Generalized $n$-gons and Chebychev Polynomials

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Two sequences of orthogonal polynomials are given whose weight functions consist of an absolutely continuous part and two point masses. Combinatorial proofs of the orthogonality relations are given. The polynomials include natural $q$-analogs of the Chebychev polynomials. The technique uses association schemes of generalized $n$-gons to find approximating discrete orthogonality relations. The Feit-Higman Theorem is a corollary of these orthogonality relations for the polynomials.

1. INTRODUCTION

Recently there has been much interest in discrete orthogonal polynomials. The theory of association schemes, as developed by Delsarte [5], has transferred certain combinatorial problems to properties of the polynomials. This also works in reverse. For example, the weights in the discrete orthogonality relation can be found by counting appropriate sets. Thus, a combinatorial proof of orthogonality is given.

For absolutely continuous weight functions, a group theoretic proof of orthogonality can sometimes be given. The ultraspherical polynomials arise naturally in the theory of spherical harmonics as functions in distinct irreducible representations of the rotation group. So the orthogonality of polynomials arises from the orthogonality of the irreducible representations. The absolutely continuous weight function is related to the measure on the group. In fact, the combinatorial case often can be reduced to the group case by letting finite groups act on finite sets [6].

There has been little attention, however, to weight functions which have jumps and an absolutely continuous part simultaneously. There are known explicit examples [1, 2]. In this paper, we shall give a combinatorially motivated proof of such an orthogonality relation. Our polynomials are generalizations of Chebychev polynomials. One such example is Theorem 5.1 (ii), which we state here.

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Theorem. For any real number $s \neq 0$ with $|s| < 1$, let

$$\tilde{p}_j(x) = (s + 1) T_j(x) + (s - 1)(1 + x) U_{j-1}(x),$$

where $T_j$ and $U_{j-1}$ are Chebychev polynomials of the 1st and 2nd kinds. Then

$$\int_{-1}^{1} \tilde{p}_j(x) \tilde{p}_l(x) w(x) \, dx + \frac{\pi(1 - s)}{s(1 + s)} \tilde{p}_j(x_1) \tilde{p}_l(x_1) = 0 \quad \text{for } j \neq l,$$

where $w(x) = \left(\frac{(1 - x)/(1 - 2sx + s^2)}{(1 - x^2)^{-1/2}}\right)$ and $x_1 = (s^2 + 1)/2s$.

Our main idea for a combinatorial proof of this theorem (and the related results in Section 5) is to first find a discrete orthogonality relation for $\tilde{p}_0, ..., \tilde{p}_n$. Letting $n \to \infty$ will give the mixed weight functions. This idea does make sense. For a regular $n$-gon, an association scheme is easily defined which gives a discrete orthogonality relation for the Chebychev polynomials $T_0, ..., T_d$ ($d = \lfloor n/2 \rfloor$). The details are given in Section 2. As $n \to \infty$, this relation approaches the usual absolutely continuous orthogonality relation. Formally, for large $n$, a regular $n$-gon is not much different from a circle.

For a generalization we shall consider generalized $n$-gons in Section 3. An association scheme is defined and the discrete orthogonality relations are found. Again the $n \to \infty$ case gives the mixed weight functions. This process also gives two possible natural $q$-analogs of Chebychev polynomials. One that was considered by Cartier is given in Section 6.

Also in this paper we shall give an application of the discrete orthogonality relations. In Section 4 we shall give another proof of the Feit–Higman theorem [8] on the nonexistence of generalized $n$-gons. To show that they do not exist, we shall show that the $L^2$-norm is not a rational number. This proof is closely related to one given by Kilmoyer and Solomon [9].

It may seem paradoxical that we use orthogonality relations based upon combinatorial structures that do not exist to derive the mixed orthogonality relations. At each stage, the orthogonality remains true even though the generalized $n$-gons do not exist. Some might be content to call this combinatorial proof fake. Nevertheless, the author believes it is certainly combinatorially motivated.

2. Regular $n$-gons

In preparation for the generalized $n$-gons, we find the polynomials and orthogonality relations for the regular $n$-gons. This material can be found in [3] or [5].
Let \( X_n \ (n \geq 3) \) be a regular \( n \)-gon. We consider \( X_n \) as a graph with \( n \) nodes. There is a natural distance function on \( X_n \) given by the graph. This distance function clearly makes \( X_n \times X_n \) into an association scheme \([5]\) with \( d + 1 = \lceil n/2 \rceil + 1 \) classes. The valencies for this scheme are

\[
v_0 = 1, \quad v_i = 2, \quad i = 1, \ldots, d - 1,
\]

and

\[
v_d = 1, \quad \text{for } n \text{ even,}
\]

\[= 2, \quad \text{for } n \text{ odd.}
\]

To find the eigenfunctions of any metrically regular association scheme \( X_n \) with \( d + 1 \) classes and distance function \( \rho \), we need to solve the equation

\[
\lambda p(k) = a_k p(k + 1) + b_k p(k) + c_k p(k - 1), \quad k = 0, 1, \ldots, d, \quad (2.1)
\]

where

\[
a_k = |\{x \in X_n : \rho(x_1, x) = 1, \rho(x_0, x) = k + 1\}|, \quad (2.2)
\]

\[
b_k = |\{x \in X_n : \rho(x_1, x) = 1, \rho(x_0, x) = k\}|, \quad (2.3)
\]

\[
c_k = |\{x \in X_n : \rho(x_1, x) = 1, \rho(x_0, x) = k - 1\}|, \quad (2.4)
\]

for some fixed \( x_0 \) and \( x_1 \) with \( \rho(x_0, x_1) = k \). Because the association scheme is metrically regular, \( a_k, b_k, \) and \( c_k \) are independent of \( x_0 \) and \( x_1 \) with \( \rho(x_0, x_1) = k \). The right-hand side of (2.1) just means that we move 1 away from a fixed point \( (x_i) \) that is \( k \) away from an initial fixed point \( (x_0) \). The resulting points must be \( k - 1, k, \) or \( k + 1 \) from \( x_i \).

If \( a_k \neq 0 \) for \( k = 0, 1, \ldots, d \), then (2.1) defines \( p(k) \) as a polynomial of degree \( k \) in \( \lambda \), \( p(k) \equiv p(k, \lambda) \). The final equation \((k = d \text{ of } (2.1))\) gives the \( d + 1 \) distinct values of \( \lambda \).

For any association scheme we have orthogonality relations \([5]\).

**Proposition 2.5.** Let \( p(k, \lambda_j) \) be the eigenvalues of the association scheme \( X \times X \) normalized by \( p(0, \lambda_j) = 1 \). Then

(i) \[
\frac{1}{|X|} \sum_{k=0}^{d} p(k, \lambda_j) p(k, \lambda_j) v_k - \delta_{jl} / m_j, \quad \text{and}
\]

(ii) \[
\frac{1}{|X|} \sum_{k=0}^{d} p(j, \lambda_k) p(l, \lambda_k) m_k = \delta_{jl} / v_j,
\]

where \( v_j \) and \( m_j \) are the valencies and multiplicities, respectively.
Next we write Eq. (2.1) for the regular \( n \)-gon. They are:

\[
\begin{align*}
\lambda p(0) &= 2p(1) \\
\lambda p(k) &= p(k + 1) + p(k - 1), \quad k = 1, 2, \ldots, d - 1, \\
\lambda p(d) &= 2p(d - 1), \quad \text{if } \ n \text{ is even,} \\
&= p(d) + p(d - 1), \quad \text{if } \ n \text{ is odd.}
\end{align*}
\] (2.6)  (2.7)  (2.8)

Clearly (2.7) implies (for \( p(0) = 1, \ p(1) = \lambda/2 \))

\[ p(k) = T_k(\lambda/2), \quad k = 0, 1, \ldots, d, \] (2.9)

where \( T_k(x) \) is the Chebychev polynomial of the 1st kind,

\[ T_k(\cos \theta) = \cos k\theta. \] (2.10)

The values for \( \lambda \) are given by (2.8). They are

\[ \lambda_j = 2 \cos(2\pi j/n), \quad j = 0, 1, \ldots, d. \] (2.11)

So

\[ p(k, \lambda) = T_k(\cos(2\pi j/n)) = \cos(2\pi jk/n). \] (2.12)

This is well known [3, p. 17].

The orthogonality relation from Proposition 2.5(i) is

\[
\frac{1}{n} + \frac{2}{n} \sum_{k=1}^{d} \cos(2\pi jk/n) \cos(2\pi lk/n) = \begin{cases} 
0, & j \neq l, \\
\frac{1}{2}, & j = l \neq 0, \\
1, & j = l = 0.
\end{cases}
\] (2.13)

for \( n \) odd, and

\[
\frac{1}{n} \sum_{k=1}^{d-1} \cos(2\pi jk/n) \cos(2\pi lk/n) = \begin{cases} 
0, & j \neq l, \\
\frac{1}{2}, & d \neq j = l \neq 0, \\
1, & j = l = 0 \text{ or } d,
\end{cases}
\] (2.14)

for \( n \) even. Because \( p(k, \lambda_j) = p(j, \lambda) \) the orthogonality from Proposition 2.5(ii) is the same as the above orthogonality. Also note that the
multiplicities $m_j$ are identical to the valencies $v_j$. In this case $m_j$ is the dimension of the corresponding irreducible representation of the dihedral group $D_n$.

As $n \to \infty$, (2.13) and (2.14) become

$$\int_0^1 \cos(j\pi x) \cos(l\pi x) \, dx = \delta_{jl}/2, \quad j, l = 0, 1, \ldots, \tag{2.15}$$

or

$$\int_{-1}^1 T_j(y) T_l(y) \frac{dy}{\sqrt{1 - y^2}} = \pi \delta_{jl}/2, \quad j, l = 0, 1, \ldots. \tag{2.16}$$

This is the absolutely continuous orthogonality relation for Chebyshev polynomials. (In (2.15) and (2.16) we must have $(j, l) \neq (0, 0)$.) We have obtained it from a limit of discrete orthogonality relations.

3. Generalized $n$-gons

In this section we carry out all of the calculations of section 2 for generalized $n$-gons. Before defining the association scheme, we repeat the definition of a generalized $n$-gon given in [9].

**Definition 3.1.** An incidence structure $I$ is a triple $(P, L, F)$ of points $P$, lines $L$, and flags $F \subseteq P \times L$. If $(p, l)$ is a flag we say $p$ is on $l$ or $l$ is on $p$.

**Definition 3.2.** For an incidence structure $I$, the distance between any $h, c \in P \cup L$ ($\rho(h, c)$) is the length of the shortest chain $b = x_0, x_1, \ldots, x_k = c$, where $x_i \in P \cup L$, and $x_{i-1}$ and $x_i$ are incident for $i = 1, 2, \ldots, k - 1$.

**Definition 3.3.** A generalized $n$-gon is an incidence structure $I = (P, L, F)$ such that:

(i) there are $s + 1$ points on every line,

(ii) there are $t + 1$ lines on every point,

(iii) $\rho(b, c) \leq n$ for all $b, c \in P \cup L$,

(iv) if $\rho(b, c) < n$, there is a unique chain of length $\rho(b, c)$ from $b$ to $c$, and

(v) for any $b \in P \cup L$, there is a $c \in P \cup L$ such that $\rho(b, c) = n$.

For a generalized $n$-gon $(P, L, F)$, it is clear that if $x_0, x_1 \in P$, $\rho(x_0, x_1)$ is even. We define a graph $G$ whose nodes are the points in $P$. Two points
$x_0, x_1 \in P$ are connected by an edge if $\rho(x_0, x_1) = 2$. In fact, for $x_0, x_1 \in P$, $\rho(x_0, x_1)$ is twice the distance of $x_0$ to $x_1$ in the graph $G$. We show that $G$ is metrically regular [3, p. 137], so it defines an association scheme [5]. To do this we merely verify that the constants $a_k, b_k, c_k$ given in (2.2), (2.3), and (2.4) really are independent of $x_0$ and $x_1$.

**Proposition 3.4.** Let $x_0, x_1 \in P$ with $\rho(x_0, x_1) = 2k$. Then if $d = \lfloor n/2 \rfloor$ (the maximum distance),

(i) $a_0 = s(t + 1), a_k = st, k = 1, \ldots, d - 1, a_d = 0$,

(ii) $b_0 = 0, b_k = s - 1, k = 1, \ldots, d - 1, b_d = (s - 1)(t + 1), \text{ for } n \text{ even},$ 
$= s(t + 1), \text{ for } n \text{ odd},$

(iii) $c_0 = 0, c_k = 1, k = 1, \ldots, d - 1, c_d = t + 1, \text{ for } n \text{ even},$ 
$= 1, \text{ for } n \text{ odd}.$

In particular $a_k, b_k, c_k$ depend only upon $\rho(x_0, x_1) = 2k$, and not upon the choice of $x_0$ and $x_1$.

**Proof.** First assume $0 < 2k < n - 1$ so that $\rho(x_0, x_1) = 2k$ is not maximal. There is a unique chain $C$ from $x_0$ to $x_1$ of length $2k$ because $2k < n$. This produces one point $p$ such that $\rho(x_0, p) = 2k - 2$ and $\rho(p, x_1) = 2$. If a different point $q$ existed such that $\rho(x_0, q) = 2k - 2$ and $\rho(q, x_1) = 2$, we would have another chain $C' \neq C$ from $x_0$ to $x_1$ of length $2k$. Thus $c_k = 1$. If $l$ is the line in $C$ such that $\rho(x_0, l) = 2k - 1$, then $p, x_1 \in l$. The remaining $(s + 1) - 2$ points on $l$ all must be distance $2k$ from $x_0$ (by uniqueness of $C$). So $b_k = s - 1$. Clearly $a_k + b_k + c_k$ is the number of points adjacent to $x_1$. This is $s(t + 1)$, so $a_k = s(t + 1) - s = st$.

Just as in (2.8) we handle the last value of $k$ differently depending upon the parity of $n$. If $n$ is even, let $2k = n$. Any line $l$ on $x_1$ must satisfy $\rho(x_0, l) = 2k - 1$. Each line $l$ contains a unique and distinct point $q$ such that $\rho(x_0, q) = 2k - 2$. So $a_d = 0, b_d = (s - 1)(t + 1)$, and $c_d = t + 1$ for $n$ even. If $n$ is odd let $2k = n - 1$. A similar argument gives $a_d = 0, b_d = s(t + 1) - 1$, and $c_d = 1$.

From Proposition 3.4 we can easily compute the valencies of the association scheme.

**Proposition 3.5.** The valencies of the association scheme of points of a generalized $n$-gon are

$$v_0 = 1, \quad v_k = s(t + 1)(st)^{k-1}, \quad k = 1, \ldots, d - 1.$$
and
\[ v_d = s(st)^{d-1}, \quad \text{for } n \text{ even}, \]
\[ = s(t + 1)(st)^{d-1}, \quad \text{for } n \text{ odd}. \]

**Proof.** By considering those points whose distance from a fixed point is either \( k \) or \( k + 1 \) we have \( a_k v_k = c_{k+1} v_{k+1} \). The values of \( a_k \) and \( c_k \) given in Proposition 3.4 yield the stated values for \( v_k \).

If \( n \) is odd, it is well known that \( s = t \), \([9]\). This follows easily by finding disjoint paths from an antipodal pair \((p, l)\), \( p(p, l) = n \). We shall use this fact later.

Proposition 3.4 also gives the three-term recurrence for the polynomials. It is
\[
\lambda p(k) = stp(k + 1) + (s - 1) p(k) + p(k - 1), \quad k = 1, \ldots, d - 1. \tag{3.6}
\]

where \( p(0) = 1 \) and \( p(1) = s / (t + 1) \). This is easily solved with the generating function
\[
G(z) = \sum_{k=0}^{\infty} p(k) z^k.
\]

We find that
\[
G(z) = \left(1 - \frac{(\lambda - s + 1) z}{st} + \frac{\lambda z}{s(t + 1)}\right)\left(1 - \frac{(\lambda - s + 1) z}{st} + \frac{z^2}{st}\right). \tag{3.7}
\]

So
\[
p(k, \lambda) = (st)^{-k/2} [T_k(x) + (xs(t - 1) + (st)^{1/2}(s - 1))/(s(t + 1)) U_{k-1}(x)], \tag{3.8}
\]

where \( x = (\lambda - s + 1)/2(st)^{1/2} \) and \( U_k(x) \) is the Chebychev polynomial of the 2nd kind. \([7]\). We can check that if \( s = t = 1 \), then \( x = \lambda / 2 \) and \( p(k, \lambda) \) reduces to \( T_k(x) \) as in Section 2.

The values of the eigenvalue \( \lambda \) are determined by the final equation \((k = d)\). A trivial eigenvalue is \( \lambda = s(t + 1) \) with \( p(k) \equiv 1 \), because \( a_k + b_k + c_k = s(t + 1) \). The final equation is
\[
\lambda p(d) = (s - 1)(t + 1) p(d) + (t + 1) p(d - 1) \quad \text{for } n \text{ even}, \tag{3.9}
\]
\[
\lambda p(d) = [s(t + 1) - 1] p(d) + p(d - 1) \quad \text{for } n \text{ odd}. \tag{3.10}
\]
It is clear that (3.9) and (3.10) are equivalent to

\[
\begin{align*}
\lambda p(d) &= s(t + 1) p(d + 1) \quad \text{for } n \text{ even}, \\
p(d) &= p(d + 1) \quad \text{for } n \text{ odd}.
\end{align*}
\]

The \( d + 1 \) roots of this polynomial equation are the eigenvalues.

**Theorem 3.13.** The eigenvalues of the association scheme of points of a generalized \( n \)-gon are:

1. \( \lambda_0 = s(t + 1), \quad \lambda_j = 2 \sqrt{st} \cos \theta_j + s - 1, \quad \theta_j = \pi j/d, \quad j = 1, \ldots, d - 1, \) \( \lambda_d = -t - 1 \) for \( n \) even, and
2. \( \lambda_0 = s(s + 1), \quad \lambda_j = 2s \cos \theta_j + s - 1, \quad \theta_j = 2\pi j/n, \quad j = 1, \ldots, d, \) for \( n \) odd. (Recall \( s = t \) here.)

**Proof.** We have already mentioned that \( \lambda_0 \) is always an eigenvalue. Next we check that \( x = \cos \theta_j \) in (3.8) satisfies (3.11) and (3.12). This is easily done with \( T_k(\cos \theta) = \cos k\theta \) and \( U_{k-1}(\cos \theta) = \sin k\theta/\sin \theta. \) We omit the details. For \( n \) even we need to verify \( \lambda_d = -t - 1 \) words with \( p(k) = (-s)^{-k}. \) It is also clear that the given eigenvalues are distinct.

In order to write the orthogonality relations of Proposition 2.5, we need to find the multiplicities \( m_j. \) They are easily found from explicit formula (3.8) for \( p(k, \lambda_j) \) and the valencies \( v_j \) in Proposition 3.5.

**Corollary 3.14.** Let \( n \) be even. Then:

1. \( 1/m_0 = 1, \)
2. \( 1/m_j = (d(1 + t)/|P|2t)[1 + ((s(t - 1) \cos \theta_j + (s - 1) \sqrt{st})/s(t + 1))/\sin^2 \theta_j], \) if \( j = 1, \ldots, d - 1, \theta_j = \pi j/d, \) and
3. \( 1/m_d = (1/|P|)[1 + ((t + 1)/(s - t))(1 - (t^{d-1}/s^{d-1})) + t^{d-1}/s^{d-1}]. \)

**Corollary 3.15.** Let \( n \) be odd. Then:

1. \( 1/m_0 = 1, \) and
2. \( 1/m_j = (1/|P|)((s + 1)(2d + 1)/4s)[1 + ((s - 1)^2/(s + 1)^2) - (1 + \cos \theta_j)^2/\sin^2 \theta_j)], \) where \( j = 1, \ldots, d \) and \( \theta_j = 2\pi j/(2d + 1). \)

**4. THE FEIT–HIGMAN THEOREM**

In this brief section the connection between the Feit–Higman theorem and the orthogonality relation is indicated. In fact, this proof is essentially
equivalent to one of Kilmoyer and Solomon [9]. They computed the characters of the Hecke algebra of a generalized dihedral group. The orthogonality relation gave them the theorem. We use the same idea.

The multiplicity $m_j$ is always an integer, so $1/m_j$ is rational. Suppose $n$ is even. In Corollary 3.14, the only possible nonrational numbers are $\sqrt{st}\cos\theta_j$ and $\sin^2\theta_j$. Computing $1/m_1 + 1/m_{d-1}$ we see that $[i(s-1)^2 + s(t-1)^2]/\sin^2\theta_1$ is rational. Just as in [9], this forces $s = t = 1$ or $d = 1, 2, 3, 4, 6$. The exclusion of $n = 12$ and restriction on $n = 6$ or 8 are carried out exactly as in [9], by considering $1/m_j - 1/m_{d-j}$. The odd case for $n$ is completed with Corollary 3.15.

5. The Limiting Measures

In this section we let $n \to \infty$ in the orthogonality relations for $p(k, \lambda_j)$ in Section 3. Even though the generalized $n$-gons do not exist, these relations remain correct.

It is easier to first consider the dual orthogonality in Proposition 2.5(ii). The polynomials $p(j)$ and $p(l)$ are both evaluated at $\lambda_k$. We interpret the finite sum as a Riemann approximation of an integral. We use $|P| = (1 + s)(1 - (st)^d)/(1 - st)$. The first and last terms of the sum can lead to jumps in the weight function. First we state the result for $n$ odd ($s = t$) from Corollary 3.15.

**Theorem 5.1.** For any real number $s \neq 0$ let

$$p_j(x) = (s + 1) T_j(x) + (s - 1)(1 + x) U_{j-1}(x).$$

Then if $j \neq 0$ or $l \neq 0$:

(i) $\int_{-1}^1 \tilde{p}_j(x) \tilde{p}_l(x) w(x) \, dx = \pi \delta_{jl}$, where $|s| > 1$, $w(x) = ((1 - x)/(1 - 2sx + s^2))(1 - x^2)^{-1/2}$, $\int_{-1}^1 w(x) \, dx = \pi/s(s + 1)$, and

(ii) $\int_{-1}^1 \tilde{p}_j(x) \tilde{p}_l(x) w(x) \, dx + (\pi(1 - s)/s(s + 1)) \tilde{p}_j(x_1) \tilde{p}_l(x_1) = \pi \delta_{jl}$, where $|s| < 1$, $w(x)$ is given in (i), and $x_1 = (s^2 + 1)/2s$.

It is easily seen that Theorem 5.1 reduces to the orthogonality relation for $T_j$ if $s = 1$. Similarly for $s = -1(\tilde{p}_j(x_1) = s^l(s + 1))$, we have the orthogonality relation for $U_j$.

For $n$ even, we use the values of $m_j/|P|$ that are given in Corollary 3.14. Again we check $j = 0$ and $j = d$ for jumps.

**Theorem 5.2.** For any real numbers $s$ and $t$ such that $st > 0$, let $r_j(x) = s(t + 1) T_j(x) + [xs(t - 1) + \sqrt{st}(s - 1)] U_{j-1}(x)$. Then if $j \neq 0$ or $l \neq 0$: 

}\]
(i) \[ \int_{-1}^{1} r_j(x) r_l(x) w(x) \, dx = \pi(t + 1)^2 s^2 \delta_{jl}/2, \] where \(|s| < |t|, \ st > 1,\) and \( w(x) = \left[ 1 + \left( s(t - 1)x + (s - 1) \sqrt{s t} \right)^2 / s^2 (t + 1)^2 (1 - x^2) \right]^{-1} (1 - x^2)^{-1/2} \). \[ \text{moreover, } \int_{-1}^{1} w(x) \, dx = \pi(t + 1)/2t, \]

(ii) for \( st < 1 \) the jump \( (\pi(t + 1)(1 - st)/2t(s + 1)) r_j(x_l) r_l(x_l) \) is added to the left-hand side of (i), where \( x_l = (1 + st)/2 \).

(iii) for \( |s| > |t| \) the jump \( (\pi(t + 1)(s - t)/2t(s + 1)) r_j(x_2) r_l(x_2) \) is added to the left-hand side of (i), where \( x_2 = -(s + 1)/2 \).

Cases (ii) and (iii) can simultaneously occur.

The polynomials \( r_j(x) \) are special cases of the \( q = 0 \) \( \Phi_3 \) polynomials [2, Eq. (4.29)]. We take \( c = d = 0, a = 1/\sqrt{st}, \) and \( b = -|s| \sqrt{st}/s, \) because \( w(x) \) can be rewritten as

\[
w(x) = \frac{(t + 1)^2 \sqrt{1 - x^2}}{t^2(1 - 2x/\sqrt{st} + 1/st)(1 + 2 \sqrt{s/t x + s/t})}, \quad s > 0. \tag{5.3}
\]

\[
w(x) = \frac{(t + 1)^2 \sqrt{1 - x^2}}{t^2(1 - 2x/\sqrt{st} + 1/st)(1 - 2 \sqrt{s/t x + s/t})}, \quad s < 0. \tag{5.4}
\]

In fact

\[
r_j(x) = stp_j(x; a, b, 0, 0 | 0), \quad j \geq 1, \tag{5.5}
\]

\[
r_0(x) = s(t + 1) \neq stp_0(x) = st. \tag{5.6}
\]

The jumps for \( st < 1 \) and \( |s| > |t| \) correspond exactly to \( |a| > 1 \) and \( |b| > 1. \) The values of these jumps can be negative. For example, in Theorem 5.2(ii), if \(-1 < s < 0 \) and \(-1 < t < 0, \) then

\[
(t + 1)(1 - st)/t(s + 1) < 0.
\]

This does not conflict with Favard's theorem, for if we took \( \hat{r}_0(x) = 1, \) and \( \hat{r}_j(x) = r_j(x)/st, j \geq 1, \) we would have

\[
2x\hat{r}_j(x) = \hat{r}_j(x) + (t + 1)/tt\hat{r}_0(x) \tag{5.7}
\]

\[
2x\hat{r}_j(x) = \hat{r}_{j+1}(x) + \hat{r}_{j-1}(x). \quad j \geq 2. \tag{5.8}
\]

So we need \( (t + 1)/t > 0, \) or \( t < -1, \ t > 0 \) for a positive measure. In the combinatorial situation \( s \) and \( t \) are positive integers and the jumps are positive.

It was mentioned in [1] that in the natural \( q \)-analog of Chebychev polynomials, the \( q \) totally disappears. Thus there was no natural \( q \)-analog. We offer the polynomials \( p_j \) of Theorem 5.1 with \( s = q \) as a possible \( q \)-analog. This is motivated by the double coset correspondence between Weyl...
groups and Chevalley groups. This correspondence has produced other $q$-analogs [10]. Another possible $q$-analog is given in the next section.

We were unable to use the dual orthogonality relations of Proposition 2.5(i) to find any nontrivial results. Ismail has pointed out that the results of this section can be obtained by techniques of Pollaczek and a theorem of Nevai. Geronimus [11] apparently was the first one to study these polynomials (without the jumps).

6. Another Association Scheme

We have another association scheme by considering points and lines simultaneously. Put $X = P \cup L$ and let $p$ be the distance. We must have $s = t$ in this case. Then

\[ \lambda p(0) = (s + 1) p(1), \]  
\[ \lambda p(k) = sp(k + 1) + p(k - 1), \quad k = 1, 2, \ldots, n - 1, \]

and

\[ \lambda p(n) = (s + 1) p(n - 1). \]

We state the results:

**Theorem 6.4.** For the association scheme of points and lines of a generalized $n$-gon $(s = t)$,

(i) the eigenfunctions are

\[ p(k, \lambda_0) = 1, \quad \lambda_0 = s + 1, \quad p(k, \lambda_n) = (-1)^n, \quad \lambda_n = -s - 1. \]

\[ p(k, \lambda_j) = s^{-k/3}[2T_k(x) + (s - 1) U_k(x)]/(s + 1), \]

where $\lambda_j = 2 \sqrt{s} x_j = 2 \sqrt{s} \cos \theta_j$, $\theta_j = \pi j/n$, for $j = 1, 2, \ldots, n - 1$.

(ii) the valencies are given by $v_j = (s + 1)s^{j - 1}, j = 1, 2, \ldots, n - 1, v_0 = 1, \quad v_n = s^{n-1}$,

(iii) the multiplicities are given by

\[ |P \cup L|/m_j = \frac{2n}{(s + 1)} \left[ 1 + \frac{(s - 1)^2}{4s \sin^2 \theta_j} \right], \quad j = 1, 2, \ldots, n - 1, \]

and $m_n = m_0 = 1$. 
Corollary 6.5. For any real number \( s > 0 \) let \( v_j(x) = 2T_j(x) + (s - 1) U_j(x) \). Then if \( j \neq 0 \) or \( l \neq 0 \):

(i) \( \int_{-1}^1 v_j(x) v_l(x) w(x) \, dx = \pi \delta_{jl}/2 \), if \( s > 1 \), where \( w(x) = (1 - x^2)/(s + 1)^2 - 4sx^2(1 - x^2)^{-1/2} \), \( \int_{-1}^1 w(x) \, dx = \pi/2s(s + 1) \), and

(ii) \( \int_{-1}^1 v_j(x) v_l(x) w(x) \, dx + \pi(1 - s)(v_j(x_1) v_l(x_1) + v_j(x_2) v_l(x_2))/4s(s + 1) = \pi \delta_{jl}/2 \), if \( s < 1 \), \( w(x) \) is given in (i), and \( x_1 = (s + 1)/2 \sqrt{s} = -x_2 \).

The polynomials \( v_j(x) \) are explicitly given in [1]. They are

\[ v_j(x) = s^2 C_j(x; 1/s \mid 0)/(s - 1). \]  

(See the other references given there.) Cartier [4] had previously given the orthogonality relation Corollary 6.5(i) by considering trees. These trees corresponded to the homogeneous space \( G/B \), for a group with \( (B, N) \) pair whose Weyl group is the infinite dihedral group. An \( \infty \)-gon is such a tree. Our \( n \to \infty \) case has recovered this case.

Our other \( q \)-analog of Chebychev polynomials is given by \( v_j(x) \) with \( s = q \). The trees considered by Cartier furnish a geometry for a \( q \)-analog of a circle. Instead of a \( q \)-analog of the rotations of a circle, we have the \( q \)-analog of \( D_\infty \), acting on an \( \infty \)-gon. This suggests considering \( q \)-analogs of affine Weyl groups for higher dimensional \( q \)-spheres. For example, the building associated with affine \( A_2 \) might give a \( q \)-analog of \( S^2 \). An approximating technique does not work here, however. There are many possible choices for \( q \)-ultraspherical polynomials [2]. Realizing one choice as spherical harmonics on a \( q \)-sphere would be very interesting.

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References

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