Linear Algebra and its Applications 432 (2010) 556–565

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Spanning tree invariants, loop systems and doubly stochastic matrices

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ARTICLE INFO

Article history:
Received 26 March 2009
Accepted 1 September 2009
Available online 1 October 2009

Submitted by R.A. Brualdi

AMS classification:
15A18
15A51
37B10

Keywords:
Spanning tree
Loop system
Doubly stochastic
Markov chain

ABSTRACT

The spanning tree invariant of Lind and Tuncel [12] is observed in the context of loop systems of Markov chains. For $n = 1, 2, 3$ the spanning tree invariants of the loop systems of a Markov chain determined by an irreducible stochastic $(n \times n)$-matrix P coincide if and only if P is doubly stochastic and, in this case, the common value of the spanning tree invariants of the loop systems is n.

1. Introduction

Lind and Tuncel introduce the *spanning tree invariant* as an invariant of block isomorphism of Markov chains [12]. Its definition is as follows. Let P be a (finite and irreducible) stochastic matrix. For every subdigraph H of the underlying digraph $D(P) = (V(P), E(P))$ induced by P, let

$$wt_P(H) = \prod_{e \in E(H)} wt_P(e), \tag{1.1}$$

where $wt_P(e) = P_{xy}$ is the transition probability of the edge $e = xy \in E(P)$ with x and y the initial and terminal vertices of e (since P is stochastic, $\sum_{y \in V(P)} P_{xy} = 1$ for all $x \in V(P)$). For every vertex $u \in V(P)$, let

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.09.002

$$\mathcal{S}(u) = \{T | T \text{ is a spanning tree rooted at } u\}$$

(a subdigraph T is a *spanning tree rooted at u* if u has no outgoing edges in T, every vertex $v \in V(P) \setminus \{u\}$ has a unique outgoing edge in T and there exists a unique path in T from v to u). The *local spanning tree invariant at u* is

$$\tau_u(P) = \sum_{T \in \mathcal{S}(u)} wt_P(T) \tag{1.2}$$

and the *spanning tree invariant* is

$$\tau(P) = \sum_{u \in V(P)} \tau_u(P). \tag{1.3}$$

Our original motivation for looking at the spanning tree invariant was to determine if it was an invariant of *almost isomorphism*, an equivalence relation introduced in [5]. We know from [12] that if the spectrum of P is $\{\lambda_1 = 1, \lambda_2, \ldots, \lambda_n\}$, then

$$\tau(P) = \prod_{k=2}^{n} (1 - \lambda_k)$$

and this implies that the spanning tree invariant is determined by the (*stochastic*) *zeta function* (see [1,14,4])

$$\zeta_P(t) = \frac{1}{\det(I - tP)}.$$

It is well known that almost isomorphism does not preserve the zeta function (see [5,8]). Hence almost isomorphism does not preserve the spanning tree invariant. Another way to see this is by means of *loop systems* (see [5,8,9]). We describe them formally using the semiring $\mathbf{R}_+$ which is defined as follows. Let $(\mathbb{R}_+, \times)$ be the multiplicative group of the positive real numbers. Let $\mathbb{Z}[\mathbb{R}_+]$ be the integral group ring on $\mathbb{R}_+$ and let $\mathbb{Z}_+[\mathbb{R}_+]$ be the semiring consisting of those elements with nonnegative integral coefficients. Let $\mathbf{R} = \mathbb{Z}[\mathbb{R}_+][[t]]$ be the ring of power series with coefficients in $\mathbb{Z}[\mathbb{R}_+]$ and let $\mathbf{R}_+$ be the semiring consisting of those elements with coefficients $\mathbb{Z}_+[\mathbb{R}_+]$. The *first return loop system at* $u \in V(P)$ is described by the power series $f^{(u)} \in \mathbf{R}_+$ defined by

$$1 - f^{(u)}(t) = \frac{\det(I - tP)}{\det(I - tQ^{(u)})},$$

where $Q^{(u)}$ results from P by removing the row and column corresponding to u. For any $n > 1$, the nth coefficient is

$$\sum_{\rho \in \mathbb{R}_+} a_n^{(\rho)}[\rho],$$

where a_n^{ρ} equals the number of first return loops to u with weight ρ (zero is the constant term, an instance of what is known in positive K-theory as the no $\mathbb{Z}_+$-cycles condition of Boyle and Wagoner [6]). This fact comes from carrying out the weights on the following form of the zeta function of a shift of finite type defined by a nonnegative integral matrix A,

$$\frac{1}{\det(I - tA)} = \prod_{\gamma} (1 - t^{|\gamma|})^{-1}$$

with γ ranging over all periodic orbits of length $|\gamma|$ (see [13]).

The power series $f^{(u)}$ determines a countable state Markov chain called a *loop system* (we let $P^{(u)}$ be its transition matrix and henceforth we may identify it with $f^{(u)}$). It is a Markov chain on the *loop digraph* $D(f^{(u)})$ with vertex set $V(f^{(u)})$ and edge set $E(f^{(u)})$ determined by first setting a distinguished vertex that we also denote by $u \in V(f^{(u)})$ and then, for each ρ which appears in $f^{(u)}$ and every $n \geqslant 1$,

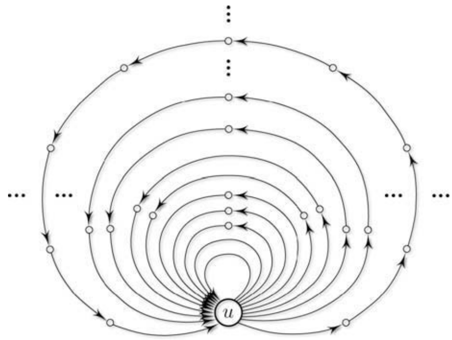

Fig. 1. A loop system at u.

$a_n^{(\rho)}$ simple directed cycles at u of length $n \geqslant 1$, all vertex disjoint except for u, with the weight of the outgoing edges of u being ρ and 1 the weight of all other edges. Fig. 1 describes a loop system with two first return loops of length one, three of length two, two of length three and so forth; the weights are not illustrated (clearly P yields at most one first return loop of length one, the illustration shows more to put this in the context of matrices over more general rings as in [8,14]). Loop systems are important in studying almost isomorphisms, in particular, because an irreducible and strongly positive recurrent Markov shift is always almost isomorphic to its loop systems via finitary isomorphisms which are magic word isomorphisms and hence have finite (exponentially fast) coding time (see also [5,8,18]).

If P were an infinite stochastic matrix, then (1.1), (1.2) and (1.3) would make sense only as limits. We assume that P is finite, but $P^{(u)}$ is infinite. Still, it is simple to determine the spanning trees on the loop digraph $D(f^{(u)})$ and therefore to obtain for the loop system $f^{(u)}$ an expression for the spanning tree invariant $\tau(f^{(u)})$ (see Lemma 2.1). In general a Markov chain possesses loop systems with distinct spanning tree invariants (see Theorem 2.3). Let the *spanning tree invariant spectrum* be $\{\tau(f^{(u)})\}_{u \in V(P)}$ with $f^{(u)}$ the loop system at $u \in V(P)$. The following is a natural question.

Question 1.4. When does the spanning tree invariant spectrum consist of a singleton?

If $|V(P)| \leqslant 3$, then the spanning tree invariants of the loop systems coincide if and only if P is *doubly stochastic* (i.e. if and only if $\sum_{x \in V(P)} P_{xy} = 1$ for all $y \in V(P)$), and in this case $|V(P)|$ is the common value of the spanning tree invariants. We think that this holds in general for irreducible systems, and that it is equivalent to a condition that we call *the doubly stochastic condition on spanning tree invariants*, namely

$$\frac{\tau(P) - \tau_u(P)}{\tau_u(P)} = \frac{\tau(P) - \tau_v(P)}{\tau_v(P)} \quad \forall u, v \in V(P). \tag{1.5}$$

On their spanning tree invariant Lind and Tuncel comment: “Since spanning trees are maximal subgraphs without loops, this is in some sense an operation orthogonal to recurrent behavior”. (They also point out that it is possible to construct finer invariants e.g. by the matrix of powers P^t, see [14].) In this note we use the spanning tree invariant spectrum to detect symmetries like double stochasticity.

There exists a general interest in doubly stochastic processes and the literature is extensive. A fundamental fact on doubly stochastic matrices is due to Birkhoff who showed in [3] that the set of doubly stochastic matrices is a polytope with the permutation matrices as extreme points. Recent work on doubly stochastic matrices includes problems on (inverse) eigenvalues [10,11,15,19,17,16], tridiagonal matrices [7,20] and trees [21,2].

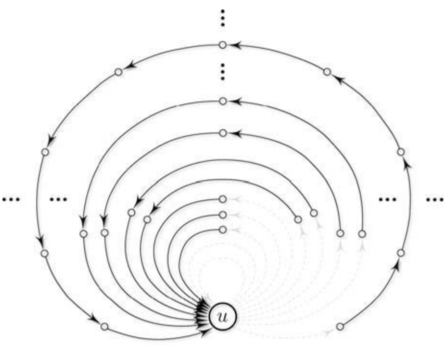

Fig. 2. In a loop system at u, the weight of the spanning tree rooted at $r = u$ is 1.

2. The spanning tree invariant of loop systems

Henceforth P is a finite stochastic irreducible matrix such that the adjacency matrix of the underlying Markov shift has Perron value greater than one.

Lemma 2.1. *Let $f^{(u)} \in \mathbf{R}_+$ be a loop system of P. Write*

$$f^{(u)}(t) = \sum_{k=1}^{\infty} \sum_{\rho \in \mathbb{R}_+} a_k^{(\rho)}[\rho]t^k.$$

Then

$$\tau(f^{(u)}) = 1 + \sum_{k=2}^{\infty} \sum_{\rho \in \mathbb{R}_+} (k-1)a_k^{(\rho)}\rho.$$

Proof. For each vertex $r \in V(f^{(u)})$ in the loop graph defined by $f^{(u)}$, there is one and only one spanning tree rooted at r and is as follows:

1. If the root r is the distinguished vertex u, then the tree is the one that results from removing all edges that start at u. Clearly, in this case, the weight of the tree is 1 (see Fig. 2).
2. If the root r is not the distinguished vertex u, then r belongs to a unique loop γ at u of some length $k \geqslant 2$ and weight ρ (γ possesses $k-1$ of this kind of vertices). The tree is the one that results from removing all edges that start at u except for the edge that belongs to γ, and in its place remove the only edge that starts at r. Clearly, in this case, the weight of the tree is ρ (see Fig. 3).

The result follows. $\square$

Observation 2.2. If $Q^{(u)}$ results from P by removing the row and column corresponding to u, then

$$f^{(u)} = P_{uu}t + \sum_{i,j \neq u} \sum_{k=2}^{\infty} P_{ui}(Q^{(u)})_{ij}^{k-2} P_{ju} t^k$$

and thus

$$\tau(f^{(u)}) = 1 + \sum_{i,j \neq u} \sum_{k=2}^{\infty} (k-1) P_{ui}(Q^{(u)})_{ij}^{k-2} P_{ju}.$$

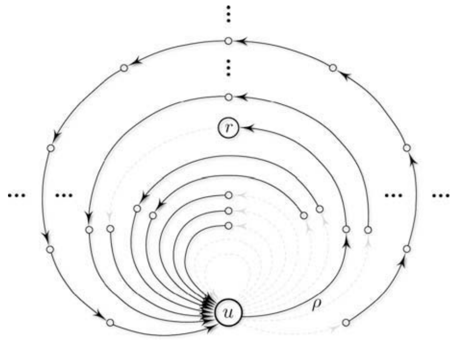

Fig. 3. In a loop system at u, the weight of the spanning tree rooted at $r \neq u$ is ρ.

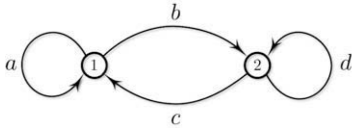

Fig. 4. A 2×2 Markov chain.

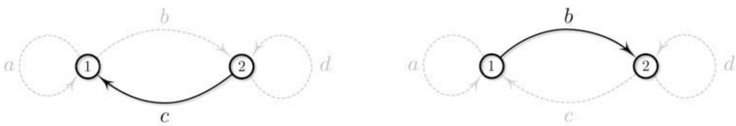

Fig. 5. The spanning trees of a 2×2 Markov chain.

Now, if $\lambda_{Q^{(u)}}$ is the Perron value of $Q^{(u)}$, then $\lambda_{Q^{(u)}} < 1$ because $Q^{(u)} < P$. It follows that there exists $c > 0$ such that $(Q^{(u)})^k_{ij} < c\lambda^k_{Q^{(u)}}$ (see e.g. [13]). Hence $\tau(f^{(u)}) < \infty$ (more generally, it can be shown that if an infinite matrix is strongly positive recurrent [5,9], then its spanning tree invariant is finite).

Theorem 2.3. *Let P be an irreducible stochastic (2×2)-matrix and let $f^{(1)}, f^{(2)} \in \mathbf{R}_+$ be the loop systems at 1 and 2. Then $\tau(f^{(1)}) = \tau(f^{(2)})$ if and only if P is doubly stochastic and in this case $\tau(f^{(1)}) = \tau(f^{(2)}) = 2$.*

Proof. Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an irreducible (so $b > 0$ and $c > 0$) stochastic matrix. It determines a 2×2 Markov chain (see Fig. 4). The local spanning tree invariants of P are as follows (see Fig. 5):

$$\tau_1(P) = c \quad \text{and} \quad \tau_2(P) = b.$$

We will show th at the spanning tree invariants of the loop systems coincide if and only if (1.5) holds, that is, if and only if

$$\frac{\tau_2(P)}{\tau_1(P)} = \frac{\tau_1(P)}{\tau_2(P)} \tag{2.4}$$

and that this is equivalent to P being doubly stochastic.

We let $Q^{(1)} = (d)$ and $Q^{(2)} = (a)$ to write the power series that describe the loop systems

$$f^{(1)}(t) = [a]t + \sum_{n=0}^{\infty} \left(b(Q^{(1)})^n c \right) t^{n+2}$$

and

$$f^{(2)}(t) = [d]t + \sum_{n=0}^{\infty} \left(c(Q^{(1)})^n b \right) t^{n+2}.$$

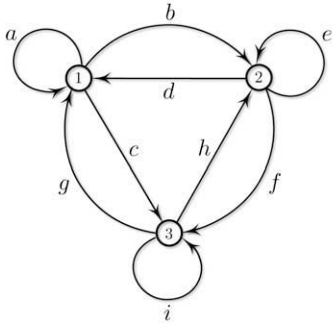

Fig. 6. A 3 × 3 Markov chain.

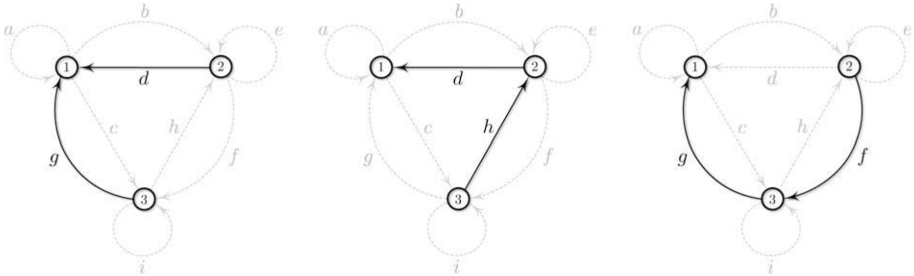

Fig. 7. The spanning trees rooted at 1 that define the local spanning tree invariant $\tau_1(P)$.

Since $(Q^{(1)})^n = (d^n)$ and $(Q^{(2)})^n = (a^n)$, Lemma 2.1 implies

$$\tau(f^{(1)}) = 1 + \sum_{n=0}^{\infty}(n+1)bd^nc = 1 + bc\sum_{n=0}^{\infty}(n+1)d^n = 1 + \frac{bc}{(1-d)^2} = 1 + \frac{b}{c}$$

and

$$\tau(f^{(2)}) = 1 + \sum_{n=0}^{\infty}(n+1)ca^nb = 1 + bc\sum_{n=0}^{\infty}(n+1)a^n = 1 + \frac{bc}{(1-a)^2} = 1 + \frac{c}{b},$$

so that $\tau(f^{(1)}) = \tau(f^{(2)})$ if and only if $b/c = c/b$, that is, if and only if (2.4) holds. In this case, $b^2 = c^2$ and hence $b = c$ and $a = d$ and hence P is doubly stochastic. Clearly $\tau(f^{(1)}) = \tau(f^{(2)}) = 2$. □

Theorem 2.4. *Let P be an irreducible stochastic (3×3)-matrix and let $f^{(k)} \in \mathbf{R}_+$ be the loop systems at $k = 1, 2, 3$. Then $\tau(f^{(1)}) = \tau(f^{(2)}) = \tau(f^{(3)})$ if and only if P is doubly stochastic and in this case $\tau(f^{(1)}) = \tau(f^{(2)}) = \tau(f^{(3)}) = 3$.*

Proof. Let

$$P = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

be an irreducible stochastic matrix (see Fig. 6). The local spanning tree invariants of P are as follows (see Fig. 7):

$$\tau_1(P) = dg + dh + fg,$$
$$\tau_2(P) = bg + bh + ch,$$
$$\tau_3(P) = bf + cd + cf.$$

We will show that the spanning tree invariants of the loop systems coincide if and only if (1.5) holds, that is, if

$$\frac{\tau_2(P)+\tau_3(P)}{\tau_1(P)}=\frac{\tau_1(P)+\tau_3(P)}{\tau_2(P)}=\frac{\tau_1(P)+\tau_2(P)}{\tau_3(P)} \tag{2.6}$$

and that this is equivalent to P being doubly stochastic.

For each $k=1,2,3$, let $Q^{(k)}$ be the matrix that results from P by removing the kth row and column of P. Their eigenvalues are

$$x_1=\frac{1}{2}\left(e+i-\sqrt{\alpha}\right)\quad\text{and}\quad y_1=\frac{1}{2}\left(e+i+\sqrt{\alpha}\right),$$

with $\alpha=e^2-2ei+i^2+4fh\geqslant 0$,

$$x_2=\frac{1}{2}\left(a+i-\sqrt{\beta}\right)\quad\text{and}\quad y_2=\frac{1}{2}\left(a+i+\sqrt{\beta}\right),$$

with $\beta=a^2-2ai+i^2+4cg\geqslant 0$ and

$$x_3=\frac{1}{2}\left(a+e-\sqrt{\gamma}\right)\quad\text{and}\quad y_3=\frac{1}{2}\left(a+e+\sqrt{\gamma}\right)$$

with $\gamma=a^2-2ae+e^2+4bd\geqslant 0$. Diagonalizing we obtain

$$(Q^{(1)})^n=\begin{pmatrix}-\frac{e-i-\sqrt{\alpha}}{2\sqrt{\alpha}}x_1^n+\frac{e-i+\sqrt{\alpha}}{2\sqrt{\alpha}}y_1^n & -\frac{f}{\sqrt{\alpha}}x_1^n+\frac{f}{\sqrt{\alpha}}y_1^n\\ -\frac{h}{\sqrt{\alpha}}x_1^n+\frac{h}{\sqrt{\alpha}}y_1^n & \frac{e-i+\sqrt{\alpha}}{2\sqrt{\alpha}}x_1^n-\frac{e-i-\sqrt{\alpha}}{2\sqrt{\alpha}}y_1^n\end{pmatrix},$$

$$(Q^{(2)})^n=\begin{pmatrix}-\frac{a-i-\sqrt{\beta}}{2\sqrt{\beta}}x_2^n+\frac{a-i+\sqrt{\beta}}{2\sqrt{\beta}}y_2^n & -\frac{c}{\sqrt{\beta}}x_2^n+\frac{c}{\sqrt{\beta}}y_2^n\\ -\frac{g}{\sqrt{\beta}}x_2^n+\frac{g}{\sqrt{\beta}}y_2^n & \frac{a-i+\sqrt{\beta}}{2\sqrt{\beta}}x_2^n-\frac{a-i-\sqrt{\beta}}{2\sqrt{\beta}}y_2^n\end{pmatrix},$$

$$(Q^{(3)})^n=\begin{pmatrix}-\frac{a-e-\sqrt{\gamma}}{2\sqrt{\gamma}}x_3^n+\frac{a-e+\sqrt{\gamma}}{2\sqrt{\gamma}}y_3^n & -\frac{b}{\sqrt{\gamma}}x_3^n+\frac{b}{\sqrt{\gamma}}y_3^n\\ -\frac{d}{\sqrt{\gamma}}x_3^n+\frac{d}{\sqrt{\gamma}}y_3^n & \frac{a-e+\sqrt{\gamma}}{2\sqrt{\gamma}}x_3^n-\frac{a-e-\sqrt{\gamma}}{2\sqrt{\gamma}}y_3^n\end{pmatrix}$$

and then the loop systems are described by

$$f^{(1)}(t)=at+\sum_{n=0}^{\infty}\left(b(Q^{(1)})^n_{11}d+b(Q^{(1)})^n_{12}g+c(Q^{(1)})^n_{21}d+c(Q^{(1)})^n_{22}g\right)t^{n+2},$$

$$f^{(2)}(t)=et+\sum_{n=0}^{\infty}\left(d(Q^{(2)})^n_{11}b+d(Q^{(2)})^n_{12}h+f(Q^{(2)})^n_{21}b+f(Q^{(2)})^n_{22}h\right)t^{n+2},$$

$$f^{(3)}(t)=it+\sum_{n=0}^{\infty}\left(g(Q^{(3)})^n_{11}c+g(Q^{(3)})^n_{12}f+h(Q^{(3)})^n_{21}c+h(Q^{(3)})^n_{22}f\right)t^{n+2}.$$

Let

$$A_1=-bd\left(\frac{e-i-\sqrt{\alpha}}{2\sqrt{\alpha}}\right)-bg\frac{f}{\sqrt{\alpha}}-cd\frac{h}{\sqrt{\alpha}}+cg\left(\frac{e-i+\sqrt{\alpha}}{2\sqrt{\alpha}}\right),$$

$$B_1=bd\left(\frac{e-i+\sqrt{\alpha}}{2\sqrt{\alpha}}\right)+bg\frac{f}{\sqrt{\alpha}}+cd\frac{h}{\sqrt{\alpha}}-cg\left(\frac{e-i-\sqrt{\alpha}}{2\sqrt{\alpha}}\right),$$

$$A_2=-db\left(\frac{a-i-\sqrt{\beta}}{2\sqrt{\beta}}\right)-dh\frac{c}{\sqrt{\beta}}-fb\frac{g}{\sqrt{\beta}}+fh\left(\frac{a-i+\sqrt{\beta}}{2\sqrt{\beta}}\right),$$

$$B_2 = db\left(\frac{a-i+\sqrt{\beta}}{2\sqrt{\beta}}\right) + dh\frac{c}{\sqrt{\beta}} + fb\frac{g}{\sqrt{\beta}} - fh\left(\frac{a-i-\sqrt{\beta}}{2\sqrt{\beta}}\right),$$
$$A_3 = -gc\left(\frac{a-e-\sqrt{\gamma}}{2\sqrt{\gamma}}\right) - gf\frac{b}{\sqrt{\gamma}} - hc\frac{d}{\sqrt{\gamma}} + hf\left(\frac{a-e+\sqrt{\gamma}}{2\sqrt{\gamma}}\right),$$
$$B_3 = gc\left(\frac{a-e+\sqrt{\gamma}}{2\sqrt{\gamma}}\right) + gf\frac{b}{\sqrt{\gamma}} + hc\frac{d}{\sqrt{\gamma}} - hf\left(\frac{a-e-\sqrt{\gamma}}{2\sqrt{\gamma}}\right)$$

(observe that $A_1 + B_1 = bd + cg$, $A_2 + B_2 = db + fh$ and $A_3 + B_3 = gc + hf$). Using Lemma 2.1, the spanning tree invariants of the loop systems are

$$\tau(f^{(1)}) = 1 + \frac{A_1}{(1-x_1)^2} + \frac{B_1}{(1-y_1)^2}, \tag{2.7}$$
$$\tau(f^{(2)}) = 1 + \frac{A_2}{(1-x_2)^2} + \frac{B_2}{(1-y_2)^2}, \tag{2.8}$$
$$\tau(f^{(3)}) = 1 + \frac{A_3}{(1-x_3)^2} + \frac{B_3}{(1-y_3)^2}. \tag{2.9}$$

Since $a = 1 - b - c$, $e = 1 - d - f$ and $i = 1 - g - h$, simplification yields

$$\frac{A_1}{(1-x_1)^2} + \frac{B_1}{(1-y_1)^2} = \frac{\tau_2(P) + \tau_3(P)}{\tau_1(P)}, \tag{2.10}$$
$$\frac{A_2}{(1-x_2)^2} + \frac{B_2}{(1-y_2)^2} = \frac{\tau_1(P) + \tau_3(P)}{\tau_2(P)}, \tag{2.11}$$
$$\frac{A_3}{(1-x_3)^2} + \frac{B_3}{(1-y_3)^2} = \frac{\tau_1(P) + \tau_2(P)}{\tau_3(P)}. \tag{2.12}$$

Hence the spanning tree invariants of the loop systems coincide if and only if (2.6) holds. We will show that this happens precisely when P is doubly stochastic. We have that

$$0 = \frac{\tau_2(P) + \tau_3(P)}{\tau_1(P)} - \frac{\tau_1(P) + \tau_3(P)}{\tau_2(P)} = \frac{\tau_2(P) - \tau_1(P)}{\tau_1(P)\tau_2(P)}\tau(P),$$
$$0 = \frac{\tau_2(P) + \tau_3(P)}{\tau_1(P)} - \frac{\tau_1(P) + \tau_2(P)}{\tau_3(P)} = \frac{\tau_3(P) - \tau_1(P)}{\tau_1(P)\tau_3(P)}\tau(P),$$
$$0 = \frac{\tau_1(P) + \tau_3(P)}{\tau_2(P)} - \frac{\tau_1(P) + \tau_2(P)}{\tau_3(P)} = \frac{\tau_3(P) - \tau_2(P)}{\tau_2(P)\tau_3(P)}\tau(P).$$

This yields the following system of equations

$$bg + bh + ch - dg - dh - fg = 0, \tag{2.13}$$
$$bf + cd + cf - dg - dh - fg = 0, \tag{2.14}$$
$$bf + cd + cf - bg - bh - ch = 0. \tag{2.15}$$

From (2.13) and (2.14) we get the following system of equations on g and h,

$$(b - d - f)g + (b + c - d)h = 0,$$
$$(d + f)g + dh = cd + bf + cf.$$

Its solution is $g = b + c - d$ and $h = -b + d + f$ and therefore $g + h = c + f$. Similarly, (2.14) and (2.15) yield the system of equations on d and f,

$$(c-g-h)d+(b+c-g)f=0,$$
$$cd+(b+c)f=bg+bh+ch.$$

Its solution is $d=b+c-g$ and $f=-c+g+h$ and therefore $d+f=b+h$. Finally, (2.15) and (2.13) yield the following system of equations on b and c,

$$(f-g-h)b+(d+f-h)c=0,$$
$$(g+h)b+hc=dg+dh+fg.$$

Its solution is $b=d+f-h$ and $c=-f+g+h$ and therefore $b+c=d+g$. Then

$$P=\begin{pmatrix}1-b-c & b & c\\ d & 1-d-f & f\\ g & h & 1-g-h\end{pmatrix}$$

is doubly stochastic because $g+h=c+f$, $d+f=b+h$ and $b+c=d+g$. The spanning tree invariants of the loop systems are given by (2.7), (2.8) and (2.9). We show that Eqs. (2.10), (2.11) and (2.12) are all equal to 2. Starting with (2.10), we have

$$\begin{aligned}\frac{b(f+g+h)+c(d+f+h)}{dg+dh+fg} &= \frac{b(2f+c)+c(2d+2f-b)}{cd+df+fg}\\ &= 2\frac{bf+cd+cf}{cd+df+fg}=2\frac{cd+df+fg}{cd+df+fg}=2.\end{aligned}$$

For (2.11), we have

$$\begin{aligned}\frac{f(b+c+g)+d(c+g+h)}{bg+bh+ch} &= \frac{f(d+2g)+d(2g+2h-f)}{bg+dh+gh}\\ &= 2\frac{dg+dh+fg}{bg+dh+gh}=2\frac{bg+dh+gh}{bg+dh+gh}=2.\end{aligned}$$

Finally (2.12) is

$$\begin{aligned}\frac{g(b+d+f)+h(b+c+d)}{bf+cd+cf} &= \frac{g(2b+h)+h(2b+2c-g)}{bc+bf+ch}\\ &= 2\frac{bg+bh+ch}{bc+bf+ch}=2\frac{bc+bf+ch}{bc+bf+ch}=2. \qquad \square\end{aligned}$$

Acknowledgment

We are sincerely grateful to the anonymous referee for reading the original manuscript and making suggestions to improve the paper.

References

[1] M. Artin, B. Mazur, On periodic points, Ann. Math. 81 (1969) 82–99.
[2] M.H. Ahmadi, J.H. Baek, S.G. Hwang, Permanents of doubly stochastic trees, Linear Algebra Appl. 370 (2003) 15–24.
[3] G. Birkhoff, Three observations on linear algebra, Univ. Nac. Tucumn. Rev. Ser. A 5 (1946) 147–151.
[4] M. Boyle, A zeta function for homomorphisms of dynamical systems, J. London Math. Soc. 40 (2) (1989) 335–368.
[5] M. Boyle, J. Buzzi, R. Gómez, Almost isomorphism of countable state Markov shifts, J. für die reine und angewandte Math. (Crelles's J.) 592 (2006) 23–47.
[6] M. Boyle, J.B. Wagoner, Modern Dynamical Systems and Applications, Cambridge Univ. Press, Cambridge, 2004, pp. 45–66.
[7] G. Dahl, Tridiagonal doubly stochastic matrices, Linear Algebra Appl. 290 (2004) 197–208.
[8] R. Gómez, Positive K-theory for finitary isomorphisms of countable state Markov chains, Erogd. Th. Dynam. Syst. 23 (2003) 1485–1504.
[9] B.M. Gurevich, S. Savchenko, Thermodynamical formalism for symbolic Markov chains with a countable number of states (Russian), Uspekhi Mat. Nauk 53 (1998) 3–106., translation in Russian Math. Surveys. 53 (199) 245–344.

[10] S.G. Hwang, S.S. Pyo, The inverse eigenvalue problem for symmetric doubly stochastic matrices, Linear Algebra Appl. 379 (2004) 77–83.
[11] I. Kaddoura, B. Mourad, On a conjecture concerning the inverse eigenvalue problem of 4×4 symmetric doubly stochastic matrices, Int. Math. Forum 3 (31) (2008) 1513–1519.
[12] D. Lind, S. Tuncel, A spanning tree invariant for Markov shifts, Codes, Systems and Graphical Models, IMA Vol. Math. Appl., vol. 123, Minneapolis, MN, 1999.
[13] D. Lind, B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, 1995.
[14] B. Marcus, S. Tuncel, The weight-per-symbol polytope and scaffolds of invariants associated with Markov chains, Ergodic Theory Dynam. Syst. 11 (1991) 129–180.
[15] J. Mashreghi, R. Rivard, On a conjecture about the eigenvalues of doubly stochastic matrices, Linear and Multilinear Algebra 55 (5) (2007) 491–498.
[16] B. Mourad, An inverse problem for symmetric doubly stochastic matrices, Inv. Prob. 19 (2003) 821–831.
[17] B. Mourad, A note on the boundary of the set where the decreasingly ordered spectra of symmetric doubly stochastic matrices lie, Linear Algebra Appl. 419 (2006) 643–647.
[18] W. Parry, Finitary isomorphisms with finite expected coding-lengths, Bull. London Math. Soc. 11 (1979) 170–176.
[19] R. Pereira, M.A. Vali, Inequalities for the spectra of symmetric doubly stochastic matrices, Linear Algebra Appl. 419 (2006) 643–647.
[20] S.Z. Song, Y.B. Jun, Minimum permanents of tridiagonal doubly stochastic matrices, Linear and Multilinear Algebra 50 (4) (2002) 301–306.
[21] X.D. Zhanga, J.X. Wub, Doubly stochastic matrices of trees, Appl. Math. Lett. 18 (2005) 339–343.