Bounds on the average connectivity of a graph

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Received 4 May 1999; received in revised form 27 June 2002; accepted 5 August 2002

Abstract

In this paper, we consider the concept of the average connectivity of a graph, defined to be the average, over all pairs of vertices, of the maximum number of internally disjoint paths connecting these vertices. We establish sharp bounds for this parameter in terms of the average degree and improve one of these bounds for bipartite graphs with perfect matchings. Sharp upper bounds for planar and outerplanar graphs and cartesian products of graphs are established. Nordhaus–Gaddum-type results for this parameter and relationships between the clique number and chromatic number of a graph are also established.

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Keywords: Average connectivity; Average degree; Planar graphs; Outerplanar graphs; Cartesian products; Nordhaus–Gaddum bounds; Chromatic number; Clique number

1. Introduction

The best known measure of reliability of a graph is its \textit{connectivity}, defined to be the minimum number of vertices whose deletion results in a disconnected or trivial graph (the latter applying only to complete graphs). As the connectivity is a worst-case measure, it does not always reflect what happens throughout the graph. For example, a tree and the graph obtained by appending an end-vertex to a complete graph both have connectivity 1. Nevertheless, for large order the latter graph is far more reliable than the former. Interest in the vulnerability and reliability of networks such as transportation and communication networks, has given rise to a host of other measures of reliability see for example [1]. In this paper we investigate a new measure for the reliability of graph...
a graph, the *average connectivity*, recently introduced by Beineke et al. [2]. Whereas other global measures of reliability, such as the toughness and integrity of a graph, are NP-hard, the average connectivity can be computed in polynomial time, making it much more attractive for applications.

Let $G = (V, E)$ be a graph of order $p$ and let $u$ and $v$ be two distinct vertices of $G$. The *connectivity* between $u$ and $v$, $\kappa(u, v)$ or $\kappa_G(u, v)$, is the maximum number of pairwise internally disjoint $u-v$ paths in $G$. The average connectivity $\bar{\kappa}(G)$ is defined as the average of the connectivities between all pairs of vertices of $G$, that is,

$$\bar{\kappa}(G) = \left(\frac{p}{2}\right)^{-1} \sum_{\{u, v\} \subseteq V} \kappa(u, v).$$

While the (ordinary) connectivity is the minimum number of vertices whose removal separates at least one connected pair of vertices, the average connectivity is a measure for the expected number of vertices that have to be removed to separate a randomly chosen pair of vertices.

In order to avoid fractions, we shall frequently work with the *total connectivity* $K(G)$ of $G$, defined by

$$K(G) = \sum_{\{u, v\} \subseteq V} \kappa(u, v).$$

From these definitions we have $K(G) = (\frac{p^2}{2})\bar{\kappa}(G)$. The notation we use in this paper is as follows. We consider only simple undirected graphs with no multiple edges. For a graph $G$ we denote the vertex set, edge set, order, and size by $V(G)$, $E(G)$, $p(G)$ and $q(G)$, respectively. The degree of a vertex $v$ in $G$ is denoted by $\deg_G v$. We use $\delta(G)$ for the minimum degree of $G$. The average degree of $G$, $\bar{d}(G)$, is defined as $(\sum_{v \in V} \deg_G v)/p$. If the graph $G$ is understood, we drop the subscript (or argument) $G$. For other notions and notation we refer the reader to the textbook [3].

### 2. Average connectivity and average degree

We first present upper and lower bounds on the average connectivity of a graph in terms of its order and average degree. We will use the following result which was established in [2].

**Theorem 1.** Let $G$ be a graph with $p$ vertices and $q \geq p$ edges. Then

$$\bar{\kappa}(G) \leq \frac{2q}{p} - \frac{r(p-r)}{p(p-1)},$$

where $r = 2q - p \lfloor 2q/p \rfloor$. Moreover, equality holds if and only if for every two vertices $u$ and $v$ of $G$,

1. $|\deg_G u - \deg_G v| \leq 1$ and
2. $\bar{\kappa}_G(u, v) = \min\{\deg_G u, \deg_G v\}$. 


Clearly, the connectivity of a graph does not exceed its minimum degree. As the average degree of a graph $G$ is given by $2|E(G)|/|V(G)|$, Theorem 1 shows that the average degree is an upper bound for the average connectivity of a graph.

In order to obtain a lower bound for the average connectivity in terms of the average degree, let $G$ be a graph and $u$ and $v$ vertices of $G$. For $i = 1, 2$ let $\kappa_i(u, v)$ denote the number of $u - v$ paths of length $i$ in $G$. Moreover, let $\kappa_3(u, v)$ denote the number of induced $u - v$ paths of length 3 in $G$ or in $G - uv$ if $uv \in E(G)$.

It is easy to see that for any two distinct vertices $u, v$ of $G$ the inequality $\kappa(u, v) \geq \kappa_1(u, v) + \kappa_2(u, v) + \kappa_3(u, v)$ holds. This fact will be used to obtain lower bounds for $\kappa$. 

**Theorem 2.** Let $G$ be a graph of order $p$ with average degree $\bar{d}$. Then

$$\frac{\bar{d}^2}{p - 1} \leq \kappa(G) \leq \bar{d},$$

and both bounds are sharp for positive integer values of $\bar{d}$.

**Proof.** Since $G$ has average degree $\bar{d}$,

$$\sum_{\{u,v\} \subset V} \kappa_1(u, v) = q(G) = \frac{1}{2} p \bar{d}.$$

Since every vertex $w$ of $G$ is the centre vertex of

$$\left(\frac{\text{deg } w}{2}\right)$$

paths of length 2, we have

$$\sum_{\{u,v\} \subset V} \kappa_2(u, v) = \sum_{w \in V} \left(\frac{\text{deg } w}{2}\right).$$

By the convexity of the function $f(x) = x(x - 1)/2$, the last sum is bounded below by $p\bar{d}(\bar{d} - 1)/2$. Hence

$$K(G) \geq \sum_{\{u,v\} \subset V} (\kappa_1(u, v) + \kappa_2(u, v)) \geq \frac{1}{2} p \bar{d} + p \frac{\bar{d}(\bar{d} - 1)}{2} = \frac{\bar{d}^2 p}{2}.$$

Division by $\binom{p}{2}$ yields the lower bound. As mentioned prior to this theorem, the upper bound is an immediate consequence of Theorem 1. The lower bound in Theorem 2 is sharp. To see this, let $\bar{d}$ be an integer and $p$ a multiple of $\bar{d} + 1$. Then the lower bound is attained by the graph $G = [p/(\bar{d} + 1)]K_{\bar{d} + 1}$. The upper bound is achieved by any $\bar{d}$-regular graph which is also $\bar{d}$-connected. \(\Box\)
Corollary 3. Let $G$ be a graph of order $p$ and size $q$. Then
\[ \kappa(G) \geq \frac{4q^2}{p^2(p-1)}. \]

For bipartite graphs we can improve the lower bound of Theorem 2, by a factor of about 2, provided the graph contains a perfect matching.

Theorem 4. Let $G$ be a bipartite graph of order $p$ with average degree $\bar{d}$. If $G$ has a perfect matching, then
\[ \kappa(G) \geq \frac{2\bar{d}^2 - \bar{d}}{p-1}. \]

Proof. Let $p = 2n$. Let $M = \{u_i,v_i \mid 1 \leq i \leq n\}$ be a perfect matching of $G$ such that $\{u_1,u_2,\ldots,u_n\}$ and $\{v_1,v_2,\ldots,v_n\}$ are the partite sets of $G$. Let $d_i$ and $e_i$ denote the degrees of $u_i$ and $v_i$, respectively. As in the proof of Theorem 2, we have
\[ \sum_{\{u,v\} \subset V} (\kappa_1(u,v) + \kappa_2(u,v)) = \frac{pd\bar{d}}{2} + \sum_{i=1}^{n} \left( \binom{d_i}{2} + \binom{e_i}{2} \right) + (q-p/2). \]
Since every $u_i,v_i \in M$ is the middle edge of $(d_i-1)(e_i-1)$ paths of length 3, and since for each edge $u_i,v_j \notin M$, we have an additional path $v_i,u_i,v_j,u_j$ of length 3, we have
\[ \sum_{\{u,v\} \subset V} \kappa_3(u,v) \geq \sum_{i=1}^{n} ((d_i-1)(e_i-1)) + (q-p/2). \]

Hence
\[ K(G) \geq \sum_{\{u,v\} \subset V} (\kappa_1(u,v) + \kappa_2(u,v) + \kappa_3(u,v)) \]
\[ \geq \frac{pd\bar{d}}{2} + \sum_{i=1}^{n} \left( \binom{d_i}{2} + \binom{e_i}{2} + (d_i-1)(e_i-1) \right) + (q-p/2) \]
\[ = \frac{pd\bar{d}}{2} + \frac{1}{2} \sum_{i=1}^{n} (d_i + e_i)^2 - \frac{3}{2} \sum_{i=1}^{n} (d_i + e_i) + q. \]

The last sum equals $2q = pd\bar{d}$ and the middle sum is minimized if the numbers $d_i + e_i$ have the same value for all $i$. Hence
\[ K(G) \geq \frac{pd\bar{d}}{2} + \frac{p}{4}(2\bar{d})^2 - \frac{3}{2}pd\bar{d} + \frac{p\bar{d}}{2} \]
\[ = \frac{p}{2}(2\bar{d}^2 - \bar{d}). \]
Division by $\binom{p}{2}$ yields the theorem. \[ \square \]

Since every nonempty regular bipartite graph has a perfect matching, we obtain the following corollary.
Corollary 5. If \( G \) is a \( \delta \)-regular bipartite graph of order \( p \), then \( \kappa(G) \geq 2(\delta^2 - \delta)/(p - 1) \).

Let \( \hat{d} \) be an integer and \( p \) a multiple of \( 2\hat{d} \). Then the above bounds are sharp as shown by the graph \( G = (p/(2\hat{d}))K_{\hat{d},\hat{d}} \).

3. Average connectivity of planar and outerplanar graphs

A graph is planar if it can be drawn in the plane without its edges crossing. A planar graph is outerplanar if it can be drawn in the plane so that all its vertices are on the boundary of the exterior region. When referring to a planar graph we will assume that we have associated with it an embedding in the plane so that no two of its edges cross. Similarly, when referring to an outerplanar graph we will assume that we have associated with it an embedding in the plane so that all vertices lie on the boundary of the exterior region of this embedding.

It is a well-known fact that if \( G \) is a planar graph on \( p \) vertices and \( q \) edges, then \( q \leq 3p - 6 \). Moreover, if \( G \) is a planar graph on \( p \) vertices, then \( q = 3p - 6 \) if and only if \( G \) is a maximal planar graph. In this case the boundary of every region is a triangle. We now determine sharp upper and lower bounds on the average connectivity of both planar and outerplanar graphs.

Theorem 6. If \( G \) is a maximal planar graph on \( p \geq 3 \) vertices, then

\[
\bar{\kappa}(G) \leq 6 + \frac{156 - 24p}{p(p - 1)}.
\]

Moreover, this bound is sharp for all \( p \geq 14 \) and \( p \equiv 2 \pmod{6} \).

Proof. Let \( G \) be a maximal planar graph on \( p \) vertices. Since \( q = 3p - 6 \), it follows, from Theorem 1, that

\[
\bar{\kappa}(G) \leq \frac{6p - 12}{p} + \frac{(p - 12)(-12)}{p(p - 1)} = 6 + \frac{156 - 24p}{p(p - 1)}.
\]

To show that this bound is sharp, form \( G_k \) from \( k \) disjoint 6-cycles \( C_1, C_2, \ldots, C_k \), where \( C_i : v_{i,1}, v_{i,2}, \ldots, v_{i,6}, v_{i,1} \), by adding the edges \( v_{i,j}v_{i+1,j} \), for \( 1 \leq i \leq k - 1 \) and \( 1 \leq j \leq 6 \), as well as the edges \( v_{i,j}v_{i+1,j-1} \), for \( 1 \leq i \leq k - 1 \) and \( 2 \leq j \leq 6 \) and \( v_{i,1}v_{i+1,6} \), for \( 1 \leq i \leq k - 1 \). Finally, add two vertices \( v_0 \) and \( v_{k+1} \) and join \( v_0 \) to the vertices of \( C_1 \) and \( v_{k+1} \) to the vertices of \( C_k \). The graph \( G_3 \) is shown in Fig. 1. The graph \( G_k \) is a maximal planar graph with \( 6k + 2 \) vertices of which the vertices of \( C_1 \) and \( C_k \) have degree 5 and all other vertices have degree 6. For \( k \geq 6 \) it can be shown, by a tedious but straightforward argument, that for all vertices \( u \) and \( v \) of \( G_k \), \( \kappa(u, v) = \min \{\deg u, \deg v\} \). Since \( p = 6k + 2 \),

\[
\bar{\kappa}(G_k) = 6 + \frac{156 - 24p}{p(p - 1)}.
\]
Remark 7. Since every maximal planar graph on at least four vertices is 3-connected, it follows that 3 is a lower bound for the average connectivity of such a maximal planar graph. To see that this bound is asymptotically sharp, consider the graph $H_k$, obtained from the disjoint union of 3-cycles $C_i : a_i, b_i, c_i, a_i$, $1 \leq i \leq k$ by adding edges $a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}$ and, $a_ib_{i+1}, b_ic_{i+1}, c_ia_{i+1}$. for $i = 1, 2, \ldots, k - 1$. It is easy to verify that $H_k$ is maximal planar and that for large $k$, the connectivity between almost every pair of distinct vertices of $H_k$ is 3. Hence $\kappa(H_k)$ approaches 3 as $k$ tends to infinity.

In contrast to maximal planar graphs, for which the average connectivity of two maximal planar graphs of the same order need not be the same, maximal outerplanar graphs of the same order always have the same average connectivity, as the next result shows. Moreover, while there are maximal planar graphs whose average connectivity approaches their average degree (as the order gets large), the average connectivity of maximal outerplanar graphs approaches half the average degree as the order gets large.

**Theorem 8.** Let $G$ be a maximal outerplanar graph of order $p \geq 4$. Then

$$\kappa(G) = 2 + \frac{2p - 6}{p(p - 1)}.$$

**Proof.** Let $G$ be a maximal outerplanar graph of order $p \geq 4$ and let $C : v_1, v_2, \ldots, v_p, v_1$ be the boundary of the exterior region of $G$. Let $v_i$ and $v_j$ be any two distinct vertices of $G$ where $i < j$. Then the two paths $P_1 : v_i, v_{i+1}, \ldots, v_j$ and $P_2 : v_i, v_{j-1}, \ldots, v_j$, subscripts modulo $p$, are two internally disjoint $v_i - v_j$ paths of $G$. So $\kappa(v_i, v_j) \geq 2$. Suppose first that $v_i$ and $v_j$ are adjacent on $C$. We may assume that $v_{i+1} = v_j$. The case where $v_{i-1} = v_j$ can be handled in a similar manner. Let $v_r$ be the last vertex on $P_2 - v_j$ that is
adjacent with \( v_i \). Then \( v_i \) and \( v_j \) are not connected in \( G - v_r - v_i v_j \). Hence \( \kappa(v_i, v_j) = 2 \) in this case.

Suppose that \( v_i \) and \( v_j \) are not adjacent on \( C \). Let \( v_r \) be the last vertex on \( P_1 - v_j \) that is adjacent with \( v_i \) and let \( v_s \) be the last vertex on \( P_2 - v_j \) that is adjacent with \( v_i \).

Then \( v_i \) and \( v_j \) are not connected in \( G - \{v_j, v_r \} - v_i v_j \) if \( v_i v_j \notin E(G) \) or \( G - \{v_j, v_r \} - v_i v_j \) if \( v_i v_j \notin E(G) \). So in this case \( \kappa(v_i, v_j) = 3 \) or 2, depending on whether \( v_i v_j \) is an edge of \( G \) or not, respectively. So \( \kappa(v_i, v_j) = 2 \) if \( v_i \) and \( v_j \) are adjacent on \( C \) or if \( v_i \) and \( v_j \) are nonadjacent in \( G \). Moreover, \( \kappa(v_i, v_j) = 3 \) if \( v_i v_j \) is a chord of \( C \). As a maximal outerplanar graphs has \( 2p - 3 \) edges of which \( p - 3 \) are chords of \( C \),

\[
K(G) = \left\lfloor \left( \frac{p}{2} \right) - (p - 3) \right\rfloor 2 + (p - 3)3.
\]

Hence, \( \bar{\kappa}(G) = 2 + \frac{2p - 6}{p(p - 1)} \).

As the average degree of a maximal outerplanar graph of order \( p \) is \( (4p - 6)/p \), which approaches 4 as \( p \) gets large, it follows that the average connectivity of a maximal outerplanar graph approaches half the average degree of such a graph.

### 4. Average connectivity of Cartesian products of graphs

Recall that the cartesian product of two graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 \times G_2 \) is the graph with vertex set \( V(G_1) \times V(G_2) \) and edge set \( \{(u, v)(u', v')|u = u' \text{ and } vv' \in E(G_2), \text{ or } v = v' \text{ and } uu' \in E(G_1)\} \). For a vertex \( u \) of \( G_1 \) (\( v \) of \( G_2 \)) let \( G_2^u \) (respectively, \( G_1^v \)) denote the copy of \( G_2 \) (respectively, \( G_1 \)) that replaces \( u \) (respectively \( v \)) in \( G_1 \times G_2 \). The goal here is to present sharp upper and lower bounds on the average connectivity of products of graphs. Let \( G_1 \) and \( G_2 \) be graphs with \( p_1 \) and \( p_2 \) vertices and \( q_1 \) and \( q_2 \) edges, respectively. Let \( r_1 = 2q_1 - p_1 \lfloor 2q_1/p_1 \rfloor \) and let \( r_2 = 2q_2 - p_2 \lfloor 2q_2/p_2 \rfloor \). Then \( 2q_1 = s_1 p_1 + r_1 \) and \( 2q_2 = s_2 p_2 + r_2 \) for some integers \( s_1 \) and \( s_2 \) and \( 0 \leq r_1 < p_1 \) and \( 0 \leq r_2 < p_2 \). It follows, from Theorem 1, that \( \bar{\kappa}(G_1) \leq 2q_1/p_1 - r_1(p_1 - r_1)/p_1(p_1 - 1) \) and \( \bar{\kappa}(G_2) \leq 2q_2/p_2 - r_2(p_2 - r_2)/p_2(p_2 - 1) \). Let \( p \) and \( q \) be the number of vertices and edges of \( G_1 \times G_2 \). Then \( p = p_1 p_2 \) and \( q = p_2 q_1 + p_1 q_2 \). Since \( 2q_i = s_i p_i + r_i \) for \( i = 1, 2 \), it follows that \( 2q_1 p_2 = s_1 p_1 p_2 + r_1 p_2 \) and \( 2q_2 p_1 = s_2 p_2 p_1 + r_2 p_1 \). Therefore, \( 2q = 2(q_1 p_2 + q_2 p_1) = s_1 p_1 p_2 + s_2 p_2 p_1 + r_1 p_2 + r_2 p_1 \). Let

\[
r = \begin{cases} 
  r_1 p_2 + r_2 p_1 & \text{if } r_1 p_2 + r_2 p_1 < p_1 p_2, \\
  r_1 p_2 + r_2 p_1 - p_1 p_2 & \text{otherwise}.
\end{cases}
\]

Then, by Theorem 1,

\[
\bar{\kappa}(G_1 \times G_2) \leq \frac{2q}{p} - \frac{r(p - r)}{p(p - 1)} = \frac{2p_2 q_1 + 2p_1 q_2}{p_1 p_2} - \frac{r(p_1 p_2 - r)}{p_1 p_2 (p_1 p_2 - 1)}
\]

Hence, \( \bar{\kappa}(G) = 2 + \frac{2p - 6}{p(p - 1)} \).
From Theorem 1, it follows that if equality holds in (1), then $G_1 \times G_2$ is almost regular. It is not difficult to see that if $G_1 \times G_2$ is regular or almost regular, then at least one of $G_1$ and $G_2$ is regular and the other is regular or almost regular.

Suppose that $G_1$ and $G_2$ are both optimally connected and that $G_1$ is $d_1$-regular or almost $d_1$-regular and suppose that $G_2$ is $d_2$-regular. Then necessarily every vertex of $G_1 \times G_2$ has degree $d_1 + d_2$ or $d_1 + d_2 + 1$ and hence $G_1 \times G_2$ is regular or almost regular. We now show that $G_1 \times G_2$ is optimally connected. Let $(u,v)$ and $(u',v')$ be two vertices of $G_1 \times G_2$. To show that $G_1 \times G_2$ is optimally connected we consider three cases.

Case 1: Suppose $v = v'$. Then $(u,v)$ and $(u',v')$ both belong to $G_i^1$ in $G_1 \times G_2$. Since $G_1$ is optimally connected, there exist $t = \min\{\deg_{G_i}u, \deg_{G_i}u'\}$ internally disjoint $u-u'$ paths in $G_1$ and hence in $G_i^1$. Let $P_1, P_2, \ldots, P_t$ denote these paths. Let $v_1, v_2, \ldots, v_{d_2}$ be the neighbors of $v$ in $G_2$. For each $i$, $1 \leq i \leq d_2$, let $Q_i$ be a path obtained by taking the edge $(u,v)(u,v_i)$ followed by a $(u,v_i) - (u',v_i)$ path in $G_i^0$ and then by the edge $(u',v_i)(u',v)$. Then, $P_1, P_2, \ldots, P_t, Q_1, Q_2, \ldots, Q_{d_2}$ is a collection of $t + d_2 = \min\{\deg_{G_i \times G_2}(u,v), \deg_{G_i \times G_2}(u',v')\}$ internally disjoint $(u,v) - (u',v')$ paths in $G_1 \times G_2$.

Case 2: Suppose $u = u'$. Then $(u,v)$ and $(u',v')$ both belong to $G_2^2$ in $G_1 \times G_2$. Using an argument similar to the one used in Case 1 we can show that $\kappa_{G_1 \times G_2}((u,v),(u',v')) = \min\{\deg_{G_1 \times G_2}(u,v), \deg_{G_1 \times G_2}(u',v')\}$.

Case 3: Suppose $u \neq u'$ and $v \neq v'$. Let $t = \min\{\deg_{G_i}u, \deg_{G_i}u'\}$. Then $t = d_1$ or $t = d_1 + 1$. Since $G_1$ is optimally connected, there exist $t$ internally disjoint $u-u'$ paths in $G_1$. Let $P_1, P_2, \ldots, P_t$ be $t$ internally disjoint $(u,v) - (u',v')$ paths in $G_i^1$ and suppose $P_1$ is the shortest such path.

Since $G_2$ is optimally connected, there exist $d_2$ internally disjoint $v-v'$ paths say $Q_1, Q_2, \ldots, Q_{d_2}$ in $G_2$. Suppose $Q_i: v, v_i, v_{i+1}, \ldots, v_{i+d_2}, v'$ for $1 \leq i \leq d_2$. For $i = 1, 2, \ldots, d_2 - 1$, let $Q_i'$ be the $(u,v) - (u',v')$ path in $G_1 \times G_2$ obtained by taking the edge $(u,v)(u,v_i)$ followed by the $(u,v_i) - (u',v_i)$ path in $G_i^1$ and then by the path $(u',v_i),(u',v_{i+1}), \ldots, (u',v_{i+d_2}),(u',v')$. Let $Q_i'$ be obtained by taking the path $(u,v), (u,v_{d_2+1}), (u,v_{d_2+2}), \ldots, (u,v')$ followed by the $(u,v') - (u',v')$ path in $G_i^0$ that corresponds to the path $P_i$ in $G_i^1$. Let $(u_j, v)$ be the vertex of $P_j$, $2 \leq j \leq t$, that precedes $(u',v)$ on this path. For $j = 2, 3, \ldots, t$ let $P_j'$ be constructed by taking $P_j - (u,v)$ followed by the path $(u_j,v),(u_j,v_{d_2+1}),(u_j,v_{d_2+2}), \ldots, (u_j,v_{d_2+d_2}),(u_j,v')$. Finally, let $P_1'$ be obtained by taking $P_1$ followed by the path $(u,v'), (u',v_{d_2+1}), (u',v_{d_2+2}), \ldots, (u',v_{d_2+d_2}), (u',v')$. Then $P_1', P_2', \ldots, P_t', Q_1', Q_2', \ldots, Q_{d_2}'$ is a collection of $t + d_2 = \min\{\deg_{G_1 \times G_2}(u,v), \deg_{G_1 \times G_2}(u',v')\}$ internally disjoint $(u,v) - (u',v')$ paths in $G_1 \times G_2$. Hence $G_1 \times G_2$ is optimally connected.

We now summarize the result obtained above in a theorem.
Theorem 9. Let $G_1$ and $G_2$ be connected graphs of orders $p_1$ and $p_2$ and sizes $q_1$ and $q_2$, respectively. Let $r_i = 2q_i - p_i\lfloor 2q_i/p_i \rfloor$ for $i = 1, 2$. Then
\[
\kappa(G_1 \times G_2) \leq \frac{2q_1}{p_1} + \frac{2q_2}{p_2} - \frac{r(p_1p_2 - r)}{p_1p_2(p_1p_2 - 1)}
\]
\[
= \tilde{d}(G_1) + \tilde{d}(G_2) - \frac{r(p_1p_2 - r)}{p_1p_2(p_1p_2 - 1)},
\]
where
\[
r = \begin{cases} 
    r_1p_2 + r_2p_1 & \text{if } r_1p_2 + r_2p_1 < p_1p_2, \\
    r_1p_2 + r_2p_1 - p_1p_2 & \text{otherwise}.
\end{cases}
\]
Moreover, the bound is sharp if one of $G_1$ and $G_2$ is regular and if the other graph is regular or almost regular and $G_1$ and $G_2$ are both optimally connected.

We now turn to lower bounds for $\kappa(G_1 \times G_2)$ where $G_1$ and $G_2$ are both connected graphs of orders $p_1$ and $p_2$ and sizes $q_1$ and $q_2$, respectively. Let $(u, v)$ and $(u', v')$ be two distinct vertices of $G_1 \times G_2$. Suppose first that $v = v'$. Then $\kappa_{G_1 \times G_2}((u, v), (u', v')) \geq \kappa_{G_1}(u, u') + \deg_{G_2} v$, as we now show. There exist $\kappa_G(u, u') = t$ internally disjoint $(u, v) - (u', v)$ paths, say $P_1, P_2, \ldots, P_t$, in $G_1$. Let $d = \deg_{G_2} v$ and suppose $v_1, v_2, \ldots, v_d$ are the neighbours of $v$ in $G_2$. Let $Q_i$ be a $(u, v) - (u', v')$ path of $G_1 \times G_2$ constructed by taking the edge $(u, v)(u, v_i)$ followed by some $(u, v_i) - (u', v_i)$ path in $G_1^u$ and the edge $(u', v_i)(u', v')$. Then $P_1, P_2, \ldots, P_t, Q_1, Q_2, \ldots, Q_d$ is a collection of $\kappa_{G_1}(u, u') + \deg_{G_2} v$ internally disjoint $(u, v) - (u', v')$ paths in $G_1 \times G_2$.

Similarly, if $u = u'$, then $\kappa_{G_1 \times G_2}((u, v), (u', v')) \geq \kappa_{G_2}(v, v') + \deg_{G_1} u$. Suppose now that $u \neq u'$ and $v \neq v'$. Then it can be shown, as in Case 3 of the previous theorem, that there exist at least $\kappa_{G_1}(u, u') + \kappa_{G_2}(v, v')$ internally disjoint $(u, v) - (u', v')$ paths in $G_1$. If we now sum $\kappa_{G_1 \times G_2}((u, v), (u', v'))$, over all pairs of vertices $(u, v), (u', v')$ of $G_1 \times G_2$, it follows that the total connectivity of $G_1 \times G_2$ is bounded below as follows:
\[
K(G_1 \times G_2) \geq p_2K(G_1) + 2q_2 \binom{p_1}{2} + p_1K(G_2) + 2q_1 \binom{p_2}{2} + p_2(p_2 - 1)K(G_1) + p_1(p_1 - 1)K(G_2).
\]
So
\[
\kappa(G_1 \times G_2) \geq \kappa(G_1)\frac{p_2(p_1 - 1)}{p_1p_2 - 1} + \kappa(G_2)\frac{p_1(p_2 - 1)}{p_1p_2 - 1}
\]
\[
+ \frac{4q_2 \binom{p_1}{2} + 4q_1 \binom{p_2}{2}}{p_1p_2(p_1p_2 - 1)}.
\]
This bound is attained, for example, if $G_1$ is the $(n - 1)$-cube and $G_2$ is $K_2$. We have thus established the following theorem.
Theorem 10. Let \( G_1 \) and \( G_2 \) be connected graphs of order \( p_1 \) and \( p_2 \) and size \( q_1 \) and \( q_2 \), respectively. Then

\[
\kappa(G_1 \times G_2) \geq \kappa(G_1) \frac{p_2(p_1-1)}{p_1 p_2 - 1} + \kappa(G_2) \frac{p_1(p_2-1)}{p_1 p_2 - 1} + \frac{4q_2 \binom{p_1}{2} + 4q_1 \binom{p_2}{2}}{p_1 p_2 (p_1 p_2 - 1)}.
\]

Moreover, this bound is sharp.

5. Nordhaus–Gaddum-type results

The following lemma, which will be needed in the proof of Theorem 12, gives an estimate of how close a degree sequence of a graph of order \( p \) can be to the sequence \( p - 1, p - 2, p - 3, \ldots, 0 \). Its proof makes use of the necessary condition, of the well-known Erdős–Gallai condition, for a sequence of nonnegative integers to be the degree sequence of some graph.

Lemma 11. Let \( G \) be a graph of order \( p \) with nonincreasing degree sequence \( d_1, d_2, \ldots, d_p \). Let \( e_i = |d_i - (p - i)| \) for \( i = 1, 2, \ldots, p \). Then

\[
\sum_{i=1}^{p} e_i \leq \left\lfloor \frac{p}{2} \right\rfloor.
\]

Proof. Let the vertices of \( G \) be \( v_1, v_2, \ldots, v_p \), where \( \deg_G v_i = d_i \) for \( i = 1, 2, \ldots, p \). Let \( 1 \leq k \leq p - 1 \) and define \( V_1 = \{v_1, v_2, \ldots, v_k\} \) and \( V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_p\} \). The graph induced by \( V_1 \) cannot have more than \( \binom{k}{2} \) edges. If the degree sum of the vertices in \( V_1 \) exceeds \( 2\binom{k}{2} \), then the difference is a lower bound for the number of edges joining vertices in \( V_1 \) to vertices in \( V_2 \). Hence we have

\[
\sum_{i=1}^{k} d_i - 2 \binom{k}{2} \leq \sum_{i=k+1}^{p} d_i.
\]

With \( d_i \geq p - i - e_i \) for \( i = 1, \ldots, k \) and \( d_i \leq p - i + e_i \) for \( i = k+1, \ldots, p \), we obtain

\[
\sum_{i=1}^{k} (p - i) - \sum_{i=1}^{k} e_i - 2 \binom{k}{2} \leq \sum_{i=k+1}^{p} (p - i) + \sum_{i=k+1}^{p} e_i
\]

or equivalently

\[
\sum_{i=1}^{p} e_i \geq \sum_{i=1}^{k} (p - i) - 2 \binom{k}{2} - \sum_{i=k+1}^{p} (p - i) = -2k^2 + \frac{p-p^2}{2} + 2pk.
\]

Since this is true for all \( k \), and as this expression is maximized when \( k = \lfloor p/2 \rfloor \) and takes on the value \( \lfloor p/2 \rfloor \), the desired result follows. \( \square \)
\textbf{Theorem 12.} Let $G$ be a graph of order $p$ and let $\bar{G}$ be its complement. Then
\[
p - 1 \geq \kappa(G) + \kappa(\bar{G}) \geq \begin{cases} 
\frac{4p - 5}{6} + \frac{1}{2p - 2} & \text{if } p \text{ is even,} \\
\frac{4p - 5}{6} + \frac{1}{2p} & \text{if } p \text{ is odd.}
\end{cases}
\]
Both bounds are sharp.

\textbf{Proof.} Let $G$ be a graph of order $p$. For every two vertices $u$ and $v$ of $G$, $\kappa_G(u, v) \leq \min\{\deg_G u, \deg_G v\} \leq \deg_G u$ and $\kappa_\bar{G}(u, v) \leq \min\{\deg_\bar{G} u, \deg_\bar{G} v\} \leq \deg_\bar{G} u = p - 1 - \deg_G u$. Hence $\kappa_G(u, v) + \kappa_\bar{G}(u, v) \leq p - 1$. The upper bound of the theorem now follows and the complete graph $\bar{G} = K_p$ shows that this bound is sharp.

We now establish the lower bound. Let $d_1 \geq d_2 \geq \cdots \geq d_p$ be the degree sequence of $G$. Let $v_1, v_2, \ldots, v_p$ be the vertices of $G$ such that $d_i = \deg_G v_i$ for $1 \leq i \leq p$. We first prove that for vertices $v_i, v_j$ with $i < j$
\[
2\kappa_1(v_i, v_j, G) + 2\kappa_1(v_i, v_j, \bar{G}) + \kappa_2(v_i, v_j, G)
\]
\[
+ \kappa_2(v_i, v_j, \bar{G}) + 2\kappa_3(v_i, v_j, G) + 2\kappa_3(v_i, v_j, \bar{G}) \geq p - d_i + d_j.
\]
(2)
We assume that $v_iv_j \notin E(G)$ (otherwise analogous arguments apply). Since each vertex in $V(G) - v_i - v_j$, that is adjacent to both $v_i$ and $v_j$ in $G$ (or $\bar{G}$), yields a path of length 2 in $G$ (or $\bar{G}$, respectively), we have
\[
\kappa_2(v_i, v_j, G) + \kappa_2(v_i, v_j, \bar{G}) = |N_G(v_i) \cap N_G(v_j)| + |N_\bar{G}(v_i) \cap N_\bar{G}(v_j)|.
\]
(3)
Let $N_G(v_j) - N_G(v_i) = \{w_1, w_2, \ldots, w_r\}$ (possibly $r = 0$). Since $\deg_G v_i \geq \deg_G v_j$ there exist at least $r$ distinct vertices $u_1, u_2, \ldots, u_r \in N_G(v_i) - N_G(v_j)$. For $1 \leq l \leq r$ we define $P_l = v_i, u_l, w_l, v_j$ and $\tilde{P}_l = v_i, w_l, u_l, v_j$. If $u_lw_l$ is an edge of $G$, then $P_l$ is a path in $G$. If $u_lw_l$ is an edge in $\bar{G}$, $\tilde{P}_l$ is a path in $\bar{G}$. Hence
\[
\kappa_3(v_i, v_j, G) + \kappa_3(v_i, v_j, \bar{G}) \geq |N_G(v_j) - N_G(v_i)|.
\]
(4)
Since $|N_G(v_j) - N_G(v_i)| = |N_\bar{G}(v_i) - N_\bar{G}(v_j)| - 1$, Eq. (4) yields,
\[
\kappa_3(v_i, v_j, G) + \kappa_3(v_i, v_j, \bar{G}) \geq |N_G(v_j) - N_G(v_i)| - 1.
\]
(5)
Adding Eqs. (3)–(5) gives
\[
\kappa_2(v_i, v_j, G) + \kappa_2(v_i, v_j, \bar{G}) + 2\kappa_3(v_i, v_j, G) + 2\kappa_3(v_i, v_j, \bar{G})
\]
\[
\geq |N_G(v_j) \cap N_G(v_i)| + |N_G(v_j) \cap N_\bar{G}(v_i)| + |N_G(v_j) - N_G(v_i)|
\]
\[
+ |N_\bar{G}(v_j) - N_\bar{G}(v_i)| - 1
\]
\[
= \deg_G v_j + \deg_\bar{G} v_i - 1
\]
\[
= p - 2 + d_j - d_i.
\]
This inequality in conjunction with the fact that $\kappa_1(v_i, v_j, G) + \kappa_1(v_i, v_j, \bar{G}) = 1$ gives (2). We now count the total number of paths of length 2 in $G$ and $\bar{G}$. Every vertex $v_i$
of \( G \) gives rise to \( \binom{d_i}{2} \) paths of length 2 in \( G \) and to 
\[
\binom{p - 1 - d_i}{2}
\]
paths of length 2 in \( \tilde{G} \). Thus
\[
\sum_{i<j}(\kappa_2(v_i, v_j, G) + \kappa_2(v_i, v_j, \tilde{G})) = \sum_{i=1}^{p} \left( \binom{d_i}{2} + \binom{p - 1 - d_i}{2} \right).
\]
(6)
By definition of \( \kappa_i \) we have 
\[
\kappa_1(v_i, v_j) \geq \kappa_1(v_i, v_j) + \kappa_2(v_i, v_j) + \kappa_3(v_i, v_j).
\]
Summing Eq. (2) over all pairs \( i \) and \( j \) with \( i < j \) and adding this to (6) yields
\[
2 \sum_{i<j}(\kappa(v_i, v_j, G) + \kappa(v_i, v_j, \tilde{G})) \geq \sum_{i<j}(2\kappa_1(v_i, v_j, G) + 2\kappa_1(v_i, v_j, \tilde{G})) + \kappa_2(v_i, v_j, G) + \kappa_2(v_i, v_j, \tilde{G})
\]
\[
+ 2\kappa_3(v_i, v_j, G) + 2\kappa_3(v_i, v_j, \tilde{G})) + \sum_{i<j}(\kappa_2(v_i, v_j, G) + \kappa_2(v_i, v_j, \tilde{G}))
\]
\[
\geq \sum_{i<j}(p - d_i + d_j) + \sum_{i=1}^{p} \left( \binom{d_i}{2} + \binom{p - 1 - d_i}{2} \right)
\]
\[
= \frac{1}{2} \ p^2(p - 1) - \sum_{i=1}^{p} (p - 2i + 1)d_i
\]
\[
+ \sum_{i=1}^{p} \left( d_i^2 + \frac{1}{2}(p - 1)^2 - (p - 1)d_i - \frac{1}{2}(p - 1) \right)
\]
\[
= \frac{p(p - 1)(4p - 5)}{6} + \sum_{i=1}^{p} (d_i - (p - i))^2.
\]
(7)
For \( i = 1, \ldots, p \) let \( e_i = |d_i - (p - i)| \). By Lemma 11 we have \( \sum_{i=1}^{p} e_i \geq \lfloor p/2 \rfloor \). Hence the sum in (7) is minimized if \( e_i = 1 \) for \( \lfloor p/2 \rfloor \) values of \( i \) and \( e_i = 0 \) for the remaining \( \lceil p/2 \rceil \) values of \( i \). Hence
\[
2(K(G) + K(\tilde{G})) \geq \frac{p(p - 1)(4p - 5)}{6} + \left\lfloor \frac{p}{2} \right\rfloor,
\]
which establishes the right inequality of the theorem.

In order to prove that this inequality is sharp, consider the graphs \( G_p \) defined recursively as follows: Let \( G_1 = K_1, G_2 = K_2 \), and for \( p \geq 3 \), \( G_p = (G_{p-2} \cup K_1) + K_1 \). Hence \( G_p \) is obtained from \( G_{p-2} \) by adding two new vertices \( w_{p-1} \) and \( w_p \), where \( w_{p-1} \) is adjacent only to \( w_p \) and \( w_p \) is adjacent to each vertex of \( G_p \). Simple calculations show
that

\[ K(G_p) = \begin{cases} 
\frac{p}{24}(4p^2 - 3p + 2) & \text{if } p \text{ is even}, \\
\frac{p + 1}{24}(4p^2 - 7p + 3) & \text{if } p \text{ is odd}.
\end{cases} \]

Since, for \( p \geq 2 \), the graph \( \tilde{G}_p \) is isomorphic to \( G_{p-1} \cup K_1 \), it follows that

\[ \tilde{k}(G_p) + \tilde{k}(\tilde{G}_p) = \begin{cases} 
\frac{4p - 5}{6} + \frac{1}{2p} & \text{if } p \text{ is even}, \\
\frac{4p - 5}{6} + \frac{1}{2p} & \text{if } p \text{ is odd},
\end{cases} \]

as desired. \( \square \)

As is often the case, the upper and lower bounds for the product of \( \tilde{k}(G) \) and \( \tilde{k}(\tilde{G}) \) can be derived from the bounds for the sum.

**Corollary 13.** If \( G \) is a graph of order \( p \) then

\[ 0 \leq \tilde{k}(G)\tilde{k}(\tilde{G}) \leq \frac{(p-1)^2}{4}. \]

The lower bound is sharp for every \( p \geq 1 \) and the upper bound is sharp for every \( p \) with \( p \equiv 1 \pmod{4} \).

**Proof.** The first inequality follows from the fact that \( \tilde{k}(G) \geq 0 \) and \( \tilde{k}(\tilde{G}) \geq 0 \). The complete graph \( G = K_p \) shows that this bound is sharp.

The second bound follows from the upper bound in Theorem 12. Let \( p \equiv 1 \pmod{4} \). Let \( G_p \) be the graph with vertex set \( \{v_1, v_2, \ldots, v_p\} \) where \( v_i \) and \( v_j \) are adjacent if \( |i - j| \leq (p-1)/4 \), i.e., \( G_p \) is a circulant and so is \( \tilde{G}_p \). It is easy to verify that both \( G_p \) and \( \tilde{G}_p \) are \( (p-1)/2 \)-regular and \( (p-1)/2 \)-connected. Hence

\[ \tilde{k}(G_p) = \tilde{k}(\tilde{G}_p) = \frac{p-1}{2}. \]

Hence the upper bound holds with equality for \( G_p \) and \( \tilde{G}_p \). \( \square \)

### 6. Clique number and chromatic number

In this section we present upper and lower bounds on the average connectivity of a graph of (a) given order and clique size and (b) given order and chromatic number.

**Theorem 14.** Let \( G \) be a graph of order \( p \) and clique number \( k \). Then

\[ \frac{k(k-1)^2}{p(p-1)} \leq \tilde{k}(G) \leq \frac{k-1}{k-1}. \]  \hspace{1cm} (8)

The lower bound is sharp for all \( p \) and the upper bound for \( p \) a multiple of \( k \).
Proof. We first prove the lower bound. Let $H$ be a complete subgraph of order $k$ in $G$. Then $K(H) \leq K(G)$. Hence

$$\tilde{\kappa}(G) = \left(\frac{p}{2}\right)^{-1} K(G) \geq \left(\frac{p}{2}\right)^{-1} K(H) = \frac{k(k-1)^2}{p(p-1)},$$

as desired. For any given $p \geq k$, the graph consisting of a clique of order $k$ and $p-k$ isolated vertices shows that the bound is sharp.

The upper bound follows from Theorem 2 and the fact that, by Turán’s Theorem, the average degree of $G$ is at most $\left\lfloor \frac{k-1}{k} \right\rfloor p$. The $k$-partite Turán graph of order $p$ shows that this bound is best possible if $p$ is a multiple of $k$. □

**Theorem 15.** Let $G$ be a graph of order $p$ and chromatic number $k$. Then

$$\frac{k(k-1)^2}{p(p-1)} \leq \tilde{\kappa}(G) \leq \frac{k-1}{k} p.$$ (9)

The lower bound is sharp for all $p$ and the upper bound is attained for all $p$ a multiple of $k$.

Proof. Let $G$ be a graph with chromatic number $k$. We first prove the lower bound for $\tilde{\kappa}(G)$. Let $H$ be a $k$-critical subgraph of $G$. Then we have

$$K(H) \geq \sum_{\{u,v\} \subset V(H)} (\kappa_1(u,v,H) + \kappa_2(u,v,H)) = q(H) + \sum_{v \in V(H)} \left(\frac{\deg_H(v)}{2}\right).$$

Since $H$ is $k$-critical, $H$ has minimum degree at least $k-1$ and order $p(H) \geq k$. Therefore

$$K(H) \geq \frac{p(H)(k-1)}{2} + p(H) \binom{k-1}{2} \geq \frac{k(k-1)}{2} + \frac{k(k-1)(k-2)}{2} = (k-1) \binom{k}{2}. $$

Hence

$$\tilde{\kappa}(G) \geq K(H) \left(\frac{p(G)}{2}\right)^{-1} \geq (k-1) \binom{k}{2} \left(\frac{p(G)}{2}\right)^{-1},$$

as desired. The graph of order $p$ containing a complete graph of order $k$ and $p-k$ isolated vertices shows that the bound is best possible.

The upper bound follows from Theorem 14. The balanced complete $k$-partite Turán graph of order $p$, for $p$ a multiple of $k$, shows that the bound is best possible. □

**References**