Barrelledness in $\ell_\infty(\Omega, X)$ subspaces

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Abstract

Given a nonempty set $\Omega$ and a normed space $X$, we prove several barrelledness properties of some subspaces of $\ell_\infty(\Omega, X)$, the linear space of all bounded functions of $\Omega$ into $X$ equipped with the supremum norm.

Keywords: Barrelled space; Ultrabornological space; Banach disk

1. Introduction

Throughout this paper $\Omega$ will denote a nonempty set, $X$ a normed space and $\mathbb{K}$ the scalar field of real or complex numbers. We shall deal with the linear space $\ell_\infty(\Omega, X)$ over $\mathbb{K}$ of all $X$-valued bounded functions $f: \Omega \to X$, equipped with the supremum norm, denote by $\ell_\infty(\Omega, X)_{cv}$ the linear subspace of $\ell_\infty(\Omega, X)$ of all those countably valued functions, and by $\ell_\infty(\Omega, X)_{cs}$ the linear subspace of $\ell_\infty(\Omega, X)$ consisting of all functions of countable support. As usual, $\ell_\infty(X)$ shall stand for $\ell_\infty(\mathbb{N}, X)$, that is, the linear space over $\mathbb{K}$ of all bounded sequences in $X$ provided with the supremum norm. Clearly $\ell_\infty(X) = \ell_\infty(X)_{cs} = \ell_\infty(X)_{cv}$ and $\ell_\infty(\Omega, X)_{cv}$ is dense in $\ell_\infty(\Omega, X)$ whenever $X$ is separable.

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Let us recall that a locally convex space $E$ is barrelled [7] if each barrel (i.e., each absorbing closed absolutely convex subset) of $E$ is a neighbourhood of 0; or equivalently, if each linear mapping with closed graph from $E$ into any Banach (i.e., complete normed) or Fréchet (i.e., complete metrizable locally convex) space is continuous [8]. $E$ is ultrabornological [8] if each absolutely convex set in $E$ which absorbs the Banach disks of $E$ (i.e., the closed absolutely convex bounded subsets $U$ of $E$ whose linear span $\text{sp}(U)$ become a Banach space when provided with the norm defined by its Minkowski functional) is a neighbourhood of 0 in $E$. $E$ is totally barrelled [11] if given an arbitrary sequence of closed absolutely convex subsets covering $E$, one of them is a neighbourhood of 0. And $E$ is totally barrelled [11] if given a sequence of vector subspaces of $E$ that covers $E$, one of them is barrelled and its closure is finite codimensional in $E$. In general (cf. [5,6]),

$$\text{UBL} \Rightarrow \text{totally barrelled} \Rightarrow \text{barrelled}.$$ 

The following three results are well known.

**Theorem 1.1** [1, Theorem 1]. $\ell_\infty(\Omega, X)$ is barrelled whenever $X$ is barrelled and either $|\Omega|$ or $|X|$ is a nonmeasurable cardinal.

**Theorem 1.2** [2, Theorem 2]. $\ell_\infty(\Omega, X)_{cv}$ is ultrabornological if and only if $X$ is ultrabornological.

**Theorem 1.3** [3, Theorem 2]. $\ell_\infty(\Omega, X)_{cs}$ is UBL if and only if $X$ is UBL.

From Theorem 1.1 it follows that the normed space $\ell_\infty(\Omega, X)_{cs}$ is barrelled if and only if $X$ is barrelled. In particular, $\ell_\infty(X)$ is barrelled if and only if $X$ is barrelled. And from Theorem 1.2 it follows that (i) the space $\ell_\infty(X)$ is ultrabornological if and only if $X$ is ultrabornological, and (ii) assuming that $X$ is separable and each functional on $\ell_\infty(\Omega, X)$ is null whenever it is bounded on every Banach disk and vanishes on $\ell_\infty(\Omega, X)_{cv}$, then $\ell_\infty(\Omega, X)$ is ultrabornological if and only if $X$ is ultrabornological. On the other hand, Theorem 1.3 provides that $\ell_\infty(X)$ is UBL if and only if $X$ is UBL. From the above the following questions do arise.

**Problem 1.** Does the fact of $X$ being ultrabornological imply that $\ell_\infty(\Omega, X)_{cs}$ is ultrabornological?

**Problem 2.** Does the fact of $X$ being UBL imply that $\ell_\infty(\Omega, X)_{cv}$ is UBL?

**Problem 3.** Does the fact of $X$ being ultrabornological and $|\Omega|$ or $|X|$ a nonmeasurable cardinal imply that $\ell_\infty(\Omega, X)$ is ultrabornological?

We have not been able to solve Problem 3. However we shall prove in Theorem 2.3 bellow that the space $\ell_\infty(\Omega, X)_{cs}$ is ultrabornological if and only if $X$ is ultrabornological, hence solving Problem 1 in the positive. In our Theorem 3.2 we show that if $X$ is UBL then $\ell_\infty(\Omega, X)_{cv}$ is the locally convex hull of a family of UBL subspaces, which is “close” to
becoming a positive solution of Problem 2. Finally in Section 4 we demonstrate that the linear subspace of $\ell_\infty(\Omega, X)$ consisting of all those functions with totally bounded range is barrelled if and only if $X$ is barrelled, and that if $X$ is UBL then the linear subspace of $\ell_\infty(\Omega, X)$ of all those functions which are uniform limits of sequences of functions with finite-dimensional ranges is UBL whenever it is totally barrelled.

2. On the ultrabornological property in $\ell_\infty(\Omega, X)_{cs}$

Given a nonempty subset $\Gamma$ of $\Omega$, $\ell_\infty(\Gamma, X)_{cs}$ will stand for the subspace of $\ell_\infty(\Omega, X)_{cs}$ of all those functions (countably) supported in $\Gamma$. As usual, given a normed space $Y$, $BY$ will denote its unit ball.

Lemma 2.1. Let $V$ be an absolutely convex set in $\ell_\infty(X)$. If $V$ absorbs the Banach disks of $\ell_\infty(X)$, then there is $m \in \mathbb{N}$ such that $mV \supseteq B_{\ell_\infty(\mathbb{N}\setminus\{1, \ldots, m\}, X)}$.

Proof. Assume that the lemma is not true. Then for each $n \in \mathbb{N}$ there is some $x_n \in \ell_\infty(\mathbb{N}\setminus\{1, \ldots, n\}, X) \setminus nV$ with $\|x_n\| \leq 1$. Since for each $\xi = (\xi_n) \in \ell_1$, $\sum_{n=1}^\infty \xi_n x_n$ converges to some $x_\xi \in \ell_\infty(\hat{X})$ and

$$x_\xi(j) = \sum_{i=1}^\infty \xi_i x_i(j) = \sum_{i=1}^{j-1} \xi_i x_i(j) \in X$$

for each $j \in \mathbb{N}$, then $x_\xi \in \ell_\infty(X)$ and we consider $\varphi : \ell_1 \to \ell_\infty(X)$ defined by $\varphi(\xi) = x_\xi$. Hence $D := \varphi(B_{\ell_1})$ is a Banach disk in $\ell_\infty(X)$ and, by hypothesis, there exists $k \in \mathbb{N}$ such that $D \subseteq kV$. Therefore $x_k \in kV$, a contradiction.  

Lemma 2.2. $\ell_\infty(X)$ is ultrabornological if and only if $X$ is ultrabornological.

Proof. If $\ell_\infty(X)$ is ultrabornological, it is clear that $X$ is ultrabornological. Conversely, assume that $X$ is ultrabornological and let $V$ be an absolutely convex set in $\ell_\infty(X)$ which absorbs the Banach disks of $\ell_\infty(X)$. According to Lemma 2.1 there exists $m \in \mathbb{N}$ such that $mV \supseteq B_{\ell_\infty(\mathbb{N}\setminus\{1, \ldots, m\}, X)}$. Since

$$\ell_\infty(X) = \ell_\infty([1, \ldots, m], X) \oplus \ell_\infty(\mathbb{N}\setminus\{1, \ldots, m\}, X),$$

it suffices to show that $V$ absorbs the closed unit ball of $\ell_\infty([1, \ldots, m], X)$. But the latter is true for $\ell_\infty([1, \ldots, m], X) \simeq X^m$ is ultrabornological and therefore $V$ absorbs the Banach disks of $\ell_\infty([1, \ldots, m], X)$.  

Theorem 2.3. The space $\ell_\infty(\Omega, X)_{cs}$ is ultrabornological if and only if $X$ is ultrabornological.
Proof. Let \( V \) be an absolutely convex set in \( \ell_\infty(\Omega, X)_{cs} \) which absorbs the Banach disks of \( \ell_\infty(\Omega, X)_{cs} \). Let us show firstly that there exists a countable (perhaps empty) set \( \Delta \) such that \( V \) absorbs the closed unit ball of \( \ell_\infty(\Omega \setminus \Delta, X)_{cs} \). Otherwise take \( f_1 \in \ell_\infty(\Omega, X)_{cs} \) such that \( \|f_1\|_\infty = 1 \) and \( f_1 \notin V \). Since \( \Delta_1 := \text{supp} \ f_1 \) is countable, there must be \( f_2 \in \ell_\infty(\Omega \setminus \Delta_1, X)_{cs} \) such that \( \|f_2\|_\infty = 1 \) and \( f_2 \notin 2V \). Then set \( \Delta_2 := \text{supp} \ f_2 \) and choose \( f_3 \in \ell_\infty(\Omega \setminus (\Delta_1 \cup \Delta_2), X)_{cs} \) such that \( \|f_3\|_\infty = 1 \) and \( f_3 \notin 3V \). Going on we obtain a bounded sequence \( \{f_n\} \) in \( \ell_\infty(\Omega, X)_{cs} \) and a pairwise disjoint sequence \( \{\Delta_n\} \) of countable sets in \( \Omega \) such that \( \Delta_n := \text{supp} \ f_n \) and \( f_n \notin nV \) for each \( n \in \mathbb{N} \). Since the \( f_i \) are disjointly supported, the pointwise limit \( f_\infty(\omega) := \sum_{i=1}^{\infty} \xi_i f_i(\omega) \) belongs to \( X \) for each \( \omega \in \Omega, \xi \in \ell_1 \). This assures that \( \{f_\infty(\cdot) : \xi_i \in \ell_1, \|\xi\|_1 \leq 1\} \) is a Banach disk contained in \( \ell_\infty(\Omega, X)_{cs} \).

And given that \( V \) absorbs the Banach disks of \( \ell_\infty(\Omega, X)_{cs} \), there is some \( k \in \mathbb{N} \) such that \( f_k \notin kV \), a contradiction.

If \( \Delta \) is finite, then \( \ell_\infty(\Delta, X)_{cs} = \ell_\infty(\Delta, X) \) is isomorphic to \( X^\Delta \) and therefore ultrabornological. If \( \Delta \) is (countable) infinite, then \( \ell_\infty(\Delta, X)_{cs} = \ell_\infty(\Delta, X) \) is isomorphic to \( \ell_\infty(X) \) and consequently, by Lemma 2.2, it is ultrabornological. In either case, \( V \) absorbs the closed unit ball of \( \ell_\infty(\Delta, X)_{cs} \), hence \( V \) is a neighbourhood of 0 in \( \ell_\infty(\Omega, X)_{cs} \).

3. Conditions for \( \ell_\infty(\Omega, X)_{cv} \) to be UBL

In the next result \( \Pi(\Omega) \) will stand for the family of all countable partitions of \( \Omega \) and given \( \pi = \{A_n : n \in \mathbb{N}\} \in \Pi(\Omega) \) let us denote by \( \ell_\infty(\pi, X) \) the linear subspace of \( \ell_\infty(\Omega, X)_{cv} \) formed by all those countably valued functions which take a constant value on each set \( A_n \) of \( \pi \). It is plain that

(i) \( \ell_\infty(\pi, X) \) is linearly isometric to \( X^n \) for some \( n \in \mathbb{N} \) or to \( \ell_\infty(X) \) depending on whether \( \pi \) is finite or infinite.

(ii) \( \ell_\infty(\Omega, X)_{cv} = \bigcup \{\ell_\infty(\pi, X) : \pi \in \Pi(\Omega)\} \).

Lemma 3.1. \( \ell_\infty(\Omega, X)_{cv} \) is the locally convex hull of \( \{\ell_\infty(\pi, X) : \pi \in \Pi(\Omega)\} \).

Proof. Assume the statement is false and there is an absolutely convex set \( V \) in \( \ell_\infty(\Omega, X)_{cv} \) which absorbs the bounded sets but \( V \) is not a neighbourhood of 0. Proceeding as in the proof of [4, Theorem 4.3], but using [2, Lemma 1 and Theorem 1] instead of [4, Lemmas 4.1 and 4.2] one can obtain a pairwise disjointly supported bounded sequence \( \{f_n\} \) in \( \ell_\infty(\Omega, X)_{cv} \) such that \( f_n \notin nV \) for each \( n \in \mathbb{N} \). If \( \{x_{n,m} : m \in J_n\} \), with \( J_n \subseteq \mathbb{N} \), is the range of non-null values of \( f_n \), then set \( A_{n,m} := f_n^{-1}(x_{n,m}) \) for each \( m \in J_n \). Since the sequence \( \{f_n\} \) is pairwise disjointly supported, setting \( A_0 = \Omega \setminus \bigcup_{n \in \mathbb{N}} \bigcup_{m \in J_n} A_{n,m} \), then \( \pi := \{A_0, A_{n,m} : m \in J_n, n \in \mathbb{N}\} \in \Pi(\Omega) \) and \( f_n \in \ell_\infty(\pi, X) \) for each \( n \in \mathbb{N} \). Finally, the fact that \( V \) absorbs the closed unit ball of \( \ell_\infty(\pi, X) \) and the sequence \( \{f_n\} \) is bounded imply that \( f_k \in kV \) for some \( k \in \mathbb{N} \), a contradiction.

Theorem 3.2. If \( X \) is UBL, then \( \ell_\infty(\Omega, X)_{cv} \) is the locally convex hull of a family of UBL spaces.
Proof. If $X$ is UBL, then $\ell_\infty(X)$ and $X^n$ are UBL for each $n \in \mathbb{N}$. Then each subspace $\ell_\infty(\pi, X)$ is UBL and the conclusion follows from the previous lemma.

4. The subspace of functions with totally bounded range

Let $TB(\Omega, X)$ be the subspace of $\ell_\infty(\Omega, X)$ consisting of all functions with totally bounded range and $F(\Omega, X)$ be the linear subspace of $\ell_\infty(\Omega, X)$ of all those functions which are the uniform limit of a sequence of functions with finite-dimensional range.

Theorem 4.1. The space $TB(\Omega, X)$ is barrelled if and only if $X$ is barrelled.

Proof. Given $f \in TB(\Omega, X)$, let $K_f$ be the closure of $f(\Omega)$ in $\hat{X}$ and denote by $f^\beta$ the Stone–Čech extension $f^\beta: \beta\Omega \to K_f$ of $f: \Omega \to K_f$, the latter considered as a continuous mapping from $\Omega$, equipped with the discrete topology, into the compact topological space $K_f$. Then the linear operator $S: TB(\Omega, X) \to C(\beta\Omega, \hat{X})$ defined by $Sf = f^\beta$ embeds $TB(\Omega, X)$ isometrically into $C(\beta\Omega, \hat{X})$, since

$$\|Sf\|_\infty = \sup_{\omega \in \beta\Omega} \|f^\beta(\omega)\| = \sup_{t \in \Omega} \|f(t)\| = \|f\|_\infty.$$ 

Seeing $C(\beta\Omega, X)$ as a topological subspace of $S(TB(\Omega, X))$, one has that $C(\beta\Omega, X) \hookrightarrow TB(\Omega, X) \hookrightarrow C(\beta\Omega, \hat{X})$.

If $X$ is barrelled, then $C(\beta\Omega, X)$ is barrelled according to a well-known result of Mendoza [9]. And since $C(\beta\Omega, X)$ is a dense subspace of $C(\beta\Omega, \hat{X})$, we conclude that $TB(\Omega, X)$ is barrelled.

We end by giving a necessary and sufficient condition for $F(\Omega, X)$ to be UBL.

Theorem 4.2. Let $X$ be an UBL space. If $F(\Omega, X)$ is totally barrelled, then $F(\Omega, X)$ is UBL.

Proof. Let $\{W_n\}$ be a sequence of closed absolutely convex sets in $F(\Omega, X)$ covering $F(\Omega, X)$ and identify $\ell_\infty(\Omega) \otimes_\pi X$ with the topological subspace of $\ell_\infty(\Omega, X)$ of all functions of finite-dimensional range. Since the projective topology on $\ell_\infty(\Omega) \otimes X$ is stronger than the injective one, each $W_p$ is also closed in $\ell_\infty(\Omega) \otimes_\pi X$. So, using the fact that $\ell_\infty(\Omega) \otimes_\pi X$ is UBL (cf. [5, Theorem 1.6.5]), there is some $p \in \mathbb{N}$ such that $W_p$ absorbs the closed unit ball of $\ell_\infty(\Omega) \otimes_\pi X$. This implies that $sp(W_p) \supseteq \ell_\infty(\Omega) \otimes_\pi X$ and, consequently, $sp(W_p)$ is a dense linear subspace of $F(\Omega, X)$. Hence, according to [10, Theorem 4.1], there is a subsequence $\{W_{n_j}\}$ of $\{W_n\}$ covering $F(\Omega, X)$ such that $sp(W_{n_j})$ is dense in $F(\Omega, X)$ for each $j \in \mathbb{N}$. Since $F(\Omega, X)$ is assumed to be totally barrelled, there is $j \in \mathbb{N}$ such that $sp(W_{n_j})$ is barrelled. This implies that $sp(W_{n_j})$ is a closed linear subspace of $F(\Omega, X)$, which guarantees that $sp(W_{n_j}) = F(\Omega, X)$. Since $F(\Omega, X)$ is barrelled, $W_{n_j}$ must absorb the closed unit ball of $F(\Omega, X)$ and therefore $F(\Omega, X)$ is UBL.
References