslant 2\,|A_1 - A_2|$.

Since $\tilde{\Omega}$ is dense in Ω we infer that u is Lipschitz with the constant 2 in the whole set Ω. Therefore $|Du| < 3$ for a.e. $x \in \Omega \setminus \tilde{\Omega}$, i.e., $|Du| < f$ in this set. This shows that no solution of the equation $|Du| = f$ a.e. in Ω exists.

4. DIFFERENTIAL INCLUSIONS WITH GRADIENT EXTREMAL POINTS

In this section we give the proof of Theorem 1.6. Then we show that the choice $(x, u) \to \text{gr extr}\, U(x, u)$ is optimal to solve the differential inclusions. We also discuss which progress can be made in the general case of continuous multi-valued functions and we prove Theorem 1.7.

To apply the general reduction principles to the case of Theorem 1.6 we have to establish first

LEMMA 4.1. *Assume that U is a compact convex set with nonempty interior. Then its interior can be reduced to the set* $\overline{\text{gr extr}\, U}$.

Proof. To prove the lemma we have to show that given $A \in \text{int}\, U$ and $\delta > 0$ there is a piece-wise affine function $u \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with the properties:

(1) $Du \in \text{int}\, U$ a.e. in Ω,

(2) $\|\text{dist}(Du, \overline{\text{gr extr}\, U})\|_{L^1(\Omega)} \leqslant \delta$.

Without loss of generality we can assume that $A = 0$. To each point $F \in \partial U$ we can associate an integer number ind F which is dimension of the smallest face (of ∂U) containing F. It is clear that $F \in \text{extr}\, U$ if and only if ind $F = 0$.

Let $\varepsilon > 0$. Consider the set $U^\varepsilon := \{(1-\varepsilon)\, v : v \in U\}$.

Take a matrix $B \in \mathbf{R}^{m \times n}$ with rank $B = 1$. Then there exist $t_1 < 0$, $t_2 > 0$ such that $A_i := t_i B \in \partial U^\varepsilon$ $(i = 1, 2)$ and $tB \in \operatorname{int} U^\varepsilon$ for $t \in]t_1, t_2[$. By Lemma 3.1 we can find a piece-wise affine function $u \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ the gradient of which assumes finitely many values and satisfies

$$Du \in U^\varepsilon \text{ a.e.}, \qquad \operatorname{meas}\{x \in \Omega : Du(x) \neq A_i,\, i = 1, 2\} < \varepsilon_1 < \varepsilon. \tag{4.1}$$

In the case $A_1 \notin \operatorname{gr\,extr} U^\varepsilon$ we can isolate a face $U_1 \subset \partial U^\varepsilon$ such that $A_1 \in \operatorname{re\,int} U_1$ (in this case ind A_1 is equal to dimension of U_1). We can also find a matrix B_1 with rank $B_1 = 1$ such that for some $t_3 < 0$, $t_4 > 0$ we have

$$A_3 := A_1 + t_3 B_1 \in \partial\{\operatorname{re\,int} U_1\}, \qquad A_4 := A_1 + t_4 B_1 \in \partial\{\operatorname{re\,int} U_1\},$$

and $A_1 + tB_1 \in \operatorname{re\,int} U_1$ for $t \in]t_3, t_4[$.

Applying Lemma 3.1 to the set $\Omega_1 := \{x \in \Omega : Du = A_1\}$ we can find a piece-wise affine function $\phi \in l_{A_1} + W_0^{1,\infty}(\Omega_1; \mathbf{R}^m)$ such that $D\phi \in \operatorname{int} U$ a.e. in Ω_1 and for $u_1 := u + \phi$ we have

$$\operatorname{meas}\{x \in \Omega_1 : Du_1 \neq A_3 \text{ or } Du_1 \neq A_4\} < \varepsilon_2, \qquad \text{where} \qquad 0 < \varepsilon_2,\ \varepsilon_1 + \varepsilon_2 < \varepsilon.$$

In this case

$$\operatorname{meas}\{x \in \Omega : Du_1 \notin \{A_2, A_3, A_4\}\} < \varepsilon. \tag{4.2}$$

Note that $\max\{\operatorname{ind} A_3, \operatorname{ind} A_4\} < \operatorname{ind} A_1 \leqslant mn$. If one of the points A_i $(i \in \{2, 3, 4\})$ still does not belong to the set $\operatorname{gr\,extr} U^\varepsilon$ then we can continue the same process in the set $\Omega_i = \{x \in \Omega : Du = A_i\}$. In this case we can no more guarantee that the gradients of the perturbations stays in the set U^ε. However we can select such a perturbation with the gradient staying in the set int U.

It is clear that we need at most mn iterations to achieve the points of the set $\operatorname{gr\,extr} U^\varepsilon$. The final function $u \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ is piece-wise affine with the gradient assuming finitely many values. Moreover, following (4.1), (4.2) we can choose u in such a way that $\operatorname{meas}\{x \in \Omega : Du(x) \notin \operatorname{gr\,extr} U^\varepsilon\} \leqslant \varepsilon$.

Since $\varepsilon > 0$ can be taken arbitrary small the claim of Lemma 4.1 is proved. Q.E.D.

To apply Corollary 1.4 we need to establish lower semicontinuity of the mapping $(x, u) \to \operatorname{gr\,extr} U(x, u)$.

LEMMA 4.2. *Assume that* $U: \bar{\Omega} \times \mathbf{R}^m \to 2^{\mathbf{R}^{m \times n}}$ *is a continuous multi-valued mapping whose values are convex compact sets.*

Then the multi-valued mapping $(x, u) \to \overline{\text{gr extr } U(x, u)}$ *is lower semicontinuous, i.e., if* $v_0 \in \overline{\text{gr extr } U(x_0, u_0)}$ *and* $(x_k, u_k) \to (x_0, u_0)$, $k \to \infty$, *then there exist* $v_k \in \overline{\text{gr extr } U(x_k, u_k)}$ *such that* $v_k \to v_0$ *as* $k \to \infty$.

Proof. It is enough to show that the mapping $(x, u) \to \text{gr extr } U(x, u)$ is lower semicontinuous.

Recall that to each point $v \in \partial U$ of a convex set U we can assign an integer number $\text{ind}(v)$, which is dimension of the smallest face h of ∂U containing v (in this case $v \in \text{re int } h$).

Let v_0 be a gradient extremum point of the set $U(x_0, u_0)$. Assume that there exists a sequence $(x_k, u_k) \to (x_0, u_0)$ and $\varepsilon > 0$ such that for each $k \in \mathbf{N}$ the set $B((x_0, u_0), \varepsilon)$ does not contain extremum points of $U(x_k, u_k)$. Define

$$I := \inf\{\liminf_{k \to \infty} \text{ind}(\tilde{v}_k) : \tilde{v}_k \to v_0, \tilde{v}_k \in \partial U(x_k, u_k)\}. \tag{4.3}$$

Switching, if necessary, to a subsequence we can find a sequence $v_k \in \partial U(x_k, u_k)$ such that $v_k \to v_0$ and $\text{ind}(v_k) = I \geqslant 1$ for all sufficiently large $k \in \mathbf{N}$.

Let $V_k \subset \partial U(x_k, u_k)$ be the face of dimension $\text{ind}(v_k)$ which contains v_k, $k \in \mathbf{N}$. We claim that for all sufficiently large $k \in \mathbf{N}$ the face V_k does not contain rank-one connections. Otherwise we can find a subsequence (not relabeled) each element of which contains a rank-one direction a_k with $|a_k| = 1$, $a_k \to a_0$. Moreover there exists a $\delta > 0$ such that

$$v_k \in [v_k - a_k\delta, v_k + a_k\delta] \subset V_k, \qquad k \in \mathbf{N}. \tag{4.4}$$

If the claim (4.4) fails then there exists a subsequence v_k (not relabeled) and $\tilde{v}_k \in \partial(\text{re int } V_k)$ such that $v_k - \tilde{v}_k \to 0$. Then $\text{ind}(\tilde{v}_k) < \text{ind}(v_k)$ for all sufficiently large k and this contradicts (4.3). Therefore (4.4) holds.

In view of (4.4) we have $v_0 \in [v_0 - a_0\delta, v_0 + a_0\delta] \subset U(x_0, u_0)$, where $\text{rank}(a_0) = 1$. This contradicts the assumption $v_0 \in \text{gr extr } U(x_0, u_0)$. The contradiction proves that V_k does not contain rank-one connections if k is sufficiently large.

Therefore $v_k \in \text{gr extr } U(x_k, u_k)$ for all sufficiently large $k \in \mathbf{N}$. This proves that in case v_0 can not be approximated by extremum points of $U(x_k, u_k)$ it still can be approximated by gradient extremum points of these sets. The proof of the lemma is complete. Q.E.D.

Proof of Theorem 1.6. This will be reduced to the verification of the assumptions of Corollary 1.4.

Let $A \in \text{int } U(x_0, u_0)$, and let $\varepsilon > 0$. Without loss of generality we can assume $A = 0$.

To meet the requirement of Corollary 1.4 we can take the set $U_\delta := (1-\delta)\, U(x_0, u_0)$ with $\delta > 0$ so small that

$$\operatorname{dist}(\overline{\text{gr extr } U_\delta}; \overline{\text{gr extr } U(x_0, u_0)}) < \varepsilon/2.$$

By Lemma 4.1 U_δ can be reduced to the set $\overline{\text{gr extr } U_\delta}$.

In view of convexity and continuity of the function $(x, u) \to U(x, u)$ the inclusion $U_\delta \subset U(x, u)$ holds for all (x, u) sufficiently close to (x_0, u_0). Moreover, lower semicontinuity of the mufti-valued function $(x, u) \to K(x, u) := \overline{\text{gr extr } U(x, u)}$ is the content of Lemma 4.2.

Since all the requirements of Corollary 1.4 hold the claim of Theorem 1.6 follows.

Q.E.D.

Now we want to show that the function $(x, u) \to \text{gr extr } U(x, u)$ is an optimal choice to resolve the differential inclusions. Then we discuss the general case, i.e. we allow nonconvex sets $U(x, u)$.

To treat the convex case we will use the following auxiliary lemma.

LEMMA 4.3. *Let U be a compact and convex subset of $\mathbf{R}^{m\times n}$ with nonempty interior. Let K be a compact subset of U such that for each $A \in \operatorname{int} U$ we can find a sequence $u_j \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with the property*

$$\int_\Omega \operatorname{dist}(A + Du_j(x), K)\, dx \to 0, \qquad j \to \infty.$$

Then $\text{gr extr } U \subset K$.

This result was proved in [Z1]. The key ingredient of the proof is the observation that given a linear subspace V of $\mathbf{R}^{m\times n}$ without rank-one connections and given $A \in V$ the estimate

$$\int_\Omega |D\phi(x) - \operatorname{Pr}_V D\phi(x)|^2\, dx \geqslant c \int_\Omega |D\phi(x)|^2\, dx, \qquad c > 0,$$

holds for every function $\phi \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$, where $\operatorname{Pr}_V D\phi$ is the projection of the vector $D\phi$ on the space V (see [BFJK]; the result also follows from Theorem 3 in [Ta], see also [Se], [DP]).

THEOREM 4.4. *Let $U: \Omega \to 2^{\mathbf{R}^{m\times n}}$ be a continuous multi-valued function whose values are compact convex sets with nonempty interior. Let also $K: \Omega \to 2^{\mathbf{R}^{m\times n}}$ be a lower semicontinuous and compact multi-valued function with $K(\cdot) \subset U(\cdot)$.*

If for a.e. $x \in \Omega$, *each* $A \in \operatorname{int} U(x)$, *and each* $\varepsilon > 0$ *the problem*

$$A + D\phi(\cdot) \in K(\cdot), \qquad \phi \in W_0^{1,\infty}(B(x,\varepsilon); \mathbf{R}^m) \tag{4.5}$$

has a solution, then gr extr $U(\cdot) \subset K(\cdot)$ *a.e. in* Ω.

Remark. It follows from the proof that the analogous result (with an additional requirement $\|\phi\|_C \leqslant \varepsilon$) holds if U: $\Omega \times \mathbf{R}^m \to 2^{\mathbf{R}^{m \times n}}$ is a compact, convex and lower semicontinuous function and if U and K in addition have the following property: for each $\delta > 0$ there exists a subset Ω_δ of Ω with *meas*$(\Omega \backslash \Omega_\delta) \leqslant \delta$ such that the restrictions of U and K to $\Omega_\delta \times \mathbf{R}^m$ are continuous.

Proof of Theorem 4.4. Note that there exists a sequence Ω_k of compact subsets of Ω such that meas$(\Omega \backslash \Omega_k) \to 0$, $k \to \infty$, and the restriction of K to Ω_k is continuous in the Hausdorff metric, cf. [CV].

Fix $k \in \mathbf{N}$ and fix a Lebesgue point x_0 of Ω_k. We assert that there exists a sequence $u_k \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that

$$\operatorname{dist}(Du_k(\cdot), K(x_0)) \to 0 \quad \text{in} \quad L^1 \quad \text{as} \quad k \to \infty.$$

In fact by (4.5) for each $\varepsilon > 0$ we can find a function $\phi_\varepsilon \in l_A + W_0^{1,\infty}(B(x_0, \varepsilon); \mathbf{R}^m)$ such that $D\phi_\varepsilon(\cdot) \in K(\cdot)$ a.e.. Since x_0 is a Lebesgue point of Ω_k and the restriction of K to Ω_k is continuous we infer

$$\int_{B(x_0,\varepsilon)} \operatorname{dist}(D\phi_\varepsilon(x), K(x_0))\, dx / \operatorname{meas} B(x_0, \varepsilon) \to 0, \qquad \varepsilon \to 0.$$

Then we can apply the Vitaly covering argument to construct a family $u_\varepsilon \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with the property

$$\operatorname{dist}(Du_\varepsilon(\cdot), K(x_0)) \to 0 \quad \text{in} \quad L^1, \qquad \varepsilon \to 0.$$

Lemma 4.3 implies that gr extr $U(x_0) \subset K(x_0)$. Therefore the inclusion gr extr $U(\cdot) \subset K(\cdot)$ holds a.e. in Ω. Q.E.D.

To treat the general case (without requiring convexity of $U(\cdot)$) one has to establish an effective characterization of those subsets of U to which U can be reduced.

The result of [Z2] says that given a compact set U one can always find the smallest subset $K \subset \partial U$ which "generates" U. More precisely for each $A \in \operatorname{int} U$ one can find a sequence of perturbations $\phi_k \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that $\operatorname{dist}(A + D\phi_k, K) \to 0$ a.e. in Ω and each set K' having the same property contains K as a subset. It is *not known*, however, whether the sequence ϕ_k can be selected to satisfy the inclusion $A + D\phi_k \in U$. Moreover it is not known how the sets $K \subset \partial U$ depend on parameters.

However we can apply Corollary 1.4 to establish the following abstract result. We say that a compact set U with nonempty interior can be *properly reduced* to a set $K \subset \partial U$ if for each $A \in \operatorname{int} U$ and each $\varepsilon > 0$ there exists a piece-wise affine function $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that

$$\operatorname{dist}(A + D\phi, (\mathbf{R}^{m \times n} \backslash \operatorname{int} U)) \geqslant \delta > 0 \text{ a.e.},$$

$$\int_\Omega \operatorname{dist}(A + D\phi(x), K)\, dx \leqslant \varepsilon. \tag{4.6}$$

THEOREM 4.5. *Assume that* $U: \Omega \times \mathbf{R}^{m \times n} \to 2^{\mathbf{R}^{m \times n}}$ *is a continuous compact multi-valued function such that for each* $(x_0, u_0) \in \Omega \times \mathbf{R}^{m \times n}$ *and each* $v \in \operatorname{int} U(x_0, u_0)$ *there exists a neighborhood of* v *which belongs to all sets* $U(x, u)$ *with* (x, u) *sufficiently close to* (x_0, u_0).

Let $K: \Omega \times \mathbf{R}^{m \times n} \to 2^{\mathbf{R}^{m \times n}}$ *be a lower semicontinuous compact function such that for each* $(x, u) \in \Omega \times \mathbf{R}^{m \times n}$ *the set* $U(x, u)$ *can be properly reduced to the set* $K(x, u)$.

Then for each piece-wise affine function $f \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ *with* $Df(\cdot) \in \operatorname{int} U(\cdot, f(\cdot))$ *a.e. and each* $\eta > 0$ *there exists a solution of the problem*

$$Du(\cdot) \in K(\cdot, u(\cdot)) \text{ a.e. in } \Omega,$$

$$u \in W^{1,\infty}(\Omega; \mathbf{R}^m), \qquad u|_{\partial\Omega} = f|_{\partial\Omega}, \qquad \|u - f\|_{L^\infty} \leqslant \eta.$$

Proof. It suffices to apply Corollary 1.4 with $V(x, u) = \operatorname{int} U(x, u)$ instead of U. To verify the main hypothesis of Corollary 1.4 one uses the fact that, for $\delta > 0$, the set $S = \{v : \operatorname{dist}(v, \mathbf{R}^{m \times n} \backslash U(x_0, u_0)) \geqslant \delta\}$ is compact. Hence $S \subset U(x, u)$ for all (x, u) sufficiently close to (x_0, u_0) and the argument is easily concluded. Q.E.D.

Theorem 1.7 is an easy corollary of Theorem 4.5.

Proof of Theorem 1.7. It is enough to treat the case of the linear boundary data f, i.e., $f = l_v$. Moreover without loss of generality we can assume that $v \in \operatorname{int} U(x, l_v(x))$ everywhere in Ω, otherwise we can switch to an open subset $\tilde{\Omega}$ of Ω such that $v \in K(x, l_v(x))$ a.e. in $\Omega \backslash \tilde{\Omega}$, $v \in \operatorname{int} U(x, l_v(x))$ everywhere in $\tilde{\Omega}$. The latter holds because of continuity of the mapping $(x, u) \to K(x, u)$.

In order to verify the assumptions of Theorem 4.5 we use the following facts (we always assume (1.2)).

(i) $(A, B) \to U_{A, B}$ is upper semicontinuous (this follows immediately from the description of U as a level set, see [Sv] or [MSv1])

(ii) $\partial(\operatorname{int} U_{A, B}) = \partial U_{A, B}$ (see [MSv1], Lemma 5.1)

(iii) if $F \in \operatorname{int} U_{A,B}$, then $F \in \operatorname{int} U_{A',B'}$ for all (A', B') close to (A, B) (see [MSv1, Corollary 5.2)

(iv) $\operatorname{int} U_{A,B}$ can be reduced to $SO(2)\,A \cup SO(2)\,B$ (see e.g., [MSv1], Lemma 3.2).

Now it follows from (i)–(iii) that the maps $(A, B) \to U_{A,B}$ and $(x, u) \to U(x, u) = U_{A(x,u),\,B(x,u)}$ are continuous. In connection with (ii)–(iv) this shows that $U_{A,B}$ can be properly reduced to $SO(2)(A) \cup SO(2)(B)$.

Q.E.D.

5. COMPARISON WITH THE BAIRE CATEGORY APPROACH

In this section we discuss difference between the Baire category method developed in particular by the Italian school (see e.g. [C], [B], [BF], [DeBP], [DM1]–[DM4] and papers mentioned therein) and our method of constructing sequences of approximate solutions converging strongly in $W^{1,1}$-norm, which is based on Gromov's idea (whose theory of convex integration greatly generalizes earlier work of Nash and Kuiper on the embedding problem).

Recall that the Baire category approach for solving differential inclusions

$$L(Du) = 0 \text{ a.e. in } \Omega, \qquad u|_{\partial\Omega} = f|_{\partial\Omega}$$

consists in proving that the sets of approximate solutions, i.e. of those admissible functions u that $\int_\Omega |L(Du(x))|\, dx < \varepsilon$, are open and dense in the $L^\infty(\Omega; \mathbf{R}^m)$-norm. Then a Baire category argument allows to conclude that the set of solutions is dense in the L^∞-norm in the set of admissible functions.

The advantage of the method is that it reduces the problem to the construction of approximate solutions. On the other hand one has to verify openness in L^∞ of the set of approximate solutions, which is a rather restrictive property.

For a more specific comparison with our approach we first recall the notion of quasiconvexity introduced by Morrey, cf. [Mo].

Definition 5.1. Let U be a bounded subset of $\mathbf{R}^{m \times n}$, let $L\colon U \to \mathbf{R}$ be continuous and bounded from below, and let $L(v) = \infty$ for $v \notin U$. We say that L is quasiconvex at a point $A \in U$ if for each piece-wise affine function $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ such that $A + D\phi \in U$ a.e. in Ω the inequality

$$\int_\Omega L(A + D\phi(x))\, dx \geqslant L(A) \operatorname{meas} \Omega$$

holds.

The function L^{qc} is called the quasiconvexification of L if for each $A \in U$ we have

$$L^{qc}(A) := \inf_{\phi} \frac{1}{\text{meas } \Omega} \int_{\Omega} L(A + D\phi(x))\, dx,$$

where $\phi \in W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ are piece-wise affine functions such that $A + D\phi \in U$ a.e. in Ω.

It is easy to show that L^{qc} is a quasiconvex function.

A typical result available by the Baire category method is

THEOREM 5.2 [DM1, Thm.2.1]. *Let* $\Omega \subset \mathbf{R}^n$ *be an open set, and let* $\phi \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ *and* $L: \mathbf{R}^{m \times n} \to \mathbf{R}$ *satisfy the following hypotheses*:

$$L \text{ is quasiconvex}; \tag{5.1}$$

there exists a compact convex set U *such that* $U \subset \{\xi \in \mathbf{R}^{m \times n} : L(\xi) \leqslant 0\}$;

$$(L^-)^{qc} = 0 \text{ on int } U, \text{ where } L^- = -L \text{ on } U \text{ and } +\infty \text{ otherwise}; \tag{5.3}$$

$$D\phi \text{ is compactly contained in int } U. \tag{5.4}$$

Then there exists $u \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ *such that*

$$\begin{aligned} L(Du(x)) &= 0, \qquad && \text{a.e. } x \in \Omega, \\ u(x) &= \phi(x), && x \in \partial\Omega. \end{aligned} \tag{5.5}$$

Moreover $Du(x) \in U$ *a.e.*

Here the authors define the set of the admissible functions as

$$V := \{u \in \phi + W_0^{1,\infty}(\Omega; \mathbf{R}^m) : Du(x) \in U \text{ a.e. in } \Omega\}.$$

and the sets of approximate solutions as

$$V_k = \left\{ u \in V : \int_{\Omega} L^-(Du(x))\, dx < \frac{1}{k} \right\}.$$

Convexity of U allows to approximate original functions by admissible piecewise affine ones in $W^{1,\infty}$-norm, see [DM1, Sect. 6]. Moreover it implies completeness of V in the L^∞-norm.

The requirement of quasiconvexity of L allows to obtain openness of V_k in the $L^\infty(\Omega; \mathbf{R}^m)$-norm since integral functionals with quasiconvex integrands are sequentially weak* lower semicontinuous in $W^{1,\infty}(\Omega; \mathbf{R}^m)$.

Moreover quasiconvexity of integrands is just a characterization of this property of integral functionals [Mo]. Therefore the requirement (5.1) is necessary for sequential weak* upper semicontinuity of the integral functional with integrand L^-, that means that this condition is optimal for applying the Baire category arguments (since we need openness of V_k). Note that density of the sets V_k follows from the identity $(L^-)^{qc}=0$ on int U. Then the set $\bigcap_k V_k$ is dense in V and contains only solutions of the equation (5.5).

Note that continuity and quasiconvexity of $L|_U \leqslant 0$ imply that the set $K := \{\xi \in U : L(\xi)=0\}$ is generally larger than the set $\overline{\text{gr extr } U}$. First, it follows from [Z1] that $\overline{\text{gr extr } U} \subset K$, see also Section 4. Moreover, if $A \in \overline{\text{gr extr } U}$ and there are $B_1, B_2 \in \overline{\text{gr extr } U}$ with $A \in]B_1, B_2[$, rank$(B_2 - B_1)=1$, then $[B_1, B_2] \subset K$. This follows from continuity of L and Lemma 3.1. The set of such A might be nonempty in the case $n \geqslant 3$, but other points of $]B_1, B_2[$ may not lie in the set $\overline{\text{gr extr } U}$ (see the example of the set U based on Proposition 5.3). Therefore K is generally larger than the set $\overline{\text{gr extr } U}$.

Another interesting idea to modify the Baire category argument was proposed recently in [DM3, Section 4], see also [DM4, Section 6]. There the authors proved Theorem 1.2 under the additional requirement that K has the property: for each $\varepsilon > 0$ and each $A \in U$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $u \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ satisfies $\|\text{dist}(Du(\cdot), K)\|_{L^1} \leqslant \delta$ then for each sequence $\phi_k \in l_A + W_0^{1,\infty}(\Omega; \mathbf{R}^m)$ with $D\phi_k \in U$ a.e. and $\phi_k \rightharpoonup^* u$ in $W^{1,\infty}(\Omega; \mathbf{R}^m)$ the inequality

$$\limsup_{k \to \infty} \|\text{dist}(D\phi_k; K)\|_{L^1} \leqslant \varepsilon \tag{5.6}$$

holds.

Given a piece-wise affine function ϕ with $D\phi \in (U \cup K)$ the set V of admissible functions is defined as the closure of the set of all piece-wise affine functions

$$u \in \phi + W_0^{1,\infty}(\Omega; \mathbf{R}^m), \qquad Du \in (U \cup K),$$

in the $L^\infty(\Omega; \mathbf{R}^m)$-norm. It is clear that V is a complete metric space in the L^∞-metric.

The authors consider the standard abstract lower semicontinuous extension of the functional $u \in V \to -\int_\Omega \text{dist}(Du(x), K)\, dx$, which is

$$I(u) := \inf \left\{ \liminf_{j \to \infty} - \int_\Omega \text{dist}(Du_j(x), K)\, dx : u_j \rightharpoonup^* u,\ u_j \in V \right\}.$$

We have that if $u \in V$ and $I(u)=0$ then $Du \in K$ a.e. in Ω.

The sets

$$V_k := \{u \in V : I(u) > -1/k\}$$

of approximate solutions are automatically open in the L^∞ topology since the functional $I(u)$ is sequentially lower semicontinuous in this topology. Density of the set V_k follows from the requirement (5.6). In fact by (5.6) the set V_k contains all functions $u \in V$ with

$$-\!\!\!\!\!\!\int_\Omega \operatorname{dist}(Du(x), K)\, dx \leqslant \delta,$$

where $\delta = \delta(1/k) > 0$. Since the latter set is dense in V by the assumptions of Theorem 1.2 and the Vitaly covering argument (see Section 2) we infer that all V_k, $k \in \mathbf{N}$, are dense in V. The Baire category argument allows to conclude that the set $\bigcap_k V_k$, which consists of solutions $f \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ of the differential inclusion

$$Df \in K, \qquad f = \phi \quad \text{in } \partial\Omega,$$

is dense in V (in the L^∞-norm).

Note that in this construction the authors exploit the fact that to apply the Baire category argument it is enough to deal with neighborhoods of the functional $u \to -\!\!\!\!\!\!\int_\Omega \operatorname{dist}(Du(x), K)\, dx$ at zero, i.e. it is enough to require stability in the L^∞-norm of those approximate solutions which have the gradients sufficiently close to K in the integral norm.

In the latter result one does not specify the structure of the set U. However K should have special structure which in the case of convex U gives the same result as Theorem 5.2 stated above.

Some improvements of the Baire category approach are still possible. In the case of convex U one can, e.g., try to use upper semicontinuous quasiconvex integrands L like in the original approach due to A. Bressan (see [B], [BF]), where the scalar case was completely treated in the case of continuous multi-valued functions. However the construction of such integrands might be a bit tricky. It is also possible to use more flexible integrands which give functionals lower semicontinuous in a class of functions smaller than all admissible Lipschitz functions (like rank-one convex integrands and the functions given by iterative application of Lemma 3.1 and their limits). In any case the requirement of openness of the sets of approximate solutions in the L^∞-norm requires a special structure of U and K, which we can avoid by dealing with strongly convergent approximate solutions as in Theorem 1.2.

Theorem 1.3 shows how to develop our method in the case of nonhomogeneous differential inclusions and allows to remove the quasiconvexity requirement (i.e., the requirement that $L(x, u, \cdot)$ is quasiconvex), which is responsible for openness of the approximate solutions in L^∞, in the results contained in the papers [DM2–DM4].

The case of convex sets is the best studied in literature and it is easier to show the difference in the constructions described above in this case. We will exploit a well-known fact that in the case $n \geqslant 3$ the set of extremum points $\operatorname{extr} S$ of a compact convex subset S of $\mathbf{R}^n$ can be nonclosed. More specifically we will need an example of a set S with the properties described in Proposition 5.3. Then the set U in question will be

$$U := \{v \in \mathbf{R}^{3\times 2} : (v_{11}, v_{21}, v_{31}) \in S, v_{i2} \in [0, 1], i \in \{1, 2, 3\}\}.$$

Let $f\colon [0, 1] \to [0, 1]$ be a decreasing concave function such that $f(0) = 1, f(1) = 0$, and f is affine in each interval $I_k :=]1/2^k, 1/2^{k-1}[, k \in \mathbf{N}$. Let d_k denote the value of f' in I_k and assume $d_k < d_{k+1}$, $d_k \to 0$ as $k \to \infty$.

Consider another function $g\colon [0, 1] \to [0, 1]$ such that $g(0) = 1$ and $g' = d_{k+1}$ in I_k, $k \in \mathbf{N}$. Then $g > f$ everywhere in $]0, 1]$.

Consider the sets

$$S_- := \{(v_1, v_2, v_3) : 0 \leqslant v_1 \leqslant 1, v_2 = -1, 0 \leqslant v_3 \leqslant f(v_1)\},$$

$$S_+ := \{(v_1, v_2, v_3) : 0 \leqslant v_1 \leqslant 1, v_2 = 1, 0 \leqslant v_3 \leqslant f(v_1)\},$$

$$S_0 := \{(v_1, v_2, v_3) : 0 \leqslant v_1 \leqslant 1, v_2 = 0, 0 \leqslant v_3 \leqslant g(v_1)\}.$$

The set $S \subset \mathbf{R}^3$ is defined as the convex hull of the set $S_- \cup S_+ \cup S_0$.

PROPOSITION 5.3. *We have*

$$\{(0, 1, 1), (0, -1, 1), (1/2^k, 1, f(1/2^k)), (1/2^k, -1, f(1/2^k)), (1/2^k, 0, g(1/2^k)), k \in \mathbf{N}\} \subset \operatorname{extr} S.$$

However no point of the set $[(0, -1, 1), (0, 1, 1)] \setminus \{(0, 0, 1)\}$ *belongs to the set* $\overline{\operatorname{extr} S}$.

Proof. It is obvious that the points

$$(0, 1, 1), (0, -1, 1), (1/2^k, 1, f(1/2^k)), (1/2^k, -1, f(1/2^k)), \qquad k \in \mathbf{N},$$

belong to the set $\operatorname{extr} S$. To prove the proposition we also have to show that $(1/2^k, 0, g(1/2^k)) \in \operatorname{extr} S$, $k \in \mathbf{N}$, and

$$([(0, -1, 1)(0, 1, 1)] \setminus \{(0, 0, 1)\}) \cap \overline{\operatorname{extr} S} = \varnothing.$$

Fix $k \in \mathbf{N}$. Let $a = (1/2^{k-1}, 0, g(1/2^{k-1}))$. Note that

$$d_k = f' \text{ in } I_k,\ d_k = g' \text{ in } I_{k-1}.$$

Consider the plane H_k^- which contains the segments $J_k^- := \{[x, -1, f(x)] : x \in I_k\}$, $J_k^0 := \{[x, 0, g(x)] : x \in I_{k-1}\}$ (there exists such a plane since the segments are parallel).

Since the functions f, g are concave we infer that the set S lies below H_k^-. Moreover $H_k^- \cap S = S_k^-$, where S_k^- is the convex hull of the set $J_k^- \cup J_k^0$. Since a is an extremum point of the set S_k^- it is also an extremum point of the set S.

To show that each point $b \in (\,](0, 1, 1), (0, -1, 1)[\,\backslash\{(0, 0, 1)\}\,)$ does not lie in the set $\overline{\operatorname{extr} S}$ consider a sequence $b_j \to b$. We will show that $b_j \notin \operatorname{extr} S$ for all sufficiently large $j \in \mathbf{N}$. If b_j is sufficiently close to b and the first coordinate of b_j is zero, then $b_j \in \{(0, x, y) : -1 < x < 1, 0 < y \leqslant 1\}$ and b_j can not be an extremum point of the latter set. Another possibility to stay in the set extr S is $b_j \in (\bigcup_k (H_k^+ \cup H_k^-)) \cap S$, i.e. $b_j \in \bigcup_k (S_k^- \cup S_k^+)$. However all extremum points of the sets S_k^+, S_k^- have the second coordinate equal to 1, -1 or 0. This shows that $b_j \notin \operatorname{extr} S$ for all sufficiently large $j \in \mathbf{N}$. This proves the claim. Q.E.D.

Very recently a new type of arguments was suggested by B. Kirchheim [Ki]. He observed that one can use the set of all uniform limits f of piecewise affine admissible functions to define a function $L^\infty \to L^1$ $(f \to Df)$. This function is continuous in a dense set since it can be obtained as a pointwise limit of continuous functions (we can take e.g. mollifications of Df with radius $\varepsilon > 0$ instead of Df and send ε to 0). It can be shown that the points of continuity are solutions of the differential inclusion.

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