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Homology of algebras of families of pseudodifferential operators[☆]

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Abstract

We compute the Hochschild, cyclic, and periodic cyclic homology groups of algebras of families of Laurent complete symbols on manifolds with corners. We show in particular that the spectral sequence associated with Hochschild homology degenerates at E^2 and converges to Hochschild homology. As a byproduct, we identify the space of residue traces on fibrations by manifolds with corners. In the process, we prove some structural results about algebras of complete symbols on manifolds with corners.

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0. Introduction

Some of the main tools in the applications of non-commutative geometry to index theory and other areas of mathematics are the Hochschild and periodic cyclic homology groups. Hochschild homology, for example, can be used to understand the residue trace introduced by Guillemin and Wodzicki [14,49]. Other higher residue cocycles appear when studying more complicated singular spaces. See [11] for example.

In this paper, we study the Hochschild homology of certain algebras of complete symbols. Recall that an algebra of complete symbols is the quotient of the algebra of all pseudodifferential operators by the ideal of regularizing (or order $-\infty$),

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operators. Previously, results in this direction were obtained in [3,7,19,28,33,39], and by Wodzicki [50] (unpublished). See also [35].

Our algebras of complete symbols can be obtained as algebras of complete symbols on differentiable groupoids [20,31,32,38]. For this class of examples, it has been shown in [3] that the periodic cyclic homology can be computed, without any further assumption on the groupoid under consideration, in terms of the Laurent cohomology spaces of the cosphere bundle of the associated Lie algebroid. The Hochschild homology groups of algebras of complete symbols on differentiable groupoids, however, cannot be described in general in a simple, uniform way for all differentiable groupoids. Finding the right language in which to express these Hochschild homology groups seems to be a problem in itself—clearly an interesting one.

Let $\pi : M \rightarrow B$ be a fibration with the base B smooth (no corners), but whose fibers (and also M) are allowed to have corners. On M we consider a $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle $E \rightarrow M$, to which we associate the algebra $\mathcal{A}_{\mathcal{G}}(M|B; E)$ of complete symbols of smooth families of pseudodifferential operators acting between sections of E along the fibers of $M \rightarrow B$ and with at most Laurent singularities at the faces (see Section 3 for precise definitions). The precise construction of this algebra is done using groupoids, see Section 3, but the resulting algebra depends only on $\pi : M \rightarrow B$, and not on the groupoid \mathcal{G} used to define it, as long as the groupoid \mathcal{G} satisfies assumptions (15) and (16) of Section 3. In particular, one can take \mathcal{G} to be the groupoid that defines the families b -calculus [27] (this is recalled in Section 6).

In the present paper, we determine the Hochschild, cyclic, and periodic cyclic homology of the above algebra $\mathcal{A}_{\mathcal{G}}(M|B; E)$. Let $S_{\text{vert}}^*(M) := (T_{\text{vert}}^*M \setminus 0)/\mathbb{R}_+^*$ denote the cosphere bundle of the vertical cotangent bundle to the fibration $\pi : M \rightarrow B$. To any manifold with corners X , we functorially associate in Section 2 a space $\mathcal{L}(X)$ by replacing each face $F \subset X$ of codimension k with $F \times (S^1)^k$, the product of the unit circle with itself k -times. We denote $H_{\mathcal{G}}^q(X) := H^q(\mathcal{L}(X))$ and $H_{c,\mathcal{G}}^q(X) := H_c(\mathcal{L}(X))$, for simplicity. We shall call these groups the *Laurent cohomology* groups, respectively the *compactly supported Laurent cohomology* groups of X . The periodic cyclic homology of the algebra $\mathcal{A}_{\mathcal{G}}(M|B; E)$ of Laurent vertical complete symbols with coefficients in the $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle E is then given by Theorem 3:

$$\text{HP}_j(\mathcal{A}_{\mathcal{G}}(M|B; E)) \simeq \bigoplus_{k \in \mathbb{Z}} H_{c,\mathcal{G}}^{j+2k}(S_{\text{vert}}^*(M) \times S^1), \quad j = 0, 1. \tag{1}$$

The Hochschild homology groups turn out to be infinite dimensional, in general, unlike the case of ordinary algebras of pseudodifferential operators (when B is reduced to a point), see [3] and the references therein. Let \mathcal{F}^* be the local coefficient system over B given by the Laurent cohomology groups of the fibers of $S_{\text{vert}}^*(M) \times S^1 \rightarrow B$ and let p be the dimension of the fibers of $\pi : M \rightarrow B$. Then

$$\text{HH}_m(\mathcal{A}_{\mathcal{G}}(M|B; E)) \simeq \bigoplus_{k+h=m} \Omega_c^h(B, \mathcal{F}^{2p-k}) \tag{2}$$

(Theorem 6). This result leads, in particular, to an explicit description of the space of residue supertraces on our algebras of families of pseudodifferential operators. For example, when M is smooth (no corners) and π has connected fibers of dimension at least 2, we obtain that

$$\mathrm{HH}^0(\mathcal{A}(M|B; E)) \simeq \mathcal{C}^{-\infty}(B) := \mathcal{C}_c^\infty(B)', \tag{3}$$

that is, that the space of supertraces on $\mathcal{A}(M|B; E)$ identifies with the space of distributions on B . (Note that in this case $\mathcal{A}_{\mathcal{F}}(M|B; E) = \mathcal{A}(M|B; E)$.) The space of traces in the general case of fibrations $\pi : M \rightarrow B$ when M has corners is obtained by replacing B in the above formula by the union of the minimal faces of M (Theorem 9).

Let us now briefly describe the contents of each section. In Section 1, we quickly recall the basic definitions of $\mathbb{Z}/2\mathbb{Z}$ -graded homologies for topologically filtered algebras and give an appropriate criterion for the convergence of the associated spectral sequences. Section 2 is devoted to the description of the algebras $\mathcal{A}_{\mathcal{F}}(M|B; E)$ of complete symbols that we are interested in. In Section 3, we introduce the assumptions on our groupoids and also prove that the resulting algebras $\mathcal{A}_{\mathcal{F}}(M|B; E)$ depend only on $\pi : M \rightarrow B$, as long as Assumptions (15) and (16) are satisfied. Section 4 is devoted to the computation of the Hochschild homology of our algebras of complete symbols. In the process, we compute several other homology groups associated to Poisson manifolds. In Section 5, we extend the main results of the previous sections to the relative case. The last section, Section 6, treats in detail a few examples. In particular, we obtain an explicit description of the space of traces on our algebras of complete symbols. Note that in this paper almost all results are formulated in the $\mathbb{Z}/2\mathbb{Z}$ -graded case, in view of some possible applications.

We hope that the results of this paper will find applications to the index theorem for families [2] or to its generalization to families of fibrations by manifolds with boundary [5].

1. Topological filtered algebras

Topologically filtered algebras were introduced in [3] to provide a natural framework for the algebras of complete symbols associated to algebras of pseudodifferential operators. In this section we review the definition of topologically filtered algebras and a few other relevant facts. The complexes computing the various homologies of these algebras have to be defined appropriately. In view of the applications that we have in mind, we have found it necessary to extend our setting to include that of $\mathbb{Z}/2\mathbb{Z}$ -graded algebras. For basic facts about pseudodifferential operators, see one of the many nice monographs available [34,42], or [43].

We begin by recalling the definitions of Hochschild and cyclic homology groups of a topological algebra \mathcal{A} . A good reference is Connes' book [10]. See also [16,22]. See [17] for the homology of $\mathbb{Z}/2\mathbb{Z}$ -graded algebras. These definitions have to be

(slightly) modified when the multiplication of our algebra is only separately continuous. We thus discuss also the changes necessary to handle the class of algebras that we are interested in, that of “topologically filtered algebras,” and then we prove some results on the homology of these algebras.

First we consider the case of a topological algebra \mathcal{A} . Here “topological algebra” has the usual meaning, that is, \mathcal{A} is a real or complex algebra, which is at the same time a locally convex space such that the multiplication $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ is jointly continuous when $\mathcal{A} \times \mathcal{A}$ is endowed with the product topology. Denote by $\widehat{\otimes}$ the projective tensor product and $\mathcal{H}_n(\mathcal{A}) := \widehat{\mathcal{A}}^{\otimes n+1}$, the completion of $\mathcal{A}^{\otimes n+1}$ in the topology of the projective tensor product. Also, we denote as usual by $\partial a \in \mathbb{Z}/2\mathbb{Z}$ the degree of an element in the $\mathbb{Z}/2\mathbb{Z}$ -graded algebra and by b' and b the Hochschild differentials:

$$\begin{aligned}
 b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \\
 b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) + (-1)^{n+\mu} a_n a_0 \otimes \cdots \otimes a_{n-1}, \quad (4)
 \end{aligned}$$

where $\mu = \partial a_n(\partial a_0 + \cdots + \partial a_{n-1})$.

The Hochschild homology groups of the algebra \mathcal{A} , denoted $\text{HH}_*(\mathcal{A})$, are then the homology groups of the complex $(\mathcal{H}_n(\mathcal{A}), b)$. By contrast, the complex $(\mathcal{H}_n(\mathcal{A}), b')$ is often acyclic (for example when \mathcal{A} has a unit). A topological algebra \mathcal{A} for which $(\mathcal{H}_n(\mathcal{A}), b')$ is acyclic is called *H-unital* (or, better, *topologically H-unital*), following Wodzicki [49].

We now define cyclic homology. Assume first that \mathcal{A} is unital. We shall use the notation of [9]. See also [16].

$$\begin{aligned}
 s(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n, \\
 t(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= (-1)^{n+\mu} a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}, \\
 B_0(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= s \sum_{k=0}^n t^k (a_0 \otimes a_1 \otimes \cdots \otimes a_n), \quad \text{and} \quad B = (1 - t)B_0, \quad (5)
 \end{aligned}$$

where $\mu = \partial a_n(\partial a_0 + \cdots + \partial a_{n-1})$, as above. Then $[b, B]_+ := bB + Bb = B^2 = b^2 = 0$, and hence, if we define

$$\mathcal{C}(\mathcal{A})_n = \bigoplus_{k \geq 0} \mathcal{H}_{n-2k}(\mathcal{A}), \quad (6)$$

$(\mathcal{C}(\mathcal{A}), b + B)$, is a complex, called *the cyclic complex* of \mathcal{A} , whose homology is by definition the *cyclic homology* of \mathcal{A} , as introduced in [9] and [45]. The cyclic homology groups of the algebra \mathcal{A} are denoted $\text{HC}_n(\mathcal{A})$. For an algebra \mathcal{A} possibly without unit, one considers the algebra with an adjoined unit \mathcal{A}^+ and then the cyclic homology of \mathcal{A} is, by definition, the kernel of the map $\text{HC}_n(\mathcal{A}^+) \rightarrow \text{HC}_n(\mathbb{C})$ induced

by the augmentation morphism $\mathcal{A}^+ \rightarrow \mathbb{C}$. It is known that the two definitions are equivalent if \mathcal{A} has a unit.

Consideration of the natural periodicity morphism $\mathcal{C}_n(\mathcal{A}) \rightarrow \mathcal{C}_{n-2}(\mathcal{A})$ easily shows that cyclic and Hochschild homology are related by a long exact sequence

$$\cdots \rightarrow \text{HH}_n(\mathcal{A}) \xrightarrow{I} \text{HC}_n(\mathcal{A}) \xrightarrow{S} \text{HC}_{n-2}(\mathcal{A}) \xrightarrow{B} \text{HH}_{n-1}(\mathcal{A}) \xrightarrow{I} \cdots, \quad (7)$$

with the maps I , B , and S explicitly determined. The map S is also called the *periodicity operator*. See [9,23]. This exact sequence exists whether or not \mathcal{A} is endowed with a topology.

Recall that an algebra \mathcal{A} with a given topology, is a *topologically filtered algebra* if there exists an increasing multi-filtration $F_n^m \mathcal{A} \subset \mathcal{A}$,

$$F_n^m \mathcal{A} \subset F_{n'}^{m'} \mathcal{A}, \quad \text{if } n \leq n', \quad m \leq m',$$

by closed, complemented subspaces, satisfying the following properties:

- (1) $\mathcal{A} = \bigcup_{n,m} F_n^m \mathcal{A}$;
- (2) the union $\mathcal{A}_n := \bigcup_m F_n^m \mathcal{A}$ is a closed subspace such that

$$F_n^m \mathcal{A} = \mathcal{A}_n \cap \left(\bigcup_j F_j^m \mathcal{A} \right);$$

- (3) multiplication maps $F_n^m \mathcal{A} \otimes F_{n'}^{m'} \mathcal{A}$ to $F_{n+n'}^{m+m'} \mathcal{A}$;
- (4) the maps

$$F_n^m \mathcal{A} / F_{n-j}^m \mathcal{A} \otimes F_{n'}^{m'} \mathcal{A} / F_{n'-j}^{m'} \mathcal{A} \rightarrow F_{n+n'}^{m+m'} \mathcal{A} / F_{n+n'-j}^{m+m'} \mathcal{A}$$

induced by multiplication are continuous;

- (5) the quotient $F_n^m \mathcal{A} / F_{n-j}^m \mathcal{A}$ is a nuclear Frechet space in the induced topology;
- (6) the natural map

$$F_n^m \mathcal{A} \rightarrow \lim_{\leftarrow} F_n^m / F_{n-j}^m \mathcal{A}, \quad j \rightarrow \infty$$

is a homeomorphism; and

- (7) the topology on \mathcal{A} is the strict inductive limit of the subspaces $F_n^m \mathcal{A}$, as $n \rightarrow \infty$ (recall that $F_n^m \mathcal{A}$ is assumed to be closed in $F_{n+1}^{m+1} \mathcal{A}$).

(The above definition is a simplified version of the original definition in [3].)

For topologically filtered algebras, the multiplication is not necessarily jointly continuous, and the definition of the Hochschild and cyclic homologies using the projective tensor product of the algebra \mathcal{A} with itself, as above, is not very useful. For this reason, we change the definition of the space $\mathcal{H}_m(\mathcal{A})$ as follows.

Consider

$$F'_p = \lim_{m \rightarrow \infty} \sum_{k_0 + \dots + k_q = p} \widehat{\otimes}_{j=0}^q F_{k_j}^m \mathcal{A}, \tag{8}$$

(projective tensor products) which defines an increasing sequence of subspaces (i.e. filtration) of $\widehat{\mathcal{A}}^{\otimes q+1}$. We use this filtration to define $\mathcal{H}_q(\mathcal{A})$ as a completion. Namely,

$$F_p \mathcal{H}_q(\mathcal{A}) := \lim_{\leftarrow} F'_p / F'_{p-j} \quad \text{and} \quad \mathcal{H}_q(\mathcal{A}) := \bigcup_p F_p \mathcal{H}_q(\mathcal{A}) \tag{9}$$

where $j \rightarrow \infty$ in the projective limit. The operator B extends to a well defined map $B: \mathcal{H}_q(\mathcal{A}) \rightarrow \mathcal{H}_{q+1}(\mathcal{A})$, which allows us to define the cyclic complex and the cyclic homology of the algebra \mathcal{A} as the homology of the complex $(\mathcal{C}_*(\mathcal{A}), b + B)$, with $\mathcal{C}_q(\mathcal{A}) := \bigoplus \mathcal{H}_{q-2k}(\mathcal{A})$, as for topological algebras.

For any topologically filtered algebra, we denote

$$\text{Gr}(\mathcal{A}) := \bigoplus_n \mathcal{A}_n / \mathcal{A}_{n-1}$$

the *graded algebra* associated to \mathcal{A} , where \mathcal{A}_n is the union $\bigcup_{m,n} F_{m,n}^m \mathcal{A}$, as before. Its topology is that of an inductive limit of Frechet spaces:

$$\text{Gr}(\mathcal{A}) \simeq \lim_{N,m \rightarrow \infty} \bigoplus_{n=-N}^N F_n^m \mathcal{A} / F_{n-1}^m \mathcal{A},$$

which makes sense by (2) in the definition of the topologically filtered algebra \mathcal{A} .

For the algebras like $\text{Gr}(\mathcal{A})$, we need yet a third way of topologizing its iterated tensor products. For our purposes, the correct definition is then

$$\mathcal{H}_q(\text{Gr}(\mathcal{A})) \simeq \lim_{N,m \rightarrow \infty} \left(\bigoplus_{n=-N}^N F_n^m \mathcal{A} / F_{n-1}^m \mathcal{A} \right)^{\widehat{\otimes}^{q+1}}.$$

The Hochschild homology of $\text{Gr}(\mathcal{A})$ is the homology of the complex $(\mathcal{H}_*(\text{Gr}(\mathcal{A})), b)$. The operator B again extends to a map $B: \mathcal{H}_q(\text{Gr}(\mathcal{A})) \rightarrow \mathcal{H}_{q+1}(\text{Gr}(\mathcal{A}))$ and we can define the cyclic homology of $\text{Gr}(\mathcal{A})$ as above. The operators S, B and I associated to $\mathcal{H}_q(\text{Gr}(\mathcal{A}))$ are the graded operators associated with the corresponding operators (also denoted S, B and I) on $\mathcal{H}_q(\mathcal{A})$.

The Hochschild and cyclic complexes of the algebra $\text{Gr}(\mathcal{A})$ decompose naturally as direct sums of complexes indexed by $p \in \mathbb{Z}$. For example, $\mathcal{H}_q(\text{Gr}(\mathcal{A}))$ is the direct sum of the subspaces $\mathcal{H}_q(\text{Gr}(\mathcal{A}))_p$, where

$$\mathcal{H}_q(\text{Gr}(\mathcal{A}))_p = \lim_{m,N \rightarrow \infty} \bigoplus_{k_j} \left(\widehat{\otimes}_{j=0}^q F_{k_j}^m \mathcal{A} / F_{k_j-1}^m \mathcal{A} \right),$$

where $k_0 + k_1 + \dots + k_q = p$ and $-N \leq k_j \leq N$.

The corresponding subcomplexes of the cyclic complex are defined similarly. We denote by $\mathrm{HH}_*(\mathrm{Gr}(\mathcal{A}))_p$ and $\mathrm{HC}_*(\mathrm{Gr}(\mathcal{A}))_p$ the homologies of the corresponding complexes (Hochschild and, respectively, cyclic).

The following two results are well-known consequences of standard results in homological algebra (for topologically filtered algebras they were proved in [3]).

Lemma 1. *Let \mathcal{A} be a topologically filtered algebra. Then the natural filtrations on the Hochschild and cyclic complexes of \mathcal{A} define spectral sequences $\mathrm{EH}_{k,h}^r$ and $\mathrm{EC}_{k,h}^r$ such that*

$$\mathrm{EH}_{k,h}^1 \simeq \mathrm{HH}_{k+h}(\mathrm{Gr}(\mathcal{A}))_k \quad \text{and} \quad \mathrm{EC}_{k,h}^1 \simeq \mathrm{HC}_{k+h}(\mathrm{Gr}(\mathcal{A}))_k.$$

Moreover, the periodicity morphism S induces a morphism $S' : \mathrm{EC}_{k,h}^r \rightarrow \mathrm{EC}_{k,h-2}^r$ of spectral sequences. For $r = 1$, the morphism S' is the graded map associated to the periodicity operator $S : \mathrm{HC}_n(\mathrm{Gr}(\mathcal{A})) \rightarrow \mathrm{HC}_{n-2}(\mathrm{Gr}(\mathcal{A}))$ and the natural filtration of the groups $\mathrm{HC}_n(\mathrm{Gr}(\mathcal{A}))$.

Proof. The filtration $F_p \mathcal{H}_q(\mathcal{A})$ of the complex computing the Hochschild homology of \mathcal{A} gives rise to a spectral sequence $(E^r)_{r \geq 1}$ with

$$E_{k,h}^1 = \mathrm{H}_{k+h}(F_k \mathcal{H}(\mathcal{A}) / F_{k-1} \mathcal{H}(\mathcal{A})),$$

by standard homological algebra. By the definition of the Hochschild complex of $\mathrm{Gr}(\mathcal{A})$, we have:

$$\mathrm{H}_{k+h}(F_k \mathcal{H}_*(\mathcal{A}) / F_{k-1} \mathcal{H}_*(\mathcal{A})) \simeq \mathrm{HH}_{k+h}(\mathrm{Gr}(\mathcal{A}))_k.$$

This completes the proof for Hochschild homology. The proof for cyclic homology is similar. \square

In our considerations below, we shall need the following classical result, which was proved for topologically filtered algebras in [3]. Due to the importance of this result in what follows and for the convenience of the reader, we include a proof of it.

Theorem 1. *Fix an integer N and $a \geq 1$. Let \mathcal{A} be a topologically filtered algebra such that $\mathrm{EH}_{k,h}^a(\mathcal{A}) = 0$, for all $k < N$ and all h . Then the spectral sequence $\mathrm{EH}_{k,h}^r = \mathrm{EH}_{k,h}^r(\mathcal{A})$ defined in Lemma 1 converges to $\mathrm{HH}_{k+h}(\mathcal{A})$. More precisely, we have*

$$\mathrm{HH}_j(\mathcal{A}) \simeq \bigoplus_{k=N}^{\infty} \mathrm{EH}_{k,j-k}^{\infty}.$$

A similar result holds for the cyclic homology spectral sequence.

Proof. We have

$$\mathcal{H}_q(\mathcal{A}) = \varprojlim_{\leftarrow} \mathcal{H}_q(\mathcal{A}) / F_p \mathcal{H}_q(\mathcal{A}). \tag{10}$$

This enables to write, for every fixed q , the well-known associated \lim^1 exact sequence (see [3, Lemma 6], for example)

$$0 \rightarrow \varprojlim^1 H_{q+1}(\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A})) \rightarrow H_q(\mathcal{H}(\mathcal{A})) \rightarrow \varprojlim H_q(\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A})) \rightarrow 0.$$

Let $E_{k,h}^r(p)$ be the spectral sequence associated to the homology of the filtered complex $(\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A}), b)$. Then $E_{k,h}^r(p)$ converges because it is a translation of a first quadrant spectral sequence. Therefore, the homology groups $H_q(\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A}))$ are endowed with a filtration $\mathcal{F}_t(p)$ (= the image of the homology of the complex $F_t\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A})$) so that

$$\mathcal{F}_t(p)/\mathcal{F}_{t-1}(p) \simeq E_{t,q-t}^\infty(p). \tag{11}$$

Moreover, we have the following non-natural isomorphism

$$H_q(\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A})) \simeq \bigoplus_{t+s=q} E_{t,s}^\infty(p). \tag{12}$$

Furthermore, the spectral sequence $E_{k,h}^r(p)$ satisfies:

$$E_{k,h}^r(p) = \begin{cases} 0 & \text{if } k \leq p \text{ and} \\ \text{EH}_{k,h}^r & \text{if } k > p + r. \end{cases} \tag{13}$$

Consider now the projective system

$$A_n := H_q(\mathcal{H}(\mathcal{A})/F_{N-na}\mathcal{H}(\mathcal{A})), \quad B_n := \mathcal{F}_{N-na+a}(N-na), \quad \text{and} \quad C_n := A_n/B_n.$$

Then the ker-coker lemma [1] for the short exact sequence

$$0 \rightarrow \Pi B_n \hookrightarrow \Pi A_n \rightarrow \Pi C_n \rightarrow 0$$

gives rise to the following well-known exact sequence:

$$0 \rightarrow \varprojlim B_n \rightarrow \varprojlim A_n \rightarrow \varprojlim C_n \rightarrow \varprojlim^1 B_n \rightarrow \varprojlim^1 A_n \rightarrow \varprojlim^1 C_n \rightarrow 0.$$

(See [3, Lemma 7], for example).

By conditions (11) and (13), the natural map $A_{n+1} \rightarrow A_n$ restricts to the zero map $B_{n+1} \rightarrow B_n$ and it induces an isomorphism $C_{n+1} \rightarrow C_n$, for $n \geq 2$. Therefore we get:

$$\varprojlim^1 A_n = 0 \quad \text{and} \quad \varprojlim A_n = C_{n_0}, \quad \forall n_0 \geq 2.$$

And hence, finally,

$$\text{HH}_q(\mathcal{A}) \simeq C_{n_0} = \bigoplus_{l \geq N} E_{l,q-l}^\infty \simeq \bigoplus_{l \in \mathbb{Z}} \text{EH}_{l,q-l}^\infty. \quad \square$$

The above isomorphism is not natural, in general, but comes from a filtration $F_k \text{HH}_q(\mathcal{A})$ of $\text{HH}_q(\mathcal{A})$ whose subquotients $F_k \text{HH}_q(\mathcal{A}) / F_{k-1} \text{HH}_q(\mathcal{A})$ identify naturally with $\text{EH}_{k,q-k}^\infty$, see [25,26].

It is useful to mention here that the composite map

$$\text{HH}_q(\mathcal{A}) \xrightarrow{\text{I}} \text{HC}_q(\mathcal{A}) \xrightarrow{\text{B}} \text{HH}_{q+1}(\mathcal{A})$$

preserves the filtrations and hence it induces natural maps

$$(\text{B} \circ \text{I})^{(r)} : \text{EH}_{k,h}^r(\mathcal{A}) \rightarrow \text{EH}_{k,h+1}^r(\mathcal{A}).$$

For $r = 1$, this map is the composition of the corresponding morphisms

$$\text{HH}_q(\text{Gr}(\mathcal{A})) \rightarrow \text{HC}_q(\text{Gr}(\mathcal{A})) \rightarrow \text{HH}_{q+1}(\text{Gr}(\mathcal{A}))$$

for the graded algebra of \mathcal{A} .

2. Algebras of complete symbols

We now introduce the algebras of complete symbols that we study in this paper.

We shall follow the standard notation for groupoids and Lie algebroids, using the conventions of [20]. In particular, if \mathcal{G} is a differentiable groupoid with space of units M , then $d, r : \mathcal{G} \rightarrow M$ denote the domain and range maps, respectively, so that the composition gg' of two elements $g, g' \in \mathcal{G}$ is defined if, and only if, $d(g) = r(g')$.

We shall also follow [3] for some specific constructions involving manifolds with corners, some of which are recalled below. As in that paper, we are interested in certain specific groupoid algebras associated to manifolds with corners. If \mathcal{G} is a differentiable groupoid with space of units M and $E \rightarrow M$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle, then we shall denote by

$$\Psi^\infty(\mathcal{G}; E) = \bigcup_{m \in \mathbb{Z}} \Psi^m(\mathcal{G}; E)$$

the algebra of pseudodifferential operators on \mathcal{G} acting on sections of the vector bundle r^*E . We also define

$$\Psi^{-\infty}(\mathcal{G}; E) := \bigcap_{m \in \mathbb{Z}} \Psi^m(\mathcal{G}; E)$$

(see [20] or [38] for definitions). These two algebras are naturally $\mathbb{Z}/2\mathbb{Z}$ -graded.

We shall denote by $\mathcal{O}(M)$ the space of smooth functions on the interior of M that have only Laurent singularities at the boundary faces. If every hyperface H of M has a defining function x_H , then $\mathcal{O}(M)$ is the ring generated by $\mathcal{C}^\infty(M)$ and x_H^{-1} . Let then

$$\mathcal{A}(\mathcal{G}; E) := \Psi^\infty(\mathcal{G}; E) / \Psi^{-\infty}(\mathcal{G}; E) \quad \text{and} \quad \mathcal{A}_\varphi(\mathcal{G}; E) = \mathcal{O}(M) \mathcal{A}(\mathcal{G}; E).$$

The $\mathbb{Z}/2\mathbb{Z}$ -grading on E then provides us with a natural $\mathbb{Z}/2\mathbb{Z}$ -grading on the algebras $\mathcal{A}(\mathcal{G}; E)$ and $\mathcal{A}_\varphi(\mathcal{G}; E)$ too.

Proposition 1. *Assume that \mathcal{G} and M are as above and that M is σ -compact. Then the quotients $\mathcal{A}(\mathcal{G}; E)$ and $\mathcal{A}_\varphi(\mathcal{G}; E)$ are topologically filtered algebras.*

Proof. Let $M = \bigcup K_m$ be an exhaustion of M with compact sets (that is, $K_m \subset \text{int}(K_{m+1})$). Also, let x_1, \dots, x_k be defining functions of the hyperfaces of M and $f = x_1 \dots x_k$. We define $F_n^m \mathcal{A}_\varphi(\mathcal{G}; E)$ to be generated by $f^{-m_+}(P + \Psi^{-\infty}(\mathcal{G}; E))$, $m_+ = \max\{0, m\}$, where $P \in \Psi^n(\mathcal{G}; E)$ is such that its distribution kernel is contained in $K_m \times K_m$.

The proof then is exactly the same as the one for $E = \mathbb{C}$ in [3]. \square

Let $A(\mathcal{G})$ be the Lie algebroid of \mathcal{G} (see [20]) and let $S^*(\mathcal{G})$ be the sphere bundle of $A^*(\mathcal{G})$, that is, the set of unit vectors in the dual of the Lie algebroid of \mathcal{G} , and denote $H_c^{[q]} = \bigoplus_{k \in \mathbb{Z}} H_c^{q+2k}$ (singular cohomology with compact support and coefficients in \mathbb{C}).

Theorem 2. *Assume that the base M is σ -compact, then the periodic cyclic homology of the algebra $\mathcal{A}(\mathcal{G}; E)$ is given by*

$$\text{HP}_q(\mathcal{A}(\mathcal{G}; E)) \simeq H_c^{[q]}(S^*(\mathcal{G}) \times S^1). \tag{14}$$

Proof. An argument similar to that of Lemma 5 in Section 4 shows that the Hochschild homology is unchanged by introducing the extra vector bundle E and the $\mathbb{Z}/2\mathbb{Z}$ -grading. Standard homological algebra arguments then show that the same is true for cyclic and periodic cyclic homology. The result follows then from the case $E = \mathbb{C}$ that was proved in [3]. \square

To state the result for the algebra $\mathcal{A}_\varphi(\mathcal{G}; E)$, we need first to recall a construction from [3] that will be used several times in what follows.

Let P be a manifold with corners. We shall assume that P has embedded faces, for simplicity. Then $\mathcal{L}(P)$ is a space naturally associated to P and defined as follows. Consider for each face F of P the space $F \times (S^1)^k$, where k is the codimension of the face. We establish a one-to-one correspondence between the canonical k copies of the unit circle in $(S^1)^k$ and the faces F' of P containing F , of dimension one higher than that of F . We then identify the points of the disjoint union $\bigcup F \times (S^1)^k$ as follows. If $F \subset F'$ and F' corresponds to the variable $\theta_i \in S^1$ we identify $(x, \theta_1, \dots, \theta_{i-1}, 1, \theta_{i+1}, \dots, \theta_k) \in F \times (S^1)^k$ to the point $(x, \theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_k) \in F' \times (S^1)^{k-1}$ (same x). The resulting quotient space is by definition $\mathcal{L}(P)$. This construction extends to the case when the faces are not necessarily embedded by localization.

By construction, there exists a continuous map $p_{\mathcal{G}}: \mathcal{L}(P) \rightarrow P$. Let $J_{\varepsilon} = S^1 \cup [1, 1 + \varepsilon) \subset \mathbb{C}$, for some $\varepsilon > 0$, with S^1 identified with the subset $\{|z| = 1\}$ of the complex plane. Then the space $\mathcal{L}(P)$ is locally modeled by $J_{\varepsilon}^k \times \mathbb{R}^{n-k}$, above each point of P belonging to an open face of codimension k .

Suppose now that $P \rightarrow B$ is a fibration by manifolds with corners with B smooth. Let Q be the typical fiber of the fibration $P \rightarrow B$. Then we obtain by the above construction, a fibration $\mathcal{L}(P) \rightarrow B$, with typical fiber the locally compact space $\mathcal{L}(Q)$.

In [3], the periodic cyclic homology of several algebras of complete symbols was computed. These results include our algebras $\mathcal{A}_{\varphi}(\mathcal{G}; E)$, when E is trivial. The result is the same in general.

Theorem 3. *For $j = 0, 1$, we have*

$$\mathrm{HP}_j(\mathcal{A}_{\varphi}(\mathcal{G}; E)) \simeq \mathrm{H}_c^{[j]}(\mathcal{L}(S^*(\mathcal{G})) \times S^1).$$

Proof. When E is trivial one-dimensional, this result was proved in [3]. The general case is proved in the same way, using the same argument as in the proof of Lemma 5, which shows that the Hochschild homology of the algebras $\mathcal{A}_{\varphi}(\mathcal{G}; E)$ does not depend on the bundle E , thanks to the Morita invariance of Hochschild homology. \square

Assume now that $\pi: M \rightarrow B$ be a fibration of smooth manifolds (no corners) and $\mathcal{G} = M \times_B M := \{(m_1, m_2), \pi(m_1) = \pi(m_2)\}$ be the fibered product groupoid. More precisely, the structural maps of \mathcal{G} are defined by $d(m_1, m_2) = m_2$, $r(m_1, m_2) = m_1$, and $(m_1, m_2)(m_2, m_3) = (m_1, m_3)$. Then the algebra $\Psi^{\infty}(\mathcal{G}; E)$ identify with the algebra of smooth families of pseudodifferential operators along the fibers of $\pi: M \rightarrow B$ that have compactly supported Schwartz kernel. Also, note that $\mathcal{A}(\mathcal{G}; E) \simeq \mathcal{A}_{\varphi}(\mathcal{G}; E)$, because there are no corners (or boundaries). We shall denote by $\mathcal{A}(M|B; E) = \mathcal{A}(\mathcal{G}; E)$ the algebra associated to this groupoid.

Corollary 1. *For any fibration $\pi: M \rightarrow B$ of smooth manifolds (without boundary), we obtain*

$$\mathrm{HP}_j(\mathcal{A}(M|B; E)) \simeq \mathrm{H}_c^{[j]}(S_{\mathrm{vert}}^*(M) \times S^1), \quad j = 0, 1.$$

This leads to a complete determination of the periodic cyclic homology of the algebras $\mathcal{A}(\mathcal{G}; E) := \Psi^{\infty}(\mathcal{G}; E) / \Psi^{-\infty}(\mathcal{G}; E)$ and $\mathcal{A}_{\varphi}(\mathcal{G}; E) := \mathcal{O}(M)\mathcal{A}(\mathcal{G}; E)$. The result is moreover easily expressed in a uniform manner for all differentiable groupoids \mathcal{G} . The Hochschild homology of these algebras seems to be more difficult to compute. Finding the groups $\mathrm{HH}_*(\mathcal{A}_{\varphi}(\mathcal{G}; E))$, in general, seems to depend on finding the right language in which to express the result. Needless to say, finding the right language to express the groups $\mathrm{HH}_*(\mathcal{A}_{\varphi}(\mathcal{G}; E))$ and then determining them is a worthy problem.

We shall determine the groups $\mathrm{HH}_*(\mathcal{A}_{\mathcal{G}}(\mathcal{G}; E))$ for a class of groupoids that, roughly speaking, consists of families of groupoids of the kind considered in [3]. We now proceed to describe this class in detail.

3. Families of manifolds with corners

To describe the class of differentiable groupoids \mathcal{G} for which we shall determine the groups $\mathrm{HH}_*(\mathcal{A}_{\mathcal{G}}(\mathcal{G}; E))$, we first describe the assumptions on the space of units of \mathcal{G} . We shall denote the space of units of \mathcal{G} by M , where M is a differentiable manifold, possibly with corners, as before. We shall assume that there exists a smooth manifold (without corners) B and a map $\pi: M \rightarrow B$ that makes M a differentiable fiber bundle over B with fiber F . We regard this fiber bundle as being the fiber bundle associated to a principal bundle with structure group $\mathrm{Diffeo}(F)$, the group of diffeomorphisms of F that map faces to faces. From now on and throughout the paper, we shall denote $n = \dim(M)$, $q = \dim(B)$. Also, we shall denote by p the dimension of the fibers of $\pi: M \rightarrow B$, so, in particular, $n = p + q$.

Assumptions. Fix M as above. We shall now describe our three assumptions on the groupoid \mathcal{G} .

Our *first assumption* on \mathcal{G} is that for any arrow $g \in \mathcal{G}$, the domain and range of g are in the same fiber of $\pi: M \rightarrow B$, that is,

$$\pi(d(g)) = \pi(r(g)), \quad \forall g \in \mathcal{G}. \quad (15)$$

The intuitive meaning of this condition is that the natural action of $\Psi^\infty(\mathcal{G}; E)$ on $\mathcal{C}_c^\infty(M)$ via the vector representation [18,38] is given by families of operators acting on the fibers of π .

Let $T_{\mathrm{vert}}M$ be the vertical tangent bundle to the fibration $\pi: M \rightarrow B$. Denote as above by $\mathcal{O}(M)$ the space of smooth functions on the interior M_0 of M that have only Laurent (or rational) type singularities at the faces of M . Let us denote by $\varrho: A(\mathcal{G}) \rightarrow TM$ the anchor map of the Lie algebroid of \mathcal{G} .

Our *second assumption* on \mathcal{G} is that the map $\varrho_\Gamma: \Gamma(A(\mathcal{G})) \rightarrow \Gamma(T_{\mathrm{vert}}M)$ defined by ϱ induces an isomorphism

$$\mathcal{O}(M) \otimes_{C^\infty(M)} \Gamma(A(\mathcal{G})) \simeq \mathcal{O}(M) \otimes_{C^\infty(M)} \Gamma(T_{\mathrm{vert}}M), \quad (16)$$

of vector spaces. Clearly the above map preserves the Lie bracket, so we get an isomorphism of Lie algebras also.

Our next assumption on \mathcal{G} is a local triviality condition on the algebra $\mathcal{A}_{\mathcal{G}}(\mathcal{G}; E)$. To state this assumption, we need to introduce some notation. For any open set $V \subset B$, we denote by \mathcal{G}_V the reduction of \mathcal{G} to $\pi^{-1}(V)$. Our previous assumptions on \mathcal{G} give that $\mathcal{G}_V = (\pi \circ d)^{-1}(V)$. Similarly, for every point $b \in B$, we denote by \mathcal{G}_b the reduction of \mathcal{G} to $\pi^{-1}(b)$. Again, our assumptions give us that $\mathcal{G}_b = (\pi \circ d)^{-1}(b) =$

$(\pi \circ r)^{-1}(b)$. Let us observe that

$$\mathcal{O}(\pi^{-1}(V)) \otimes (\Psi^\infty(V \times \mathcal{G}_b; E) / \Psi^{-\infty}(V \times \mathcal{G}_b; E))$$

has a natural filtration and a natural completion to a topologically filtered algebra, denoted

$$\mathcal{O}(\pi^{-1}(V)) \otimes_{\text{tf}} (\Psi^\infty(V \times \mathcal{G}_b; E) / \Psi^{-\infty}(V \times \mathcal{G}_b; E)).$$

Our *third and last assumption* on \mathcal{G} is the following. For any $b \in B$, we assume the existence of an open neighborhood $V \subset B$ of b and a $\mathcal{C}^\infty(B)$ -linear isomorphism

$$\begin{aligned} \mathcal{A}_{\mathcal{G}}(\pi^{-1}(V)|V; E|_V) &:= \mathcal{O}(\pi^{-1}(V))(\Psi^\infty(\mathcal{G}_V; E) / \Psi^{-\infty}(\mathcal{G}_V; E)) \\ &\simeq \mathcal{O}(\pi^{-1}(V)) \otimes_{\text{tf}} (\Psi^\infty(V \times \mathcal{G}_b; E) / \Psi^{-\infty}(V \times \mathcal{G}_b; E)) \end{aligned} \quad (17)$$

of topologically filtered algebras, where $V \times \mathcal{G}_b$ is the product groupoid, with V consisting of just units and the operations being defined pointwise.

The three assumptions above, Eqs. (15)–(17) are not completely independent, as we shall see shortly. We do not impose in this section these assumptions on our groupoid \mathcal{G} . Each result below will specify which assumptions are needed. However, beginning with the next section, we shall use all three assumptions on \mathcal{G} .

Lemma 2. *Assume that (16) is satisfied. Then the morphism ϱ above induces an isomorphism*

$$\mathcal{O}(M) \otimes_{C^\infty(M)} C^\infty(A^*(\mathcal{G})) \simeq \mathcal{O}(M) \otimes_{C^\infty(M)} C^\infty(T_{\text{vert}}^*M)$$

of Poisson algebras.

Proof. Let X, Y , and $Z \in \Gamma(A(\mathcal{G}))$. Then X, Y , and Z define functions (denoted by the same letter) $X, Y, Z : A^*(\mathcal{G}) \rightarrow \mathbb{R}$. Assume $Z = [X, Y]$. Then the Poisson bracket on $C^\infty(A^*(\mathcal{G}))$ is uniquely determined by $\{X, Y\} = Z$. The equation $\varrho([X, Y]) = [\varrho(X), \varrho(Y)]$ shows that the natural map

$$C^\infty(A^*(\mathcal{G})) \rightarrow C^\infty(T_{\text{vert}}^*M)$$

is a Poisson map. The proof is completed by including $\mathcal{O}(M)$ -coefficients. \square

See [47] for some basic facts about Poisson manifolds.

Let $M_0 := M \setminus \partial M$ be, as above, the interior of M and let $T_{\text{vert}}M_0$ be the vertical tangent bundle to the smooth fibration $M_0 \rightarrow B$. Our second assumption, Eq. (16), implies, in particular, that the anchor map ϱ restricts to an isomorphism

$$A(\mathcal{G})|_{M_0} \simeq T_{\text{vert}}M_0, \quad (18)$$

of vector bundles.

We now discuss the relation between our three assumptions on \mathcal{G} . It turns out that these assumptions do not play equal roles. In fact, the second assumption implies the third one, and, under some weak assumptions on \mathcal{G} (d -connectivity) it also implies the first assumption. The following considerations are however somewhat independent from the rest of the paper, and, for the purpose of Hochschild homology computations, the reader can just ignore some of the results below, but instead impose all three assumptions on \mathcal{G} .

First, let us notice that, in the same spirit as the above lemma, we get an isomorphism of the algebras of differential operators corresponding to \mathcal{G} and to M . More precisely, let $\text{Diff}(M, \mathcal{G})$ be the algebra of differential operators on M generated by $\mathcal{C}^\infty(M)$ and $\Gamma(A(\mathcal{G}))$. Similarly, let $\text{Diff}(M)$ be the algebra of differential operators on M generated by $\mathcal{C}^\infty(M)$ and $\Gamma(TM)$. The Poincaré–Birkhoff–Witt theorem of [38] shows that the anchor map ϱ then gives rise to a morphism

$$\varrho_{\text{Diff}} : \text{Diff}(M, \mathcal{G}) \rightarrow \text{Diff}(M). \quad (19)$$

Proposition 2. *Assume that the map $\varrho_\Gamma : \Gamma(A(\mathcal{G})) \rightarrow \Gamma(TM)$ defined by ϱ is injective, then our second assumption on \mathcal{G} , Eq. (16), is equivalent to the fact that $\varrho_{\text{Diff}} : \text{Diff}(M, \mathcal{G}) \rightarrow \text{Diff}(M)$ induces an isomorphism*

$$\mathcal{O}(M) \text{Diff}(M, \mathcal{G}) \rightarrow \mathcal{O}(M) \text{Diff}(M).$$

Proof. The space of vector fields on a manifold coincides with the space of first order differential operators without constant term (i.e. that send the function constant equal to 1 to 0). Thus, the isomorphism

$$\mathcal{O}(M) \text{Diff}(M, \mathcal{G}) \simeq \mathcal{O}(M) \text{Diff}(M)$$

is equivalent to the fact that $\mathcal{O}(M)\Gamma(A(\mathcal{G}))$ maps surjectively onto $\mathcal{O}(M)\Gamma(TM)$. Since this map is injective by assumption, the result follows. \square

The algebras $\mathcal{A}_\varphi(\mathcal{G}; E)$ turn out to depend only on $\pi : M \rightarrow B$.

Theorem 4. *The algebras $\mathcal{A}_\varphi(\mathcal{G}; E)$ are independent of \mathcal{G} , as long as assumptions (15) and (16) are satisfied.*

Proof. Assume E is trivial, for simplicity. Let π_0 be the restriction of π to the interior of M and let $\Psi_{\text{prop}}^\infty(M|B)$ be the algebra of smooth, properly supported families of operators acting on the fibers of

$$\pi_0 : M_0 := M \setminus \partial M \rightarrow B.$$

Recall [18,38] that the vector representation $\pi_v : \Psi^\infty(\mathcal{G}) \rightarrow \text{End}(\mathcal{C}_c^\infty(M_0))$ is defined uniquely by

$$(\pi_v(P)f) \circ r = P(f \circ r).$$

Assumption (15) shows that π_v factors through a morphism $\Psi^\infty(\mathcal{G}) \rightarrow \Psi_{\text{prop}}^\infty(M|B)$. Assumption (16) then implies that $\mathcal{A}_{\mathcal{G}}(\mathcal{G})$ identifies with a subalgebra of $\mathcal{B} := \Psi_{\text{prop}}^\infty(M|B)/\Psi_{\text{prop}}^{-\infty}(M|B)$.

We now argue that Proposition 2 and asymptotic completeness imply that the image of $\mathcal{O}(M)\Psi^\infty(\mathcal{G})$ in \mathcal{B} is independent of \mathcal{G} . Indeed, it is enough to check that the image of $\mathcal{O}(M)\Psi^m(\mathcal{G}) \rightarrow \mathcal{B}$ is independent of \mathcal{G} , for any m . Let $D \in \mathcal{O}(M)\text{Diff}(M; \mathcal{G})$ be an elliptic differential operator in $\mathcal{O}(M)\Psi^k(\mathcal{G})$, for some fixed $k \geq 1$. Let Q be a parametrix of D . Then Proposition 2 implies that

$$\mathcal{O}(M)\text{Diff}(M; \mathcal{G})[Q] = \mathcal{O}(M)\text{Diff}(M)[Q]. \tag{20}$$

Let \mathcal{G}_1 be another differentiable groupoid satisfying assumptions (15) and (16), then

$$\mathcal{O}(M)\text{Diff}(M; \mathcal{G}_1)[Q] = \mathcal{O}(M)\text{Diff}(M; \mathcal{G})[Q],$$

by using Eq. (20) twice. Because $\mathcal{O}(M)(\Psi^m(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})) \rightarrow \mathcal{B}$ is continuous and injective and the image of operators of order at most m of $\mathcal{O}(M)\text{Diff}(M; \mathcal{G})[Q]$ is dense in $\mathcal{O}(M)\Psi^m(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$, we obtain that the closure of the range of $\mathcal{O}(M)\Psi^m(\mathcal{G})$ in \mathcal{B} does not depend on \mathcal{G} . By looking at the complete symbols of the images of $\mathcal{O}(M)(\Psi^m(\mathcal{G}))$ and $\mathcal{O}(M)\Psi^m(\mathcal{G}_1)$ in \mathcal{B} and using the asymptotic completeness of the algebras of pseudodifferential operators $\mathcal{O}(M)(\Psi^\infty(\mathcal{G}))$ and $\mathcal{O}(M)\Psi^\infty(\mathcal{G}_1)$, we obtain that the actual range of $\Psi^m(\mathcal{G})$ in \mathcal{B} is independent of \mathcal{G} , as desired. \square

In view of the above result, we shall denote $\mathcal{A}_{\mathcal{G}}(M|B; E) = \mathcal{A}_{\mathcal{G}}(\mathcal{G}; E)$, if \mathcal{G} is a groupoid satisfying the first two assumptions, Eqs. (15) and (16), of this section.

Let us recall that \mathcal{G} is *d-connected* if, and only if, all the sets $\mathcal{G}_x := d^{-1}(x)$ are connected.

Corollary 2. *Suppose \mathcal{G} is a differentiable groupoid with units M . Then assumption (16) implies assumption (17). If \mathcal{G} is also d-connected, then (16) implies also assumption (15).*

Proof. By Theorem 4, it is enough to check (17) for any fixed groupoid \mathcal{G} satisfying (16). In particular, we can choose \mathcal{G} to be locally a product, in which case (17) is trivially satisfied. (For example, we could take $\mathcal{G} = \mathcal{G}_{M,b}$, the b -groupoid defined in Section 6.)

Let X_1, \dots, X_m be sections of $A(\mathcal{G})$. We shall write $\varrho(X_j)$ for $\varrho_{\Gamma}(X_j)$ (in our case ϱ_{Γ} is an inclusion). Then

$$\pi(\exp(\varrho(X_1)) \dots \exp(\varrho(X_m))x) = \pi(x),$$

for all $x \in M$. The assumption that \mathcal{G} be d -connected is equivalent to the assumption that, for any $g \in \mathcal{G}$, there exist X_1, \dots, X_m as above such that

$$r(g) = \exp(\varrho(X_1)) \dots \exp(\varrho(X_m))d(g),$$

see [24]. \square

It also follows from the above discussion that it is enough for our computations to consider a “typical” algebra for each fibration $\pi : M \rightarrow B$. There are several choices of “typical” algebras, in general. One possible choice, the “ b -calculus” [27], as well as the result of our computations for this algebra, will be described in Section 6.

4. Hochschild homology for families

In this section, we compute the Hochschild homology groups of the algebras $\mathcal{A}_{\mathcal{G}}(M|B; E) = \mathcal{A}_{\mathcal{G}}(\mathcal{G}; E)$ introduced in the previous section. Recall that these algebras are algebras of complete symbols associated with a groupoid \mathcal{G} with units M and a fibration $\pi : M \rightarrow B$ by manifolds with corners satisfying the assumptions of Equations (15) and (16). The results of this section are already interesting when the manifold M has no boundary. Recall that $n = \dim(M)$, $q = \dim(B)$, and $n = p + q$.

In addition to helping us eliminate our third assumption on \mathcal{G} , Eq. (17), the introduction of the Laurent-type factors also simplifies the calculations, as in [3,28]. When B is reduced to a point $*$, this also ensures that the Hochschild homology of $\mathcal{A}_{\mathcal{G}}(M) := \mathcal{A}_{\mathcal{G}}(M|*)$ is finite dimensional. For example, the dimension of the space of traces on $\mathcal{A}_{\mathcal{G}}(M|*)$ is the number of minimal faces of M [3]. Moreover, the “cone algebras” described for example in [40] are more closely related to the algebras $\mathcal{A}_{\mathcal{G}}(M|B; E)$ than to the algebras $\mathcal{A}(M|B; E)$. See also [21,39].

Our computations will use the Poisson structure of $A^*(\mathcal{G})$ and, more precisely, the “homogeneous Laurent–Poisson homology” of $A^*(\mathcal{G})\setminus 0$, where $A^*(\mathcal{G})\setminus 0$ is the dual of the Lie algebroid of \mathcal{G} , with the zero section removed. The homogeneous Laurent–Poisson homology of $A^*(\mathcal{G})\setminus 0$ is defined below and will be identified in terms of the “homogeneous, vertical Laurent–de Rham cohomology” of the fibration $A^*(\mathcal{G})\setminus 0 \rightarrow B$ (this cohomology is also defined below). The homogeneous Laurent–Poisson homology and the homogeneous vertical Laurent–de Rham cohomology are natural analogues of the Poisson and, respectively, de Rham cohomology, which are obtained, roughly speaking, by introducing Laurent type singularities at the corners of M and by considering homogeneous forms (on $A^*(\mathcal{G})\setminus 0$, for example). See [30,48,51] for more on Poisson homology.

We begin with the definition of the groups $H_{c,\varphi}^{i,j}(A^*(\mathcal{G})\setminus 0|B)_l$, the homogeneous, vertical Laurent–de Rham cohomology of the fibration $A^*(\mathcal{G})\setminus 0 \rightarrow B$. Then we shall discuss Poisson homology and its variant, the homogeneous Laurent–Poisson homology.

Let X be a manifold with corners and let $\pi_0 : X \rightarrow B$ be a fiber bundle with B smooth. Let us call the sections of $T_{\text{vert}}X$ vertical vector fields, as it is customary. Then the sections of the dual T_{vert}^*X are called vertical differential forms. There exists a natural (i.e. independent of any choices) differential operator

$$d_{\text{vert}} : \Gamma(A^k T_{\text{vert}}^* X) \rightarrow \Gamma(A^{k+1} T_{\text{vert}}^* X), \tag{21}$$

the *vertical de Rham differential*.

Every vertical vector field on X is also a vector field on X in the usual sense. On the other hand, a form on X restricts to a vertical form on X . Moreover, every vertical form on X is the restriction of a form on X , but we cannot choose that form in a canonical way. A convenient way to choose extensions of vertical forms is to consider a splitting of TX into vertical and horizontal parts. We shall hence fix from now an isomorphism (or splitting)

$$\Theta : TX \simeq T_{\text{vert}}X \oplus \pi_0^*TB. \tag{22}$$

The splitting Θ of Eq. (22) gives rise to an embedding $\Theta_k : \Gamma(A^k T_{\text{vert}}^* X) \rightarrow \Omega^k(X)$. More generally, we get isomorphisms

$$A^k T^* X \simeq \bigoplus_{i+j=k} A^i T_{\text{vert}}^* X \otimes \pi_0^* A^j T^* B.$$

Let $\Omega^{i,j}(X) := \Gamma(X, A^i T_{\text{vert}}^* X \otimes \pi_0^* A^j T^* B)$. Then $\Omega^k(X) \simeq \bigoplus_{i+j=k} \Omega^{i,j}(X)$, and we also have isomorphisms

$$\Omega^{i,0}(X) \otimes_{\mathcal{C}^\infty(B)} \Omega^j(B) \ni \omega \otimes \eta \rightarrow \omega \wedge \pi_0^* \eta \in \Omega^{i,j}(X). \tag{23}$$

The embedding Θ_k can then be used to define a map $d_{\text{vert}} : \Omega^{i,0}(X) \rightarrow \Omega^{i+1,0}(X)$ (using the same notation for the differential is unlikely to cause any confusion in our case). We extend then d_{vert} to a map

$$d_{\text{vert}} : \Omega^{i,j}(X) \rightarrow \Omega^{i+1,j}(X),$$

by using the isomorphisms of Eq. (23) above and setting

$$d_{\text{vert}}(\omega \wedge \pi_0^* \eta) = d_{\text{vert}}(\omega) \wedge \pi_0^* \eta$$

if $\eta \in \Omega^{i,0}(X)$ and $\omega \in \Omega^j(B)$. Clearly $d_{\text{vert}}^2 = 0$. The extension d_{vert} that we obtain depends on the splitting Θ of Eq. (22). The isomorphism class of the resulting complex, however, does not depend on Θ .

Let us denote by $\Omega_{\varphi}^k(X)$ the space of k -differential forms on the interior of X that have only rational (or Laurent) singularities near the corners. We shall sometimes

call forms with these properties *Laurent-differential forms*. The above definitions and properties extend to $\Omega_{\mathcal{L}}^k(X)$ as follows. Let $\Omega_{\mathcal{L}}^{i,j}(X) := \mathcal{O}(X)\Gamma(X, \mathcal{A}^i T_{\text{vert}}^* X \otimes \pi_0^* \mathcal{A}^j T^* B)$. Then

$$\Omega_{\mathcal{L}}^k(X) \simeq \bigoplus_{i+j=k} \Omega_{\mathcal{L}}^{i,j}(X),$$

and, as before, we obtain a differential

$$d_{\text{vert}} : \Omega_{\mathcal{L}}^{i,j}(X) \rightarrow \Omega_{\mathcal{L}}^{i+1,j}(X).$$

We shall denote by

$$H_{\mathcal{L}}^{i,j}(X) = \ker(d_{\text{vert}}) / d_{\text{vert}} \Omega_{\mathcal{L}}^{i-1,j}(X)$$

the homology of the above complex. Similarly, if compactly supported forms are considered, we obtain a complex whose homology we denote by $H_{c,\mathcal{L}}^{i,j}(X)$.

Define the horizontal differential $d_{\text{hor}} : \Omega_{\mathcal{L}}^{i,j}(X) \rightarrow \Omega_{\mathcal{L}}^{i,j+1}(X)$ as the component of bidegree $(0, 1)$ of d . Then $\partial := d - d_{\text{vert}} - d_{\text{hor}}$ is known to be a differential and to have bidegree $(-1, 2)$. See for instance [44].

The equality $d^2 = 0$ is equivalent to the following relations:

$$d_{\text{vert}} d_{\text{hor}} + d_{\text{hor}} d_{\text{vert}} = 0, \quad d_{\text{hor}}^2 + \partial d_{\text{vert}} + d_{\text{vert}} \partial = 0, \quad \partial^2 = 0,$$

$$d_{\text{vert}}^2 = 0, \quad \text{and} \quad \partial d_{\text{hor}} + d_{\text{hor}} \partial = 0.$$

The vertical Laurent–de Rham cohomology can be computed in a fairly explicit way. Indeed, let \mathcal{F}^k be the local coefficient system determined by the Laurent cohomology groups of the fibers of $X \rightarrow B$. Thus \mathcal{F}^k is a canonically flat vector bundle over B whose fiber at $b \in B$ is

$$\mathcal{F}^k(b) = H_{c,\mathcal{L}}^k(\pi_0^{-1}(b)) := H_{c,\mathcal{L}}^k(\pi_0^{-1}(b)|b). \tag{24}$$

Let $\Omega_c^k(B)$ be the space of compactly supported k -forms on B .

Proposition 3. *Using the above notation, we have that*

$$H_{c,\mathcal{L}}^{k,h}(X|B) \simeq \Omega_c^h(B) \otimes_{\mathcal{O}_{\infty}(B)} \Gamma(\mathcal{F}^k) =: \Omega_c^h(B; \mathcal{F}^k), \tag{25}$$

the space of compactly supported h -forms on B with values in \mathcal{F}^k . In particular, the vertical Laurent–de Rham cohomology groups $H_{c,\mathcal{L}}^{k,h}(X|B)$ are independent of the splitting $TX \simeq T_{\text{vert}} X \oplus \pi_0^* TB$ used to define them, see Eq. (22).

In addition, the action induced by the horizontal de Rham differential d_{hor} on $H_{c,\mathcal{L}}^{k,h}(X|B)$ is isomorphic under (25) to the de Rham differential on B with coefficients in the locally constant sheaf \mathcal{F}^k .

Proof. The above formula is checked right away when $\pi : X \rightarrow B$ is a trivial fiber bundle (i.e. $X = B \times F$) by using the Künneth formula for the tensor product of complexes of nuclear vector spaces, [15]. Moreover, an automorphism of the trivial fiber bundle $X = B \times F$ does not affect the isomorphism of the proposition. A partition of unity argument then completes the proof. \square

Let $A^*(\mathcal{G})\setminus 0$ be obtained from the vector bundle $A^*(\mathcal{G})$, as before, by removing the zero section. We are interested in the above constructions when $X = A^*(\mathcal{G})\setminus 0$ and $\pi_0 : A^*(\mathcal{G})\setminus 0 \rightarrow B$ is obtained from the composition of the maps $A^*(\mathcal{G}) \rightarrow M$ and $\pi : M \rightarrow B$. More precisely, for us, the relevant cohomology groups are the cohomology groups obtained by considering homogeneous forms. Let then $\Omega_{rc,\mathcal{G}}^{i,j}(A^*(\mathcal{G})\setminus 0)_l$ be the space of l -homogeneous forms in

$$\Omega_{\mathcal{G}}^{i,j}(A^*(\mathcal{G})\setminus 0) = \mathcal{O}(M)\Gamma(A^*(\mathcal{G})\setminus 0, A^l T_{\text{vert}}^*(A^*(\mathcal{G})\setminus 0) \otimes \pi_0^* A^j T^* B)$$

whose support project onto a compact subset of M . Here the homogeneity is considered with respect to the natural action of \mathbb{R}_+^* on $A^*(\mathcal{G})\setminus 0$ by dilations. We denote then by

$$H_{c,\mathcal{G}}^{i,j}(A^*(\mathcal{G})\setminus 0|B)_l$$

the homology of the complex $\Omega_{rc,\mathcal{G}}^{i,j}(A^*(\mathcal{G})\setminus 0)_l$ with respect to the vertical de Rham differential d_{vert} . We shall call these groups the *homogeneous, vertical Laurent–de Rham cohomology groups of $A^*(\mathcal{G})$* .

Similar constructions and definitions are obtained with $T_{\text{vert}}^* M$ in place of $A^*(\mathcal{G})$. Our second assumption on the groupoid \mathcal{G} , Eq. (16), gives that the two cohomologies are isomorphic.

Lemma 3. *The anchor map $q : A(\mathcal{G}) \rightarrow T_{\text{vert}} M$ induces a natural isomorphism*

$$H_{c,\mathcal{G}}^{i,j}(A^*(\mathcal{G})\setminus 0|B)_l \simeq H_{c,\mathcal{G}}^{i,j}(T_{\text{vert}}^* M \setminus 0|B)_l.$$

These groups vanish if $l \neq 0$ and, for $l = 0$, we have

$$\begin{aligned} H_{c,\mathcal{G}}^{i,j}(A^*(\mathcal{G})\setminus 0|B)_0 &\simeq H_{c,\mathcal{G}}^{i,j}(T_{\text{vert}}^* M \setminus 0|B)_0 \\ &\simeq H_{c,\mathcal{G}}^{i,j}(S_{\text{vert}}^*(M) \times S^1|B) \simeq H_{c,\mathcal{G}}^{i,j}(S^*(\mathcal{G}) \times S^1|B). \end{aligned}$$

Proof. The map q induces an isomorphism of the corresponding complexes, by Eq. (16) and the definition of the spaces $\Omega_{rc,\mathcal{G}}^{i,j}(A^*(\mathcal{G})\setminus 0)_l$. The vanishing of the groups $H_{c,\mathcal{G}}^{i,j}(T_{\text{vert}}^* M \setminus 0|B)_l$ for $l \neq 0$ follows from the homotopy invariance of de Rham cohomology. The computation of the 0-homogeneous cohomology spaces is elementary, see for instance [3,7,28]. \square

We can thus replace $A^*(\mathcal{G})$ with T_{vert}^*M for the rest of our computations of the homogeneous, vertical Laurent–de Rham cohomology groups of $A^*(\mathcal{G})$.

The homogeneous, vertical Laurent–de Rham cohomology can be computed using a method similar to the one we used to determine the non-homogeneous homology. Indeed, let \mathcal{F}^k be the local coefficient system determined by the Laurent cohomology groups of the fibers of $\pi_0 : S_{\text{vert}}^*(M) \times S^1 \rightarrow B$. Thus \mathcal{F}^k is a canonically flat vector bundle over B whose fiber at $b \in B$ is

$$\mathcal{F}^k(b) = H_{c,\mathcal{L}}^k(\pi_0^{-1}(b)) = H_c^k(\mathcal{L}(\pi_0^{-1}(b))). \tag{26}$$

Proposition 4. *Using the above notation, we have that*

$$H_{c,\mathcal{L}}^{k,h}(A^*(\mathcal{G}) \setminus 0|B)_0 \simeq \Omega_c^h(B) \otimes_{\mathcal{C}^\infty(B)} \Gamma(\mathcal{F}^k) =: \Omega_c^h(B; \mathcal{F}^k) \tag{27}$$

(recall that for $l \neq 0$, the groups $H_{c,\mathcal{L}}^{k,h}(A^*(\mathcal{G}) \setminus 0|B)_l$ vanish). The horizontal de Rham differential d_{hor} induces a differential on $H_{c,\mathcal{L}}^{k,h}(X|B)$ which is isomorphic under (27) to the de Rham differential on B with coefficients in the locally constant sheaf \mathcal{F}^k .

Proof. The proof is completely similar to that of Proposition 3 and Lemma 3. \square

Let us now introduce the Poisson homology groups that we are interested in. The following considerations apply to any regular Poisson structure. Recall that the Poisson structure on $A^*(\mathcal{G})$ is defined by a two tensor

$$G \in \mathcal{C}^\infty(A^*(\mathcal{G}), A^2TA^*(\mathcal{G}))$$

so that $\{f, g\} = i_G(df \wedge dg)$. Clearly, the tensor G must satisfy some non-trivial conditions for the map $\{, \}$ to satisfy the Jacobi identity. These conditions turn out to be equivalent to $[G, G]_{SN} = 0$, where $[,]_{SN}$ is the Schouten–Nijenhuis bracket [46]. The formula for the Poisson bracket is determined in terms of the Lie algebra structure on the space of sections of $A(\mathcal{G})$. (This was recalled in Lemma 2.)

Let $i_G : \Omega^k(A^*(\mathcal{G})) \rightarrow \Omega^{k-2}(A^*(\mathcal{G}))$ be the contraction with the tensor G . It satisfies $i_G : \Omega^{i,j}(A^*(\mathcal{G})) \rightarrow \Omega^{i-2,j}(A^*(\mathcal{G}))$. Then we obtain as in [5] a differential

$$\delta := i_G \circ d - d \circ i_G : \Omega^k(A^*(\mathcal{G})) \rightarrow \Omega^{k-1}(A^*(\mathcal{G})). \tag{28}$$

Explicitly, for any $(f_0, \dots, f_k) \in \mathcal{C}^\infty(A^*(\mathcal{G}))^{k+1}$, the differential δ is given by the formula

$$\begin{aligned} \delta(f_0 df_1 df_2 \dots df_k) &= \sum_{1 \leq j \leq k} (-1)^{j+1} \{f_0, f_j\} df_1 \dots \widehat{df_j} \dots df_k \\ &+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} f_0 d\{f_i, f_j\} df_1 \dots \widehat{df_i} \dots \widehat{df_j} \dots df_k. \end{aligned} \tag{29}$$

Let \mathcal{G} be a differentiable groupoid, as above, satisfying Assumptions (15) and (16), we can then use the splitting Θ of Eq. (22). One then shows that δ decomposes into the sum of two bihomogeneous differentials [46]

$$\delta = \delta_{\text{vert}} + \alpha,$$

where $\delta_{\text{vert}} := i_G \circ d_{\text{vert}} - d_{\text{vert}} \circ i_G$ is the vertical Poisson differential. The vertical Poisson differential has bidegree $(-1, 0)$. The extra term α has bidegree $(-2, +1)$ and is in fact given by [4]:

$$\alpha = i_G \circ d_{\text{hor}} - d_{\text{hor}} \circ i_G.$$

In particular, the commutator $[i_G, \delta]$ is trivial. It is straightforward to see that the restriction of δ_{vert} to vertical differential forms is given by the following local expression

$$\begin{aligned} & \delta_{\text{vert}}(f_0 d_{\text{vert}} f_1 d_{\text{vert}} f_2 \dots d_{\text{vert}} f_k) \\ &= \sum_{1 \leq j \leq k} (-1)^{j+1} \{f_0, f_j\} d_{\text{vert}} f_1 \dots \widehat{d_{\text{vert}} f_j} \dots d_{\text{vert}} f_k \\ &+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} f_0 d_{\text{vert}} \{f_i, f_j\} d_{\text{vert}} f_1 \dots \widehat{d_{\text{vert}} f_i} \dots \widehat{d_{\text{vert}} f_j} \dots d_{\text{vert}} f_k. \end{aligned} \quad (30)$$

The above formula determines δ_{vert} on $\Omega^{j,0}(A^*(\mathcal{G}))$. To determine δ_{vert} in general, we can use the following lemma.

Lemma 4. *Let $\alpha \in \Omega^i(A^*(\mathcal{G}))$, let $\beta \in \Omega^j(B)$, and let $\pi_0 : A^*(\mathcal{G}) \rightarrow B$ be the composite projection. Then*

$$\delta(\alpha \wedge \pi_0^*(\beta)) = \delta(\alpha) \wedge \pi_0^*(\beta) \quad \text{and} \quad \delta_{\text{vert}}(\alpha \wedge \pi_0^*(\beta)) = \delta_{\text{vert}}(\alpha) \wedge \pi_0^*(\beta).$$

Proof. It is enough to check the first equation when $\beta = g$ or $\beta = dg$, for some smooth function g on B . Our claim then follows from the fact that $\{f, g \circ \pi_0\} = 0$, for any smooth function f on M and from the explicit formula for δ , Eq. (29).

The equation for $\delta_{\text{vert}} := i_G \circ d_{\text{vert}} - d_{\text{vert}} \circ i_G$ follows from the equation for δ by checking bidegrees. \square

The formula of Eq. (29) is valid also when M has corners and it is easy to check that the differential δ is homogeneous of degree -1 with respect to the action of \mathbb{R}_+^* on $A^*(\mathcal{G}) \setminus 0$. Let $\Omega^k(A^*(\mathcal{G}) \setminus 0)_l$ be the space of k -forms on $A^*(\mathcal{G}) \setminus 0$ that are homogeneous of order l . We hence obtain a differential

$$\delta : \Omega^k(A^*(\mathcal{G}) \setminus 0)_l \rightarrow \Omega^{k-1}(A^*(\mathcal{G}) \setminus 0)_{l-1}.$$

Let $\Omega_{rc}^k(A^*(\mathcal{G})\setminus 0)_l$ be the subspace of $\Omega^k(A^*(\mathcal{G})\setminus 0)_l$ consisting of forms whose support projects onto a compact subset of M , as before. Because δ preserves the support, it maps the space $\Omega_{rc}^k(A^*(\mathcal{G})\setminus 0)_l$ to $\Omega_{rc}^{k-1}(A^*(\mathcal{G})\setminus 0)_{l-1}$.

The same result holds with δ_{vert} and we have:

$$\delta_{\text{vert}} : \Omega_{rc,\mathcal{L}}^k(A^*(\mathcal{G})\setminus 0)_l \rightarrow \Omega_{rc,\mathcal{L}}^{k-1}(A^*(\mathcal{G})\setminus 0)_{l-1}.$$

We obtain in this way a direct sum of complexes $(\mathcal{P}^k)_{k \in \mathbb{Z}}$

$$\mathcal{P}^k : \quad 0 \rightarrow \mathcal{P}_{2p+q}^k \xrightarrow{\delta} \mathcal{P}_{2p+q-1}^k \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{P}_{-k}^k \rightarrow 0, \tag{31}$$

where $\mathcal{P}_l^k = \Omega_{rc,\mathcal{L}}^{k+l}(A^*(\mathcal{G})\setminus 0)_l$.

We shall denote the homology groups of the above complex by

$$H_{\mathcal{L},k+l}^\delta(A^*(\mathcal{G})\setminus 0|B)_l := \frac{\ker(\delta : \mathcal{P}_l^k \rightarrow \mathcal{P}_{l-1}^k)}{\delta(\mathcal{P}_{l+1}^k)}.$$

In the same way we define the vertical *homogeneous Laurent–Poisson homology* groups using δ_{vert} instead of δ and denote them by

$$H_{\mathcal{L},k+l}^{\delta_{\text{vert}}}(A^*(\mathcal{G})\setminus 0|B)_l := \frac{\ker(\delta_{\text{vert}} : \mathcal{P}_l^k \rightarrow \mathcal{P}_{l-1}^k)}{\delta_{\text{vert}}(\mathcal{P}_{l+1}^k)}.$$

Furthermore, we define for any (i, j)

$$\mathcal{P}_l^{i,j} := \Omega_{rc,\mathcal{L}}^{i+l,j}(T_{\text{vert}}^*M \setminus 0)_l.$$

From the results of Section 3, we deduce that

$$\mathcal{P}_l^k \simeq \bigoplus_{i+j=k} \mathcal{P}_l^{i,j}.$$

Note that with respect to the splitting (22), we have:

$$\delta_{\text{vert}} : \mathcal{P}_l^{i,j} \rightarrow \mathcal{P}_{l-1}^{i,j},$$

and the vertical Laurent Poisson homology can be computed by fixing (i, j) and restricting to each $\mathcal{P}^{i,j} := \bigoplus_l \mathcal{P}_l^{i,j}$. However, the extra term α does not preserve $\mathcal{P}^{i,j}$, and sends $\mathcal{P}^{i,j}$ to $\mathcal{P}^{i-1,j+1}$.

The relevance for us of Poisson homology, in general, and of homogeneous Laurent–Poisson homology, in particular, is that they are related to the Hochschild homology groups of the algebras $\mathcal{A}_{\mathcal{L}}(M|B; E) := \mathcal{O}(M)(\Psi^\infty(\mathcal{G}; E)/\Psi^{-\infty}(\mathcal{G}; E))$ introduced in the previous section, where E is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle. The following lemma makes this connection precise.

Lemma 5. *The algebra $\mathcal{A}_{\mathcal{G}}(M|B; E)$ is topologically H -unital. The EH^2 term of the spectral sequence associated to $\mathcal{A}_{\mathcal{G}}(M|B; E)$ by Lemma 1 is given by*

$$\text{EH}_{k,h}^2 \simeq \text{H}_{\mathcal{L},k+h}^{\delta}(A^*(\mathcal{G})\backslash 0|B)_k.$$

Proof. When E is a trivial one-dimensional vector bundle and \mathcal{G} is an arbitrary groupoid, the above proposition was proved in [3, Proposition 7].

The extension to a non-trivial vector bundle and the $\mathbb{Z}/2\mathbb{Z}$ -graded case is obtained as a consequence of Künneth formula as follows.

Let us recall that the trace $\text{Tr} : M_N(\mathbb{C}) \rightarrow \mathbb{C}$ defines a morphism of complexes from the Hochschild complex of $M_N(A)$ to the Hochschild complex of A :

$$\text{Tr}_*(m_0 \otimes b_0 \otimes \cdots \otimes m_k \otimes b_k) := \text{Tr}(m_0 m_1 \dots m_k) b_0 \otimes b_1 \otimes \cdots \otimes b_k, \quad (32)$$

where $m_j \in M_N(\mathbb{C})$ and $b_j \in A$, so that $m_j \otimes b_j \in M_N(\mathbb{C}) \otimes A \simeq M_N(A)$.

Let us assume first that E is trivial of rank N (i.e. $E = \underline{\mathbb{C}}^N$) with trivial grading. Then

$$\mathcal{A}_{\mathcal{G}}(M|B; E) = M_N(\mathcal{A}_{\mathcal{G}}(M|B; \underline{\mathbb{C}})).$$

The result then follows from the invariance of Hochschild homology under Morita equivalence and a comparison of the canonical Hochschild homology spectral sequences associated to $\mathcal{A}_{\mathcal{G}}(M|B; E)$ and $\mathcal{A}_{\mathcal{G}}(M|B; \underline{\mathbb{C}})$ using the above morphism of Hochschild complexes (defined by the trace).

Assume now that E is trivially graded, but not necessarily trivial, as a vector bundle. Then the graded algebra of $\mathcal{A}_{\mathcal{G}}(M|B; E)$ is the algebra generated by the homogeneous sections of the lift of $\text{End}(E)$ to $A^*(\mathcal{G})\backslash 0$. We claim that the statement that we need to prove is local in the following sense. All these Hochschild homology groups are the spaces of global sections of certain sheaves and the morphisms between them are induced by morphisms of sheaves. It is known then that a morphism of sheaves that is locally an isomorphism is also globally an isomorphism. We use now this argument and the fact that E is locally trivial. We obtain that all these algebras will have the same Hochschild homology as that of the algebra corresponding to a trivial line bundle, with the isomorphism again induced by the trace. The spectral sequence of Lemma 1 then tells us that the Hochschild homology of the algebra $\mathcal{A}_{\mathcal{G}}(M|B; E)$ is independent of E .

In general, let $E = E_+ \oplus E_-$ be the decomposition of E into the direct sum of the $+1$ and, respectively, -1 eigenvalue of the grading automorphism. As above, we observe that the statement of the lemma is again local, so we can assume that $E = E_+ \oplus E_-$ is such that both E_+ and E_- are trivial bundles. Denote by N the rank of E . Then $\mathcal{A}_{\mathcal{G}}(M|B; E) \simeq M_N(\mathcal{A}_{\mathcal{G}}(M|B))$, as before, except that now the grading is not necessarily trivial, but is induced by conjugation with a matrix in $M_N(\mathbb{C})$. Our lemma then follows from the following general fact.

Let A be a (topologically filtered) algebra A and N an integer. Assume that the grading automorphism of the algebra $M_N(A)$ is given by conjugation with a matrix

in $M_N(\mathbb{C})$. Then

$$\mathrm{HH}_*(M_N(A)) \simeq \mathrm{HH}_*(A).$$

This follows, for example, from the Künneth formula in $(\mathbb{Z}/2\mathbb{Z}$ -graded) Hochschild homology (see [17]). \square

As with the homogeneous, vertical de Rham homology, we can replace $A^*(\mathcal{G})$ in $H_{\mathcal{L},k}^\delta(A^*(\mathcal{G})\backslash 0|B)_l$ with T_{vert}^*M , the vertical cotangent bundle to $\pi : M \rightarrow B$.

Lemma 6. *The anchor map $\varrho : A(\mathcal{G}) \rightarrow T_{\mathrm{vert}}M$ induces an isomorphism*

$$H_{\mathcal{L},k}^\delta(A^*(\mathcal{G})\backslash 0|B)_l \simeq H_{\mathcal{L},k}^\delta(T_{\mathrm{vert}}^*M\backslash 0|B)_l.$$

Proof. This follows from Lemma 2 and the explicit formula for the Poisson bracket, Eq. (29). \square

We now proceed as usual and construct a chain map $*_{\mathrm{vert}}$ from the complex that defines vertical Poisson homology to the complex that defines de Rham cohomology. The chain map $*_{\mathrm{vert}}$ is, in a certain sense, a vertical symplectic $*$ -operator. It corresponds to the canonical symplectic forms on the cotangent spaces of the fibers of $\pi : M \rightarrow B$. Denote by $M_b := \pi^{-1}(b)$, $b \in B$, and by ω_b the symplectic form on T^*M_b . There exists then a 2-form ω on $T_{\mathrm{vert}}^*M := \bigcup_{b \in B} T^*M_b$ that restricts on each fiber T^*M_b to the form ω_b . This form is certainly not unique. There will be, however, a unique form $\omega \in \Omega_{\mathcal{L}}^{2,0}(T_{\mathrm{vert}}^*M)$ with this property, because restriction defines an isomorphism from $\Omega_{\mathcal{L}}^{2,0}(T_{\mathrm{vert}}^*M)$ to the space $\mathcal{O}(M)\Gamma(A^2(T_{\mathrm{vert}}^*(T_{\mathrm{vert}}^*M)))$. We shall call this form ω the *vertical symplectic form* of T_{vert}^*M . It depends on the splitting of Eq. (22).

The vertical symplectic volume form on T_{vert}^*M is defined by analogy to be $\mathrm{vol}_{\mathrm{vert}}(M) := \omega^p/p!$. Next, we define $*_{\mathrm{vert}} : \Omega_{\mathcal{L}}^{k,0}(T_{\mathrm{vert}}^*M) \rightarrow \Omega_{\mathcal{L}}^{2p-k,0}(T_{\mathrm{vert}}^*M)$ by the equation

$$\beta \wedge (*_{\mathrm{vert}}\alpha) = (\beta, \alpha)_\omega \cdot \mathrm{vol}_{\mathrm{vert}}(M), \quad \forall \alpha, \beta \in \Omega_{\mathcal{L}}^{k,0}(T_{\mathrm{vert}}^*M),$$

where $(,)_\omega$ is the bilinear form induced by the symplectic form. Then we obtain that $*_{\mathrm{vert}}(\Omega_{\mathcal{L}}^{k,0}(T_{\mathrm{vert}}^*M)_l) = \Omega_{\mathcal{L}}^{2p-k,0}(T_{\mathrm{vert}}^*M)_{l+p-k}$. Finally, to define

$$*_{\mathrm{vert}} : \Omega_{\mathcal{L}}^{i,j}(T_{\mathrm{vert}}^*M) \rightarrow \Omega_{\mathcal{L}}^{2p-i,j}(T_{\mathrm{vert}}^*M)$$

in general, it is enough to define $*_{\mathrm{vert}}\alpha$ when $\alpha = \eta \wedge \pi_0^*\beta$, with $\eta \in \Omega_{\mathcal{L}}^{i,0}(M)$ and $\beta \in \Omega^j(B)$, where $\pi_0 : T_{\mathrm{vert}}^*M \rightarrow B$ is the induced projection. We set then

$$*_{\mathrm{vert}}(\alpha) = *_{\mathrm{vert}}(\eta \wedge \pi_0^*\beta) := *_{\mathrm{vert}}(\eta) \wedge \pi_0^*\beta. \tag{33}$$

Similarly, we obtain again that $*_{\mathrm{vert}}(\Omega_{\mathcal{L}}^{i,j}(T_{\mathrm{vert}}^*M)_l) = \Omega_{\mathcal{L}}^{2p-i,j}(T_{\mathrm{vert}}^*M)_{l+p-i}$.

The usual properties of the symplectic $*$ -operator in relation to de Rham and Poisson homology extend to $*_{\text{vert}}$.

Proposition 5. *Let $*_{\text{vert}}$ be the operator defined above. Then*

- (1) $*_{\text{vert}}^2 = \text{id}$;
- (2) $(-1)^{i+1} d_{\text{vert}} \circ *_{\text{vert}} = *_{\text{vert}} \circ \delta_{\text{vert}}$ on $\Omega^{i,j}(T_{\text{vert}}^*M)$.

We can extend the range of both formulas to include homogeneous forms or forms with Laurent type singularities.

Proof. Both formulas are well known when B is reduced to a point [6]. Using a version with parameters of this particular case, we obtain that the two formulas are correct on $\Omega^{i,0}(T_{\text{vert}}^*M)$.

For the general case, let $\alpha \in \Omega^{i,0}(T_{\text{vert}}^*M)$ be of the form $\alpha = \eta \wedge \pi_0^* \beta$, where $\eta \in \Omega_{\mathcal{G}}^{i,0}(M)$ and $\beta \in \Omega_c^j(B)$, and $\pi_0 : T_{\text{vert}}^*M \rightarrow B$ is the induced projection. Then, using Eq. (33), we obtain

$$*_{\text{vert}}^2(\alpha) = *_{\text{vert}}^2(\eta) \wedge \pi_0^* \beta = \alpha. \tag{34}$$

Similarly, using the definition of d_{vert} , Lemma 4, and Eq. (33), we obtain

$$\begin{aligned} (-1)^{i+1} d_{\text{vert}} \circ *_{\text{vert}}(\alpha) &= (-1)^{i+1} d_{\text{vert}} \circ *_{\text{vert}}(\eta) \wedge \pi_0^* \beta \\ &= *_{\text{vert}} \circ \delta_{\text{vert}}(\eta) \wedge \pi_0^* \beta = *_{\text{vert}} \circ \delta_{\text{vert}}(\alpha). \end{aligned}$$

This is enough to complete the proof. \square

We are ready now to determine the homogeneous, Laurent–Poisson homology groups of $A^*(\mathcal{G})$. Recall that p denotes the dimension of the fibers of $M \rightarrow B$ and q is the dimension of the manifold B . We set for any fixed $k \in \mathbb{Z}$,

$$K^{j,l} := \mathcal{P}_{l-j}^{k-j,j},$$

so that

$$\delta_{\text{vert}} : K^{j,l} \rightarrow K^{j,l-1} \quad \text{and} \quad \alpha : K^{j,l} \rightarrow K^{j+1,l}.$$

To compute the homogeneous Laurent δ -homology of $A^*(\mathcal{G}) \setminus 0$, we use that the complex splits into subcomplexes (\mathcal{P}^k, δ) . Thus we can fix the integer $k \in \mathbb{Z}$ and define a filtration of the above bicomplex $K^{j,l}$ by

$$F_h := \bigoplus_{l \in \mathbb{Z}, j \leq h} K^{j,l}.$$

Proposition 6. *The spaces $H_{\mathcal{G},k+l}^\delta(A^*(\mathcal{G}) \setminus 0|B)_l$ and $H_{\mathcal{G},k+l}^{\delta_{\text{vert}}}(A^*(\mathcal{G}) \setminus 0|B)_l$ are isomorphic.*

Proof. Recall that we have

$$\delta = \delta_{\text{vert}} + \alpha \quad \text{and} \quad \delta_{\text{vert}} \circ \alpha + \alpha \circ \delta_{\text{vert}} = 0.$$

We then use for any fixed k the decomposition $\mathcal{P}^k \simeq \bigoplus_{i+j=k} \mathcal{P}^{i,j}$ into a finite double complex and the decreasing filtration F_h defined above. This yields a spectral sequence $(E^r)_{r \geq 1}$ which converges to the δ -homology by classical homological arguments. The E^1 term of this spectral sequence is given by

$$E_{u,v}^1 = H_{\mathcal{L}}^{2p-v-k+u,u}(T_{\text{vert}}^* M \setminus 0 | B)_{p-k+u}.$$

But again by a homotopy argument, the homogeneous vertical de Rham cohomology space $H_{\mathcal{L}}^{2p-v-k+u,u}(T_{\text{vert}}^* M \setminus 0 | B)_{p-k+u}$ is trivial unless $u = k - p$. Therefore, we get:

$$E_{u,v}^1 = 0 \quad \text{if } v \neq -k - p.$$

Hence for any $r \geq 1$, we see that $d^r = 0$ and the spectral sequence collapses at E^1 . The proof is thus complete. \square

Theorem 5. *The homogeneous, Laurent–Poisson homology groups of $A^*(\mathcal{G})$ are given by*

$$H_{\mathcal{L},k}^{\delta}(A^*(\mathcal{G}) \setminus 0 | B)_l \simeq H_{c,\mathcal{L}}^{p-l,k-l-p}(T_{\text{vert}}^* M \setminus 0 | B)_0.$$

Proof. The vertical symplectic Hodge operator $*_{\text{vert}}$ yields isomorphisms

$$*_{\text{vert}} : \bigoplus_{i+j=k} \Omega_{\mathcal{L}}^{i,j}(T_{\text{vert}}^* M \setminus 0)_l \rightarrow \bigoplus_{i+j=k} \Omega_{\mathcal{L}}^{2p-i,j}(T_{\text{vert}}^* M \setminus 0)_{l+p-i}$$

which intertwine the δ_{vert} and d_{vert} differentials (up to a sign). Proposition 6 shows then that

$$H_{\mathcal{L},k}^{\delta}(A^*(\mathcal{G}) \setminus 0 | B)_l \simeq \bigoplus_{i+j=k} H_{c,\mathcal{L}}^{2p-i,j}(T_{\text{vert}}^* M \setminus 0 | B)_{l+p-i}.$$

But for $l + p - i \neq 0$, the cohomology spaces $H_{c,\mathcal{L}}^{2p-i,j}(T_{\text{vert}}^* M \setminus 0 | B)_{l+p-i}$ are trivial by the homotopy invariance of de Rham cohomology. Hence the only non-trivial term is:

$$H_{c,\mathcal{L}}^{p-l,k-l-p}(T_{\text{vert}}^* M \setminus 0 | B)_0,$$

and this completes the proof. \square

We can now apply the results of Section 1 together with Theorem 5.

Proposition 7. *The EH^2 -term of the spectral sequence associated in Lemma 1 to the Hochschild homology of the algebra $\mathcal{A}_{\mathcal{L}}(M|B; E)$ is given by:*

$$\text{EH}_{k,h}^2 \simeq \text{H}_{c,\mathcal{L}}^{p-k,h-p}(S_{\text{vert}}^*(M) \times S^1|B),$$

where $S_{\text{vert}}^*(M)$ is the sphere bundle of the vertical cotangent bundle T_{vert}^*M .

Proof. Denote by δ the Poisson differential on the vertically symplectic fibration $T_{\text{vert}}^*M \rightarrow B$. Lemma 5, Proposition 6 and Theorem 5 give by straightforward computation:

$$\text{EH}_{k,h}^2 \simeq \text{H}_{c,\mathcal{L}}^{p-k,h-p}(T_{\text{vert}}^*M \setminus 0|B)_0.$$

Now, a classical argument shows that [28]:

$$\text{H}_{c,\mathcal{L}}^{p-k,h-p}(T_{\text{vert}}^*M \setminus 0|B)_0 \simeq \text{H}_{c,\mathcal{L}}^{p-k,h-p}(S_{\text{vert}}^*(M) \times S^1|B),$$

which completes the proof. \square

In the following lemma, we shall denote by \otimes_{if} the completion of the tensor product of two algebras in the unique natural way that makes the completed tensor product a topologically filtered algebra. See also Eq. (17) where \otimes_{if} was used before.

Lemma 7. *Assume that the fibration $\pi: M \rightarrow B$ and the bundle of algebras $\mathcal{A}_{\mathcal{L}}(M|B; E)$ are trivial; that is, assume that $M = B \times F$ and $\mathcal{A}_{\mathcal{L}}(M|B; E) \simeq \mathcal{A}_{\mathcal{L}}(F; E) \otimes_{\text{if}} \mathcal{C}_c^\infty(B)$ as topologically filtered algebras. Then the spectral sequence associated (in Lemma 1) to the Hochschild homology of the algebra $\mathcal{A}_{\mathcal{L}}(M|B; E)$ collapses at EH^2 and converges. Moreover, we have*

$$\text{HH}_k(\mathcal{A}_{\mathcal{L}}(M|B; E)) \simeq \bigoplus_{i+j=k} \text{H}_{c,\mathcal{L}}^{2p-i}(S^*(F) \times S^1) \otimes \Omega_c^j(B).$$

Proof. We know from [3] that $\text{HH}_j(\mathcal{A}_{\mathcal{L}}(F; E)) \simeq \text{H}_c^{2p-j}(\mathcal{L}(S^*(F) \times S^1))$. The usual shuffle map g [25] induces a morphism of complexes

$$g: \mathcal{H}(\mathcal{C}_c^\infty(B)) \otimes \mathcal{H}(\mathcal{A}_{\mathcal{L}}(F; E)) \rightarrow \mathcal{H}(\mathcal{C}_c^\infty(B) \otimes_{\text{if}} \mathcal{A}_{\mathcal{L}}(F; E)),$$

which preserves the filtrations. The Künneth formula for Hochschild cohomology shows that this morphism induces an isomorphism on the E^1 -term of the corresponding spectral sequences. Thus g induces an isomorphism on all E^r -terms. This shows that the spectral sequence associated to $\mathcal{A}_{\mathcal{L}}(M|B; E) = \mathcal{C}_c^\infty(B) \otimes_{\text{if}} \mathcal{A}_{\mathcal{L}}(F; E)$ degenerates at $E^2 = \text{EH}^2$.

Theorem 1 and an application of the usual Künneth formula then give that

$$\begin{aligned} \mathrm{HH}_k(\mathcal{A}_{\mathcal{G}}(M|B; E)) &\simeq \bigoplus_{i+j=k} \mathrm{HH}_i(\mathcal{C}_c^\infty(B)) \otimes \mathrm{HH}_j(\mathcal{A}_{\mathcal{G}}(F; E)) \\ &\simeq \bigoplus_{i+j=k} \Omega_c^i(B) \otimes \mathrm{H}_{c, \mathcal{G}}^{2p-j}(S^*(F) \times S^1). \quad \square \end{aligned} \tag{35}$$

We now extend the above lemma to more general groupoids.

Lemma 8. *The spectral sequence EH^r associated to the Hochschild homology of $\mathcal{A}_{\mathcal{G}}(M|B; E)$ by Lemma 1 degenerates at EH^2 and converges to its Hochschild homology.*

Proof. Denote $\mathcal{A} = \mathcal{A}_{\mathcal{G}}(M|B; E)$ in this proof, for simplicity. The differential b of the Hochschild complex of \mathcal{A} is $\mathcal{C}^\infty(B)$ -linear, if $f \in \mathcal{C}^\infty(B)$ acts on $a_0 \otimes \cdots \otimes a_n$ by $f(a_0 \otimes \cdots \otimes a_n) := (fa_0) \otimes \cdots \otimes a_n$. The filtrations of the Hochschild complex are also preserved by the multiplication operators with functions $f \in \mathcal{C}^\infty(B)$. This shows that the spectral sequence associated to the Hochschild homology of $\mathcal{A}_{\mathcal{G}}(M|B; E)$ by Lemma 1 consists of $\mathcal{C}^\infty(B)$ -modules.

Denote $E_{k,h}^2(\mathcal{A}) = \mathrm{EH}_{k,h}^2$ in this lemma, to stress the dependence of our spectral sequence on the algebra \mathcal{A} . To prove that $E_{k,h}^2(\mathcal{A}) = 0$ or that $d^r = 0$ for this spectral sequence, it is enough to check that $f E_{k,h}^2(\mathcal{A}) = 0$ (or $fd^r = 0$) for any f in a trivializing open subset V of M (that is, a subset satisfying the conditions of our third assumption on the algebra $\mathcal{A} = \mathcal{A}_{\mathcal{G}}(M|B; E)$, Eq. (17)), more precisely, satisfying that

$$\begin{aligned} \mathcal{A}_V &:= \mathcal{O}(\pi^{-1}(V))(\Psi^\infty(\mathcal{G}_V; E)/\Psi^{-\infty}(\mathcal{G}_V; E)) \\ &\simeq \mathcal{O}(\pi^{-1}(V)) \otimes_{\mathcal{O}f}(\Psi^\infty(V \times \mathcal{G}_b; E)/\Psi^{-\infty}(V \times \mathcal{G}_b; E)) \end{aligned} \tag{36}$$

is a $\mathcal{C}^\infty(B)$ -linear isomorphism. Then $f E_{k,h}^2(\mathcal{A}) = f E_{k,h}^2(\mathcal{A}_V)$. By Lemma 7, $E_{k,h}^r(\mathcal{A}_V) = 0$ if $k < -p$, and hence $E_{k,h}^r(\mathcal{A}_V) = 0$ if $k < -p$. (Recall that p is the dimension of the fibers of $M \rightarrow B$.) Similarly, $fd^r = 0$ if $r \geq 2$.

This proves that the spectral sequence in Hochschild homology associated to the algebra $\mathcal{A}_{\mathcal{G}}(M|B; E)$ by Lemma 1 degenerates at EH^2 . It also proves that the assumptions of Theorem 1 are satisfied, so the spectral sequence $\mathrm{EH}_{k,h}^r$ converges to Hochschild homology. \square

Now we can state and prove the main theorem of this section. Let \mathcal{G} be a differentiable groupoid whose space of units is a manifold with corners M which is the total space of a fibration $\pi : M \rightarrow B$, as before. Denote by \mathcal{F}^j the local coefficient system defined by

$$\mathcal{F}^j(b) := \mathrm{HH}_{2p-j}(\mathcal{A}_{\mathcal{G}}(\pi^{-1}(b))) \simeq \mathrm{H}_{c, \mathcal{G}}^j(\pi_0^{-1}(b)) \simeq \mathrm{H}_c^j(\mathcal{L}(\pi_0^{-1}(b))) \tag{37}$$

with $\pi_0 : S_{\mathrm{vert}}^*(M) \times S^1 \rightarrow B$ the natural projection.

Theorem 6. Assume that B is smooth (without corners) and that \mathcal{G} satisfies the assumptions (15) and (16). Let $\mathcal{A}_{\mathcal{G}}(M|B; E)$ be the algebra of Laurent type complete symbols on \mathcal{G} with coefficients in the $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle E . Then

$$\mathrm{HH}_m(\mathcal{A}_{\mathcal{G}}(M|B; E)) \simeq \bigoplus_{k+h=m} \Omega_c^h(B, \mathcal{F}^{2p-k}), \tag{38}$$

where \mathcal{F}^j are the sheaves defined in Eq. (37).

Proof. We apply Lemma 8 and deduce that:

$$\mathrm{HH}_m(\mathcal{A}_{\mathcal{G}}(M|B; E)) \simeq \bigoplus_{i+j=m} \mathrm{EH}_{i,j}^2.$$

The computation of EH^2 was carried out in Proposition 7 and the result is:

$$\mathrm{EH}_{i,j}^2 \simeq \mathrm{H}_{c,\mathcal{G}}^{p-i,j-p}(S_{\mathrm{vert}}^*(M) \times S^1|B).$$

On the other hand, Proposition 3 applied to $X = S_{\mathrm{vert}}^*(M) \times S^1$ gives:

$$\mathrm{H}_{c,\mathcal{G}}^{p-i,j-p}(S_{\mathrm{vert}}^*(M) \times S^1|B) \simeq \Omega_c^{j-p}(B, \mathcal{F}^{p-i}).$$

Therefore we get

$$\mathrm{HH}_m(\mathcal{A}_{\mathcal{G}}(M|B; E)) \simeq \bigoplus_{i+j=m} \Omega_c^{j-p}(B, \mathcal{F}^{p-i}).$$

The conclusion follows by setting $i = k$ and $j = h$. \square

Remark 1. The above proposition has to be modified only slightly if B is also a manifold with corners. For example, when B is compact, the result remains true if we replace $\Omega_c^j(B)$ with $\mathcal{O}(B)\Omega^j(B)$.

5. The relative case

Let again \mathcal{G} be a groupoid with corners satisfying the assumptions (15) and (16) with respect to the fibration $\pi : M \rightarrow B$ of the manifold with corners M over the smooth manifold B . Let X be a union of faces of M . We shall denote by \mathcal{I}_X the ideal of smooth functions on M that vanish to infinite order on X .

We shall consider in this section the algebra of Laurent complete symbols on \mathcal{G} which vanish to infinite order over X and which represent pseudodifferential operators acting on sections of the $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle E . This algebra is denoted by $\mathcal{A}_{\mathcal{G}}(M|B, X|B; E)$. Thus we have:

$$\mathcal{A}_{\mathcal{G}}(M|B, X|B; E) := \mathcal{O}(M)\mathcal{I}_X\mathcal{A}(M|B; E).$$

Note that if $X = \emptyset$, then we recover the algebras $\mathcal{A}_{\mathcal{L}}(M|B; E)$ studied in the previous sections. The proof of Proposition 3 in [3] extends to show that the algebras $\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)$ are topologically filtered algebras.

For any fibrations $Y \rightarrow M \rightarrow B$, we denote by p_Y the projection $Y \rightarrow M$ and by π_Y the composite projection $Y \rightarrow B$. When Y is a manifold with corners, we shall denote by $p_{\mathcal{L}}$ the projection $\mathcal{L}(Y) \rightarrow M$ which is the composite map of $\mathcal{L}(Y) \rightarrow Y$ and $Y \rightarrow M$. This last notation is intended to simplify the statements of this section. (Recall that the spaces $\mathcal{L}(M)$ were introduced before Theorem 3.)

Let again $S_{\text{vert}}^*(M)$ be the quotient bundle of $T_{\text{vert}}^*M \setminus 0$ by the radial action of \mathbb{R}_+^* . In [3], the computations of periodic cyclic homology recover the case of our algebra $\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)$. The result is as follows:

Theorem 7. [3] *For $q = 0, 1$, we have:*

$$\text{HP}_q(\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)) \simeq H_c^{[q]}(\mathcal{L}(S_{\text{vert}}^*M) \times S^1 \setminus p_{\mathcal{L}}^{-1}(X)).$$

Proof. Again by a Morita equivalence argument we can forget the bundle E . We apply Proposition 5 in [3] and obtain for our groupoid \mathcal{G} :

$$\text{HP}_q(\mathcal{A}_{\mathcal{L}}(M|B, X|B)) \simeq H_c^{[q]}(\mathcal{L}(S^*(\mathcal{G})) \times S^1 \setminus p_{\mathcal{L}}^{-1}(X)).$$

As before,

$$H_c^{[q]}(\mathcal{L}(S^*(\mathcal{G})) \times S^1 \setminus p_{\mathcal{L}}^{-1}(X)) \simeq H_{c, \mathcal{L}}^{[q]}((A^*(\mathcal{G}) \setminus 0) \setminus p_{A^*(\mathcal{G}) \setminus 0}^{-1}(X))_0.$$

From Assumption (16), we thus deduce as in Lemma 6 that:

$$H_{c, \mathcal{L}}^{[q]}((A^*(\mathcal{G}) \setminus 0) \setminus p_{A^*(\mathcal{G}) \setminus 0}^{-1}(X))_0 \simeq H_{c, \mathcal{L}}^{[q]}((T_{\text{vert}}^*M \setminus 0) \setminus p_{T_{\text{vert}}^*M \setminus 0}^{-1}(X))_0.$$

The space $H_{c, \mathcal{L}}^{[q]}((T_{\text{vert}}^*M \setminus 0) \setminus p_{T_{\text{vert}}^*M \setminus 0}^{-1}(X))_0$ is again isomorphic to the space $H_c^{[q]}(\mathcal{L}(S_{\text{vert}}^*M) \times S^1 \setminus p_{\mathcal{L}}^{-1}(X))$, and this completes the proof. \square

Let us state now the corresponding results for Hochschild homology. Let \mathcal{F}_X^l be the coefficient system over B given for any $b \in B$ by the relative cohomology space

$$\begin{aligned} \mathcal{F}_X^l(b) &:= H_{\mathcal{L}}^l(S^*(\pi^{-1}(b)) \times S^1, p_{S_{\text{vert}}^*M \times S^1}^{-1}(X)) \\ &:= H^l(\mathcal{L}(S^*(\pi^{-1}(b))) \times S^1, p_{\mathcal{L}}^{-1}(\pi^{-1}(b))). \end{aligned}$$

Theorem 8. *The Hochschild homology spaces of the algebra $\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)$ are given by:*

$$\text{HH}_m(\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)) \simeq \bigoplus_{k+h=m} \Omega_c^h(B, \mathcal{F}_X^{2p-k}).$$

Proof. We can again assume that the graded vector bundle E is trivial, one-dimensional. The proof of this theorem is similar to the proof of Theorem 6, replacing cohomology by relative cohomology and using excision. More precisely, the natural filtration of the topologically filtered algebra $\mathcal{A}_{\mathcal{G}}(M|B, X|B)$ by the order of the symbols gives rise to a spectral sequence for Hochschild homology given by Lemma 1. This spectral sequence was studied in [3] where the E^2 -term was identified with the homogeneous Laurent–Poisson relative homology of the Poisson manifold $A^*(\mathcal{G})\setminus 0$. Indeed, we have [3, Proposition 7]:

$$EH_{k,h}^2 \simeq H_{\mathcal{L},k+h}^{\delta}((A^*(\mathcal{G})\setminus 0)\setminus p_{A^*(\mathcal{G})\setminus 0}^{-1}(X))_k.$$

Now using the immediate extension of Lemma 6 to the relative case, we obtain:

$$H_{\mathcal{L},k+h}^{\delta}((A^*(\mathcal{G})\setminus 0)\setminus p_{A^*(\mathcal{G})\setminus 0}^{-1}(X))_k \simeq H_{\mathcal{L},k+h}^{\delta}((T_{\text{vert}}^*M\setminus 0)\setminus p_{T_{\text{vert}}^*M\setminus 0}^{-1}(X))_k.$$

The vertical symplectic Hodge operator $*_{\text{vert}}$ preserves the forms vanishing above X , therefore we deduce using the proof of Proposition 7 that:

$$EH_{k,h}^2 \simeq H_{c,\mathcal{L}}^{p-k,h-p}((S_{\text{vert}}^*(M) \times S^1)|B, p_{\mathcal{L}}^{-1}(X)|B).$$

The arguments ensuring the degeneracy of the spectral sequence at the second level in the proof of Lemma 8 again obviously extend to the relative case. In addition, we can apply Theorem 1 to deduce the convergence of the spectral sequence to Hochschild homology. Hence we finally obtain:

$$\begin{aligned} \text{HH}_m(\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)) &\simeq \bigoplus_{k+h=m} EH_{k,h}^2 \\ &\simeq \bigoplus_{k+h=m} H_{c,\mathcal{L}}^{p-k,h-p}((S_{\text{vert}}^*(M) \times S^1)|B, p_{\mathcal{L}}^{-1}(X)|B). \end{aligned}$$

To end the proof we simply observe that Proposition 3 extends straightforward to the relative case. \square

6. Examples and applications

We begin by providing a construction of a groupoid \mathcal{G} satisfying the assumptions (15) and (16), for any fibration $\pi : M \rightarrow B$, where B is a smooth manifold (no corners) and M is a manifold, possibly with corners. For each fibration π as above we shall construct a canonical groupoid $\mathcal{G}_{M,b}$, the “ b -groupoid,” satisfying the assumptions (15) and (16).

Our examples are obtained by first defining the Lie algebroid of \mathcal{G} , and then by integrating it. (See [12,36] for general results on the integration of Lie algebroids.) Let $\mathcal{V}_b(M|B)$ be the space of vertical vector fields on M that are tangent to all faces of M . Then $\mathcal{V}_b(M|B)$ is a Lie algebra with respect to the Lie bracket of vector fields and is also a projective $\mathcal{C}^{\infty}(M)$ -module. By the Serre–Swan theorem [17] there exists

a vector bundle $A_b \rightarrow M$ such that

$$\mathcal{V}_b(M|B) \simeq \Gamma(A_b),$$

naturally. (See [29].)

The procedure in [36] then provides us with groupoids $\mathcal{G}_{M,b}$ whose Lie algebroid are isomorphic to A_b . The minimal groupoid with this property is obtained as follows. For each face F of M , consider the interior of that face, $F_0 := F \setminus \partial F$. Assume F has codimension k . Define then \mathcal{G}'_F to be the groupoid associated to an equivalence relation: two units are connected by an arrow if, and only if, they are equivalent. The equivalence relation that we consider is that $x, y \in F_0$ are equivalent if, and only if, they belong to the same connected component of a set of the form $\pi^{-1}(b) \cap F_0$. Let

$$\mathcal{G}_{M,b} = \bigcup \mathcal{G}'_F \times \mathbb{R}^k, \quad (36)$$

the union being a disjoint union, and with the induced groupoid structure. Then it can be checked directly that the charts provided in [36] define a smooth structure on \mathcal{G} such that $A(\mathcal{G}_{M,b}) \simeq A_b$. This structure must then be unique [36]. See also [8,12,31].

Let \mathcal{F} be the locally constant sheaf (or coefficient system) that associates to $b \in B$ the complex vector space with basis the minimal faces of $\pi^{-1}(b)$. (All faces of a manifold with corners are connected, by definition.)

Theorem 9. *Let $\pi : M \rightarrow B$ be as above. Then*

$$\mathrm{HH}_0(\mathcal{A}_{\mathcal{G}}(M|B; E)) \simeq C_c^\infty(B, \mathcal{F}),$$

the space of compactly supported sections of the sheaf \mathcal{F} . The space of traces of $\mathcal{A}_{\mathcal{G}}(M|B; E)$ identifies with the dual of this space:

$$\mathrm{HH}^0(\mathcal{A}_{\mathcal{G}}(M|B; E)) \simeq C^{-\infty}(B, \mathcal{F}) =: C_c^\infty(B, \mathcal{F})'.$$

Proof. This follows from Theorem 6. \square

In the particular case when M is smooth, our construction simplifies and we obtain $\mathcal{G}_{M,b} = M \times_B M$. Then the algebra $\Psi^\infty(\mathcal{G})$ consists of differentiable families of pseudodifferential operators along the fibers of $M \rightarrow B$. Similarly, let us consider $\Psi^\infty(M|B; E)$, the algebra of smooth families of pseudodifferential operators along the fibers with coefficients in the $\mathbb{Z}/2\mathbb{Z}$ -graded vector bundle E , introduced in [1]. Let $\mathcal{A}(M|B; E) := \Psi^\infty(M|B; E)/\Psi^{-\infty}(M|B; E)$ be the algebra of vertical complete symbols. Then $\mathcal{A}(M|B; E) = \mathcal{A}_{\mathcal{G}}(M|B; E)$, and we obtain

Corollary 3. *Assume $\mathcal{G} = M \times_B M$, with M smooth (without corners) and let $\mathcal{A}(M|B; E) = \mathcal{A}_{\mathcal{G}}(M|B; E)$ be the algebra of families of complete symbols along the fibers of $\pi : M \rightarrow B$, as above. Also, let \mathcal{F}^j be the locally constant sheaf given by the*

cohomology of the fibers of $S_{\text{vert}}^*(M) \times S^1 \rightarrow B$. Then we have:

$$\text{HH}_m(\mathcal{A}(M|B; E)) \simeq \bigoplus_{k+h=m} \Omega_c^h(B, \mathcal{F}^{2p-k}).$$

In particular, for $m = 0$ and provided that the fibers of $M \rightarrow B$ are connected and have dimension ≥ 2 , this isomorphism becomes

$$\text{HH}_0(\mathcal{A}(M|B; E)) \simeq \mathcal{C}_c^\infty(B).$$

Therefore, the space of traces is given by

$$\text{HH}^0(\mathcal{A}(M|B; E)) \simeq \mathcal{C}^{-\infty}(B) := \mathcal{C}_c^\infty(B)', \tag{37}$$

the space of distributions on the base manifold B .

Let ω be as before the vertical symplectic form on $T_{\text{vert}}^*M \rightarrow B$. The isomorphism of Eq. (37) can be made more explicit as follows. Let \mathcal{R} be the radial vector field on the fibers of $T_{\text{vert}}^*M \rightarrow B$ and $\alpha = i_{\mathcal{R}}(\omega^p/p!)$ the corresponding Liouville form on the fibers of $S_{\text{vert}}^*(M) \rightarrow B$. Let STr be the graded trace on the endomorphisms of the fibers of E , that is $\text{STr}(A) = \text{Tr}(\gamma \circ A)$, where γ is the involution defining the grading. Also, let π_* be the fiberwise integration on the fibers of S_{vert}^*M . Then, for any $a = \sum_{j \leq m} a_j$, we set

$$\tau_\mu(a) := \langle \mu, \pi_* \text{STr}(a_{-p})\alpha \rangle. \tag{38}$$

This formula defines a super (or graded) trace on $\mathcal{A}(M|B; E)$ such that $\mu \rightarrow \tau_\mu$ is the isomorphism described in Corollary 3, for $m = 0$.

Fix a quantization function

$$q: \bigcup S^s(T_{\text{vert}}^*M; \text{End}(E)) \rightarrow \bigcup \Psi^s(M|B; E), \tag{39}$$

where $S^s(T_{\text{vert}}^*M; \text{End}(E))$ denotes classical vertical symbols of order s . The function q is thus assumed to be continuous and to satisfy $\sigma_s(q(a)) = a$ if a is a symbol of order s . Let ρ be a positive symbol on T_{vert}^*M , such that $\rho(\xi) = |\xi|$, for $|\xi| \geq 1$. We consider a family $D(z)$ such that

$$D(z) = q(\rho^z)B_1(z) + R(z), \tag{40}$$

where B_1 is a holomorphic function on \mathbb{C} with values in $\Psi^0(M|B; E)$, $B_1(0) = 1$, $R(z)$ is a holomorphic function on \mathbb{C} with values in $\Psi^{-\infty}(M|B; E)$ and satisfying $R(0) = 0$. Then $D(0) = 1$.

Proposition 8. *Let μ be a distribution on B and $A \in \Psi^m(M|B; E)$ and $D(z)$ be as above. Then the function*

$$z \rightarrow F_A(z) := \langle \mu, \text{STr}_b(AD(z)) \rangle$$

is well defined for $\operatorname{Re}(z) < -m - p$, where p is the dimension of the fibers of $\pi : M \rightarrow B$. The function F_A extends to a meromorphic function on \mathbb{C} , with at most simple poles at the integers. For $z = 0$, the residue of this function is up to constant $\tau_\mu(A)$.

Proof. This is proved as in the classical case when $D(z)$ is given by the complex powers of a positive elliptic operator (see [13,41]). One can follow the approach from [37], for example. \square

We remark that we used above only a weak result from [36] on the integration of Lie algebroids (namely Theorem 2). Let us use this opportunity however to mention that there is a missing assumption in the general gluing theorem of [36] (i.e. Theorem 3 of that paper). Here is the corrected version.

Theorem 10. *Let A be a Lie algebroid on a manifold with corners M . Suppose that M has an A -invariant stratification $M = \bigcup S$ such that, for each stratum S , the restriction A_S is integrable and let \mathcal{G}_S be d -simply connected differential groupoids such that $A(\mathcal{G}_S) \simeq A_S$. Then A is integrable if, and only if, the exponential map $\operatorname{Exp} : A \rightarrow \mathcal{G} = \bigcup \mathcal{G}_S$ is injective on an open neighborhood of the zero section of A for some (equivalently, for any) connection on A . Moreover, if these conditions are satisfied, then the disjoint union $\mathcal{G} = \bigcup \mathcal{G}_S$ is naturally a differentiable groupoid such that $A(\mathcal{G}) \simeq A$.*

The above condition on the injectivity of the map Exp is seen to be necessary in view of the work of Crainic and Fernandes [12], and also from some earlier results of Weinstein. The map Exp introduced in [36] seems to be essential for both the results of that paper and for the results of [12]. The second author would like to thank M. Crainic for pointing out a possible problem with the original statement of the above theorem.

Let us mention for completeness the result for the computation of the cyclic homology of the algebras $\mathcal{A}_\varphi(M|B; E)$. Note first that

$$\operatorname{HC}_j(\mathcal{A}_\varphi(M|B; E)) \simeq \operatorname{HP}_j(\mathcal{A}_\varphi(M|B; E)) \quad \text{for } j \geq 2p + 2,$$

from the Connes SBI-exact sequence that related Hochschild and cyclic homology, because the Hochschild homology of $\mathcal{A}_\varphi(M|B; E)$ vanishes above this rank.

Proposition 9. *The spectral sequence $\operatorname{EC}_{k,h}^r$ associated to the cyclic homology complex of $\mathcal{A}_\varphi(M|B; E)$ has E^1 -term given by $\operatorname{EC}_{k,h}^1 = \Omega_{rc,\varphi}^{k+h}(S_{\operatorname{vert}}^* M)$, if $k \neq 0$, and by*

$$\operatorname{EC}_{0,h}^1 = \Omega^h(S_{\operatorname{vert}}^* M \times S^1) / d\Omega^{h-1}(S_{\operatorname{vert}}^* M \times S^1) \oplus \bigoplus_{j>0} \operatorname{H}^{h-2j}(S_{\operatorname{vert}}^* M \times S^1).$$

The d_1 differential is induced by the Poisson differential and the spectral sequence converges to the cyclic homology of the algebra $\mathcal{A}_\varphi(M|B; E)$.

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