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Blocks of Lie Superalgebras of Type $W(n)$

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0. INTRODUCTION

Let $\mathfrak{g}$ be a simple, finite-dimensional Lie superalgebra over $\mathbb{C}$. These have been classified by V. Kac. Unless $\mathfrak{g}$ is a Lie algebra or a Lie superalgebra of type $\mathrm{osp}(1, 2n)$, the category of finite-dimensional representations of $\mathfrak{g}$ is not semisimple; vid. [10]. This leads to a classification problem. For example, in [5], the representation theory of $\mathrm{sl}(m, n)$ is worked out by showing it is wild when $m, n \geq 2$, and by giving an explicit description of the indecomposable finite-dimensional representations of $\mathrm{sl}(1, n)$.

When $\mathfrak{g}$ is of type $W(n)$, the irreducible finite-dimensional $\mathfrak{g}$-modules are classified in [1]; in this paper, we investigate finite-dimensional indecomposable modules. We show that the category of finite-dimensional representations of $\mathfrak{g}$ is wild (i.e., classifying objects is as hard as classifying pairs of matrices; vid. Sect. 3) if $\mathfrak{g}$ is of type $W(n)$, if $n \geq 3$. More precisely, the category of finite-dimensional representations decomposes into blocks parameterized by $(\mathbb{C}/\mathbb{Z}) \times \mathbb{Z}_2$, and we show that each block is of wild type. This is done by explicitly exhibiting enough extensions between simple modules.

Second, we find the decomposition of the category of finite-dimensional representations into blocks. As an application, using an idea of Maria Gorelik, we prove that the center of the universal enveloping algebra of $\mathfrak{g}$ is trivial.

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When $n = 2$, there is a special isomorphism $W(2) \cong \mathrm{sl}(1, 2)$, in which case the representation theory is not wild, and the indecomposable representations are fully described in [5].

The results in this paper are also related to results of Nakano [9] in the finite-characteristic case, for which he shows that the restricted universal enveloping algebra has a single block, and determines the structure of projective modules.

1. PRELIMINARIES

All vector spaces and algebras we consider will be over the ground field $\mathbb{C}$. If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space, we write $\dim V = \dim V_{\bar{0}} + \varepsilon \dim V_{\bar{1}} \in \mathbb{Z}[\varepsilon]/(\varepsilon^2 - 1)$. We denote by Π the parity-change functor, so $(\Pi V)_k = V_{k+\bar{1}}$ if $k \in \mathbb{Z}_2$.

For a Lie superalgebra $\mathfrak{g}$, denote by $U(\mathfrak{g})$ its universal enveloping algebra. We are interested in the category of graded finite-dimensional representations of Lie superalgebras, with even intertwiners.

Let us define the finite-dimensional Lie superalgebra $W(n)$, where $n \geq 2$ is an integer. Let $\bigwedge[\xi] = \bigwedge[\xi_1, \dots, \xi_n]$ be the Grassmann algebra on n generators $\xi_1, \dots, \xi_n$; it is a 2^n-dimensional associative algebra, and we give it a $\mathbb{Z}$-grading (and a compatible $\mathbb{Z}_2$-grading) by setting $\deg \xi_i = 1$ for $1 \leq i \leq n$. We set $W(n) = \operatorname{Der} \bigwedge[\xi]$, the set of (super)derivations of $\bigwedge[\xi]$. It is a simple Lie superalgebra of dimension $(1 + \varepsilon)n2^{n-1}$. It inherits a $\mathbb{Z}$-grading $W(n) = \bigoplus_{k=-1}^{n-1} W_k$ from $\bigwedge[\xi]$, where W_k consists of derivations that increase the degree of a homogeneous element by k.

Denote by $\partial/\partial\xi_i$ the derivation of $\bigwedge[\xi]$ determined by $\xi_j \mapsto \delta_{ij}$. If $X \in W(n)$, then $X = \sum_{i=1}^n X(\xi_i)(\partial/\partial\xi_i)$, so $W(n) = \{\sum_{i=1}^n P_i(\xi)(\partial/\partial\xi_i) \mid P_i(\xi) \in \bigwedge[\xi]\}$. We define the Euler vector field to be $Z = \sum_{i=1}^n \xi_i(\partial/\partial\xi_i)$. Note that $W_k = \{X \in W(n) \mid [Z, X] = kX\}$.

The component W_0 is isomorphic to $\mathrm{gl}(n)$; it acts as endomorphisms of the space $\mathrm{span}\{\xi_1, \dots, \xi_n\}$ of linear functions. Let us describe the structure of $W(n)$ as a $\mathrm{gl}(n)$-module: denote by std the standard representation of $\mathrm{gl}(n)$; then there is an isomorphism $W(n) \cong \bigwedge(\mathrm{std}) \otimes \mathrm{std}^*$, where $W_k \cong \bigwedge^{k+1}(\mathrm{std}) \otimes \mathrm{std}^*$.

In addition to the isomorphism $\mathrm{gl}(n) \cong W_0$, we will fix an isomorphism $\mathrm{sl}(1, n) \cong W_{-1} \oplus W_0 \oplus \mathrm{span}\{\xi_i Z \mid i = 1, \dots, n\}$, which is compatible with the usual $\mathbb{Z}$-grading on $\mathrm{sl}(1, n)$.

We fix the Cartan subalgebra $\mathfrak{h} \subseteq \mathrm{gl}(n)$ with basis $\{\xi_i(\partial/\partial\xi_i)\}$; it is also a Cartan subsuperalgebra of $W(n)$. Weights will be written with respect to the basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of $\mathfrak{h}^*$ dual to $\{\xi_i(\partial/\partial\xi_i)\}$; thus $\mathfrak{h}^* \cong \mathbb{C}^n$.

We fix the Borel subalgebra $\mathfrak{b}_0 \subseteq \mathrm{gl}(n)$, which has positive roots $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$. The corresponding set of highest weights of finite-dimensional irreducible representations of gl(n) (modulo parity if we consider graded representations) is

$$\Lambda^+ = \{(\lambda_1, \dots, \lambda_n) \mid \forall 1 \le i < j \le n \; \lambda_i - \lambda_j \in \mathbb{Z}_{\ge 0}\}.$$

Since n is fixed, from now on we will denote $W(n)$, sl$(1, n)$, and gl(n) by W, sl, and gl, respectively. We will use the notation $W_{\ge k} = \bigoplus_{j \ge k} W_j$ and $W_{\le k} = \bigoplus_{j \le k} W_j$.

Remark. Lie superalgebra cohomology is $\mathbb{Z}_2$-graded. However, we consider only *even* maps as intertwiners in the category of $\mathfrak{g}$-modules. For example, if $\mathfrak{g}$ is any Lie superalgebra and M and N are $\mathfrak{g}$-modules, then $\mathrm{Hom}_{\mathfrak{g}}(M, N) = H^0(\mathfrak{g}; \mathrm{Hom}_{\mathbb{C}}(M, N))$, but the space of even intertwiners $M \to N$ is $(\mathrm{Hom}_{\mathfrak{g}}(M, N))_{\bar{0}}$. Similarly, only the even part of the space $\mathrm{Ext}^1_{\mathfrak{g}}(M, N) = H^1(\mathfrak{g}; \mathrm{Hom}_{\mathbb{C}}(M, N))$ should be interpreted as short exact sequences of $\mathfrak{g}$-modules.

2. KAC MODULES

The irreducible finite-dimensional representations of W are determined explicitly in [1]. Let $L(\lambda)$ denote the simple, finite-dimensional gl-module with highest weight λ and even highest-weight vector.

DEFINITION. The Kac module $K(\lambda)$ is the induced representation

$$K(\lambda) = \mathrm{ind}_{W_{\ge 0}}^{W} L(\lambda) = U(W) \otimes_{U(W_{\ge 0})} L(\lambda)$$

of W, where $W_{\ge 1}$ acts trivially on $L(\lambda)$.

The module $K(\lambda)$ is finite-dimensional, indecomposable, and has highest weight λ with respect to the Borel subalgebra $\mathfrak{b}_0 \oplus W_{\ge 1}$ of W. It therefore has a unique simple quotient, which we will denote by $S(\lambda)$. Conversely, to an irreducible finite-dimensional W-module V, associate the gl-module $V^{W_{\ge 1}}$ of $W_{\ge 1}$-invariant vectors.

For a generic weight λ, the representation $K(\lambda)$ is irreducible. If $K(\lambda)$ is not irreducible, then it has length 2. Every simple finite-dimensional W-module is isomorphic to $S(\lambda)$ or to $\Pi S(\lambda)$ for a unique $\lambda \in \Lambda^+$.

The situation for sl is parallel to that for W. We denote by ${}^{\mathrm{sl}}K(\lambda) = \mathrm{ind}^{\mathrm{sl}}_{\mathrm{sl}_0 \oplus \mathrm{sl}_1} L(\lambda)$ the Kac modules for sl; note that ${}^{\mathrm{sl}}K(\lambda)$ is the restriction of $K(\lambda)$ to sl. We shall need the following facts (see [7, Theorem 1]). There is a condition called typicality such that

$$\lambda \text{ is typical } \iff {}^{\mathrm{sl}}K(\lambda) \text{ is irreducible.}$$

Moreover, λ is typical if and only if ${}^{\mathrm{sl}}K(\lambda)$ is projective and injective in the category of finite-dimensional gl-semisimple sl-modules, and the category of finite-dimensional gl-semisimple sl-modules with typical subquotients is semisimple. The set of atypical weights is contained in a finite union of hyperplanes. We note that if $\lambda \notin \mathbb{Z}^n$, then the weight λ is typical (so, in particular, ${}^{\mathrm{sl}}K(\lambda)$ and $K(\lambda)$ are irreducible if $\lambda \notin \mathbb{Z}^n$).

Finally, we introduce certain W-modules, the big Kac modules

$$K'(\lambda) = \mathrm{ind}_{W_{-1}\oplus W_0}^{W} L(\lambda) = \mathrm{ind}_{\mathrm{sl}}^{W} \mathrm{ind}_{W_{-1}\oplus W_0}^{\mathrm{sl}} L(\lambda).$$

The $K'(\lambda)$ are indecomposable modules with highest weight λ with respect to the Borel subsuperalgebra $W_{-1} \oplus \mathfrak{b}_0$ of W. Correspondingly, there are sl-modules defined by ${}^{\mathrm{sl}}K'(\lambda) = \mathrm{ind}_{\mathrm{sl}_{-1}\oplus \mathrm{sl}_0}^{\mathrm{sl}} L(\lambda)$.

3. QUIVERS AND REPRESENTATION TYPE

A quiver is a directed graph, which consists of a set of vertices connected by various arrows (possibly including multiple arrows between two vertices, loops, etc.). Let A be a unital $\mathbb{C}$-algebra, and denote by $\mathcal{M}$ some Abelian category of modules over A. Denote by $\mathrm{Irr}\,\mathcal{M}$ the set of isomorphism classes of irreducible objects. The Ext-quiver of $\mathcal{M}$ is defined to be the quiver whose set of vertices is $\mathrm{Irr}\,\mathcal{M}$ and where the number of arrows from $[S_1]$ to $[S_2]$ is equal to $\dim \mathrm{Ext}^1_{\mathcal{M}}(S_1, S_2)$. This is a combinatorial invariant of $\mathcal{M}$, whose structure gives information about the representation type. In particular, we have the following theorem, proven in [6]:

PROPOSITION 3.1. *Let A be an algebra, and let $\mathcal{A}$ be a (not necessarily full) Abelian subcategory of the category of modules over A. Let Q be a finite subquiver of the* Ext-*quiver of $\mathcal{A}$. If Q is a connected quiver containing no path of length* 2, *then there exists a fully faithful functor from the category of representations of Q to the category $\mathcal{A}$. In particular, the set of isomorphism classes of indecomposable representations of Q embeds into the corresponding set for $\mathcal{A}$.*

The representation theory of quivers is well established (see [4] for a comprehensive overview). In particular, if the underlying graph of a quiver is not of Dynkin or of affine type, then the representation theory of the quiver is wild. More precisely, a small $\mathbb{C}$-linear Abelian category $\mathcal{M}$ is defined to be wild if there exists a full exact embedding from the category of finite-dimensional representations of $\mathbb{C}\langle x, y\rangle$, the free associative algebra on two generators, into $\mathcal{M}$. This has the consequence that the objects of $\mathcal{M}$ are unclassifiable in any finite sense. For example, if $\mathcal{M}$ is wild, then it is possible to obtain *any* finite-dimensional algebra as the endomorphisms of some object.

4. EXTENSIONS

In this section, we show the existence of certain non-split extensions between Kac modules. These will be realized as quotients of big Kac modules.

LEMMA 4.1. *Suppose that* $\lambda \neq \mu$ *and either* λ *or* μ *is typical. Then*

$$\operatorname{Ext}^1_W(K(\lambda), K(\mu)) = H^1(W, \mathrm{sl}; \operatorname{Hom}_{\mathbb{C}}(K(\lambda), K(\mu))).$$

Proof. If M is a W-module and $H^1(\mathrm{sl}; M) = 0$, then $E_2^{1,0} = 0$ in the Serre–Hochschild spectral sequence (see [2, Theorem 1.5.1]) corresponding to the pair (W, sl), which implies that $H^1(W; M) = H^1(W, \mathrm{sl}; M)$. Apply this to the module $M = \operatorname{Hom}_{\mathbb{C}}(K(\lambda), K(\mu))$. If at least one of the weights λ or μ is typical, then we have $\operatorname{Ext}^1_{\mathrm{sl}}(K(\lambda), K(\mu)) = H^1(\mathrm{sl}; M) = 0$, because $K(\lambda)$ is projective or $K(\mu)$ is injective. ∎

Note that the following results are vacuous unless $n \geq 3$ (which we will assume, from now on). If $n \geq 3$, then the sl-module W/sl is nontrivial, in which case $W/\mathrm{sl} \cong L(\varepsilon_1 + \varepsilon_2 - \varepsilon_n) \oplus W_{\geq 2}$ as a gl-module.

THEOREM 4.2. *Let* $\lambda \in \Lambda^+ \backslash \mathbb{Z}^n$, *and let* α *be a weight of* $L(\varepsilon_1 + \varepsilon_2 - \varepsilon_n)$ *such that* $\lambda + \alpha \in \Lambda^+$. *Then*

$$\begin{aligned}\dim \operatorname{Ext}^1_W(K(\lambda), K(\lambda + \alpha)) &= [(W/\mathrm{sl}) \otimes {}^{\mathrm{sl}}K(\lambda) \colon {}^{\mathrm{sl}}K(\lambda + \alpha)] \\ &= [(W/\mathrm{sl}) \otimes L(\lambda) \colon L(\lambda + \alpha)].\end{aligned}$$

Proof. The condition on λ ensures that all the Kac modules involved are sl-typical, hence simple. Therefore, by Lemma 4.1, we have $\operatorname{Ext}^1_W(K(\lambda), K(\lambda + \alpha)) = H^1(W, \mathrm{sl}; \operatorname{Hom}_{\mathbb{C}}(K(\lambda), K(\lambda + \alpha)))$. This is equal to the cohomology of the complex

$$\begin{aligned}&\operatorname{Hom}_{\mathrm{sl}}(K(\lambda), K(\lambda + \alpha)) \to \operatorname{Hom}_{\mathrm{sl}}(W/\mathrm{sl} \otimes K(\lambda), K(\lambda + \alpha)) \\ &\qquad \xrightarrow{\delta} \operatorname{Hom}_{\mathrm{sl}}\left(\bigwedge^2 (W/\mathrm{sl}) \otimes K(\lambda), K(\lambda + \alpha)\right) \to \cdots.\end{aligned}$$

The first term is zero, so there are no coboundaries, and we claim that every $f \in \operatorname{Hom}_{\mathrm{sl}}(W/\mathrm{sl} \otimes K(\lambda), K(\lambda + \alpha))$ gives a cocycle. Indeed, $(\delta f)(x) = f(dx)$, where $d\colon \bigwedge^2 (W/\mathrm{sl}) \otimes K(\lambda) \to (W/\mathrm{sl}) \otimes K(\lambda)$, and we only need to verify that there are no nonzero sl-module maps $\bigwedge^2 (W/\mathrm{sl}) \otimes K(\lambda) \to K(\lambda + \alpha)$: define the height of a weight $(\lambda_1, \ldots, \lambda_n)$ to be $\sum_{i=1}^n \lambda_i$. Then the height of any ν such that $\bigwedge^2 (W/\mathrm{sl}) \otimes K(\lambda) = \bigoplus_\nu K(\nu)$ is greater than or equal to $\mathrm{ht}(\lambda) + 2$, while $\mathrm{ht}(\lambda + \alpha) = \mathrm{ht}(\lambda) + \mathrm{ht}(\alpha) = \mathrm{ht}(\lambda) + 1$. This weight calculation shows that $K(\lambda + \alpha)$ does not occur among the $K(\nu)$.

Since all the Kac modules are typical, we can calculate the decomposition of $(W/\mathrm{sl}) \otimes {}^{\mathrm{sl}}K(\lambda)$ into a direct sum of Kac modules as the decomposition of $(W/\mathrm{sl}) \otimes L(\lambda)$ into a direct sum of gl-modules. ∎

Remark. Let $\mu, \nu \in \Lambda^+$ be typical weights. The computation of Ext^1_W as in the proof of Theorem 4.2 shows that if $\mathrm{ht}(\nu) > \mathrm{ht}(\mu)$, then $\mathrm{Ext}^1_W(K(\nu), K(\mu)) = 0$. On the other hand, this fact may be proved directly using a simple highest-weight calculation: suppose that $\mathrm{ht}(\nu) > \mathrm{ht}(\mu)$ and that $\mathrm{Ext}^1_W(K(\nu), K(\mu)) \neq 0$, and let $0 \to K(\mu) \to E \to K(\nu) \to 0$ be a non-split short exact sequence. Take a highest-weight vector $v \in K(\nu)$, which lifts to some vector $v' \in E$ of weight ν. Since $K(\nu)$ is not a direct summand, the highest-weight vector v' generates all of E. Hence E would be a quotient of the (irreducible!) Kac module $\mathrm{ind}^W_{W_{\geq 0}} L(\nu)$, which is impossible.

We also remark again that the extensions of $K(\lambda)$ by $K(\lambda + \alpha)$ described in Theorem 4.2 are quotients of the big Kac module $K' = \mathrm{ind}^W_{\mathrm{sl}} K(\lambda) \cong K'(\lambda - 2\rho_1)$, where $\rho_1 = \frac{1}{2}\sum_{i=1}^n \varepsilon_i$; this may be seen directly from the structure of K'. We have $W = W_{-1} \oplus W_0 \oplus (S_1 \oplus T_1) \oplus W_{\geq 2}$, where $S_1 = \mathrm{Ker}(\mathrm{div}|_{W_1}) \cong L(\varepsilon_1 + \varepsilon_2 - \varepsilon_n)$ and $T_1 = \mathrm{sl} \cap W_1 = \mathrm{span}\{\xi_i Z \mid 1 \leq i \leq n\}$. Using these notations, as an sl-module, we have

$$K' \cong \mathrm{Sym}(S_1 \oplus W_{\geq 2}) \otimes K(\lambda) \cong K(\lambda) \oplus \bigoplus_{\mu} K(\mu) \oplus N,$$

where $L(\lambda) \otimes S_1 = \bigoplus_\mu L(\mu)$, and N is a direct sum of Kac modules of highest weights of height greater than $\mathrm{ht}(\lambda) + 1$. By the fact in the previous paragraph, we have $\mathrm{Ext}^1_W(K(\mu), K(\lambda)) = 0$, hence $\bigoplus_\mu K(\mu) \subseteq \mathrm{Rad}\, K'$, and $\mathrm{Ext}^1_W(N, K(\mu)) = 0$, so, in fact, $\bigoplus_\mu K(\mu) \subseteq \mathrm{Rad}\, K'/\mathrm{Rad}^2 K'$.

Note that in general, even for generic highest weights, there do exist non-split extensions of $K(\lambda)$ by $K(\mu)$ with $\mathrm{ht}(\mu) - \mathrm{ht}(\lambda) \geq 2$. For example, consider the case $n = 3$; then $W/\mathrm{sl} = S_1 \oplus W_2$, with $S_1 \cong L(1, 1, -1)$ and $W_2 \cong L(1, 1, 0)$ as gl(3)-modules. A calculation involving gl(3)-modules then shows that

$$\bigwedge^2(W/\mathrm{sl}) \cong L(2, 1, -1) \oplus L(2, 1, 0) \oplus L(2, 2, -1) \oplus L(2, 1, 1).$$

Now set $\lambda = (a, a, a)$ with $a \in \mathbb{C} \setminus \mathbb{Z}$. Then

$$W/\mathrm{sl} \otimes L(\lambda) \cong L(\lambda + (1, 1, -1)) \oplus L(\lambda + (1, 1, 0)).$$

Since $L(1, 1, 0)$ does not appear as a component of $\bigwedge^2(W/\mathrm{sl})$, the argument used in the proof of Theorem 4.2 shows that $\mathrm{Ext}^1_W(K(\lambda), K(\lambda + (1, 1, 0))) \cong \mathbb{C}$.

COROLLARY 4.3. *Let $\lambda \in \Lambda^+$ be any weight, and let α be as in the statement of Theorem* 4.2. *Then* $\dim \mathrm{Ext}^1_W(K(\lambda), K(\lambda + \alpha)) \geq [(W/\mathrm{sl}) \otimes L(\lambda) \colon L(\lambda + \alpha)]$.

Proof. Consider the cohomology $H^1(W; \mathrm{Hom}(K(\lambda + t\rho_1), K(\lambda + \alpha + t\rho_1)))$ as $t \in \mathbb{C}$ varies. The complex computing this cohomology is finite-dimensional, and shifting the weights by t does not change the dimension of the components. We can therefore view it as a complex with fixed terms with a differential that depends polynomially on t. By Theorem 4.2, $\dim H^1 = [(W/\mathrm{sl}) \otimes K(\lambda)\colon K(\lambda+\alpha)]$ for generic values of t. By semicontinuity, $\dim H^1$ can only increase at special values. ∎

Note that if $\mathrm{ht}(\alpha) = 1$, then $[(W/\mathrm{sl}) \otimes L(\lambda)\colon L(\lambda+\alpha)] = [L(\varepsilon_1 + \varepsilon_2 - \varepsilon_n) \otimes L(\lambda)\colon L(\lambda+\alpha)]$.

5. BLOCKS AND WILDNESS

A block of an Abelian category $\mathcal{M}$ is defined to be an indecomposable full Abelian subcategory that is a direct summand.

We shall need several simple facts about blocks. Given a subset $\Gamma \subseteq \mathrm{Irr}\,\mathcal{M}$, we denote by $\mathcal{M}(\Gamma)$ the full subcategory of $\mathcal{M}$ consisting of objects all of whose simple subquotients are in Γ. If all objects of $\mathcal{M}$ have finite length, then it is easy to check that $\mathcal{M}$ decomposes into a direct sum of blocks, and that $\Gamma \subseteq \mathrm{Irr}\,\mathcal{M}$ is the set of vertices of a connected component of the Ext-quiver of $\mathcal{M}$ if and only if $\Gamma = \mathrm{Irr}\,\mathcal{B}$ for some block $\mathcal{B}$. All irreducible subquotients of an indecomposable object belong to the same block, and, if there exists a non-split extension $0 \to N \to E \to M \to 0$ where M and N are indecomposable, then all simple subquotients of E belong to the same block (in fact, there exists a non-split extension of some simple subquotient of M by some simple subquotient of N).

We need the following facts about weight modules over Lie superalgebras. Here $\mathfrak{g}$ is a Lie superalgebra and $\mathfrak{h}$ is a fixed Cartan subsuperalgebra; we assume for simplicity that $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$. For a $\mathfrak{g}$-module M, we denote $M^{(\alpha)} = \{v \in M \mid \forall H \in \mathfrak{h}\ \exists n \in \mathbb{Z}_+\ (H - \alpha(H))^n v = 0\}$. The module M is called a generalized weight module if $M = \bigoplus_\alpha M^{(\alpha)}$ (for instance, this is true whenever M is finite-dimensional).

LEMMA 5.1. *Let $\mathfrak{g}$ be a Lie superalgebra (with a fixed Cartan subsuperalgebra $\mathfrak{h}$), and let M be a generalized weight module. Then $M = \bigoplus_{t \in \mathfrak{h}^*/Q} M(t)$ as a $\mathfrak{g}$-module, where Q is the root lattice and $M(t) = \bigoplus\{M^{(\lambda)} \mid \lambda \in t\} \subseteq M$.*

Proof. Let M be a generalized weight module. We have the following simple fact: if $\lambda, \mu \in \mathfrak{h}^*$, then $U(\mathfrak{g})^{(\lambda)} M^{(\mu)} \subseteq M^{(\lambda+\mu)}$. An immediate consequence is that the $M(t)$, defined above, are submodules. ∎

LEMMA 5.2. *Let $\mathfrak{g}$ be as in the previous lemma, and assume that there exists a linear function $p\colon Q \to \mathbb{Z}_2$ such that, for every $\alpha \in Q$ and $0 \neq X_\alpha \in \mathfrak{g}^{(\alpha)}$, the element X_α is homogeneous of parity $p(\alpha)$. (N.b.: this should not be confused with the notation $p\colon V_{\bar{0}} \amalg V_{\bar{1}} \to \mathbb{Z}_2$ for a super-vector-space V.) Let $\mathcal{M}_t$ be the category of $\mathfrak{g}$-modules which are generalized weight modules with support contained in $t = \lambda + Q$. Then there exists a decomposition $\mathcal{M}_t = \mathcal{M}'_t \oplus \mathcal{M}''_t$, such that $M \in \mathrm{Ob}\,\mathcal{M}'_t$ if and only if $\Pi M \in \mathrm{Ob}\,\mathcal{M}''_t$.*

Proof. Let us define, for any $M \in \mathrm{Ob}(\mathcal{M}_t)$, an even map $\sigma = \sigma_M \in \mathrm{End}_{\mathfrak{g}}(M)$ such that $\sigma_M^2 = \mathrm{Id}_M$. This will give a decomposition $M = M' \oplus M''$, where $M' = \mathrm{Ker}(\sigma_M - \mathrm{Id}_M)$ and $M'' = \mathrm{Ker}(\sigma_M + \mathrm{Id}_M)$. We will verify the parity condition.

Define a shifted function $p\colon t \to \mathbb{Z}_2$ by fixing arbitrarily $\varepsilon \in \mathbb{Z}_2$ and setting $p(\lambda + \alpha) = \varepsilon + p(\alpha)$ for $\alpha \in Q$. Consider the linear map $\sigma\colon M \to M$, uniquely defined by requiring that, if $v \in M_{p(v)}^{(\mu)}$, then $\sigma(v) = (-1)^{p(\mu)-p(v)}v$. Note that, if $X_\alpha \in \mathfrak{g}^{(\alpha)}$, then $X_\alpha v \in M_{p(v)+p(\alpha)}^{(\mu+\alpha)}$, so $\sigma(X_\alpha v) = (-1)^{p(\mu)-p(v)} \times X_\alpha v = X_\alpha \sigma(v)$. Therefore, σ commutes with the action of $\mathfrak{g}$, and M breaks up into a direct sum of two submodules, as claimed.

Finally, since ΠM has the same generalized weight spaces as M, but the parity of elements is reversed, it is clear that $\sigma_{\Pi M} = -\sigma_M$, which verifies the last claim. ∎

Now, let $\mathcal{M}$ denote the category of all finite-dimensional representations of W. All objects of $\mathcal{M}$ are generalized weight modules. The set $\mathrm{Irr}\,\mathcal{M}$ is in bijection with $\Lambda^+ \times \mathbb{Z}_2$; representatives of isomorphism classes are the modules $S(\lambda)$ and $\Pi S(\lambda)$, which were defined in Section 2. Note that if $\lambda \in \Lambda^+$, then $\lambda \in 2\lambda_1\rho_1 + Q$, so the highest weights of simple finite-dimensional W-modules belong to cosets parameterized by $\mathbb{C}/\mathbb{Z}$.

For each coset $t = \lambda + Q \in \mathfrak{h}/Q$, let us fix a choice of $p\colon t \to \mathbb{Z}_2$ as in the proof of Lemma 5.2, starting from $p(\alpha) = \mathrm{ht}(\alpha)$ for $\alpha \in Q$. Given $(x, d) \in \mathbb{C}/\mathbb{Z} \times \mathbb{Z}_2$, let us define

$$\Gamma_{(x,\,d)} = \{\Pi^{p(\lambda)+d} S(\lambda) \in \mathrm{Irr}\,\mathcal{M} \mid \lambda \in 2x\rho_1 + Q\}.$$

Lemmas 5.1 and 5.2 together imply

PROPOSITION 5.3.

$$\mathcal{M} = \bigoplus_{(x,d)\in\mathbb{C}/\mathbb{Z}\times\mathbb{Z}_2} \mathcal{M}(\Gamma_{(x,\,d)}).$$

The categories $\mathcal{M}(\Gamma_{(x,\,\bar{0})})$ and $\mathcal{M}(\Gamma_{(x,\,\bar{1})})$ are equivalent via parity-reversal.

THEOREM 5.4. *The decomposition*

$$\mathcal{M} = \bigoplus_{(x,\,d)\in(\mathbb{C}/\mathbb{Z})\times\mathbb{Z}_2} \mathcal{M}(\Gamma_{(x,\,d)})$$

is the block decomposition of $\mathcal{M}$; i.e., the categories $\mathcal{M}(\Gamma_{(x,\,d)})$ are indecomposable.

Proof. We define a relation $\rightsquigarrow$ on the set of highest weights, such that $\lambda \rightsquigarrow \mu$ implies that $S(\lambda)$ and $\Pi^{p(\mu-\lambda)}S(\mu)$ are in the same $\mathcal{M}$-block. Finally, we show that if $S(\lambda), \Pi^{p(\mu-\lambda)}S(\mu) \in \Gamma_{(x,\,p(\lambda))}$, i.e., if $\lambda_1 \equiv \mu_1 (\mathrm{mod}\,\mathbb{Z})$, then we can get from λ to μ with a finite number of intermediate arrows $\rightsquigarrow$ or $\leftsquigarrow$.

If $x \notin \mathbb{Z}$, then all Kac modules in $\mathscr{M}(\Gamma_{(x,d)})$ are simple. In that case, by Theorem 4.2, there exists a nonsplit extension of $S(\lambda)$ by $\Pi S(\lambda+\alpha)$, where $\alpha = \varepsilon_i + \varepsilon_j - \varepsilon_k$, if and only if $L(\lambda+\alpha)$ occurs in the decomposition of $L(\varepsilon_1+\varepsilon_2-\varepsilon_n)\otimes L(\lambda)$ into a direct sum of gl-modules. We therefore define the relation $\lambda \rightsquigarrow \mu$ on Λ^+ so that $\lambda \rightsquigarrow \mu$ if and only if $[L(\varepsilon_1+\varepsilon_2-\varepsilon_n)\otimes L(\lambda) : L(\mu)] > 0$.

LEMMA 5.5. (1) *If* $\lambda+\nu \in \Lambda^+$ *for each weight* ν *of* $L(\varepsilon_1+\varepsilon_2-\varepsilon_n)$, *then* $\lambda \rightsquigarrow \lambda+\alpha$;

(2) *if* $\alpha = \varepsilon_i+\varepsilon_j-\varepsilon_k$ *with* i, j, *and* k *distinct*, *and if* $\lambda+\alpha\in\Lambda^+$, *then* $\lambda \rightsquigarrow \lambda+\alpha$.

Proof. Claim (1) is standard, and let us prove (2) using the Littlewood–Richardson rule (see [3, pp. 455–456]). First, we have $\lambda+\varepsilon_i+\varepsilon_j-\varepsilon_k\in\Lambda^+$ if and only if, starting from the Young diagram for λ, we obtain another Young diagram when we add one box to every row except rows i and j, to which we add two boxes, and row k, to which we add zero boxes. To apply the Littlewood–Richardson rule, we need to show that this expansion is "μ-strict" for $\mu = (2,1,\ldots,1,0)$. Assume that $i<j$; then we can fill in the numbers by putting a 1 in the leftmost box added to row i, a 2 in the leftmost box added to row j, and numbering the remaining boxes in order, starting from the top. ∎

End of the proof of Theorem 5.4. Note that all weights ν of $L(\varepsilon_1+\varepsilon_2-\varepsilon_n)$ are of the form $\nu = \varepsilon_i+\varepsilon_j-\varepsilon_k$, with $i\neq j$, so, if $\mu=\sum_i \mu_i\varepsilon_i\in\Lambda^+$ is some weight with $\mu_i-\mu_{i+1}\geq 2$ for $1\leq i<n$, then $\mu+\nu\in\Lambda^+$ for any such ν.

Next, let $\leftrightsquigarrow$ be the closure of $\rightsquigarrow$ to an equivalence relation on Λ^+. Let $\lambda\in\Lambda^+$ be any weight; we check that $\lambda \leftrightsquigarrow \lambda+\varepsilon_i$ whenever $\lambda+\varepsilon_i\in\Lambda^+$, by using Lemma 5.5 to check that there exists a diagram

$$\begin{array}{ccc} \lambda \rightsquigarrow \nu^{(1)} \rightsquigarrow \cdots \rightsquigarrow & & \nu^{(r)} \\ & & \wr\downarrow \\ \lambda+\varepsilon_i \rightsquigarrow \nu^{(1)}+\varepsilon_i \rightsquigarrow \cdots \rightsquigarrow & & \nu^{(r)}+\varepsilon_i \end{array}$$

for some weights $\nu^{(1)},\ldots,\nu^{(r)}$, which shall be chosen depending on λ.

Given the weight $\nu^{(0)}=\lambda$, choose, using Lemma 5.5, claim (2), a sequence $\nu^{(1)},\ldots,\nu^{(r)}$ with the property that $\nu^{(j)}\rightsquigarrow\nu^{(j+1)}$ and $\nu^{(j)}+\varepsilon_i \rightsquigarrow \nu^{(j+1)}+\varepsilon_i$, such that $\nu^{(r)}$ is sufficiently dominant, that is, satisfies the conditions in Lemma 5.5, claim (1). By the remark after Lemma 5.5, it will suffice to ensure that $\nu_i^{(r)}-\nu_{i+1}^{(r)}\geq 2$. For instance, let $a_k = 2(n-k-1)$ for $1\leq k\leq n-2$, and set $\nu^{(j)}=\nu^{(j-1)}+\alpha_j$, where $\alpha_j=\varepsilon_1+\varepsilon_2-\varepsilon_{n-k+1}$ for

$a_1 + \cdots + a_{k-1} < j \le a_1 + \cdots + a_k$, $1 \le k \le n-2$. Let $r = \sum_{k=1}^{n-2} a_k + 1 = n^2 - 3n + 3$; the final weight chosen will be

$$\begin{aligned}\nu^{(r)} &= \nu^{(r-1)} + \varepsilon_1 - \varepsilon_2 + \varepsilon_3 \\ &= (\lambda_1 + r)\varepsilon_1 + (\lambda_2 + r - 2)\varepsilon_2 + (\lambda_3 - 1)\varepsilon_3 + \sum_{k=4}^{n} (\lambda_k - a_{n-k+1})\varepsilon_k.\end{aligned}$$

Then each $\nu^{(j)} - \lambda \in \Lambda^+$, and the $\nu^{(j)}$ satisfy the required properties. This gives $\lambda \leftrightsquigarrow \nu^{(r)}$ and $\lambda + \varepsilon_i \leftrightsquigarrow \nu^{(r)} + \varepsilon_i$; since $\nu^{(r)}$ is sufficiently dominant, we have $\nu^{(r)} \rightsquigarrow \nu^{(r)} + \varepsilon_i$ and hence $\lambda \leftrightsquigarrow \lambda + \varepsilon_i$.

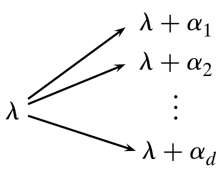
Now remove the condition that $x \notin \mathbb{Z}$. Corollary 4.3 still ensures that, if $\lambda \rightsquigarrow \lambda + \alpha$, then there exists a non-split extension of $K(\lambda)$ by $\Pi K(\lambda + \alpha)$. Since $K(\lambda)$ and $K(\lambda + \alpha)$ are indecomposable, and both $S(\lambda)$ and $\Pi S(\lambda + \alpha)$ are subquotients of such a non-split extension, it follows that $S(\lambda)$ and $\Pi S(\lambda + \alpha)$ are in the same block.

Finally, it is clear that the closure of $\rightsquigarrow$ to an equivalence relation on Λ^+ has equivalence classes which are exactly the cosets $(\lambda + Q) \cap \Lambda^+$. ∎

THEOREM 5.6. *Each block $\mathscr{M}(\Gamma_{(x,d)})$ is wild.*

Proof. Let $\lambda \in \Lambda^+$ be a weight such that $K(\lambda) \in \mathrm{Ob}(\mathscr{M}(\Gamma_{(x,d)}))$, $K(\lambda)$ is simple, $\lambda + \alpha \in \Lambda^+$ for every weight α of $L(\varepsilon_1 + \varepsilon_2 - \varepsilon_n)$, and all the $K(\lambda + \alpha)$ are also simple. For example, any sufficiently dominant weight, say, $\lambda_i \ge \lambda_{i+1} + 37$, with $\lambda_i \equiv \bar{t} \pmod{\mathbb{Z}}$, will do. Then, by Corollary 4.3, there exists a nontrivial extension of $K(\lambda) = S(\lambda)$ by $\Pi K(\lambda + \alpha) = \Pi S \times (\lambda + \alpha)$.

Therefore, the Ext-quiver of each block contains a subquiver consisting of a vertex λ with arrows from it to $\lambda + \alpha$ for each root α of $(W/\mathrm{sl})_1$. Not counting multiplicities, there are $3\binom{n}{3} + n$ such roots, namely, $\varepsilon_i + \varepsilon_j - \varepsilon_k$ with $1 \le i, j, k \le n$ and $i \ne j$. Since $n \ge 3$, we always have $d = 3\binom{n}{3} + n > 5$, and the resulting quiver is already wild (vid. Proposition 3.1):



∎

The argument in Theorem 4.3 may probably be refined to calculate *all* extensions between two (generic) Kac modules. Moreover, we would like to find a simple proof of

CONJECTURE 5.7. *All blocks not containing the trivial representation are equivalent.*

This is supported by the fact that the proof of Theorem 4.2 and Corollary 4.3 shows, at least, that the dimension of Ext^1 between two simple modules does not change when we change the block by shifting weights by generic multiples of ρ_1.

6. THE CENTER OF $U(W(n))$

We have the generalized triangular decomposition $W = W_{-1} \oplus W_0 \oplus W_{\geq 1}$, and $W_{\leq 0}$ is simultaneously a parabolic subsuperalgebra of both $\mathrm{sl}(1, n)$ and $W(n)$. The sl-module (resp. W-module) parabolically induced from a simple gl-highest-weight module $L(\lambda)$ is still denoted by ${}^{\mathrm{sl}}K'(\lambda)$ (respectively, $K'(\lambda)$).

Induction by stages gives $K'(\lambda) = U(W) \otimes_{U(\mathrm{sl})} {}^{\mathrm{sl}}K'(\lambda)$. It is easy to check that *any* simple W-module has a nonzero space of W_{-1}-invariants, so we can obtain any simple highest-weight module of W (resp. sl) as a quotient of $K'(\lambda)$ (resp. ${}^{\mathrm{sl}}K'(\lambda)$). Let ${}^{\mathrm{sl}}L(\lambda)$ be the unique irreducible quotient of ${}^{\mathrm{sl}}K'(\lambda)$.

The following result is proved in [8, Sect. 3]:

LEMMA 6.1. *Let $S \subseteq \mathfrak{h}^*$. Then, if and only if S is Zariski dense in $\mathfrak{h}^*$, we have*

$$\bigcap_{\mu \in S} \mathrm{Ann}_{U(\mathrm{sl})} {}^{\mathrm{sl}}L(\mu) = 0.$$

PROPOSITION 6.2. *For any Zariski dense subset $S \subseteq \mathfrak{h}^*$, one has*

$$\bigcap_{\mu \in S} \mathrm{Ann}_{U(W)} K'(\mu) = 0.$$

Proof. Suppose S is Zariski dense. First of all, $\mathrm{Ann}\, {}^{\mathrm{sl}}K'(\mu) \subseteq \mathrm{Ann}\, {}^{\mathrm{sl}}L(\mu)$ and, therefore, $\bigcap_\mu \mathrm{Ann}_{U(\mathrm{sl})} {}^{\mathrm{sl}}K'(\mu) = 0$ by Lemma 6.1. Next, we use the fact that $K'(\lambda)$ is induced from ${}^{\mathrm{sl}}K'(\lambda)$ to push this up to W. Applying the Poincaré–Birkhoff–Witt Theorem, produce a basis $\{X_i\}$ of $U(W)$ over $U(\mathrm{sl})$, so that, as spaces, $K'(\mu) \cong (\bigoplus_i \mathbb{C}X_i) \otimes_{\mathbb{C}} {}^{\mathrm{sl}}K'(\mu)$. Write $u \in \mathrm{Ann}_{U(W)} K'(\mu)$ as $u = \sum_i X_i Y_i$, with the $Y_i \in U(\mathrm{sl})$. Applying u to $v \in 1 \otimes {}^{\mathrm{sl}}K'(\mu) \subseteq K'(\mu)$ gives

$$\sum_i X_i Y_i v = 0,$$

and, since the X_i are linearly independent, each $Y_i v = 0$ and, therefore, $Y_i \in \mathrm{Ann}_{U(\mathrm{sl})} {}^{\mathrm{sl}}K'(\mu)$. This shows that $\mathrm{Ann}_{U(W)} K(\mu) \subseteq U(W)\mathrm{Ann}_{U(\mathrm{sl})} {}^{\mathrm{sl}}K'(\mu)$ and that

$$\bigcap_\mu \mathrm{Ann}_{U(W)} K'(\mu) \subseteq U(W) \bigcap_\mu \mathrm{Ann}_{U(\mathrm{sl})} {}^{\mathrm{sl}}K'(\mu) = 0.$$

∎

PROPOSITION 6.3. *We have $Z(W) = \mathbb{C}$, where $Z(W)$ denotes the supercenter of $U(W)$.*

This is an immediate corollary of Proposition 6.2 and

LEMMA 6.4. *If $u \in Z(W)$, then there exists a scalar $c \in \mathbb{C}$ such that $u - c \in \bigcap_{\mu\in\Gamma} \operatorname{Ann} K'(\mu)$, where $\Gamma \subseteq \mathfrak{h}^*$ is Zariski dense.*

Proof. For any $\mu \in \mathfrak{h}^*$, we have $\operatorname{End}_W K'(\mu) = \mathbb{C}$; therefore, c acts by some scalar c_μ on $K'(\mu)$. Therefore, if $\mu \in \Lambda^+$, then c also acts by multiplication by c_μ on the finite-dimensional quotient $S'(\mu)$ of $K'(\mu)$. Fix Γ to be the set of highest weights of irreducible modules in any block of the category of finite-dimensional representations of W; Theorem 5.4 implies that Γ is a Zariski-dense subset of $\mathfrak{h}^*$. Then if $\mu, \mu' \in \Gamma$, then $S'(\mu)$ and $S'(\mu')$ are in the same block, so $c_\mu = c_{\mu'}$. ∎

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