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Hypergeometric Solutions of Trigonometric KZ Equations Satisfy Dynamical Difference Equations[1]

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The trigonometric KZ equations associated to a Lie algebra $\mathfrak{g}$ depend on a param-

[illegible]
We prove that the standard hypergeometric solutions of the trigonometric KZ equations associated to sl_N also satisfy the dynamical difference equations.

Key Words: hypergeometric solutions; dynamical equations; KZ equations.

1. INTRODUCTION

Consider the simplest of hypergeometric integrals,

$$I(z, a, b) = \int_0^z t^{a-1}(t-z)^{b-1}\, dt.$$

It satisfies the differential equation

$$\frac{dI}{dz}(z, a, b) = \frac{a+b-1}{z} I(z, a, b), \tag{1}$$

the difference equation

$$I(z, a+1, b) = \frac{az}{a+b} I(z, a, b), \tag{2}$$

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and a similar difference equation with respect to b. Equations (1) and (2) are clearly compatible. More general hypergeometric integrals appear in conformal field theory as integral representations for conformal blocks; see [Ch, CF, DF, Ma, SV, V]. It is known that the integrals satisfy the KZ differential equations. The KZ equations are generalizations of (1). In [TV] a system of difference equations, generalizing Eq. (2), is proposed. The system, proposed in [TV], is compatible with the KZ equations. The difference equations were called the dynamical equations. Both the KZ differential equations and the dynamical difference equations are associated to a given Lie algebra. It was conjectured in [TV] that the hypergeometric integrals, which satisfy the KZ equations, also satisfy the dynamical equations.

In this paper we prove that the hypergeometric integrals solving the (trigonometric) KZ equations associated to sl_N also satisfy the the dynamical equations.

The trigonometric KZ equations have a rational limit called the (standard) rational KZ equations. Under this limiting procedure the dynamical difference equations turn into a system of (dynamical) differential equations compatible with the rational KZ equations. The dynamical differential equations were introduced and studied in [FMTV]; see also [TL]. In [FMTV] it was shown that the hypergeometric solutions of the rational KZ equations also satisfy the dynamical differential equations.

The paper is organized as follows. In Sections 2 and 3 we introduce notation and define the main objects of our study: the trigonometric and rational KZ equations, the dynamical equations for the Lie algebra sl_N.

In Section 4, following [Ma], we present a construction of hypergeometric solutions of the KZ equations with values in a tensor product of highest weight sl_N-modules.

The main result of the paper is Theorem 5.1. We give new formulae for hypergeometric solutions related to special (normal) orders on the set of positive roots of sl_N in Sections 6 and 8, and new formulae for the dynamical difference equations in terms of the Shapovalov form in Section 7. Both results are used in the proof of Theorem 5.1, given in Section 9.

In Appendixes A and B we adapt a theorem from [Ma] and a theorem from [EFK], respectively, to our setting. In Appendix C we give explicit formulae illustrating the main objects of our study in the case of the Lie algebra sl_3.

2. RATIONAL AND TRIGONOMETRIC KZ EQUATIONS

2.1. *Preliminaries.* Let $\mathbf{g}$ be a simple complex Lie algebra with root space decomposition $\mathbf{g} = \mathbf{h} \oplus (\bigoplus_{\alpha \in \Sigma} \mathbb{C} e_\alpha)$ where $\Sigma \subset \mathbf{h}^*$ is the set of roots.

Fix a system of simple roots $\alpha_1, ..., \alpha_r$. Let $\Sigma_\pm$ be the set of positive (negative) roots. Let $\mathbf{n}_\pm = \bigoplus_{\alpha \in \Sigma_\pm} \mathbf{g}_\alpha$. Then $\mathbf{g} = \mathbf{n}_+ \oplus \mathbf{h} \oplus \mathbf{n}_-$.

Let $(,)$ be an invariant bilinear form on $\mathbf{g}$. The form gives rise to a natural identification $\mathbf{h} \to \mathbf{h}^*$. We use this identification and make no distinction between $\mathbf{h}$ and $\mathbf{h}^*$. This identification allows us to define a scalar product on $\mathbf{h}^*$. We use the same notation $(,)$ for the pairing $\mathbf{h} \otimes \mathbf{h}^* \to \mathbb{C}$.

We use the notation $Q = \bigoplus_{k=1}^r \mathbb{Z}\alpha_k$—root lattice; $Q^+ = \bigoplus_{k=1}^r \mathbb{Z}_{\geqslant 0}\alpha_k$; $Q^\vee = \bigoplus_{k=1}^r \mathbb{Z}\alpha_k^\vee$—dual root lattice, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$; $P = \{\lambda \in \mathbf{h} \mid (\lambda, \alpha_k^\vee) \in \mathbb{Z}\}$—weight lattice; $P^+ = \{\lambda \in \mathbf{h} \mid (\lambda, \alpha_k^\vee) \in \mathbb{Z}_{\geqslant 0}\}$—cone of dominant integral weights; $\omega_k \in P^+$—fundamental weights: $(\omega_k, \alpha_l^\vee) = \delta_{k,l}$; $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma_+} \alpha = \sum_{k=1}^r \omega_k$; $P^\vee = \bigoplus_{k=1}^r \mathbb{Z}\omega_k^\vee$—dual weight lattice, where $\omega_k^\vee$—dual fundamental weights: $(\omega_k^\vee, \alpha_l) = \delta_{k,l}$.

Define a partial order on $\mathbf{h}$ putting $\mu < \lambda$ if $\lambda - \mu \in Q^+$.

For $\alpha \in \Sigma$ choose generators $e_\alpha \in \mathbf{g}_\alpha$ so that $(e_\alpha, e_{-\alpha}) = 1$. For any α, the triple

$$H_\alpha = \alpha^\vee, \qquad E_\alpha = \frac{2}{(\alpha, \alpha)} e_\alpha, \qquad F_\alpha = e_{-\alpha}$$

forms an sl_2-subalgebra in $\mathbf{g}$, $[H_\alpha, E_\alpha] = 2E_\alpha$, $[H_\alpha, F_\alpha] = -2F_\alpha$, $[E_\alpha, F_\alpha] = H_\alpha$.

The Chevalley involution τ of $\mathbf{g}$ is defined by $E_{\alpha_k} \mapsto -F_{\alpha_k}$, $F_{\alpha_k} \mapsto -E_{\alpha_k}$, $H_{\alpha_k} \mapsto -H_{\alpha_k}$, $k = 1, ..., r$. The antipode map A of $\mathbf{g}$ is defined by $g \mapsto -g$ for $g \in \{E_{\alpha_k}, F_{\alpha_k}, H_{\alpha_k}\}_{k=1}^r$.

Let $U(\mathbf{g})$ be the universal enveloping algebra of $\mathbf{g}$. The Chevalley involution, τ, extends to an involutive automorphism of $U(\mathbf{g})$ which permutes $U(\mathbf{n}_+)$ and $U(\mathbf{n}_-)$. The antipode map, A, extends to an involutive anti-automorphism of $U(\mathbf{g})$ which preserves $U(\mathbf{n}_+)$ and $U(\mathbf{n}_-)$.

Let $s_k: \mathbf{h} \to \mathbf{h}$ denote the simple reflection $s_k(\lambda) = \lambda - (\alpha_k^\vee, \lambda)\,\alpha_k$ for all $\lambda \in \mathbf{h}$, and let $\mathbb{W}$ be the Weyl group, generated by $s_1, ..., s_r$. For an element $w \in \mathbb{W}$, denote $l(w)$ the length of the minimal (reduced) presentation of w as a product of generators $s_1, ..., s_r$.

For any dual fundamental weight $\omega_k^\vee$ define an element $\omega_{[k]} = \omega_0 \omega_0^k \in \mathbb{W}$ where ω_0 (respectively, ω_0^k) is the longest element in $\mathbb{W}$ (respectively, in $\mathbb{W}^k$ generated by all simple reflections s_l preserving $\omega_k^\vee$).

Let $\lambda \in \mathbf{h}$ be a weight. Let $\mathbb{C}_\lambda$ be the one-dimensional $(\mathbf{h} \oplus \mathbf{n}_+)$-module such that $\mathbb{C}_\lambda = \mathbb{C}v_\lambda$ with $h\,v_\lambda = \lambda(h)\,v_\lambda$ for any $h \in \mathbf{h}$ and $\mathbf{n}_+\,v_\lambda = 0$. The Verma module with highest weight λ is the induced module $M_\lambda = \mathrm{Ind}_{(\mathbf{h} \oplus \mathbf{n}_+)}^{\mathbf{g}} \mathbb{C}_\lambda$. M_λ is a free $U(\mathbf{n}_-)$-module and can be identified with $U(\mathbf{n}_-)$ as a linear space by the map $U(\mathbf{n}_-) \to M_\lambda$, $u \mapsto u\,v_\lambda$ for any $u \in U(\mathbf{n}_-)$. A highest weight module of weight λ is a quotient module of the Verma module with highest weight λ. All the highest weight modules in this paper are equipped with a distinguished generator, the highest vector.

Let V_λ be a highest weight **g**-module with highest weight λ and highest weight vector v_λ. We have a weight decomposition $V_\lambda = \bigotimes_{\nu \leqslant \lambda} V_\lambda[\nu]$. Define $V_\lambda^* = \bigotimes_{\nu \leqslant \lambda} V_\lambda[\nu]^*$ as the restricted dual module to V_λ with the **g**-action $\langle g\phi, a\rangle = -\langle \phi, ga\rangle$ for every $g \in \mathbf{g}$, $a \in V_\lambda$, $\phi \in V_\lambda^*$. Then V_λ^* is a lowest weight **g**-module with the homogeneous lowest weight vector v_λ^*, such that $\langle v_\lambda^*, v_\lambda\rangle = 1$.

The Shapovalov form $S_\lambda(. , .)$ on V_λ is the unique symmetric bilinear form such that

$$S_\lambda(v_\lambda, v_\lambda) = 1, \qquad S_\lambda(gu, v) = S_\lambda(u, (A \circ \tau)(g)\, v),$$
$$\forall u, v \in V_\lambda, \forall g \in \{E_{\alpha_k}, F_{\alpha_k}\}_{k=1}^r.$$

The form $S_\lambda(. , .)$ is non-degenerate if and only if V_λ is irreducible. In particular, it is non-degenerate for generic values of λ.

2.2. *KZ Equations.* Let $\{x_k\}$ be any orthonormal basis of the Cartan subalgebra **h**. Set

$$\Omega^0 = \tfrac{1}{2} \sum_k x_k \otimes x_k, \qquad \Omega^+ = \Omega^0 + \sum_{\alpha \in \Sigma_+} e_\alpha \otimes e_{-\alpha}, \qquad \Omega^- = \Omega^0 + \sum_{\alpha \in \Sigma_+} e_{-\alpha} \otimes e_\alpha .$$

Define the Casimir operator Ω and the trigonometric R-matrix $r(z)$ by

$$\Omega = \Omega^+ + \Omega^-, \qquad r(z) = \frac{\Omega^+ z + \Omega^-}{z-1}.$$

Let $V' = V_1 \otimes \cdots \otimes V_{n+1}$, where V_j is a **g**-module. The *rational KZ operators*, $\nabla_{KZ,i}(\kappa)$, acting on a function $u(z_1, ..., z_{n+1})$ of $n+1$ complex variables with values in V' are

$$\nabla_{KZ,i}(\kappa) = \kappa \frac{\partial}{\partial z_i} - \sum_{j, j \neq i} \frac{\Omega^{(ij)}}{z_i - z_j}, \qquad i = 1, ..., n+1, \tag{3}$$

where κ is a complex parameter. The *rational KZ equations* are

$$\nabla_{KZ,i}(\kappa)\, u(z_1, ..., z_{n+1}) = 0, \qquad i = 1, ..., n+1. \tag{4}$$

The rational KZ equations are compatible, $[\nabla_{KZ,i}, \nabla_{KZ,j}] = 0$. The KZ operators commute with the **g** action on V'. Thus, the KZ operators preserve every subspace of V' consisting of all singular vectors of a given weight.

Let $V = V_1 \otimes \cdots \otimes V_n$, where V_j is a $\mathbf{g}$–module. The *trigonometric KZ operators*, $\nabla_i(\kappa, \lambda)$, with parameters $\kappa \in \mathbb{C}$ and $\lambda \in \mathbf{h}$ acting on a function $v(z_1, ..., z_n, \lambda)$ of n complex variables with values in V are

$$(5) \qquad \nabla_i(\kappa, \lambda) = \kappa z_i \frac{\partial}{\partial z_i} - \lambda^{(i)} - \sum_{j,\, j \neq i} r(z_i/z_j)^{(ij)}, \qquad i = 1, ..., n.$$

The *trigonometric KZ equations* are

$$(6) \qquad \nabla_i(\kappa, \lambda)\, v(z_1, ..., z_n, \lambda) = 0, \qquad i = 1, ..., n.$$

The trigonometric KZ equations are compatible, $[\nabla_i, \nabla_j] = 0$. The trigonometric KZ operators commute with the $\mathbf{h}$ action on V. Thus, the trigonometric KZ operators preserve every weight subspace of V.

2.3. *A Relation between Rational KZ Equations and Trigonometric KZ Equations.* For every $j = 1, ..., n+1$, let V_j be a highest weight $\mathbf{g}$-module with highest weight Λ_j and highest weight vector v_j. Set $V' = V_1 \otimes \cdots \otimes V_{n+1}$ and $V = V_1 \otimes \cdots \otimes V_n$.

Let V^*_{n+1} be the dual module to V_{n+1} with the homogeneous lowest weight vector v^*_{n+1}. Define a multi-linear map $|v^*_{n+1}\rangle\colon V' \to V$ by $y_1 \otimes \cdots \otimes y_n \otimes y_{n+1}\, |v^*_{n+1}\rangle = \langle v^*_{n+1}, y_{n+1}\rangle\, y_1 \otimes \cdots \otimes y_n$, for any $y_1 \otimes \cdots \otimes y_n \otimes y_{n+1} \in V'$.

The following well known fact, see [EFK], for example, describes the transition from rational KZ equations to trigonometric KZ equations.

PROPOSITION 2.1. *Fix a weight subspace $V'[v'] \subset V'$, $v' = \sum_{j=1}^{n+1} \Lambda_j - v_0$, where $v_0 \in Q_+$. Let $u\colon \mathbb{C}^{n+1} \to V'$ be a solution of the rational KZ equations with parameter $\kappa \in \mathbb{C}$ taking values in the subspace of $V'[v']$ consisting of all singular vectors. Set $v = \sum_{j=1}^{n} \Lambda_j - v_0$.*

Then $v(z_1, ..., z_n) = u(z_1, ..., z_n, 0)\, |v^*_{n+1}\rangle \prod_{i=1}^{n} z_i^{(\Lambda_i, \Lambda_i + 2\rho)/2\kappa}$ is a solution of the trigonometric KZ equations with values in the weight subspace $V[v] \subset V$ with parameter $\lambda = \Lambda_{n+1} + \rho + \frac{1}{2} v \in \mathbf{h}$ and the same parameter $\kappa \in \mathbb{C}$.

A proof is given in Appendix B.

3. DYNAMICAL DIFFERENCE EQUATIONS FOR $\mathbf{g} = sl_N$ [TV]

3.1. *Operators $\mathbb{B}^{\alpha}_V(\lambda)$, $\mathbb{B}_{\omega, V}(\lambda)$.* Let $\mathbf{g}$ be a simple complex Lie algebra. Fix a root $\alpha \in \Sigma_+$. Consider the sl_2 subalgebra of $\mathbf{g}$ with generators $H = H_\alpha$, $E = E_\alpha$, $F = F_\alpha$. For $t \in \mathbb{C}$, introduce

$$(7) \qquad p(t; H, E, F) = \sum_{k=0}^{\infty} F^k E^k \frac{1}{k!} \prod_{j=0}^{k-1} \frac{1}{(t - H - j)}.$$

The series $p(t, H, E, F)$ is an element of a suitable completion of $U(sl_2)$.

Let $V = V_1 \otimes \cdots \otimes V_n$ be a tensor product of highest weight $\mathbf{g}$-modules. For $\alpha \in \Sigma$, $\lambda \in \mathbf{h}$, the linear operator $\mathbb{B}_V^\alpha(\lambda)\colon V \to V$ is defined by the rule: for any $\nu \in \mathbf{h}$, and any $v \in V[\nu]$, we have

$$\mathbb{B}_V^\alpha(\lambda)\, v = p((\lambda + \tfrac{1}{2}\nu, \alpha^\vee) - 1; H_\alpha, E_\alpha, F_\alpha)\, v.$$

For every simple reflection $s_k \in \mathbb{W}$ define $\mathbb{B}_{s_k, V}(\lambda)\colon V \to V$ by $\mathbb{B}_{s_k, V}(\lambda) = \mathbb{B}_V^{\alpha_k}(\lambda)$. For every $w \in \mathbb{W}$, such that $l(w) > 1$, the operator $\mathbb{B}_{w, V}(\lambda)$ is defined by the rule. If $w = w_1 w_2$, where $w_1, w_2 \in \mathbb{W}$ and $l(w_1 w_2) = l(w_1) + l(w_2)$, then

$$\mathbb{B}_{w, V}(\lambda) = w_2^{-1}(\mathbb{B}_{w_1, V}(w_2(\lambda)))\, \mathbb{B}_{w_2, V}(\lambda).$$

3.2. *The Lie Algebra sl_N.* Let $\{e_{k,l}\}_{k,l}$, $k, l = 1, \ldots, N$, be the standard generators of the Lie algebra gl_N, $[e_{k,l}, e_{k',l'}] = \delta_{l,k'} e_{k,l'} - \delta_{k,l'} e_{l,k'}$. The Lie algebra sl_N is the Lie subalgebra of gl_N such that $sl_N = \mathbf{n}_+ \oplus \mathbf{h} \oplus \mathbf{n}_-$ where

$$\mathbf{n}_+ = \bigoplus_{1 \leqslant k < l \leqslant N} \mathbb{C}\, e_{k,l}, \qquad \mathbf{n}_- = \bigoplus_{1 \leqslant k < l \leqslant N} \mathbb{C}\, e_{l,k},$$

and $\mathbf{h} = \{\lambda = \sum_{k=1}^N \lambda_k e_{k,k} \mid \lambda_k \in \mathbb{C}, \sum_{k=1}^N \lambda_k = 0\}$.

The invariant scalar product on sl_N is defined by $(e_{k,l}, e_{k',l'}) = \delta_{k,l'} \delta_{l,k'}$.

The roots of sl_N are $\alpha_{k,l} = e_{k,k} - e_{l,l}$, $k \neq l$. The positive roots are $\{\alpha_{k,l}\}_{k<l}$. The simple roots are $\alpha_k = e_{k,k} - e_{k+1,k+1}$, $k = 1, \ldots, N-1$. For a positive root $\alpha_{k,l}$, the elements $H_{\alpha_{k,l}} = e_{k,k} - e_{l,l}$, $E_{\alpha_{k,l}} = e_{k,l}$, $F_{\alpha_{k,l}} = e_{l,k}$ generate the sl_2 subalgebra associated with $\alpha_{k,l}$.

We have $\alpha^\vee = \alpha$ for any root.

For any $k < l$, we have

$$F_{\alpha_{k,l}} = [F_{\alpha_{l-1}}, [\ldots, [F_{\alpha_{k+1}}, F_{\alpha_k}] \ldots]]. \tag{8}$$

Define the standard linear order on the set of positive roots, $\alpha_{k,l} \succ \alpha_{k',l'}$ if and only if $l > l'$, or $l = l'$ and $k > k'$.

The Weyl group $\mathbb{W}$ is the symmetric group S^N permuting coordinates of $\lambda \in \mathbf{h}$. The (dual) fundamental weights are $\omega_k = \omega_k^\vee = \sum_{h=1}^k (1 - \frac{k}{N})\, e_{h,h} - \sum_{h=k+1}^N \frac{k}{N}\, e_{h,h}$, and the permutations $w_{[k]}^{-1} \in S^N$ have the form $w_{[k]}^{-1} = \left(\begin{smallmatrix} 1 & 2 & \ldots & N-k & N-k+1 & \ldots & N \\ k+1 & k+2 & \ldots & N & 1 & \ldots & k \end{smallmatrix}\right)$, $k = 1, \ldots, N-1$.

3.3. *Dynamical Difference Equations.* The *dynamical difference equations* on a V-valued function $v(z_1, \ldots, z_n, \lambda)$ for sl_N are

$$\begin{aligned} &v(z_1, \ldots, z_n, \lambda + \kappa\omega_k^\vee) \\ &\quad = K_k(z_1, \ldots, z_n, \lambda)\, v(z_1, \ldots, z_n, \lambda), \qquad k = 1, \ldots, N-1, \end{aligned} \tag{9}$$

where

$$K_k(z_1, \ldots, z_n, \lambda) = \prod_{j=1}^{n} z_j^{(\omega_k^\vee)^{(j)}} \mathbb{B}_{\omega[k], V}(\lambda).$$

The operators $K_k(z, \lambda)$ preserve the weight decomposition of V.

THEOREM 3.1 (Theorem 17 in [TV]). *The dynamical equations* (9) *together with the trigonometric KZ equations* (6) *form a compatible system of equations. Namely,*

(10)

$$[\nabla_i(\kappa, \lambda), \nabla_j(\kappa, \lambda)] = 0, \qquad \nabla_j(\kappa, \lambda + \kappa\omega_k^\vee)\, K_k(z, \lambda) = K_k(z, \lambda)\, \nabla_j(\kappa, \lambda),$$
$$K_k(z, \lambda + \kappa\omega_l^\vee)\, K_l(z, \lambda) = K_l(z, \lambda + \kappa\omega_k^\vee)\, K_k(z, \lambda)$$

for all $i, j = 1, \ldots, n$, *and* $k, l = 1, \ldots, N-1$.

4. HYPERGEOMETRIC SOLUTIONS OF THE TRIGONOMETRIC KZ EQUATIONS FOR sl_N

We use the construction in [Ma] of hypergeometric solutions of the rational sl_N KZ equations in a tensor product of lowest weight modules. We modify this procedure in Appendix A to a construction of solutions of the rational KZ equations in a tensor product of highest weight modules; cf. [SV]. We present the result below. Then we use Proposition 2.1 to give hypergeometric solutions of the trigonometric KZ equations.

There are different constructions of hypergeometric solutions of KZ equations; cf. [Ch, CF, DF, Ma, SV, V]. One should expect that all of them give the same result, but this was never checked off as far as we know.

Let $V' = V_1 \otimes \cdots \otimes V_{n+1}$, where V_j is a highest weight sl_N module with highest weight Λ_j and highest weight vector v_j. Fix a weight subspace $V'[v'] \subset V'$, $v' = \sum_{j=1}^{n+1} \Lambda_j - \sum_{k=1}^{N-1} m_k \alpha_k$, where $\sum_{k=1}^{N-1} m_k \alpha_k \in Q^+$. Set $m = \sum_{k=1}^{N-1} m_k$ and $v_0 = \sum_{k=1}^{N-1} m_k \alpha_k$.

The function Φ'. Consider complex spaces $\mathbb{C}^{n+1}$ with coordinates z_1, $\ldots$, z_{n+1}, and $\mathbb{C}^m$ with coordinates $t_k^{(d)}$, $k = 1, \ldots, N-1$, $d \in S_k = \{1, \ldots, m_k\}$. Fix an order $\ll$ on the set of coordinates in $\mathbb{C}^m$, $(k, d) \ll (k', d')$ if and only if $k<k'$, or $k=k'$ and $d<d'$. Define a multi-valued function $\Phi'\colon \mathbb{C}^{n+1}_{z'}\times\mathbb{C}^m_t\to\mathbb{C}$

(11)
$$\Phi'(z',t)=\prod_{i<j}(z_i-z_j)^{(\Lambda_i,\Lambda_j)}\prod_{(k,d),j}(t_k^{(d)}-z_j)^{-(\alpha_k,\Lambda_j)}\prod_{(k,d)\ll(l,d')}(t_k^{(d)}-t_l^{(d')})^{(\alpha_k,\alpha_l)}.$$

The standard PBW-basis. Any order on the set Σ_+ of positive roots of sl_N induces an order on the set $\{-e_{l,k}\}_{k<l}$ which is a basis of $\mathbf{n}_-$, $-e_{l,k}$ succeeds $-e_{l',k'}$ if and only if $\alpha_{k,l}$ succeeds $\alpha_{k',l'}$. The standard order $\succ$ on Σ_+ induces the standard order $\succ$ on $\{-e_{l,k}\}_{k<l}$, $-e_{l,k}\succ -e_{l',k'}$ if and only if $l>l'$, or $l=l'$ and $k>k'$. The corresponding (standard) PBW-basis of $U(\mathbf{n}_-)$ is

$$\left\{F_{I_0}=(-1)^{\sum i_{l,k}}\frac{e_{N,N-1}^{i_{N,N-1}}}{i_{N,N-1}!}\cdots\frac{e_{2,1}^{i_{2,1}}}{i_{2,1}!}\right\},$$

where $I_0=\{i_{l,k}\}_{l>k}$ runs over all sequences of non-negative integers.

Let $F_{I_1},\ldots,F_{I_{n+1}}$ be elements of the standard PBW-basis, $I_j=\{i^j_{2,1},\ldots,i^j_{N,N-1}\}$. Set $I=(I_1,\ldots,I_{n+1})$. The corresponding "monomial" vector $F_Iv=F_{I_1}v_1\otimes\cdots\otimes F_{I_{n+1}}v_{n+1}$ lies in $V'[v']$ if

(12)
$$\sum_{j=1}^{n+1}\sum_{k=1}^{h}\sum_{l=h+1}^{N}i^j_{l,k}=m_h,\qquad\text{for all}\quad h=1,\ldots,N-1.$$

Denote $P(v_0,n+1)$ as the set of all indices I corresponding to monomial vectors in $V'[v']$. The set $\{F_Iv\}_{I\in P(v_0,n+1)}$ forms a basis of $V'[v']$ provided the tensor factors of V' are Verma modules and generates $V'[v']$ otherwise.

The rational function ϕ'. For $I\in P(v_0,n+1)$ and $h=1,\ldots,N-1$, define two index sets,

(13)
$$S(I)=\{(j,k,l,q)\mid 1\leqslant j\leqslant n+1,\,1\leqslant k<l\leqslant N,\,1\leqslant q\leqslant i^j_{l,k}\}$$
$$S_h(I)=\{s=(j,k,l,q)\in S(I)\mid k\leqslant h<l\}.$$

Condition (12) implies $|S_h(I)|=m_h$ for all $h=1,\ldots,N-1$.

For every h fix a bijection $\beta_h(I)\colon S_h(I)\to S_h$. For $s=(j,k,l,q)\in S(I)$, define rational functions

(14)
$$f^{(s)}=\prod_{h=k}^{l-2}\frac{1}{t_h^{(\beta_h(I)(s))}-t_{h+1}^{(\beta_{h+1}(I)(s))}},\qquad \phi^{(s)}=f^{(s)}\frac{1}{t_{l-1}^{(\beta_{l-1}(I)(s))}-z_j}.$$

Set

$$\phi(I) = \prod_{s \in S(I)} \phi^{(s)}, \qquad \phi'(z', t) = \sum_{I \in P(v_0, n+1)} \phi(I)\, F_I v. \tag{15}$$

EXAMPLES. Let $\mathbf{g} = sl_3$. Then $\{-e_{3,2}, -e_{3,1}, -e_{2,1}\}$ is a basis of $\mathbf{n}_-$. Let $n = 0$.

(a) $v' = \Lambda_1 - \alpha_1 - \alpha_2$,

$$\phi' = \frac{1}{(t_1^{(1)} - z_1)(t_2^{(1)} - z_1)}\, e_{3,2} e_{2,1} v_1 - \frac{1}{(t_1^{(1)} - t_2^{(1)})(t_2^{(1)} - z_1)}\, e_{3,1} v_1 .$$

(b) $v' = \Lambda_1 - 2\alpha_1 - \alpha_2$,

$$\phi' = \frac{1}{(t_1^{(1)} - t_2^{(1)})(t_2^{(1)} - z_1)(t_1^{(2)} - z_1)}\, e_{3,1} e_{2,1} v_1 - \frac{1}{(t_1^{(1)} - z_1)(t_1^{(2)} - z_1)(t_2^{(1)} - z_1)}\, e_{3,2} \frac{e_{2,1}^2}{2} v_1 .$$

The integrals. Consider the integral with values in $V'[v']$

$$u(z') = \int_{\gamma(z')} \Phi'(z', t)^{\frac{1}{\kappa}} \phi'(z', t)\, dt,$$

where $dt = \prod_{(k,d)} dt_k^{(d)}$ and $\gamma(z')$ in $\{z'\} \times \mathbb{C}_t^m$ is a horizontal family of m-dimensional cycles of the twisted homology defined by the multi-valued function $(\Phi')^{1/\kappa}$; see [Ma,SV, V].

For a positive integer m_0, let Σ_{m_0} be the symmetric group on m_0 elements. The group $\Sigma(m_1, ..., m_{N-1}) = \Sigma_{m_1} \times ... \times \Sigma_{m_{N-1}}$ acts on points $t = \{t_k^{(d)} \mid k = 1, ..., N-1, d \in S_k\}$ of $\mathbb{C}^m$ permuting coordinates in each group $\{t_k^{(d)} \mid d \in S_k\}$. Denote $\mathscr{D}(z')$ as the union of hyperplanes $\bigcup_{(k,d),j} \{t_k^{(d)} = z_j\} \cup \bigcup_{(k,d),(k+1,d')} \{t_k^{(d)} = t_{k+1}^{(d')})\}$ in $\mathbb{C}_t^m$. We always make the following assumption on $\gamma(z')$.

Assumption [Ma]. For any rational function ϕ with poles in $\mathscr{D}(z')$ and any permutation $\sigma \in \Sigma(m_1, ..., m_{N-1})$ we have $\int_{\gamma(z')} \Phi'(z', t)^{1/\kappa} \phi'(z', t)\, dt = \int_{\gamma(z')} \Phi'(z', \sigma t)^{1/\kappa} \phi'(z', \sigma t)\, dt$.

In other words we always assume that the cycle $\gamma(z')$ is skew symmetric with respect to permutations in $\Sigma(m_1, ..., m_{N-1})$.

THEOREM 4.1 Theorem 2.4 in [Ma] and Corollary 10.3 of the present paper). *The function*

$$u(z') = \int_{\gamma(z')} \Phi'(z', t)^{\frac{1}{\kappa}} \phi'(z', t)\, dt$$

takes values in the subspace of singular vectors in $V'[\nu']$ and satisfies the rational KZ equations with parameter $\kappa \in \mathbb{C}$.

The function u is called a hypergeometric solution of the rational KZ equation. Different solutions correspond to different choices of the horizontal family $\gamma(z')$.

Remark. The assumption on the cycles of integration implies that the function $u(z')$ does not depend on the choice of bijections $\{\beta_h(I)\}$.

Let $V = V_1 \otimes \cdots \otimes V_n$. Fix a weight subspace $V[\nu] \subset V$, $\nu = \sum_{j=1}^{n} \Lambda_j - \nu_0$. Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Define a function

(16)

$$\Phi(z, t; \lambda) = \prod_{i<j} (z_i - z_j)^{(\Lambda_i, \Lambda_j)} \prod_{(k, d), j} (t_k^{(d)} - z_j)^{-(\alpha_k, \Lambda_j)} \prod_{(k, d) < (l, d')} (t_k^{(d)} - t_l^{(d')})^{(\alpha_k, \alpha_l)}$$
$$\times \prod_{(k, d)} (t_k^{(d)})^{-(\alpha_k, \lambda - \rho - \nu/2)} \times \prod_{i=1}^{n} z_i^{(\Lambda_i, \lambda - \nu/2 + \Lambda_i/2)}.$$

Define a rational function $\phi(z, t)$ with values in the weight subspace $V[\nu]$,

$$\phi(z, t) = \sum_{I \in P(\nu_0, n)} \phi(I)\, F_I v.$$

Consider the integral with values in $V[\nu]$, $\int_{\gamma(z)} \Phi(z, t; \lambda)^{1/\kappa} \phi(z, t)\, dt$, where $\gamma(z)$ in $\{z\} \times \mathbb{C}^m$ is a horizontal family of m-dimensional cycles of the twisted homology defined by the multi-valued function $(\Phi)^{1/\kappa}$. Consider the union of hyperplanes $\mathscr{D} = \mathscr{D}(z_1, \ldots, z_n, 0)$ in $\mathbb{C}_t^m$. We always make the following assumption on $\gamma(z)$.

Assumption. For any rational function ϕ with poles in $\mathscr{D}$ and any permutation $\sigma \in \Sigma(m_1, \ldots, m_{N-1})$ we have $\int_{\gamma(z)} \Phi(z, t; \lambda)^{1/\kappa} \phi(z, t)\, dt = \int_{\gamma(z)} \Phi(z, \sigma t; \lambda)^{1/\kappa} \phi(z, \sigma t)\, dt$.

COROLLARY 4.2. *The function*

$$v(z; \lambda) = \int_{\gamma(z)} \Phi(z, t; \lambda)^{1/\kappa} \phi(z, t)\, dt$$

takes values in the weight space $V[\nu]$ *and satisfies the trigonometric KZ equations with parameters* $\kappa \in \mathbb{C}$ *and* $\lambda \in \mathbf{h}$.

Proof. The statement of the corollary follows from Theorem 4.1 and Proposition 2.1. ▌

The function v is called a hypergeometric solution of the trigonometric KZ equation.

5. THE MAIN RESULT

For any $k \in \{1, ..., N-1\}$ we have

$$\Phi^{\frac{1}{\kappa}}(z, t; \lambda+\kappa\omega_k^\vee) = \left(\prod_{j=1}^{n} z_j^{(\Lambda_j, \omega_k^\vee)} \prod_{d=1}^{m_k} \frac{1}{t_k^{(d)}} \right) \Phi^{\frac{1}{\kappa}}(z, t; \lambda).$$

The product $\prod_{d=1}^{m_k} (t_k^{(d)})^{-1}$ is a univalued function, which never vanishes. Therefore, we can and will consider any horizontal family of m-dimensional cycles, $\gamma(z)$ in $\{z\} \times \mathbb{C}^m$, of the twisted homology defined by the multi-valued function $(\Phi(z, t; \lambda))^{1/\kappa}$ as a horizontal family of m-dimensional cycles of the twisted homology defined by the multi-valued function $(\Phi(z, t; \lambda+\kappa\omega_k^\vee))^{1/\kappa}$. This identification implies that if we choose a hypergeometric solution of the trigonometric KZ equation, $v(z, \lambda) = \int_{\gamma(z)} \Phi(z, t; \lambda)^{1/\kappa} \phi(z, t)\, dt$ for fixed λ, we obtain a hypergeometric solutions $v(z, \lambda+\kappa\omega^\vee) = \int_{\gamma(z)} \Phi(z, t; \lambda+\kappa\omega^\vee)^{1/\kappa} \phi(z, t)\, dt$ for any $\omega^\vee \in P^\vee$.
The following theorem is the main result of this paper.

THEOREM 5.1. *Let* $V = V_1 \otimes \cdots \otimes V_n$ *be a tensor product of highest weight* sl_N *modules. Let* v *be a hypergeometric solution of the trigonometric KZ equations* (6) *with values in a weight subspace* $V[\nu] \subset V$, $\nu \in \mathbf{h}$. *Then the function* v *also satisfies the dynamical equations* (9),

$$v(z, \lambda+\kappa\omega_k^\vee) = K_k(z, \lambda)\, v(z, \lambda), \qquad k = 1, ..., N-1.$$

The theorem is proved in Section 9.

EXAMPLE. Let $\mathbf{g} = sl_2$ and $n = 1$. Denote α as the positive root of sl_2, and let $V_1 = L_p$, where p is a positive integer and L_p is the $(p+1)$-dimensional irreducible sl_2-module with highest weight $p\frac{\alpha}{2}$ and highest weight vector v_p. Consider a weight subspace $L_p[\nu]$ of L_p, where $\nu = (p-2m)\frac{\alpha}{2}$. Let $v(z_1, \lambda)$ be a hypergeometric solution of the trigonometric

KZ equation with parameters $\lambda \in \mathbf{h}$, $\kappa \in \mathbb{C}$, which takes values in $L_p[\nu]$. Up to a multiplicative constant, it has the form

(17)
$$v(z, \lambda) = z_1^{\frac{1}{2\kappa}(p-2m)(\lambda, \alpha)} I_m\left(-\frac{1}{\kappa}\left((\lambda, \alpha)-1-\frac{p-2m}{2}\right)+1, -\frac{p}{\kappa}, \frac{1}{\kappa}\right) e_{2,1}^m v_p,$$

where $I_m(a, b, c)$ is the Selberg integral, see [Me],

$$I_m(a, b, c) = \int_{0 \leqslant t_1 < \cdots < t_m \leqslant 1} \left(\prod_{k=1}^n t_k^{a-1}(1-t_k)^{b-1}\right) \prod_{1 \leqslant k < l \leqslant m} (t_l - t_k)^{2c} \, dt_1 \cdots dt_m$$
$$= \frac{1}{m!} \prod_{j=0}^{m-1} \frac{\Gamma(1+c+jc)\, \Gamma(a+jc)\, \Gamma(b+jc)}{\Gamma(1+c)\, \Gamma(a+b+(m+j-1)\, c)}.$$

The dynamical equation (9) reduces to the following equation satisfied by the Selberg integral

$$I_m(a+1, b, c) = \left(\prod_{k=1}^m \frac{a+c(m-k)}{a+b+c(2m-k-1)}\right) I_m(a, b, c).$$

Application to determinants. Let V be a finite dimensional sl_N-module, and $V[\nu]$ a weight subspace. For a positive root α fix the sl_2 subalgebra in sl_N generated by H_α, E_α, F_α. Consider V as an sl_2 module. Let $V[\nu]_\alpha \subset V$ be the sl_2-submodule generated by $V[\nu]$. Let $V[\nu]_\alpha = \bigoplus_{m \in \mathbb{Z}_\geqslant} W_m^\alpha \otimes L_{\nu+m\alpha}$ be the decomposition into irreducible sl_2-modules, where $L_{\nu+m\alpha}$ is the irreducible module with highest weight $\nu+m\alpha$ and W_m^α is the multiplicity space. Set $d_m^\alpha = \dim \mathrm{W}_\mathrm{m}^\alpha$ and

$$X_{\alpha, V[\nu]}(\lambda) = \prod_{m=0}^{\infty} \left(\prod_{j=1}^m \frac{\Gamma\left(1-\frac{1}{\kappa}\left((\lambda, \alpha)-\frac{1}{2}(\nu+j\alpha, \alpha)\right)\right)}{\Gamma\left(1-\frac{1}{\kappa}\left((\lambda, \alpha)+\frac{1}{2}(\nu+j\alpha, \alpha)\right)\right)}\right)^{d_m^\alpha},$$

where Γ is the standard gamma function.

Let $V = V_1 \otimes \cdots \otimes V_n$ be a tensor product of finite dimensional sl_N modules. Set $\Lambda_i(\lambda) = \mathrm{tr}_{V[\nu]} \lambda^{(i)}$, $\varepsilon_{i,j} = \mathrm{tr}_{V[\nu]} \Omega^{(i,j)}$, $\gamma_i = \sum_{j, j \neq i} \varepsilon_{i,j}$, where $i, j = 1, \ldots, n$. Set

$$D_{V[\nu]}(z_1, \ldots, z_n, \lambda) = \prod_{i=1}^n z_i^{\frac{1}{\kappa}\left(\Lambda_i(\lambda)-\frac{1}{2}\gamma_i\right)} \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j)^{\frac{\varepsilon_{i,j}}{\kappa}} \prod_{\alpha \in \Sigma_+} X_{\alpha, V[\nu]}(\lambda).$$

Fix a basis $v_1, \ldots, v_d$ in a weight subspace $V[\nu]$. Suppose that $u_i(z, \lambda) = \sum_{j=1}^{d} u_{i,j} v_j$, $i = 1, \ldots, d$, is a set of $V[\nu]$ valued solutions of the trigonometric KZ equations and the dynamical equations.

THEOREM 5.2 (Corollary 19 in [TV]).

$$\det(u_{i,j})_{1 \leqslant i,j \leqslant d} = C_{V[\nu]}(\lambda)\, D_{V[\nu]}(z, \lambda),$$

where $C_{V[\nu]}(\lambda)$ is a function of λ (depending also on $V_1, \ldots, V_n$, ν and κ) such that $C_{V[\nu]}(\lambda)$ is $P^\vee$-periodic, $C_{V[\nu]}(\lambda + \kappa\omega^\vee) = C_{V[\nu]}(\lambda)$ for all $\omega^\vee \in P^\vee$.

Remark. Theorem 5.1 implies that Theorem 5.2 can be applied to any set, $u_1, \ldots, u_d$, of hypergeometric solutions of the trigonometric KZ equations, thus giving a formula for the determinant of hypergeometric integrals.

Continuation of the Example. For $\mathbf{g} = sl_2$, $n = 1$, $V_1 = L_p$, we have

$$X_{\alpha, L_p}(\lambda) = \prod_{j=1}^{m} \frac{\Gamma\left(1 - \frac{1}{\kappa}\left((\lambda, \alpha) - \frac{p}{2} + m - j\right)\right)}{\Gamma\left(1 - \frac{1}{\kappa}\left((\lambda, \alpha) + \frac{p}{2} - m + j\right)\right)}, \qquad z_1^{\frac{A_1(\lambda)}{\kappa}} = z_1^{\frac{(p-2m)(\lambda, \alpha)}{2\kappa}}.$$

The determinant equals the complex function given as the expression preceding $e_{2,1}^m v_p$ in formula (17). Denote this function $u_{1,1}$. The explicit formula for the Selberg integral in terms of gamma functions implies

$$u_{1,1} = \frac{z_1^{\frac{(p-2m)(\lambda,\alpha)}{2\kappa}}}{m!} \prod_{j=0}^{m-1} \times \frac{\Gamma\left(1 + \frac{1}{\kappa}(1+j)\right) \Gamma\left(1 - \frac{1}{\kappa}\left((\lambda, \alpha) - \frac{p}{2} + m - j - 1\right)\right) \Gamma\left(\frac{j-p}{\kappa}\right)}{\Gamma\left(1 + \frac{1}{\kappa}\right) \Gamma\left(1 - \frac{1}{\kappa}\left((\lambda, \alpha) + \frac{p}{2} - j\right)\right)}.$$

Therefore, $C_{L_p[(p-2m)(\alpha/2)]}(\lambda) = (m!)^{-1} \prod_{j=0}^{m-1} \Gamma(1 + \frac{1}{\kappa}(1+j))\, \Gamma(\frac{j-p}{\kappa})(\Gamma(1 + \frac{1}{\kappa}))^{-1}$.

6. $N-1$ SPECIAL NORMAL ORDERS ON THE SET OF POSITIVE ROOTS OF sl_N

Matsuo's construction of hypergeometric solutions uses a PBW-basis of $U(sl_N)$ corresponding to the standard order of positive roots. We rewrite those solutions using PBW-bases of $U(sl_N)$ corresponding to some new $N-1$ special (normal) orders of positive roots. The special orders are defined below.

6.1 *Special Normal Orders.* A linear order on the set of positive roots Σ_+ of a simple Lie algebra $\mathbf{g}$ is called normal if for every triple of positive roots $\alpha, \alpha+\beta, \beta \in \Sigma_+$ we have either $\alpha \succ \alpha+\beta \succ \beta$, or $\beta \succ \alpha+\beta \succ \alpha$. The standard order on the set of positive roots of sl_N is a normal order.

Consider a linear order on Σ_+. A permutation σ of Σ_+ is called *an elementary transformation* if σ is a reversal of a certain sub-system $\Sigma_2 \subset \Sigma_+$, where Σ_2 has rank 2 and all elements of Σ_2 are located side by side in the system Σ_+. An elementary transformation produces a new linear order. It is known that every elementary transformation converts one normal order into another, and that any two normal orders on Σ_+ can be transformed one into the other by a composition of elementary transformations; see [AST, Z].

EXAMPLE. For a root system of type A_m (sl_{m+1}) we have only two types of elementary transformations,

$$A_1 \oplus A_1 : \cdots, \alpha, \beta, \cdots \to \cdots, \beta, \alpha, \cdots \qquad \text{if} \quad \alpha+\beta \notin \Sigma_+;$$

$$A_2 : \cdots, \alpha, \alpha+\beta, \beta, \cdots \to \cdots, \beta, \alpha+\beta, \alpha, \cdots \qquad \text{if} \quad \alpha+\beta \in \Sigma_+.$$

We compute the change of a PBW-basis under a reversal of type $A_1 \oplus A_1$ or A_2.

LEMMA 6.1. *Let $\mathbf{g}$ be a simple Lie algebra and let α and β be two positive roots. Choose arbitrary $f_\alpha \in \mathbf{g}_\alpha$ and $f_\beta \in \mathbf{g}_\beta$. Then for any triple of non-negative integers we have the following two identities in $U(\mathbf{n}_-)$,*

$$A_1 \oplus A_1 : (-1)^{a+b} \frac{f_\alpha^a}{a!} \frac{f_\beta^b}{b!} = (-1)^{a+b} \frac{f_\beta^b}{b!} \frac{f_\alpha^a}{a!};$$

$$A_2 : (-1)^{a+b+c} \frac{f_\alpha^a}{a!} \frac{[f_\alpha, f_\beta]^c}{c!} \frac{f_\beta^b}{b!} = \sum_{p=0}^{\min(a,b)} \binom{c+p}{p} (-1)^c (-1)^{a+b+c-p} \times \frac{f_\beta^{b-p}}{(b-p)!} \frac{[f_\beta, f_\alpha]^{c+p}}{(c+p)!} \frac{f_\alpha^{a-p}}{(a-p)!}.$$

Proof. The proof is a straightforward induction on a. ∎

Recall that the set Σ_+ of sl_N is $\{\alpha_{k,l}\}_{k<l}$. The standard order on Σ_+ is $\alpha_{k,l} \succ \alpha_{k',l'}$ if and only if $l > l'$, or $l = l'$ and $k > k'$.

For $h = 1, ..., N-1$, define the index sets $A_h = \{\alpha_{k,l}\}_{k \leqslant h < l}$, $B_h = \{\alpha_{k,l}\}_{l \leqslant h}$, $C_h = \{\alpha_{k,l}\}_{h<k}$. Define a linear order $\succ_h$ on Σ_+ by the following rules.

- $\alpha \succ_h \alpha' \succ_h \alpha''$ for all $\alpha \in A_h$, $\alpha' \in B_h$, $\alpha'' \in C_h$.
- The order within the index set A_h is defined by $\alpha_{k,l} \succ_h \alpha_{k',l'}$ if and only if $l < l'$, or $l = l'$ and $k > k'$ for $(k, l), (k', l') \in A_h$.
- The order within the index set B_h is the standard one.
- The order within the index set C_h is the opposite to the standard one. That is $\alpha_{k,l} \succ_h \alpha_{k',l'}$ if and only if $k < k'$, or $k = k'$ and $l < l'$ for $(k, l), (k', l') \in C_h$.

It is easy to see that $\succ_h$ is a normal order, and $\succ_{N-1}$ is the standard order. See Figs. 1 and 2 for a pictorial description.

6.2. *Hypergeometric Solutions Corresponding to the PBW-Basis of Type* $\succ_h$. Fix $h \in \{1, ..., N-1\}$. Let $V' = V_1 \otimes \cdots \otimes V_{n+1}$, where V_j is a highest weight sl_N-module with highest weight Λ_j and highest weight vector v_j. Fix a weight subspace $V'[v'] \subset V'$, $v = \sum_{j=1}^{n} \Lambda_j - \sum_{k=1}^{N-1} m_k \alpha_k$, where $v_0 = \sum_{k=1}^{N-1} m_k \alpha_k \in Q^+$. Consider the complex space $\mathbb{C}^{n+1}$ with coordinates $z_1, ..., z_{n+1}$.

The PBW-basis of $U(\mathbf{n}_-)$ *corresponding to* $\succ_h$. For any positive root $\alpha_{k,l}$, set $a_{k,l}(h)$ equal to the number of integers p, $k < p < l$, such that $\alpha_{k,p} \succ_h \alpha_{p,l}$. The number $a_{k,l}(h)$ counts the A_2 subsystems of the order $\succ_h$ which have $\alpha_{k,l}$ as their middle root and which are ordered oppositely to the standard order. Set $F_{k,l}(h) = (-1)^{a_{k,l}(h)} e_{l,k}$. $F(h) = \{-F_{k,l}(h)\}_{k<l}$ is a basis of $\mathbf{n}_-$ called the basis corresponding to $\succ_h$. Order the basis according to $\succ_h$, $-F_{k,l}(h) \succ -F_{k',l'}(h)$ if and only if $\alpha_{k,l} \succ_h \alpha_{k',l'}$. The corresponding PBW-basis of $U(\mathbf{n}_-)$ is

$$\left\{ F_{I_0}(h) = (-1)^{\sum i_{l,k}} \frac{F_{h,h+1}(h)^{i_{h+1,h}}}{i_{h+1,h}!} \cdots \frac{F_{N-1,N}(h)^{i_{N,N-1}}}{i_{N,N-1}!} \right\}, \tag{18}$$

where $I_0 = \{i_{l,k}\}_{k<l}$ runs over all sequences of positive integers.

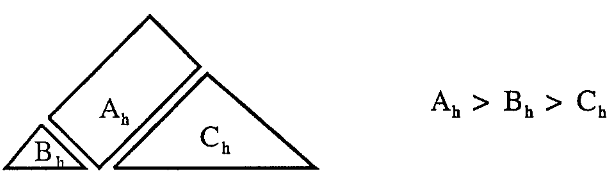

FIG. 1. General view of $\succ_h$.

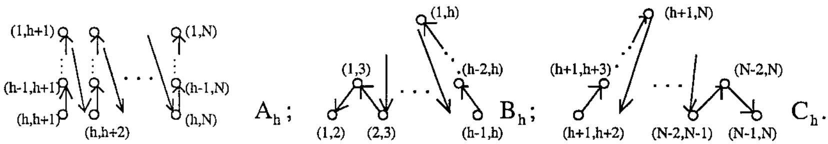

FIG. 2. The order $\succ_h$ within A_h, B_h, C_h. Arrows show immediate predecessors.

Let $\{F_{I_1}(h), ..., F_{I_{n+1}}(h)\}$ be elements of the PBW-basis of $U(\mathbf{n}_-)$ corresponding to $\succ_h$. Set $I=(I_1, \cdots, I_{n+1})$ and $F_I(h)\, v = F_{I_1}(h)\, v_1 \otimes \cdots \otimes F_{I_{n+1}}(h)\, v_{n+1}$. The vector $F_I(h)\, v$ belongs to $V'[v']$ if $I \in P(v_0, n+1)$.

The rational function $\phi(z, t; h)$. For any $I \in P(v_0, n+1)$ and any $s=(j, k, l, q) \in S(I)$, set

$$\phi_h^{(s)} = f^{(s)} \frac{(-1)^{l-1-h}}{t_h^{(\beta_h(I)(s))} - z_j} \qquad \text{if} \quad k \leqslant h < l,$$

(19) $$\phi_h^{(s)} = f^{(s)} \frac{1}{t_{l-1}^{(\beta_{l-1}(I)(s))} - z_j} \qquad \text{if} \quad k < l \leqslant h,$$

$$\phi_h^{(s)} = f^{(s)} \frac{(-1)^{l-1-k}}{t_k^{(\beta_k(I)(s))} - z_j} \qquad \text{if} \quad h < k < l,$$

where $\{\beta_p(I)\}_p$ is the set of bijections we fixed in the definition of $f^{(s)}$; see Section 4. Set

$$\phi(I, h) = \prod_{s \in S(I)} \phi_h^{(s)}, \qquad \phi(z, t; h) = \sum_{I \in P(v_0, n)} \phi(I, h)\, F_I(h)\, v.$$

Notice that $\phi(I, N-1) = \phi(I)$.

EXAMPLE. Let $\mathbf{g} = sl_3$, $n=0$. The basis $F(1) = \{-e_{3,2}, -(-e_{3,1}), -e_{2,1}\}$ of $\mathbf{n}_-$ corresponds to $\succ_1$. If $v' = \Lambda_1 - \alpha_1 - \alpha_2$, then

$$\phi'(z, t; 1) = \frac{1}{(t_1^{(1)} - z_1)(t_2^{(1)} - z_1)}\, e_{2,1} e_{3,2} v_1 + \frac{1}{(t_2^{(1)} - t_1^{(1)})(t_1^{(1)} - z_1)}\, e_{3,1} v_1 .$$

If $v' = \Lambda_1 - 2\alpha_1 - \alpha_2$, then

$$\phi' = \frac{-1}{(t_1^{(1)} - z_1)(t_1^{(2)} - z_1)(t_2^{(1)} - z_1)} \frac{e_{2,1}^2}{2}\, e_{3,2} v_1 + \frac{-1}{(t_2^{(1)} - t_1^{(1)})(t_1^{(1)} - z_1)(t_1^{(2)} - z_1)}\, e_{2,1} e_{3,1} v_1 .$$

The corresponding expressions for $\phi'(z, t; 2)$ are given in the example in Section 4, since $\succ_2$ is the standard order of positive roots of sl_3. ▌

The next theorem is our second main result.

THEOREM 6.2. *For any h, we have* $\phi(z, t; h) = \phi(z, t)$.

The proof of the theorem is in Section 8.

As a corollary, we obtain new Matsuo's type formulae for solutions of the KZ equations.

COROLLARY 6.3. (a) *Let* $u(z')$ *be the hypergeometric solution of the rational KZ equations with parameter* $\kappa \in \mathbb{C}$ *in the subspace of singular vectors of* $V'[v']$ *indicated in Theorem* 4.1. *Then, for every* $h = 1, \ldots, N-1$, *we have*

$$u(z') = \int_{\gamma(z')} \Phi'(z', t)^{1/\kappa} \Bigg(\sum_{I \in P(v_0, n+1)} \phi(I, h) \, F_I(h) \, v' \Bigg) \, dt.$$

(b) *Let* $v(z, \lambda)$ *be the hypergeometric solution of the trigonometric KZ equation with parameters* $\kappa \in \mathbb{C}$ *and* $\lambda \in \mathbf{h}$ *taking values in the weight space* $V[v]$ *of* V *indicated in Corollary* 4.2. *For every* $h = 1, \ldots, N-1$, *define a function* $v(z, \lambda; h)$

$$v(z, \lambda; h) := \int_{\gamma(z)} \Phi(z, t; \lambda)^{1/\kappa} \Bigg(\sum_{I \in P(v_0, n)} \phi(I, h) \, F_I(h) \, v \Bigg) \, dt.$$

Then $v(z, \lambda) = v(z, \lambda; h)$.

7. ADDITIVE FORM OF THE DYNAMICAL DIFFERENCE OPERATORS

7.1. *Statement of the Result.* Consider a PBW-basis $F = \{F_{I_0}\}_{I_0}$ of $U(\mathbf{n}_-)$. Let $\lambda \in \mathbf{h}$ be generic, and let M_λ be the highest weight Verma module with highest weight λ and highest weight vector v_λ. The Shapovalov form induces an isomorphism $S_\lambda \colon M_\lambda \to M_\lambda^*$. The set $\{F_{I_0} v_\lambda\}_{I_0}$ is a basis of M_λ. Let $\{(F_{I_0} v_\lambda)^*\}_{I_0}$ be the dual basis of M_λ^*. For every I_0, there exists a unique element $P_{I_0}(F, \lambda) \in U(\mathbf{n}_-)$ such that $P_{I_0}(F, \lambda)) \, v_\lambda = S_\lambda^{-1}((F_{I_0} v_\lambda)^*)$. By definition, $P_{I_0}(F, \lambda)$ is a (rational) function $\mathbf{h} \to U(\mathbf{n}_-)$, where $\lambda \mapsto P_{I_0}(F, \lambda)$.

Remark. $P_{I_0}(F, \lambda)$ is characterized by the property, $\tau(P_{I_0}(F, \lambda)) \, v_\lambda^* = (F_{I_0} v_\lambda)^*$.

The map S_λ^{-1} corresponds to an element of $M_\lambda \hat{\otimes} M_\lambda$. Identify S_λ^{-1} with this element. In terms of the basis F we have $S_\lambda^{-1} = \sum_{I_0} F_{I_0} v_\lambda \otimes P_{I_0}(F, \lambda)\, v_\lambda$.

EXAMPLE. Let $\mathbf{g} = sl_2$. Then $F = \{e_{2,1}^k\}_k$ is a basis of $U(\mathbf{n}_-)$, and

$$P_{e_{2,1}^k}(F, \lambda) = \frac{e_{2,1}^k}{k!\lambda(\lambda-1)\ldots(\lambda-k+1)}. \tag{20}$$

Explicit formula for the element S_λ^{-1} for sl_3 is given in Appendix C.

Let $V = V_1 \otimes \cdots \otimes V_n$ be a tensor product of highest weight sl_N modules and $V[v] \subset V$ a weight subspace. Recall that $\tau: U(\mathbf{g}) \to U(\mathbf{g})$ is the Chevalley involution, and $A: U(\mathbf{g}) \to U(\mathbf{g})$ is the antipode map.

The next theorem is our third main result.

THEOREM 7.1. *For every $r = 1, \ldots, N-1$, and every $v \in V[v]$, we have*

$$\mathbb{B}_{\omega[r], V}(\lambda + \rho + \tfrac{1}{2} v)\, v = \sum_{I_0 \in \mathscr{A}(r)} A(F_{I_0}(r))\, \tau(P_{I_0}(F(r), \lambda)\, v, \tag{21}$$

where $F(r) = \{F_{I_0}(r)\}_{I_0}$ is the PBW-basis of $U(\mathbf{n}_-)$ corresponding to the order $\succ_r$, and $\mathscr{A}(r) = \{I_0 = \{i_{l,k}\}_{k<l} \mid i_{l,k} = 0$ if $k > r$, or $l \leqslant r\}$.

Notice that $A(F_{I_0}(r)) \in U(\mathbf{n}_-)$ and $\tau(P_{I_0}(F(r), \lambda) \in U(\mathbf{n}_+)$, while according to the definition, given in Section 3, $\mathbb{B}_{\omega[r], V}$ is a product of elements $\mathbb{B}_V^\alpha$ each of which contains $U(\mathbf{n}_-)$ and $U(\mathbf{n}_+)$ terms. See Proposition 7.7 as well.

The theorem is proved in Section 7.4.

EXAMPLE. (a) Let $\mathbf{g} = sl_2$. Then $\mathbb{B}_{\omega[1], V}(\lambda) = \mathbb{B}^{\alpha_1}(\lambda)$.

(b) Let $\mathbf{g} = sl_3$. By definition, we have $\mathbb{B}_{\omega[1], V}(\lambda) = \mathbb{B}^{\alpha_1+\alpha_2}(\lambda)\, \mathbb{B}^{\alpha_1}(\lambda)$ and $\mathbb{B}_{\omega[2], V}(\lambda) = \mathbb{B}^{\alpha_1+\alpha_2}(\lambda)\, \mathbb{B}^{\alpha_2}(\lambda)$. Set $(\lambda, \alpha_j) = \lambda_j$, for $j = 1, 2$, and $p_k(t) = t(t-1)\ldots(t-k+1) \in \mathbb{C}[t]$ for any $k \in \mathbb{N}$. Then, for any $v \in V[v]$, we have

$$\mathbb{B}_{\omega[2], V}\left(\lambda + \rho + \frac{1}{2} v\right) v = \sum_{s,k=0}^{\infty} \frac{e_{3,1}^k e_{3,2}^s}{k!\, s!} \left(\sum_{j=0}^{k} \frac{e_{1,2}^j e_{1,3}^{k-j} e_{2,3}^{s+j} \binom{k}{j}}{p_{s+j}(\lambda_2)\, p_k(\lambda_2 + \lambda_1 + 1)} \right) v,$$

$$\mathbb{B}_{\omega[1], V}\left(\lambda + \rho + \frac{1}{2} v\right) v = \sum_{s,k=0}^{\infty} \frac{e_{3,1}^k e_{2,1}^s}{k!\, s!} \left(\sum_{j=0}^{k} \frac{(-1)^j e_{2,3}^j e_{1,3}^{k-j} e_{1,2}^{s+j} \binom{k}{j}}{p_{s+j}(\lambda_1)\, p_k(\lambda_2 + \lambda_1 + 1)} \right) v.$$

7.2. *The Universal Fusion Matrix.* Let $R(\mathbf{h})$ be the quotient field of the algebra $U(\mathbf{h})$. Then $R(\mathbf{h})$ is isomorphic to the field of rational functions over $\mathbf{h}$, using the identification of the Cartan subalgebra $\mathbf{h}$ with its dual.

Define algebras $U_1(\mathbf{h}\oplus\mathbf{n}_\pm) = U(\mathbf{h}\oplus\mathbf{n}_\pm)\otimes_{U(\mathbf{h})} R(\mathbf{h})$. Consider the space $U_1(\mathbf{h}\oplus\mathbf{n}_-)\hat{\otimes} U_1(\mathbf{h}\oplus\mathbf{n}_+)$, which is a formal series completion of $U_1(\mathbf{h}\oplus\mathbf{n}_-)\otimes U_1(\mathbf{h}\oplus\mathbf{n}_+)$. For any basis $\{a_K\otimes b_K\}_K$ of $U(\mathbf{n}_-)\otimes U(\mathbf{n}_+)$, an element of $U_1(\mathbf{h}\oplus\mathbf{n}_-)\hat{\otimes} U_1(\mathbf{h}\oplus\mathbf{n}_+)$ has the form $\sum_K a_K\psi_{1,K}\otimes b_K\psi_{2,K}$, where $\psi_{1,K}$, $\psi_{2,K}\in R(\mathbf{h})$.

Denote $\pi_{\mathbf{h}}$ as the natural projection $U_1(\mathbf{h}\oplus\mathbf{n}_-)\hat{\otimes} U_1(\mathbf{h}\oplus\mathbf{n}_+)\to R(\mathbf{h})\otimes R(\mathbf{h})$. Let $\{x_k\}$ be an orthonormal basis of $\mathbf{h}$.

THEOREM 7.2 (Proposition 1 in [ABRR]). *For every $\lambda\in\mathbf{h}$, there exists a unique solution $J(\lambda)$ of the ABRR-equation*

$$(22)\qquad \left[1\otimes\left(\lambda+\rho-\tfrac{1}{2}\sum_k x_k^2\right), J(\lambda)\right] = -\left(\sum_{\alpha\in\Sigma_+} e_{-\alpha}\otimes e_\alpha\right)J(\lambda),$$

such that $J(\lambda)$ belongs to $U_1(\mathbf{h}\oplus\mathbf{n}_-)\hat{\otimes} U_1(\mathbf{h}\oplus\mathbf{n}_+)$ and $\pi_{\mathbf{h}(J(\lambda))=1\otimes 1}$. Moreover, $F(\lambda)$ is of weight zero, and has the expansion

$$J(\lambda) = \sum_{K=0}^{\infty} a_K\otimes(b_K\psi_K(\lambda)),$$

where $a_K\in U(\mathbf{n}_-)$, $b_K\in U(\mathbf{n}_+)$, $\psi_K\in R(\mathbf{h})$, for all K, $a_0=b_0=\psi_0(\lambda)=1$.

For technical reasons a proof of the theorem is given at the end of this section.

The solution $J(\lambda)$ is called the universal fusion matrix of $U(\mathbf{g})$. Theorem 7.5 and equality (20) give a formula for the universal fusion matrix of $U(sl_2)$. See Appendix C for a similar formula in the sl_3 case.

Set $Q^\dagger(\lambda) = \sum_{K\geqslant 0} A(a_K)\, b_K\psi_K(\lambda)$, cf. [EV]. The element $Q^\dagger(\lambda)$ belongs to a formal series completion of $U(sl_N)\otimes_{U(\mathbf{h})} R(\mathbf{h})$. Its action is well defined in a tensor product of highest weight sl_N modules.

THEOREM 7.3 (Theorem 34 in [EV]). *Let w_0 be the longest element in the Weyl group $\mathbb{W}$. For any $v\in V[\nu]$, we have*

$$\mathbb{B}_{w_0,V}(\lambda+\rho-\tfrac{1}{2}\nu)\, v = Q^\dagger(\lambda)\, v.$$

Next, we present a connection between the Shapovalov form and the universal fusion matrix.

THEOREM 7.4 (Proposition 13.1 in [ES]). *We have $S_\lambda^{-1} = ((1\otimes\tau)\,J(0))(v_\lambda\otimes v_\lambda)$, where S_λ^{-1} is identified with the corresponding element of $M_\lambda\hat{\otimes} M_\lambda$.*

More explicitly, in terms of a basis F of $U(\mathbf{n}_-)$, we have

$$\sum_{I_0} F_{I_0} v_\lambda \otimes P_{I_0}(F, \lambda)\, v_\lambda = \Big(\sum_{K \geqslant 0} a_K \otimes \tau(b_K \psi_K(0)) \Big) (v_\lambda \otimes v_\lambda). \tag{23}$$

Let $F(\lambda)$ be a rational function of $\lambda \in \mathbf{h}$ with values in $R(\mathbf{h})$. Assume that the λ dependence in $F(\lambda)$ is through a finite number of inner products $\{(\lambda, \beta_j)\}_{j=1}^k$, $F(\lambda) = F((\lambda, \beta_1), \ldots, (\lambda, \beta_k))$, where $\beta_1, \ldots, \beta_k \in \mathbf{h}$. Set $F_+(\lambda) = F((\lambda, \beta_1) + \beta_1, \ldots, (\lambda, \beta_k) + \beta_k)$. The function $F_+(\lambda)$ is a rational function of $\lambda \in \mathbf{h}$ with values in $R(\mathbf{h})$.

THEOREM 7.5. *Let $F = \{F_{I_0}\}_{I_0}$ be a homogeneous basis of $U(\mathbf{n}_-)$. Set $J_+(\lambda) = \sum_K a_K \otimes (b_K(\psi_K)_+(\lambda))$. Then $J_+(\lambda)$ is well defined, and*

$$J_+(\lambda) = \sum_{I_0} F_{I_0} \otimes \tau(P_{I_0}(F, \lambda)).$$

Notice that, in particular, the theorem says that $J_+(\lambda)$ takes values in $U(\mathbf{n}_-) \hat{\otimes} U(\mathbf{n}_+)$ for any $\lambda \in \mathbf{h}$. The proof of the theorem is given at the end of this section.

We derive the following corollary from Theorems 7.3 and 7.5.

COROLLARY 7.6. *Let w_0 be the longest element in the Weyl group $\mathbb{W}$. Then*

$$\mathbb{B}_{w_0, V}(\lambda + \rho + \tfrac{1}{2} \nu)\, v = \Big(\sum_{I_0} A(F_{I_0})\, \tau(P_{I_0}(F, \lambda)) \Big) v,$$

for any homogeneous basis $F = \{F_{I_0}\}_{I_0}$ of $U(\mathbf{n}_-)$, and any homogeneous element $v \in V[\nu]$.

Indeed, by Theorem 7.3, we have

$$\mathbb{B}_{w_0, V}(\lambda + \rho + \tfrac{1}{2} \nu)\, v = Q^\dagger(\lambda + \nu)\, v = \sum_{K \geqslant 0} A(a_K)\, b_K \psi_K(\lambda + \nu)\, v.$$

Theorem 7.5 gives

$$J_+(\lambda) = \sum_{K \geqslant 0} a_K \otimes (b_K(\psi_K)_+ (\lambda)) = \sum_{I_0} F_{I_0} \otimes \tau(P_{I_0}(F, \lambda)).$$

Finally, the equality $((\lambda, \beta)+\beta)\, v=((\lambda, \beta)+(v, \beta))\, v=(\lambda+v, \beta)\, v$ for any $\lambda, \beta \in \mathbf{h}$, implies

$$\sum_{K \geqslant 0} A(a_K)\, b_K \psi_K(\lambda+v)\, v=\sum_{K \geqslant 0} A(a_K)\, b_K(\psi_K)_+\, (\lambda)\, v$$

$$=\sum_{I_0} A(F_{I_0})\, \tau(P_{I_0}(F, \lambda))\, v. \quad \blacksquare$$

Proof of Theorem 7.2. Fix $\lambda \in \mathbf{h}$. Denote $wt(\,.\,)$ as the weight function defined on homogeneous elements in $U(\mathbf{g})$ with values in $\mathbf{h}$. Fix a basis $\{a_K \otimes b_K\}_K$ of $U(\mathbf{n}_-) \otimes U(\mathbf{n}_+)$, such that a_K and b_K, $K \geqslant 0$, are homogeneous elements of $U(\mathbf{n}_-)$ and $U(\mathbf{n}_+)$, respectively. Let $J(\lambda)=\sum_{K \geqslant 0} a_K \psi_{1, K}(\lambda) \otimes b_K \psi_{2, K}(\lambda)$, where $\psi_{1, K}(\lambda), \psi_{2, K}(\lambda) \in R(\mathbf{h})$

The ABRR equation (22) is

$$\begin{aligned}(24) \quad & \sum_{K \geqslant 0} (a_K \psi_{1, K}(\lambda) \otimes b_K \psi_{2, K}(\lambda))(1 \otimes ((\lambda+\rho, wt(b_K)-\tfrac{1}{2}(wt(b_K), wt(b_K)) \\ & \quad -wt(b_K))) \\ & =-\left(\sum_{\alpha \in \Sigma_+} \sum_{K \geqslant 0} (e_{-\alpha} a_K \psi_{1, K}(\lambda) \otimes e_\alpha b_K \psi_{2, K}(\lambda)\right).\end{aligned}$$

Equality (24) presents a system of recurrence relations for $\{\psi_{1, K}(\lambda) \otimes \psi_{2, K}(\lambda)\}_{k \geqslant 0} \subset R(\mathbf{h}) \otimes R(\mathbf{h})$. Together with the initial conditions $\psi_{1,0}(\lambda)=\psi_{2,0}(\lambda)=1$, they uniquely determine $\psi_{1, K}(\lambda) \otimes \psi_{2, K}(\lambda)$ for all K. Moreover, the solution is such that $\psi_{1, K}(\lambda)=1$. Set $\psi_K(\lambda)=\psi_{2, K}(\lambda)$. $\blacksquare$

Proof of Theorem 7.5. Observe that $\psi_K(\lambda)$ is a function of λ with values in $R(\mathbf{h})$; it depends on λ via the inner products $\{(\lambda, wt(b_L(\lambda))\}$, $0 \leqslant wt(b_L(\lambda) \leqslant wt(b_K(\lambda))$. Thus, the shifted function $(\psi_K)_+(\lambda)$ is well defined for any K, and $J_+(\lambda)$ is well defined.

Write a system of recurrence relations for $J_+(\lambda)$ by shifting the system of reccurence relations (24). We have

$$\begin{aligned}(25) \quad & \sum_{K \geqslant 0} (a_K \otimes (b_K(\psi_K)_+(\lambda)))(1 \otimes ((\lambda+\rho, wt(b_K))-\tfrac{1}{2}(wt(b_K), wt(b_K)))) \\ & =-\sum_{\alpha \in \Sigma_+} \sum_{K \geqslant 0} (e_{-\alpha} a_K \otimes (e_\alpha b_K(\psi_K)_+(\lambda))).\end{aligned}$$

Equation (25) implies a system of reccurence relations for the functions $\{(\psi_K)_+(\lambda)\}_{K \geqslant 0}$. The initial condition is $(\psi_0)_+(\lambda)=1$. Therefore, for the unique solution, each $(\psi_K)_+(\lambda)$ is a complex-valued functions. The shifted universal fusion matrix $J_+(\lambda)$ takes values in $U(\mathbf{n}_-)\hat{\otimes} U(\mathbf{n}_+)$.

Next we study $J(0)$ which is connected to $\sum_{I_0} F_{I_0} \otimes \tau(P_{I_0}(F,\lambda))$ by Theorem 7.4. The ABRR reccurence relations (24) applied to $J(0)$ give

$$(26)\quad \sum_{K\geqslant 0} (a_K \otimes (b_K\psi_K(0)))(1\otimes((\rho, wt(b_K)) - \tfrac{1}{2}(wt(b_K), wt(b_K)) - wt(b_K)))$$
$$= -\sum_{\alpha\in\Sigma_+}\sum_{K\geqslant 0}(e_{-\alpha}a_K \otimes (e_\alpha b_K\psi_K(0))).$$

Consider $(1\otimes\tau)(J(0)) = \sum_{K\geqslant 0} a_K \otimes \tau(b_K\psi_K(0)) \in U(\mathbf{n}_-)\hat{\otimes} U_1(\mathbf{h}\oplus\mathbf{n}_-)$. Its action on the vector $v_\lambda \otimes v_\lambda \in M_\lambda \otimes M_\lambda$ defines an element of $M_\lambda \hat{\otimes} M_\lambda$. Notice that $\tau((\rho, wt(b_K)) - \frac{1}{2}(wt(b_K), wt(b_K)) - wt(b_K)) = ((\rho, wt(b_K)) - \frac{1}{2}(wt(b_K), wt(b_K)) + wt(b_K))$, because τ restricted to $\mathbb{C}$ is the identity and τ restricted to $\mathbf{h}$ is multiplication by -1. Since $wt(b_K)\, v_\lambda = (\lambda, wt(b_K))\, v_\lambda$, equality (26) implies

$$(27)\quad \sum_{K\geqslant 0}(a_K v_\lambda \otimes \tau(b_K\psi_K(0))\, v_\lambda)((\lambda+\rho, wt(b_K)) - \tfrac{1}{2}(wt(b_K), wt(b_K)))$$
$$= -\sum_{\alpha\in\Sigma_+}\sum_{K\geqslant 0} e_{-\alpha}a_K v_\lambda \otimes \tau(e_\alpha b_K\psi_K(0))\, v_\lambda.$$

For any K, define a complex-valued function $\psi_K(0,\lambda)$, such that $\tau(\psi_K(0))\, v_\lambda = \psi_K(0,\lambda)\, v_\lambda$. Therefore, we have $\psi_K(0,\lambda)(a_K v_\lambda \otimes \tau(b_K)\, v_\lambda) = a_K v_\lambda \otimes \tau(b_K\psi_K(0))\, v_\lambda$. Now (27) takes the form

$$(28)\quad \sum_{K\geqslant 0}(a_K v_\lambda \otimes \tau(b_K)\, v_\lambda)\, \psi_K(0,\lambda)((\lambda+\rho, wt(b_K)) - \tfrac{1}{2}(wt(b_K), wt(b_K)))$$
$$= -\sum_{\alpha\in\Sigma_+}\sum_{K\geqslant 0}(e_{-\alpha}a_K v_\lambda \otimes \tau(e_\alpha b_K)\, v_\lambda)\, \psi_K(0,\lambda).$$

Equality (28) together with the isomorphism between M_λ and $U(\mathbf{n}_-)$ implies a system of recurrence relations for the functions $\{\psi_K(0,\lambda)\}_{K\geqslant 0}$ with initial condition $\psi_0(0,\lambda) = 1$. This system of recurrence relations coincides with the systems of recurrence relations for $\{(\psi_K)_+(\lambda)\}_{k\geqslant 0}$. The initial condition in both cases is the same. Therefore the solutions coincide, i.e., for every K, $(\psi_K)_+(\lambda) = \psi_K(0,\lambda)$ and $a_K \otimes (b_K(\psi_K)_+(\lambda)) = a_K \otimes (b_K\psi_K(0,\lambda))$.

In our notation, Theorem 7.4 together with the linear isomorphism between M_λ and $U(\mathbf{n}_-)$ give, the following explicit formula, $\sum_{K\geqslant 0} a_K \otimes \tau(b_K)\,\psi_K(0,\lambda) = \sum_{I_0} F_{I_0} \otimes P_{I_0}(F,\lambda)$. Therefore, $J_+(\lambda) = \sum_{I_0} F_{I_0} \otimes \tau(P_{I_0}(F,\lambda))$, which is the statement of the corollary. ∎

7.3. *Normal Orders on the Set of Positive Roots and Multiplicative Presentations of* $\mathbb{B}_{w_0,V}(\lambda)$. Let $\omega = s_{i_m} \cdots s_{i_1}$ be a reduced decomposition of an element $\omega \in \mathbb{W}$. Set $\alpha^1 = \alpha_{i_1}$, $\alpha^p = s_{i_1} \cdots s_{i_{p-1}} \alpha_{i_p}$, for $p = 2, ..., m$.

PROPOSITION 7.7 [TV]. $\mathbb{B}_{\omega,V}(\lambda) = \mathbb{B}_V^{\alpha^m}(\lambda) \cdots \mathbb{B}_V^{\alpha^1}(\lambda)$.

Let $\omega_0 = s_{i_m} \cdots s_{i_1}$ be a reduced decomposition of the longest element $\omega_0 \in \mathbb{W}$. Then $\alpha^m \succ \cdots \succ \alpha_1$ is a normal order on Σ_+, and any normal order corresponds to a reduced decomposition of the longest element; see [Z].

COROLLARY 7.8. *If* $\alpha^m \succ \cdots \succ \alpha^1$ *is a normal order on* Σ_+, *then*

$$\mathbb{B}_{\omega_0,V}(\lambda) = \mathbb{B}_V^{\alpha^m}(\lambda) \cdots \mathbb{B}_V^{\alpha^1}(\lambda).$$

7.4. *The Proof of Theorem* 7.1. Fix $r \in \{1, ..., N-1\}$. Recall that the element $\omega_{[r]}$ is defined by $\omega_{[r]} = \omega_0 \omega_0^r \in \mathbb{W}$, where ω_0 (respectively, ω_0^r) is the longest element in $\mathbb{W}$ (respectively, in $\mathbb{W}^r$ generated by all simple reflections s_l preserving the dual fundamental weight $\omega_r^\vee$). The explicit form of $\omega_r^\vee = \sum_{k=1}^r (1 - \frac{r}{N}) e_{k,k} - \sum_{k=r+1}^N \frac{r}{N} e_{k,k}$ implies that $\mathbb{W}^r$ is generated by the simple reflections $s_1, ..., s_{r-1}, s_{r+1}, ..., s_{N-1}$. Denote $\mathbb{W}_2$ as the subgroup of $\mathbb{W}$ generated by $s_1, ..., s_{r-1}$ and denote $\mathbb{W}_1$ as the subgroup of $\mathbb{W}$ generated by $s_{r+1}, ..., s_{N-1}$. Denote R_2 as the root system with base $\alpha_1, ..., \alpha_{r-1}$, and denote R_1 as the root system with base $\alpha_{r+1}, ..., \alpha_{N-1}$. Then $\mathbb{W}_1$ is the Weyl group of R_1, and $\mathbb{W}_2$ is the Weyl group of R_2, and $\mathbb{W}^h = \mathbb{W}_2 \times \mathbb{W}_1$.

Consider the normal order $\succ_r$. Set $m_1 = (N-r-1)(N-r)/2$, $m_2 = (r-1)\,r/2$, $m_3 = r(N-r)$, $m = m_1 + m_2 + m_3$. Write explicitly the order $\succ_r$ on Σ_+, $\alpha^m \succ_r \cdots \succ_r \alpha^1$. We have $\Sigma_+ = A_r \cup B_r \cup C_r$. Moreover, $A_r = \{\alpha^m, ..., \alpha^{1+m_2+m_1}\}$, B_r is the set of positive roots for R_2, C_r is the set of positive roots for R_1, and $A_r \succ_r B_r \succ_r C_r$. Let $\omega_0 = s_{i_m} \cdots s_{i_1}$ be the reduced decomposition of ω_0 corresponding to $\succ_r$. Denote $\omega_1 = s_{i_{m_1}} \cdots s_{i_1}$, $\omega_2 = s_{i_{m_2+m_1}} \cdots s_{i_{1+m_1}}$, $\omega_3 = s_{i_m} \cdots s_{i_{1+m_2+m_1}}$. Clearly, $\omega_2 \omega_1 = \omega_0^r$, $\omega_3 = \omega_{[r]}$.

Let $\{\beta^{m_3}, ..., \beta^1\}$ be the set of positive roots corresponding to the reduced presentation $\omega_{[r]} = s_{i_m} \cdots s_{i_{1+m_2+m_1}}$. Then $\mathbb{B}_{\omega_{[r]},V}(\lambda) = \mathbb{B}_V^{\beta^{m_3}}(\lambda) \cdots \mathbb{B}_V^{\beta^1}(\lambda)$. The definition of the correspondence between normal orders and reduced presentations of ω_0 implies that $A_r = (\omega_0^r)^{-1}(\{\beta^{m_3}, ..., \beta^1\})$ as ordered sets. Thus, we have $\{\beta^{m_3}, ..., \beta^1\} = \omega_0^r(\{\alpha^m, ..., \alpha^{1+m_2+m_1}\})$.

LEMMA 7.9. *The reflection* ω_0^r *reverses* A_r, $\omega_0^r(A_r) = \{\alpha^{1+m_1+m_2}, ..., \alpha^m\}$. *Therefore,*

$$\mathbb{B}_{\omega_{[r]},V}(\lambda) = \mathbb{B}_V^{\alpha^{1+m_1+m_2}}(\lambda) \cdots \mathbb{B}_V^{\alpha^m}(\lambda).$$

The proof is a straightforward computation.

COROLLARY 7.10. $\mathbb{B}_{\omega_0, V}(\lambda) = \mathbb{B}_{\omega_1, V}(\lambda)\, \mathbb{B}_{\omega_2, V}(\lambda)\, \mathbb{B}_{\omega_{[r]}, V}(\lambda)$.

Proof. Consider the normal order $\prec^r$ on Σ_+ reverse to $\succ_r$. For any $\alpha, \beta \in \Sigma_+$ we have $\alpha \prec^r \beta$ if and only if $\alpha \succ_r \beta$. Notice that $A_r \prec^h B_r \prec^r C_r$ and the order $\prec^r$ within each of the sets A_r, B_r, C_r is reverse to $\succ_r$. Consider the presentation of $\mathbb{B}_{\omega_0, V}(\lambda)$ corresponding to $\prec^r$ given in Corollary 7.8. Apply Lemma 7.9 to obtain the statement of the corollary. ▮

Recall that $F(r) = \{F_{I_0}(r)\}_{I_0}$ is the PBW-basis of $U(\mathbf{n}_-)$ corresponding to $\succ_r$.

LEMMA 7.11. *For any homogeneous vector $v \in V[\nu]$, we have*

$$\mathbb{B}_{\omega_{[r]}, V}(\lambda+\rho+\tfrac{1}{2}\nu)\, v = \sum_{I_0 \in \mathscr{A}(r)} A(F_{I_0}(r))\, Q_{I_0}(F(r), \lambda)\, v,$$

where $\{Q_{I_0}(F(r), \lambda)\}_{I_0}$ is a subset of homogeneous elements of $U(\mathbf{n}_+)$.

Proof. For every positive root α, every $\lambda \in \mathbf{h}$, and every positive integer s, set $p_{0, \lambda, \alpha} = 1$, and $p_{s, \lambda, \alpha} = s!((\lambda+\rho, \alpha)-1)\ldots((\lambda+\rho, \alpha)-s) \in \mathbb{C}$. Use formula (7) to obtain

$$\mathbb{B}_V^{\alpha}(\lambda+\rho+\tfrac{1}{2}\nu)\, v = \sum_{s=0}^{\infty} F_\alpha^s E_\alpha^s (p_{s, \lambda, \alpha})^{-1}\, v.$$

Since by definition $A_r = \{\alpha_{r, r+1}, \alpha_{r-1, r+1}, \ldots, \alpha_{2, N}, \alpha_{1, N}\}$, see Fig. 2, Lemma 7.9 implies

$$\mathbb{B}_{\omega_{[r]}, V}(\lambda+\rho+\tfrac{1}{2}\nu)\, v = \sum_{s_{r, r+1}=0}^{\infty} \cdots \sum_{s_{1, N}=0}^{\infty} \frac{F_{\alpha_{1, N}}^{s_{1, N}} E_{\alpha_{1, N}}^{k_{1, N}} \cdots F_{\alpha_{r, r+1}}^{s_{r, r+1}} E_{\alpha_{r, r+1}}^{s_{r, r+1}}}{p_{s_{k_{1, N}}, \lambda, \alpha_{1, N}} \cdots p_{s_{r, r+1}, \lambda, \alpha_{r, r+1}}}\, v. \tag{29}$$

Next, we need to rearrange the right hand side of formula (29) by moving all elements of type $F_{k, l}$ to the left through elements of type $E_{k', l'}$, using the commutation relations of sl_N. Since $E_{\alpha_{k', l'}} = e_{k', l'}$ and $F_{\alpha_{k, l}} = e_{l, k}$, we have

(a) $[E_{\alpha_{k', l}}, F_{\alpha_{k, l}}] = E_{\alpha_{k', k}}$, if $k' < k$;

(b) $[E_{\alpha_{k', l}}, F_{\alpha_{k, l}}] = F_{\alpha_{k, k'}}$, if $k < k'$;

(c) $[E_{\alpha_{k, l'}}, F_{\alpha_{k, l}}] = -E_{\alpha_{l, l'}}$, if $l < l'$;

(d) $[E_{\alpha_{k, l'}}, F_{\alpha_{k, l}}] = -F_{\alpha_{l', l}}$, if $l' < l$;

(e) $[E_{\alpha_{k, l}}, F_{\alpha_{k, l}}] = H_{\alpha_{k, l}}$;

(f) $[E_{\alpha_{k', l'}}, F_{\alpha_{k, l}}] = 0$, if $k \neq k'$ and $l \neq l'$.

The initial ordering of the product of E's and F's in formula (29) is according to the $\succ_r$ ordering of A_r, $F_{\alpha_{k,l}}^{s_{k,l}} E_{\alpha_{k,l}}^{s_{k,l}}$ is to the left of $F_{\alpha_{k',l'}}^{s_{k'}}$, $l' E_{\alpha_{k',l'}}^{s_{k'}}$, l' if and only if $\alpha_{k',l'} \succ_r \alpha_{k,l}$. See Fig. 2 for the definition of $\succ_r$ on A_r. The set of indices $\{(k,l)\}$ is characterized by $1 \leqslant k \leqslant r < l \leqslant N$.

We first move all $F_{\alpha_{2,N}}$'s to the left through all $E_{\alpha_{1,N}}$'s, then we move all $F_{\alpha_{3,N}}$'s and so on. The last step is to move all $F_{\alpha_{r,r+1}}$'s to the left through all E's resulting from the previous steps.

Claim. After each step we have:

(*) all $F_{\alpha_{k,l}}$'s in the respective rearranged version of formula (29) satisfy $k \leqslant r < l$;

(**) if $k = k' \leqslant r < l' \leqslant l$, or $k \leqslant k' \leqslant r < l' = l$ then any $E_{\alpha_{k',l'}}$ is to the right of any $F_{\alpha_{k,l}}$.

In the initial state represented in formula (29) the claim follows from the definition of the order $\succ_r$. Now, assume that the claim is valid after p rearranging steps. Assume that we move $F_{\alpha_{k,l}}$ at the $(p+1)$st step.

If the $(p+1)$st step is of type (a) or (c), then we get an additional $\pm E_{\alpha_{k',l'}}$ with $k' < l' \leqslant r$ or $r < k' < l'$. E's of this type do not affect the claim, so it remains valid.

If the $(p+1)$st step is of type (b) or (d), then we get an additional $\pm F_{\alpha_{k',l'}}$ with $k \leqslant k' \leqslant r < l' \leqslant l$. Other F's are excluded because of condition (**). Moreover, a simple check up shows that condition (**) remains valid for the additional summand.

Condition (**) shows that the $(p+1)$st step cannot be of type (e), and after a step of type (f) the claim is trivially satisfied.

Inductive argument shows that the claim is satisfied after each step of the rearrangement. As a final result we get

$$\mathbb{B}_{\omega[r],V}(\lambda+\rho+\tfrac{1}{2}\nu)\, v = \sum_{s_{r,r+1}=0}^{\infty} \cdots \sum_{s_{1,N}=0}^{\infty} \left(\prod_{k \leqslant r < l} F_{\alpha_{k,l}}^{s_{k,l}} \right) \tilde{Q}_{\{s_{k,l}\}}(\lambda)\, v, \tag{30}$$

where each $\tilde{Q}_{\{s_{k,l}\}}(\lambda)$ is a homogeneous element of $U(\mathbf{n}_+)$. Finally, identify the set $\{s_{k,l} \in \mathbb{Z}_{\geqslant 0} \mid 1 \leqslant k \leqslant r < l \leqslant N\}$ with the set $\mathscr{A}(r) = \{s_{k,l} \in \mathbb{Z}_{\geqslant 0} \mid 1 \leqslant k < l \leqslant N,\ s_{k,l} = 0 \text{ if } r < k, \text{ or } l \leqslant r\}$. The relation $(\prod_{k \leqslant r < l} F_{\alpha_{k,l}}^{s_{k,l}}) = \pm(\prod_{k \leqslant r < l} s_{k,l}!)\, A(F_{\{s_{k,l}\}}(r))$ together with equality (30) implies the statement of the lemma. ∎

Since ω_1 is the longest element in $\mathbb{W}(N-r-1)$ and ω_2 is the longest element in $\mathbb{W}(r-1)$ we can apply Corollary 7.6 to $\mathbb{B}_{\omega_1,V}$ and $\mathbb{B}_{\omega_2,V}$. Choose

$F(r)=\{F_{I_0}(r)\}_{I_0}$ as the weighted PBW-basis of $U(\mathbf{n}_-)$. Then, for every $v \in V[\nu]$, we have

$$\mathbb{B}_{\omega_1, V}(\lambda+\rho+\tfrac{1}{2}\nu)\, v = \sum_{I_0 \in \mathscr{C}(r)} A(F_{I_0}(r))\, \tau(P_{I_0}(F(r), \lambda))\, v,$$

$$\mathbb{B}_{\omega_2, V}(\lambda+\rho+\tfrac{1}{2}\nu)\, v = \sum_{I_0 \in \mathscr{B}(r)} A(F_{I_0}(r))\, \tau(P_{I_0}(F(r), \lambda))\, v,$$

where $\mathscr{C}(r)=\{I_0=\{i_{l,k}\}_{k<l} \mid i_{l,k}=0 \text{ if } k \leqslant r\}$, and $\mathscr{B}(r)=\{I_0=\{i_{l,k}\}_{k<l} \mid i_{l,k}=0 \text{ if } l>r\}$. It is easy to see that, for any $I_1 \in \mathscr{C}(r)$, $I_2 \in \mathscr{B}(r)$, we have $[\tau(P_{I_1}(F, \lambda), A(F_{I_2})]=0$. Therefore,

(31)
$$\mathbb{B}_{\omega_1, V}(\lambda+\rho+\tfrac{1}{2}\nu)\, \mathbb{B}_{\omega_2, V}(\lambda+\rho+\tfrac{1}{2}\nu)\, v = \sum_{I_0 \in \mathscr{B}\mathscr{C}(r)} A(F_{I_0}(r))\, \tau(P_{I_0}(F(r), \lambda))\, v,$$

where $\mathscr{B}\mathscr{C}(r)=\{I_0=\{i_{l,k}\}_{k<l} \mid i_{l,k}=0 \text{ if } k \leqslant r<l\}$.

Apply Corollary 7.6 to the right hand side of Corollary 7.10, and Lemma 7.11 and formula (31) to the left hand side of Corollary 7.10. For any $v \in V$, we obtain

(32)
$$\sum_{I_0} A(F_{I_0}(r))\, \tau(P_{I_0}(F(r), \lambda))\, v$$
$$= \sum_{I_0 \in \mathscr{B}\mathscr{C}(r)} A(F_{I_0}(r))\, \tau(P_{I_0}(F(r), \lambda)) \sum_{I_0 \in \mathscr{A}(r)} A(F_{I_0}(r))\, Q_{I_0}(F(r), \lambda)\, v.$$

Set $V=M_\lambda$ the Verma module with highest weight λ. Weight considerations imply

$$\tau(P_{I_0}(F(r), \lambda))\, F_{J_0}(r)\, v_\lambda = 0, \qquad Q_{I_0}(F(r), \lambda)\, F_{J_0}(r)\, v_\lambda = 0 \quad \text{if} \quad wt(F_{J_0}(r)\, v_\lambda) \nleqslant wt(F_{I_0}(r)\, v_\lambda).$$

The defining property of $P_{I_0}(F(r), \lambda)$ gives

$$S_\lambda(\tau(P_{I_0}(F(r), \lambda))\, F_{J_0}(r)\, v_\lambda, v_\lambda) = S_\lambda(F_{J_0}(r)\, v_\lambda, A(P_{I_0}(F(r), \lambda))\, v_\lambda) = \pm\delta_{I_0, J_0}.$$

Therefore $\tau(P_{I_0}(F(r), \lambda))\, F_{I_0}(r)\, v_\lambda = \pm v_\lambda$, and $\tau(P_{I_0}(F(r), \lambda))\, F_{I_0}(r)\, v_\lambda = 0$ if $J_0 \neq I_0$ and $wt(F_{J_0}(r)\, v_\lambda) = wt(F_{I_0}(r)\, v_\lambda)$.

Finally, apply equality (32) to $F_{I_0}(r)\, v_\lambda$, for $I_0 \in \mathscr{A}$, and use induction on the partialy ordered set $\{wt(F_{I_0}(r)\, v_\lambda)\}_{I_0 \in \mathscr{A}}$ to obtain $Q_{I_0}(F(r), \lambda)\, F_{J_0}(r)\, v_\lambda = \tau(P_{I_0}(F(r), \lambda))\, F_{J_0}(r)\, v_\lambda$, for any J_0, and thus $Q_{I_0}(F(r), \lambda) = \tau(P_{I_0}(F(r), \lambda))$. The last equality and Lemma 7.11 imply the statement of Theorem 7.1 ∎

8. PROOF OF THEOREM 6.2.

8.1. *Sequence of Elementary Transformations Converting* $\succ_h$ *into* $\succ_{h-1}$. Recall that $h \in \{1, ..., N-1\}$. The following construction is very important. Namely, there exists a sequence σ of elementary transformations

$$\sigma = \{\sigma_1, ..., \sigma_M\}, \tag{33}$$

which converts the normal order $\succ_h$ into the normal order $\succ_{h-1}$. The sequence is such that, for every pair of positive integers (k, l), $1 \leqslant k < h < l \leqslant N$, the A_2 elementary transformation of type $\cdots, \alpha_{h,l}, \alpha_{k,l}, \alpha_{k,h}, \cdots \to \cdots, \alpha_{k,h}, \alpha_{k,l}, \alpha_{h,l}, \cdots$ is used exactly once in this sequence and all other elementary transformations in the sequence are of type $A_1 \oplus A_1$. The sequence is given as follows.

The construction of σ. We have $\Sigma_+ = A_h \cup B_h \cup C_h = A_{h-1} \cup B_{h-1} \cup C_{h-1}$, and
$A_h \succ_h B_h \succ_h C_h$, $A_{h-1} \succ_{h-1} B_{h-1} \succ_{h-1} C_{h-1}$. Moreover,

$$A_{h-1} = A_h + \{\alpha_{k,h}\}_{k=1}^{k=h-1} - \{\alpha_{h,l}\}_{l=h+1}^{l=N}, \qquad B_{h-1} = B_h - \{\alpha_{k,h}\}_{k=1}^{k=h-1},$$
$$C_{h-1} = C_h + \{\alpha_{h,l}\}_{l=h+1}^{l=N},$$

where $\{\alpha_{k,h}\}_{k=1}^{k=h-1} \subset B_h$, $\{\alpha_{h,l}\}_{l=h+1}^{l=N} \subset A_h$. The roots $\{\alpha_{k,h}\}_{k=1}^{k=h-1}$ are the largest elements in B_h according to $\succ_h$ and are the largest elements in A_{h-1} according to $\succ_{h-1}$. The linear order $\succ_h$ on the set $\{\alpha_{k,h}\}_{k=1}^{k=h-1}$ is the same as the linear order $\succ_{h-1}$. The roots $\{\alpha_{h,l}\}_{l=h+1}^{l=N}$ are the largest elements in C_{h-1} according to $\succ_{h-1}$. The linear order $\succ_h$ on the set $\{\alpha_{h,l}\}_{l=h+1}^{l=N}$ is the same as the linear order $\succ_{h-1}$.

The procedure changing the order $\succ_h$ into the order $\succ_{h-1}$ consists of two parts described in detail below. In part one, we move the sequence of roots $\{\alpha_{k,h}\}_{k=1}^{k=h-1} \subset B_h$ to the left through the set A_h. In part two, we move the roots $\{\alpha_{h,l}\}_{l=h+1}^{l=N} \subset A_h$ to the right through the set B_{h-1}. The result after part two is the order $\succ_{h-1}$.

Part one. The root $\alpha_{h-1,h}$ is the largest root in B_h. Thus, it immediately succeeds the set of roots A_h with respect to the order $\succ_h$. On the other hand, it is the highest element of the order $\succ_{h-1}$. We need to move $\alpha_{h-1,h}$ through all elements of the set A_h. Make $h-2$ elementary transformations of type $A_1 \oplus A_1$ moving $\alpha_{h-1,h}$ over $\alpha_{1,N}, ..., \alpha_{h-2,N}$. Then perform $\cdots, \alpha_{h,N}, \alpha_{h-1,N}, \alpha_{h-1,h}, \cdots \to \cdots, \alpha_{h-1,h}, \alpha_{h-1,N}, \alpha_{h,N}, \cdots$. Again, make $h-2$ elementary transformations of type $A_1 \oplus A_1$ moving $\alpha_{h-1,h}$ over $\alpha_{1,N-1}, ..., \alpha_{h-2,N-1}$. Then perform $\cdots, \alpha_{h,N-1}, \alpha_{h-1,N-1}, \alpha_{h-1,h}, \cdots \to \cdots, \alpha_{h-1,h}, \alpha_{h-1,N-1}, \alpha_{h,N-1}, \cdots$. After $N-h$ groups of steps of the above

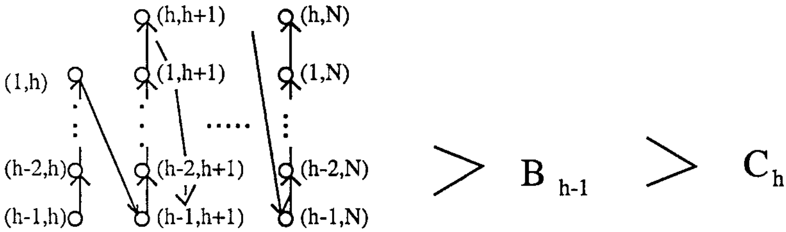

FIG. 3. The resulting order after part one.

type the root $\alpha_{h-1,h}$ is greater than any root of the set A_h. Now, the root $\alpha_{h-2,h}$ is the largest root in $B_h - \{\alpha_{h-1,h}\}$ and it immediately succeeds the set A_h in the normal order resulting after the above $N-h$ groups of steps. Repeat the procedure for the root $\alpha_{h-2,h}$, and then for the roots $\alpha_{h-3,h}, \ldots, \alpha_{1,h}$. This is the end of part one. The resulting order after part one is shown in Fig. 3. In part one we used exactly one A_2 elementary transformation of type $\cdots, \alpha_{h,l}, \alpha_{k,l}, \alpha_{k,h}, \cdots \to \cdots, \alpha_{k,h}, \alpha_{k,l}, \alpha_{h,l}, \cdots$ for every pair of positive integers (k, l), $1 \leqslant k < h < l \leqslant N$. All other elementary transformations in part one were of the type $A_1 \oplus A_1$.

Part two. We need to move the roots $\alpha_{h,N}, \ldots, \alpha_{h,h+1}$ from A_h through B_{h-1} to put them in front of C_h; see Fig. 3. We first move $\alpha_{h,N}$, then $\alpha_{h,N-1}$ and so on. It is easy to see that all elementary transformations we use in this part are of type $A_1 \oplus A_1$. ∎

8.2. *Intermediate Normal Orders between $\succ_h$ and $\succ_{h-1}$.* Consider again the sequence of elementary transformations $\sigma = \{\sigma_1, \ldots, \sigma_M\}$. Denote $\succ^p$ the normal order we obtain from $\succ_h$ after applying $\sigma_1, \ldots, \sigma_p$. Set $\succ^0 = \succ_h$. We have $\succ^M = \succ_{h-1}$. From the construction of the sequence σ it follows that A_2 elementary transformations in σ are labeled by pairs (k, l), $1 \leqslant k < h < l \leqslant N$. Namely, a pair (k, l) labels the transformation

$$\cdots, \alpha_{h,l}, \alpha_{k,l}, \alpha_{k,h}, \cdots \to \cdots, \alpha_{k,h}, \alpha_{k,l}, \alpha_{h,l}, \cdots.$$

For $p = 0, \ldots, M$, let X_p be the set of pairs (k, l) which label A_2 transformations in $\{\sigma_1, \ldots, \sigma_p\}$.

For every pair of integers (k, l), $k < l$, set $F_{k,l}(\succ^p) = F_{k,l}(h-1)$ for $(k, l) \in X_p$, and $F_{k,l}(\succ^p) = F_{k,l}(h)$ for $(k, l) \notin X_p$. The set $\{-F_{k,l}(\succ^p)\}_{k<l}$ is a basis of $\mathbf{n}_-$. Order it according to $\succ^p$. The corresponding PBW-basis of $U(\mathbf{n}_-)$ is

$$F_I(\succ^p) = (-1)^{\sum i_{l,k}} \prod_{k<l} \frac{F_{k,l}(\succ^p)^{i_{l,k}}}{i_{l,k}!}.$$

Let $\{F_{I_1}(\succ^p), \ldots, F_{I_{n+1}}(\succ^p)\}$ be elements of the PBW-basis of $U(\mathbf{n}_-)$ corresponding to $\succ^p$. Set $I=(I_1, \ldots, I_{n+1})$ and $F_I(\succ^p)\, v = F_{I_1}(\succ^p)\, v_1 \otimes \cdots \otimes F_{I_{n+1}}(\succ^p)\, v_{n+1}$. The vector $F_I(\succ^p)\, v$ belongs to $V'[v']$ if $I \in P(v_0, n+1)$

Introduce the rational function $\phi(I;\, \succ^p)$ as

$$\phi(I;\, \succ^p) = \prod_{j=1}^{n+1} \left(\prod_{(k', l') \in X_p} \prod_{q=1}^{i_{l'}} , \phi_{h-1}^{(j, k', l', q)} \times \prod_{(k, l) \notin X_p} \prod_{q=1}^{i_{l,k}} \phi_h^{(j, k, l, q)} \right).$$

8.3. *Geometric Interpretation of the Rational Functions* $\phi(I, h)$ *and* $\phi(I;\, \succ^p)$. For any pair of integers (l, k), $1 \leqslant k < l \leqslant N$, let the string of type (l, k) corresponding to $\succ_h$ be the oriented graph with $l-k$ vertices labeled by positive integers $k, \ldots, l-1$ given in Figure 4, cases (a), (c), (e). For any $z \in \mathbb{C}$, let the string of type (l, k) corresponding to $\succ_h$ grounded at z be the oriented tree with labeled vertices given in Fig. 4, cases (b), (d), (f).

Let $I_0 = \{i_{l,k}\}_{k<l}$ be a set of non-negative integers. Define the tree $T(I_0, h, z)$ as the union of strings corresponding to $\succ_h$ grounded at z, such that the string of type (l, k) grounded at z enters exactly $i_{l,k}$ times and all vertices z of all strings are then identified, see Fig. 5.

Define the tree $T(I_0, \succ^p, z)$ as follows. For every $(k, l) \in X_p$ take $i_{l,k}$ strings of type (l, k) corresponding to $\succ_{h-1}$ and grounded at z_1. For every $(k, l) \notin X_p$ take $i_{l,k}$ strings of type (l, k) corresponding to $\succ_h$ and grounded at z_1. Let $T(I_0, \succ^p, z)$ be the union of all selected strings with all vertices z_1 then identified.

For any $I = (I_1, \ldots, I_{n+1}) \in P(v_0, n+1)$, let the forest $T(I, h) = \bigsqcup_{j=1}^{n+1} T(I_j, h, z_j)$ be the disjoint union of trees, and let the forest $T(I, \succ^p) = \bigsqcup_{j=1}^{n+1} T(I_j, h, \succ^p)$ be the disjoint union of trees.

For any $p = 1, \ldots, N-1$, the number of vertices in the forest $T(I, h)$ labeled by p equals m_p. For every p fix a bijection $\beta_p^h(I)$: {the vertices of $T(I, h)$ labeled by p} $\to S_p$. To every oriented edge (w', w'') from a vertex w' labeled by an integer p' to a vertex w'' labeled by an integer p'' assign the

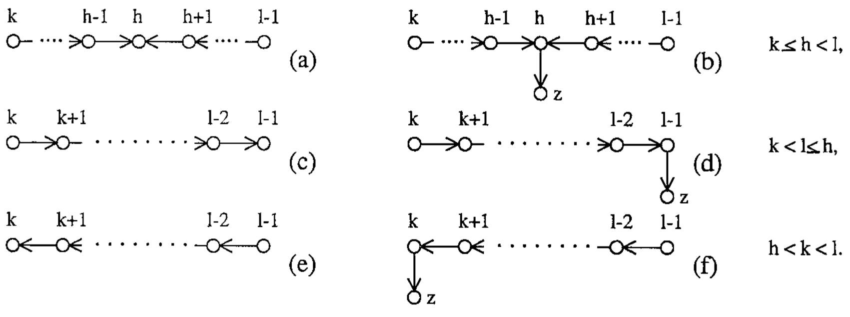

FIG. 4. Strings and grounded strings corresponding to $\succ_h$.

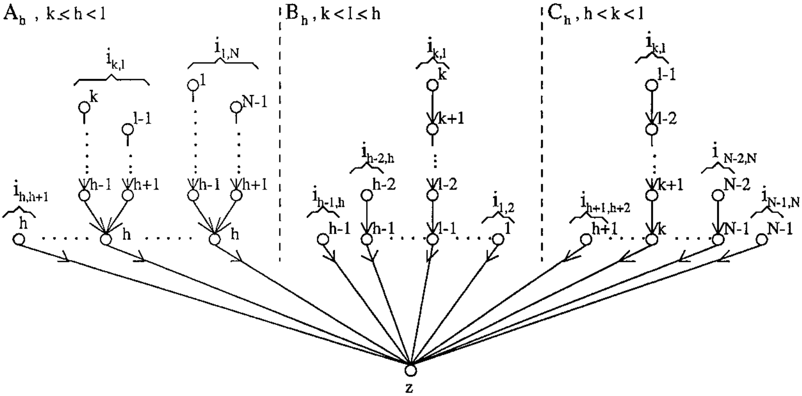

FIG. 5. The tree $T(I_0, h, z)$.

function $f(w', w'') = 1/(t_{p'}^{(\beta^h_{p'}(I)(w'))} - t_{p''}^{(\beta^h_{p'}(I)(w''))})$. To every oriented edge (w', w'') from vertex w' labeled by an integer p' to a vertex w'' labeled by a complex number z assign the function $f(w', w'') = 1/(t_{p'}^{(\beta^h_{p'}(I)(w'))} - z)$. Define rational functions corresponding to a forest (respectively, a tree, a grounded string, or a string) as the product of the functions $f(w', w'')$ over all edges (w', w'') of the forest (respectively, the tree, the grounded string, or the string). It is clear that the rational function corresponding to the forest $T(I, h)$ is exactly the function $\phi(I, h)$ defined in Section 6.

Define a rational function corresponding to the forest $T(I, \succ^p)$ analogously. The rational function corresponding to the forest $T(I, \succ^p)$ equals $\phi(I, \succ^p)$.

Consistency of the Definitions. Let $h' \in \{h-1, h\}$. Since $\succ^0 = \succ_h$, $\succ^M = \succ_{h-1}$, we have given two different definitions of the basis of $\mathbf{n}_-$ corresponding to $\succ_{h'}$, the rational function $\phi(I, h')$, and the forest $T(I, h')$; see Sections 6.2, 8.2, 8.3. We will show that these two definitions are consistent.

In the case $h' = h$, we have $X_0 = \varnothing$. Trivially, $T(I, h) = T(I, \succ^0)$, $\phi(I, h) = \phi(I, \succ^0)$, and $F_{k,l}(h) = F_{k,l}(\succ^0)$ for $k < l$.

In the case $h' = h$, we have $X_M = \{(k, l) \mid 1 \leqslant k < h < l \leqslant N\}$. We start with the basis of $\mathbf{n}_-$. Recall that $F_{k,l}(h') = (-1)^{a_{k,l}(h')} e_{l,k}$, for any h'. The following five cases exhaust all possibilities.

- If $k < h < l$, then $F_{k,l}(\succ^M) = F_{k,l}(h-1)$ by definition.
- *If* $k < l < h$, then $F_{k,l}(\succ^M) = F_{k,l}(h)$, and $a_{k,l}(h) = a_{k,l}(h-1) = 0$. Thus, $F_{k,l}(h) = F_{k,l}(h-1)$ and $F_{k,l}(\succ^M) = F_{k,l}(h-1)$.
- If $k < l = h$, then $F_{k,l}(\succ^M) = F_{k,l}(h)$, $a_{k,l}(h) = 0$, and $a_{k,l}(h-1) = l-(h-1)-1$. Thus, $F_{k,l}(h) = F_{k,l}(h-1)$ and $F_{k,l}(\succ^M) = F_{k,l}(h-1)$.

• If $h<k<l$, then $F_{k,l}(\succ^M)=F_{k,l}(h)$, $a_{k,l}(h)=a_{k,l}(h-1)=l-k-1$. Thus, $F_{k,l}(h)=F_{k,l}(h-1)$ and $F_{k,l}(\succ^M)=F_{k,l}(h-1)$.

• If $h=k<l$, then $F_{k,l}(\succ^M)=F_{k,l}(h)$, $a_{k,l}(h)=l-h-1$, and $a_{k,l}(h-1)=l-k-1$. Thus, $F_{k,l}(h)=F_{k,l}(h-1)$ and $F_{k,l}(\succ^M)=F_{k,l}(h-1)$.

Similarly, for every set of non-negative integers $I_0=\{i_{l,k}\}_{k<l}$, we compare the tree $T(I_0, z, \succ^M)$ with the tree $T(I_0, z, h-1)$. In order to construct the tree $T(I_0, z, h-1)$, for every pair (l, k), $k<l$, we took $i_{l,k}$ strings of type (l, k) grounded at z_1 which correspond to the order $\succ_{h-1}$ and then we identified all vertices z_1. In the construction of the tree $T(I_0, z, \succ^M)$, for every pair (l, k), $k<h<l$, we took $i_{l,k}$ strings of type (l, k) grounded at z_1 which correspond to the order $\succ_{h-1}$, and for every pair (l, k), $h\leqslant k<l$ or $k<l\leqslant h$, we took $i_{l,k}$ strings of type (l, k) grounded at z_1 which correspond to the order $\succ_h$. Then we identified all vertices z_1. For every pair (l, k), $h\leqslant k<l$ or $k<l\leqslant h$, the string of type (l, k) grounded at z_1 which corresponds to the order $\succ_{h-1}$ coincides with the string of type (l, k) grounded at z_1 which corresponds to the order $\succ_h$, see Fig. 4. Therefore, the tree $T(I_0, z, \succ^M)$ coincides with the tree $T(I_0, z, h-1)$. Conclude that the forest $T(I, h-1)$ coincides with the forest $T(I, \succ^M)$, and we have $\phi(I, h-1)=\phi(I, \succ^M)$.

8.5. *Proof of Theorem* 6.2. The theorem states that, for any h, we have $\phi(z, t; h)=\phi(z, t)$. The equality $\phi(z, t; N-1)=\phi(z, t)$ is valid by definition. It suffices to show that $\phi(z, t; h)=\phi(z, t; h-1)$ for every h.

First consider the case $n=0$. Recall that $n+1$ is the number of tensor factors in V'. Thus, $V'=V_1$ is a Verma module. The discussion in Section 8.4 implies that

$$\phi(z, t; h)=\sum_{I\in P(v_0, 1)} \phi(I, \succ^0)\, F_I(\succ^0)\, v_1,$$

$$\sum_{I\in P(v_0, 1)} \phi(I, \succ^M)\, F_I(\succ^M)\, v_1=\phi(z, t; h-1).$$

In order to finish the case $n=0$, it suffice to show that

$$\sum_{I\in P(v_0, 1)} \phi(I, \succ^p)\, F_I(\succ^p)\, v_1=\sum_{I\in P(v_0, 1)} \phi(I, \succ^{p+1})\, F_I(\succ^{p+1})\, v_1, \qquad (34)$$

$$\text{for} \quad p=0, \ldots, M-1.$$

Fix $p\in\{0, \ldots, M-1\}$.

Assume that σ_{p+1} is of type $A_1\oplus A_1$. We have $X_p=X_{p+1}$ and $\phi(I, \succ^p)=\phi(I, \succ^{p+1})$ for every index $I\in P(v_0, 1)$. The case $A_1\oplus A_1$ of

Lemma 6.1 implies $F_I(\succ^p) = F_I(\succ^{p+1})$ for every index $I \in P(v_0, 1)$. This implies (34).

Assume that σ_{p+1} is of type A_2 and is labeled by a pair of integers (k, l), $k < h < l$. The set X_{p+1} equals the set $X_p \cup \{(k, l)\}$. Let us check, that we can apply the A_2 case of Lemma 6.1 to the triple $F_{h,l}(\succ^p)$, $F_{k,l}(\succ^p)$, $F_{k,h}(\succ^p) \in sl_N$. Set $f_\alpha = F_{h,l}(h), f_\beta = F_{k,h}(h)$. Then $f_\alpha = F_{h,l}(\succ^p) = F_{h,l}(\succ^{p+1})$, $f_\beta = F_{k,h}(\succ^p) = F_{k,h}(\succ^{p+1})$,

$$[f_\alpha, f_\beta] = (-1)^{l-h-1}[e_{l,h}, e_{h,k}] = (-1)^{l-h-1} e_{l,k} = F_{k,l}(h) = F_{k,l}(\succ^p),$$
$$[f_\beta, f_\alpha] = (-1)^{l-(h-1)-1} e_{l,k} = F_{k,l}(h-1) = F_{k,l}(\succ^{p+1}).$$

Here we used the explicit form of the numbers $\{a_{k,l}(h)\}$, namely,

$$a_{k,l}(h) = 0 \quad \text{if} \quad k < l \leqslant h, \quad a_{k,l}(h) = l-k-1 \quad \text{if} \quad h < k < l,$$
$$a_{k,l}(h) = l-h-1 \quad \text{if} \quad k \leqslant h < l.$$

By Lemma 6.1, we have

$$\frac{F_{h,l}(\succ^p)^a}{a!}\frac{F_{k,l}(\succ^p)^c}{c!}\frac{F_{k,h}(\succ^p)^b}{b!}$$
$$= \sum_r \binom{c+r}{r}(-1)^{c-r}\frac{F_{k,h}(\succ^{p+1})^{b-r}}{(b-r)!}\frac{F_{k,l}(\succ^{p+1})^{c+r}}{(c+r)!}\frac{F_{h,l}(\succ^{p+1})^{a-r}}{(a-r)!}.$$

Use this equality to transform $\sum_{I \in P(v_0, 1)} \phi(I, \succ^p) F_I(\succ^p) v_1$. Set $a = i_{h,l}$, $b = i_{k,h}$, $c = i_{k,l}$, $I' = \{i_{p,q}\}_{(p,q) \notin \{(h,l),(k,h),(k,l)\}}$, $|I'| = \sum i_{p,q}$. We have $I = \{I', a, b, c\}$ and

$$(35) \quad \sum_{I \in P(v_0, 1)} \phi(I, \succ^p) F_I(\succ^p) v_1$$
$$= \sum_{I', d_1, d_2} (-1)^{|I'|} \sum_{\substack{a+c=d_1,\\ b+c=d_2}} (-1)^{a+b+c} \phi(\{I', a, b, c\}; \succ^p)$$
$$\times \cdots \frac{F_{h,l}(\succ^p)^a}{a!}\frac{F_{k,l}(\succ^p)^c}{c!}\frac{F_{k,h}(\succ^p)^b}{b!} \cdots v_1$$
$$= \sum_{I', d_1, d_2} (-1)^{|I'|} \sum_{\substack{a+c=d_1,\\ b+c=d_2}} \sum_r \binom{c+r}{r}(-1)^c(-1)^{a+b+c-r}$$
$$\times \phi(\{I', a, b, c\}; \succ^p)$$
$$\times \cdots \frac{F_{k,h}(\succ^{p+1})^{b-r}}{(b-r)!}\frac{F_{k,l}(\succ^{p+1})^{c+r}}{(c+r)!}\frac{F_{h,l}(\succ^{p+1})^{a-r}}{(a-r)!} \cdots v_1.$$

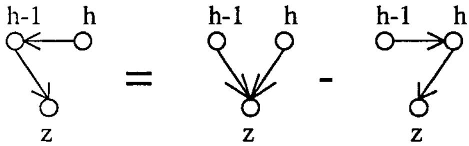

FIGURE 6

$$(36)\qquad \sum_{I\in P(v_0,1)} \phi(I, \succ^p)\, F_I(\succ^p)\, v_1$$

$$= \sum_{I',d_1,d_2} (-1)^{|I'|} \sum_{\substack{a'+c'=d_1,\\ b'+c'=d_2}} (-1)^{a'+b'+c'}\, \psi(\{I',a',b',c'\};\, \succ^p)$$

$$\times \cdots \frac{F_{k,h}(\succ^{p+1})^{b'}}{(b')!}\, \frac{F_{k,l}(\succ^{p+1})^{c'}}{(c')!}\, \frac{F_{h,l}(\succ^{p+1})^{a'}}{(a')!} \cdots v_1,$$

where $\psi(\{I',a',b',c'\};\, \succ^p) = \sum_r \binom{c'}{r}(-1)^{c'-r}\, \phi(\{I',a'+r,b'+r,c'-r\};\, \succ^p)$.
The following claim implies (34).

Claim. For every $I = \{I',a',b',c'\} \in P(v_0,1)$ we have

$$(37)\qquad \psi(\{I',a',b',c'\};\, \succ^p) = \phi(\{I',a',b',c'\};\, \succ^{p+1}).$$

The right hand side of (37) equals the product of the functions $f(w',w'')$ over all oriented edges (w',w'') of the tree $T(I,z_1,\succ^{p+1})$. The left hand side of (37) depends on rational functions $\{\phi(\{I',a'+r,b'+r,c'-r\};\, \succ^p)\}_r$ which correspond to trees $\{T(\{I',a'+r,b'+r,c'-r\},z_1,\succ^{p+1})\}_r$. For every r, the tree $T(I,z_1,\succ^{p+1})$ and the tree $T(\{I',a'+r,b'+r,c'-r\},$ $z_1,\succ^{p+1})$ differ only at the grounded strings of types (k,l), (h,l), (k,h). Namely, the tree $T(I,z_1,\succ^{p+1})$ has a' strings of type (h,l) grounded at z_1 corresponding to $\succ_h$, b' strings of type (k,h) grounded at z_1 corresponding

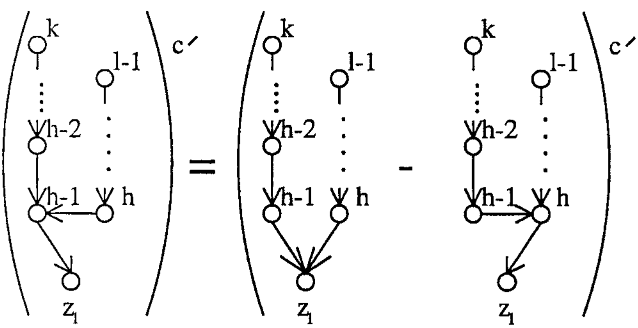

FIGURE 7

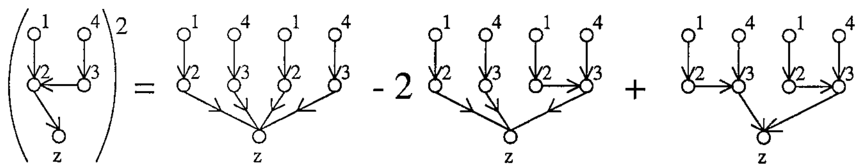

FIG. 8. Example.

to $\succ_h$ and c' strings of type (k, l) grounded at z_1 corresponding to $\succ_{h-1}$. The tree $T((I', a'+r, b'+r, c'-r), z_1, \succ^{p+1})$ has $a'+r$ strings of type (h, l) grounded at z_1 corresponding to $\succ_h$, $b'+r$ strings of type (k, h) grounded at z_1 corresponding to $\succ_h$ and $c'-r$ strings of type (k, l) grounded at z_1 corresponding to $\succ_h$.

The claim now follows from identities

$$\frac{1}{(t_h-t_{h-1})(t_{h-1}-z)}=\frac{1}{(t_h-z)(t_{h-1}-z)}-\frac{1}{(t_{h-1}-t_h)(t_h-z)}, \tag{38}$$

$$\prod_{q=1}^{c'} \phi_{h-1}^{(1,k,l,q)}=\prod_{q=1}^{c'} (\phi_h^{(1,k,h,q)}\phi_h^{(1,h,l,q)}-\phi_h^{(1,k,l,q)}) \tag{39}$$

Identity (38) has a simple tree interpretation shown in Fig. 6.

Identity (39) has a tree interpretation shown in Fig. 7. In Fig. 7 multiplication of trees means disjoint union of trees, addition is formal, and multiplication distributes through addition; see the example in Fig. 8. The claim is proved. Thus, the statement of Theorem 6.2 is proved for $n=0$. ∎

The statement of Theorem 6.2 for arbitrary n is proved as the case $n=0$ since we execute the changes of bases and the reorganization of rational functions separately for each tensor factor. There is no interference between the rational functions corresponding to different tensor factors since the grounding points of the strings are distinct. ∎

9. PROOF OF THEOREM 5.1

Let $V=M_1\otimes\cdots\otimes M_n$, where M_j is the sl_N Verma module with highest weight Λ_j and highest weight vector v_j. Set $v=v_1\otimes\cdots\otimes v_n$. Fix a weight subspace $V[\nu]\subset V$, where $\nu=\sum_{j=1}^n \Lambda_j-\sum_{k=1}^{N-1} m_k\alpha_k$, and $\sum_{k=1}^{N-1} m_k\alpha_k\in Q^+$. Recall the notation $m=\sum_{k=1}^{N-1} m_k$, $\nu_0=\sum_{k=1}^{N-1} m_k\alpha_k$. Set $\Lambda_{n+1}=\lambda-\rho-\frac{1}{2}\nu$. Let M_{n+1} be the sl_N Verma module with highest weight Λ_{n+1} and highest weight vector v_{n+1}. Consider the auxiliary space $V'=V\otimes M_{n+1}$ and its weight space $V'[\nu']$, where $\nu'=\nu+\Lambda_{n+1}$.

Vector Spaces of Rational Functions. For every $I \in P(v_0, n+1)$ fix a set of bijections $\{\beta_p(I)\}$ and define a vector space of rational functions $\Omega'(\{t_k^{(d)}\}, \{z_j\}) = \mathbb{C}\{\phi(I) \mid I \in P(v_0, n+1)\}$. These functions depend on the complex variables $t_k^{(d)}, z_j$, $k = 1, ..., N-1$, $d \in \{1, ..., m_k\}$, $j = 1, ..., n+1$. Denote Ω' the restriction $\Omega'(\{t_k^{(d)}\}, z_1, ..., z_n, 0)$. We will use the same notation $\phi(I)$ for a function in Ω'.

For every $I \in P(v_0, n)$ fix a set of bijections $\{\beta_p(I)\}$ and define a vector space of rational functions $\Omega = \mathbb{C}\{\phi(I) \mid I \in P(v_0, n)\}$. These functions depend on the complex variables $t_k^{(d)}, z_j$, $k = 1, ..., N-1$, $d \in \{1, ..., m_k\}$, $j = 1, ..., n$. Identify each index $I \in P(v_0, n)$ with the index $I' = (I, \{i_{l,k}^{(n+1)} = 0\}_{k<l}) \in P(v_0, n+1)$. Since $\phi(I') = \phi(I)$ for an appropriate choice of bijections $\{\beta_p(I')\}$, $\{\beta_p(I)\}$ we will consider Ω as a subset of Ω'.

Consider the set of rational functions with poles in $\mathscr{D}$; see Section 4. Note that it contains Ω'. Define an equivalence relation $\backsim$ on the set. Namely, let ϕ_1, ϕ_2 be two elements of the set. We write $\phi_1 \backsim \phi_2$ if $\int_{\gamma(z)} \Phi^{1/\kappa} \phi_1 \, dt = \int_{\gamma(z)} \Phi^{1/\kappa} \phi_2 \, dt$ for any horizontal family of integration cycles $\gamma(z)$, which satisfies Matsuo's assumption.

EXAMPLES (a) For a fixed $I \in P(v_0, n+1)$, the functions $\phi(I)$ corresponding to different choices of bijections $\{\beta_p(I)\}$ are equivalent under $\backsim$.

(b) If there exists a rational form η with poles in $\mathscr{D}$ such that $\Phi^{1/\kappa}(\phi_1 - \phi_2) \, dt = d_t(\Phi^{1/\kappa}\eta)$, where d_t denotes the exterior differentiation with respect to the variables $t_k^{(d)}$, then $\phi_1 \backsim \phi_2$ because $\gamma(z)$ is a family of closed cycles.

The set $\{(F_I v')^*\}_{I \in P(v_0, n+1)}$ is a basis of $(V'[v'])^*$; see the definition of $\{F_I v'\}$ in Section 4. Define a linear map $D' \colon (V'[v'])^* \to \Omega'$ by setting $D((F_I v')^*) = \phi(I)$, for all $I \in P(v_0, n+1)$. Analogously, define a linear map $D \colon (V[v])^* \to \Omega$ by setting $D((F_I v)^*) = \phi(I)$, for all $I \in P(v_0, n)$.

For indices $I \in P(v_0, n)$ and $I' = (I, \{i_{l,k}^{(n+1)} = 0\}_{k<l}) \in P(v_0, n+1)$ we have

$$D'(F_{I'} v') = D'(F_I v \otimes v_{n+1}) = \phi(I') = \phi(I) = D(F_I v). \tag{40}$$

Reformulate Theorem 6.2 as follows.

THEOREM 9.1. *For every $h = 1, ..., N-1$, we have $\sum_{I \in P(v_0, n)} D((F_I(h)\, v)^*)\, F_I(h)\, v = \phi(t, z)$. Moreover, for all I, $D((F_I(h)\, v)^*) = \phi(I, h)$.*

9.2. *The Action of sl_N on $(V')^*$.* For every set of non-negative integers $I_0 = \{i_{l,k}\}_{k<l}$, and every l', k', $k' < l'$, denote $I_0 \pm 1_{l',k'} = \{i'_{l,k}\}_{k<l}$, where $i'_{l',k'} = i_{l',k'} \pm 1$ and $i'_{l,k} = i_{l,k}$ if $(l, k) \neq (l', k')$. Let $I = (I_1, ..., I_{n+1})$, where $I_j = \{i_{l,k}^{(j)}\}_{k<l}$ is a set of non-negative integers. Let $I \pm 1_{l',k'}^{(j)}$ denote addition in the component I_j.

LEMMA 9.2 (cf. Lemma 4.2 in [Ma]). *For every I, and every $h=1, ..., N-1$, we have*

$$E_{\alpha_h}(F_I v')^* = \sum_{j=1}^{n+1} \left(\sum_{p=h+2}^{N-1} i_{p,h+1}^{(j)} (F_{I+1_{p,h}^{(j)}-1_{p,h+1}^{(j)}} v')^* - \sum_{p=1}^{h-1} i_{h,p}^{(j)} (F_{I+1_{h+1,p}^{(j)}-1_{h,p}^{(j)}} v')^* \right.$$
$$\left. + \left((\Lambda_j, \alpha_h) + \sum_{p=1}^{h-1} i_{h,p}^{(j)} - \sum_{p=1}^{h} i_{h+1,p}^{(j)} \right) (F_{I+1_{h+1,h}^{(j)}} v')^* \right).$$

The proof is a straightforward computation. We use the standard PBW-basis of $U(\mathbf{n}_-)$. ∎

LEMMA 9.3. *For every I, and every k, h, l, $1 \leqslant k \leqslant h < l \leqslant N$, we have*

$$F_{k,l}(h)(F_I(h)\, v')^* = \sum_{j=1}^{n+1} i_{l,k}^{(j)} (F_{I-1_{l,k}^{(j)}}(h)\, v')^*. \tag{41}$$

The proof is a straightforward computation. We use the PBW-basis of $U(\mathbf{n}_-)$ corresponding to $\succ_h$. ∎

Next, we will compute the image of $E_{\alpha_h}(F_I v')^* \in V'[v']$ under D'. According to Lemma 9.2, $D'(E_{\alpha_h}(F_I v')^*)$ is a sum of rational functions with complex coefficients, $\sum_{J \in P(v_0, n+1)} c_J \phi(J)$. For each index J, we can choose a set of bijections $\{\beta_p(J)\}$ such that the function $\phi(J)$ has the form $\phi(I)/(t_h - *)$, where t_h is a new variable of type h, and $*$ is any of the t variables fixed by the set of bijections $\{\beta_p(I)\}$, or a complex number in the set $\{z_1, ..., z_n, 0\}$.

LEMMA 9.4 [[Ma], Lemma 3.4]. *Let $s=(j,k,l,q) \in S(I)$. Then, we have*

(a) $$\frac{1}{t_h - z_j} \phi^{(j,k,l,q)} \backsim \phi^{(j,h,h+1,i_{h+1,h}^{(j)}+1)} \phi^{(j,k,l,q)},$$

(b) $$\left(\sum_{p=k}^{l-1} \frac{(\alpha_h, \alpha_p)}{t_h - t_p^{(\beta_p(I)(s))}} \right) \phi^{(j,k,l,q)}$$
$$\backsim \begin{cases} -\phi^{(j,h,l,i_{l,h}^{(j)}+1)} & \textit{if } h=k-1, \\ \phi^{(j,k,l,q)} \phi^{(j,h,h+1,i_{h+1,h}^{(j)}+1)} & \textit{if } h=l-1, \\ \phi^{(j,k,h+1,i_{h+1,l}^{(j)}+1)} - \phi^{(j,k,h,q)} \phi^{(j,h,h+1,i_{h+1,h}^{(j)}+1)} & \textit{if } h=l, \\ 0 & \textit{otherwise.} \end{cases}$$

COROLLARY 9.5.

$$\text{(a)}\quad \frac{d_{t_h}\Phi}{\Phi}\,\phi(I) \backsim \sum_{j=1}^{n+1}\Bigg(-\sum_{p=h+2}^{N-1} i^{(j)}_{p,h+1}\phi(I+1^{(j)}_{p,h}-1^{(j)}_{p,h+1})$$
$$+\sum_{p=1}^{h-1} i^{(j)}_{h,p}\phi(I+1^{(j)}_{h+1,p}-1^{(j)}_{h,p})$$
$$+\Bigg(-(\Lambda_j,\alpha_h)-\sum_{p=1}^{h-1} i^{(j)}_{h,p}+\sum_{p=1}^{h} i^{(j)}_{h+1,p}\Bigg)\phi(I+1^{(j)}_{h+1,h})\Bigg).$$

$$\text{(b)}\quad D'(E_{\alpha_h}(F_I v')^*) \backsim \frac{d_{t_h}\Phi}{\Phi}\,\phi(I).$$

Proof. (a) Our choice of Λ_{n+1} is consistent with the definition of the function Φ, that is, $(t_k^{(d)})^{-(\alpha_k,\,\Lambda_{n+1})}$ is the contribution of the point $z_{n+1}=0$ in Φ. Observe that

$$\frac{d_{t_h}\Phi}{\Phi}=\sum_{j=1}^{n+1}\Bigg(-(\Lambda_j,\alpha_h)\frac{d(t_h-z_j)}{(t_h-z_j)}+\sum_{k<l}\sum_{q=1}^{i^{(j)}_{l,k}}\sum_{h=k}^{l-1}(\alpha_h,\alpha_p)\frac{d(t_h-t_p^{(\beta_p(I)(s))})}{(t_h-t_p^{(\beta_p(I)(s))})}\Bigg),$$

and apply Lemma 9.4.

(b) Part (a) and Lemma 9.2 imply statement (b). ∎

COROLLARY 9.6. *For any $x\in U(\mathbf{n}_+)$ and any $u'=u\otimes u_{n+1}\in V'=V\otimes V_\lambda$, such that $xu\in V'[v']$, we have $D'(u\otimes xu_{n+1})\backsim D'(A(x)\,u\otimes u_{n+1})$, where A is the antipode map.*

Proof. It is sufficient to show that for any $h=1,\ldots,N-1$, we have $D'(E_{\alpha_h}(F_I v')^*)\backsim 0$. Equivalently, we may show $(d_{t_h}\Phi/\Phi)\,\phi(I)\backsim 0$. Introduce the logarithmic $(m-1)$-form

$$\eta_I=\prod_{s\in S(I)}\Bigg(\prod_{p=k}^{l-2}\frac{d(t_p^{(\beta_p(I)(s))}-t_{p+1}^{(\beta_{p+1}(I)(s))})}{t_p^{(\beta_p(I)(s))}-t_{p+1}^{(\beta_{p+1}(I)(s))}}\Bigg)\frac{d(t_{l-1}^{(\beta_{l-1}(I)(s))}-z_j)}{t_{l-1}^{(\beta_{l-1}(I)(s))}-z_j}.$$

We have $d_t\eta_I=0$, and

$$\kappa d_t(\Phi^{\frac{1}{\kappa}}\eta_I)=\Phi^{\frac{1}{\kappa}}\frac{d_t\Phi}{\Phi}\wedge\eta_I=\Phi^{\frac{1}{\kappa}}\frac{d_{t_h}\Phi}{\Phi}\wedge\eta_I.$$

Let i be the inclusion map $i\colon \mathbb{C}^m\to\{(z,0)\}\times\mathbb{C}^m\subset\mathbb{C}^{n+1}\times\mathbb{C}^m$, and as usual, dt denotes the standard volume form on $\mathbb{C}^m_t$. We have $i^*((d_t\Phi/\Phi)\wedge\eta_I)=\pm(d_{t_h}\Phi/\Phi)\,\phi(I)\,dt$. Thus, $(d_{t_h}\Phi/\Phi)\,\phi(I)\backsim 0$. ∎

9.3. *Proof of Theorem* 5.1. Let $v(z, \lambda)$ be the hypergeometric solution of the trigonometric KZ equation indicated in Corollary 4.2. Fix $h \in \{1, ..., N-1\}$. By definition,

$$v(z, \lambda+\kappa\omega_h^\vee) = \int_{\gamma(z)} \Phi(z, t; \lambda+\kappa\omega_h^\vee)^{1/\kappa} \phi(z, t)\, dt.$$

The function $\Phi^{1/\kappa}$ changes as

$$\Phi^{\frac{1}{\kappa}}(z, t; \lambda+\kappa\omega_h^\vee) = \left(\prod_{j=1}^{n} z_j^{(\Lambda_j, \omega_h^\vee)} \prod_{d=1}^{m_h} \frac{1}{t_h^{(d)}}\right) \Phi^{\frac{1}{\kappa}}(z, t; \lambda).$$

We study the product $[\prod_{j=1}^n z_j^{(\Lambda_j, \omega_h^\vee)} \prod_{d=1}^{m_h} (t_h^{(d)})^{-1}]\, \phi(z, t)$, using the expansion $\phi(z, t) = \sum_{I \in P(v_0, n)} \phi(I, h)\, F_I(h)\, v$ obtained in Theorem 6.2.

The identity

$$\frac{1}{(t_h^{(d)} - z_j)\, t_h^{(d)}} = \frac{1}{z_j}\left(\frac{1}{t_h^{(d)} - z_j} - \frac{1}{t_h^{(d)}}\right)$$

implies the tree interpretation shown in Fig. 9. Set $m_{h,j}(I)$ to be equal to the number of elements of the set $\{(j', k, l, q) \in S_h(I) \mid j' = j\}$. For all I, we have $\sum_{j=1}^n m_{h,j}(I) = m_h$. Use the equality of rational functions presented in Fig. 9 and the definition $\phi(I, h) = \prod_{j=1}^n \prod_{(j,k,l,q) \in S(I)} \phi_h^{(j,k,l,q)}$ to obtain

$$\left(\prod_{d=1}^{m_h} (t_h^{(d)})^{-1}\right) \phi(I, h)$$

$$= \prod_{j=1}^{n} z_j^{-m_{h,j}(I)} \left(\prod_{(j,k,l,q) \in S_h(I)} (\phi_h^{(j,k,l,q)} - \phi_h^{(n+1,k,l,q)}) \prod_{(j,k,l,q) \notin S_h(I)} \phi_h^{(j,k,l,q)}\right).$$

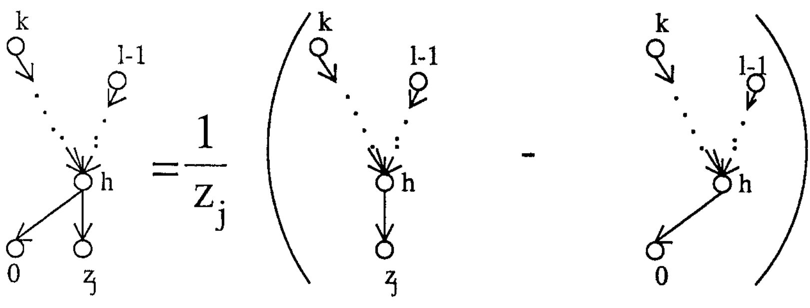

FIGURE 9

Therefore, we have

(42)

$$\left[\prod_{j=1}^{n} z_j^{(\Lambda_j,\, \omega_h^\vee)} \prod_{d=1}^{m_h} (t_h^{(d)})^{-1}\right] \phi(z, t)$$

$$= \prod_{j=1}^{n} z_j^{(\omega_h^\vee)^{(j)}} \left(\prod_{(j,k,l,q) \in S_h(I)} (\phi_h^{(j,k,l,q)} - \phi_h^{(n+1,k,l,q)}) \prod_{(j,k,l,q) \notin S_h(I)} \phi_h^{(j,k,l,q)} \right) F_I(h)\, v.$$

We see that the shift of the parameter λ reduces to the following operation on rational functions. Take each function $\phi(I, h)$ in the formula for $\phi(z, t)$, $z \in \mathbb{C}^n$, $t \in \mathbb{C}^m$, and substitute it with the rational function $\prod_{(j,k,l,q) \in S_h(I)} (\phi_h^{(j,k,l,q)} - \phi_h^{(n+1,k,l,q)}) \prod_{(j,k,l,q) \notin S_h(I)} \phi_h^{(j,k,l,q)}$, which depends on variables $(z, z_{n+1}) \in \mathbb{C}^{n+1}$, $t \in \mathbb{C}^m$. Then set $z_{n+1} = 0$. The result is a function that depends on the initial set of variables, $z \in \mathbb{C}^n$, $t \in \mathbb{C}^m$.

LEMMA 9.7. *For every $I \in P(v_0, n)$ we have*

$$\prod_{(j,k,l,q) \in S_h(I)} (\phi_h^{(j,k,l,q)} - \phi_h^{(n+1,k,l,q)}) \prod_{(j,k,l,q) \notin S_h(I)} \phi_h^{(j,k,l,q)}$$

$$= \sum_{I_0 \in \mathscr{A}(h)} D'(F_{I_0}(h)(F_I(h)\, v)^* \otimes (F_{I_0}(h)\, v_{n+1})^*).$$

Proof. We will compare additive presentations of the two sides of the equality. The left hand side, after expansion of the products, has the following tree interpretation. It is the rational function corresponding to a formal sum of forests. An element of this formal sum is a new forest produced from $T(I, h)$ as follows. We add one tree, consisting only of strings of type (l, k), $k \leqslant h < k$, corresponding to $\succ_h$ and grounded at zero. For each string of type (l, k) belonging to the new tree, one string of type (l, k) is removed from one of the trees which belong to the original forest $T(I, h)$. The coefficient assigned to the new forest is a combinatorial coefficient times a sign coefficient, namely minus one to the power equal to the number of strings which comprise the additional tree. Recall the definition of the index set $\mathscr{A}(h) = \{I_0 = \{i_{l,k}\}_{k<l} \mid i_{l,k} = 0$ if $k > h$, or $l \leqslant h\}$. Thus, the additional tree is always of the form $T(I_0, h, 0)$ for some $I_0 \in \mathscr{A}(h)$. In this notation the sign coefficient in the above description is $(-1)^{|I_0|}$. Let us give an explicit formula for the combinatorial coefficient. Consider $I_0 \in \mathscr{A}(h)$, $I = (I_1, \ldots, I_n) \in P(v_0, n)$, where $I_j = \{i_{l,k}^{(j)}\}_{k<l}$ is a set of non-negative integers. In order to remove $i_{l,k}^0$ strings of type (l, k) from the forest $T(I, h)$, we should choose numbers $i_{l,k}^{(j,0)}$, where $0 \leqslant i_{l,k}^{(j,0)} \leqslant i_{l,k}^{(j)}$ and $\sum_{j=1}^n i_{l,k}^{(j,0)} = i_{l,k}^{(0)}$. Then, for each $j = 1, \ldots, n$, we remove $i_{l,k}^{(j,0)}$ strings of

type (l, k) from the tree $T(I, h, z_j) \in T(I, h)$. The combinatorial coefficient of this choice is $\prod_{k<l} \prod_{j=1}^{n} \binom{i_{k,l}^{(j)}}{i_{k,l}^{(j,0)}}$. Therefore, the left hand side equals

(43)

$$\sum_{I_0 \in \mathscr{A}(h)} (-1)^{|I_0|} \sum_{\{i_{l,k}^{(j,0)}\}} \left(\prod_{k<l} \prod_{j=1}^{n} \binom{i_{k,l}^{(j)}}{i_{k,l}^{(j,0)}} \right) \phi\left(\left\{ I - \sum_{j=1}^{n} \sum_{k<l} i_{k,l}^{(j,0)} 1_{l,k}^{(j)}, I_0 \right\}, h \right).$$

Next, we consider the right hand side. Repeated application of Lemma 9.3 gives

$$(-1)^{|I_0|} F_{I_0}(h)(F_I(h)\, v)^* = \sum_{\{i_{l,k}^{(j,0)}\}} \left(\prod_{k<l} \prod_{j=1}^{n} \binom{i_{k,l}^{(j)}}{i_{k,l}^{(j,0)}} \right) (F_{I - \sum_{j=1}^{n} \sum_{k<l} i_{k,l}^{(j,0)} 1_{l,k}^{(j)}}(h)\, v)^*.$$

Therefore

(44)

$$F_{I_0}(h)(F_I(h)\, v)^* \otimes (F_{I_0}(h)\, v_{n+1})^*$$

$$(-1)^{|I_0|} \sum_{\{i_{l,k}^{(j,0)}\}} \left(\prod_{k<l} \prod_{j=1}^{n} \binom{i_{k,l}^{(j)}}{i_{k,l}^{(j,0)}} \right) (F_{I - \sum_{j=1}^{n} \sum_{k<l} i_{k,l}^{(j,0)} 1_{l,k}^{(j)}}(h)\, v)^* \otimes (F_{I_0}(h)\, v_{n+1})^*.$$

Finally, formula (43), equality (44), and the definition of the map D' imply the statement of the lemma. ▮

LEMMA 9.8. *For every $I \in P(v_0, n)$, and every $I_0 \in \mathscr{A}(h)$, we have*

(a) $D'(F_{I_0}(h)(F_I(h)\, v)^* \otimes (F_{I_0}(h)\, v_{n+1})^*) \backsim D(A \circ \tau(P_{I_0}(F(h), \Lambda_{n+1}))\, F_{I_0}(h)(F_I(h)\, v)^*)$.

(b) $D(A \circ \tau(P_{I_0}(F(h), \Lambda_{n+1}))\, F_{I_0}(h)(F_I(h)\, v)^*) = \sum_{J \in P(v_0, n)} \langle (F_I(h)\, v)^*, A(F_{I_0}(h))\, \tau(P_{I_0}(F(h), \Lambda_{n+1}))\, F_J(h)\, v \rangle\, D((F_J(h)\, v)^*)$.

Proof. Part (a).

$$\begin{aligned} &D'(F_{I_0}(h)(F_I(h)\, v)^* \otimes (F_{I_0}(h)\, v_{n+1})^*) \\ &\quad = D'(F_{I_0}(h)(F_I(h)\, v)^* \otimes \tau(P_{I_0}(F(h), \Lambda_{n+1}))(v_{n+1})^*) \\ &\quad \backsim D'(A \circ \tau(P_{I_0}(F(h), \Lambda_{n+1}))\, F_{I_0}(h)(F_I(h)\, v)^* \otimes (v_{n+1})^*) \\ &\quad = D(A \circ \tau(P_{I_0}(F(h), \Lambda_{n+1}))\, F_{I_0}(h)(F_I(h)\, v)^*). \end{aligned}$$

We use the property $(F_{I_0}(h)\, v_{n+1})^* = \tau(P_{I_0}(F(h), \Lambda_{n+1})(v_{n+1})^*)$ to write the first equality, Corollary 9.6 to write the second equivalence, and the identification (40) for the last step.

Part (b) is trivial. We write $A \circ \tau(P_{I_0}(F(h), \Lambda_{n+1}) F_{I_0}(h)(F_I(h)\, v)^*$ in terms of the dual basis $\{(F_J v)^*\}$ of $V[\nu]^*$, then use the linear property of the map D and the definition of sl_N-action on the dual space.

$$
\begin{aligned}
&D(A \circ \tau(P_{I_0}(F(h), \Lambda_{n+1})) F_{I_0}(h)(F_I(h)\, v)^*) \\
&\quad = \sum_{J \in P(v_0, n)} D(\langle A \circ \tau(P_{I_0}(F(h), \Lambda_{n+1})) F_{I_0}(h)(F_I(h)\, v)^*, F_J(h)\, v\rangle (F_J(h)\, v)^*) \\
&\quad = \sum_{J \in P(v_0, n)} \langle A \circ \tau(P_{I_0}(F(h), \Lambda_{n+1}) F_{I_0}(h)(F_I(h)\, v)^*, F_J(h)\, v\rangle\, D((F_J(h)\, v)^*) \\
&\quad = \sum_{J \in P(v_0, n)} \langle (F_I(h)\, v)^*, A(F_{I_0}(h))\, \tau(P_{I_0}(F(h), \Lambda_{n+1})) \\
&\quad\quad \times F_J(h)\, v\rangle\, D((F_J(h)\, v)^*). \quad \blacksquare
\end{aligned}
$$

Finally, we combine all the steps in the following computation.

$$
\begin{aligned}
&\left[\prod_{j=1}^{n} z_j^{(\Lambda_j, \omega_h^\vee)} \prod_{d=1}^{m_h} (t_h^{(d)})^{-1}\right] \phi(z, t) \\
&\quad = \prod_{j=1}^{n} z_j^{(\omega_h^\vee)^{(j)}} \sum_{I \in P(v_0, n)} \Bigg(\prod_{(j, k, l, q) \in S_h(I)} (\phi_h^{(j, k, l, q)} - \phi_h^{(n+1, k, l, q)}) \\
&\quad\quad \times \prod_{(j, k, l, q) \notin S_h(I)} \phi_h^{(j, k, l, q)} \Bigg) F_I v \\
&\quad = \prod_{j=1}^{n} z_j^{(\omega_h^\vee)^{(j)}} \sum_{I \in P(v_0, n)} \sum_{I_0 \in \mathscr{A}(h)} D'(F_{I_0}(h)(F_I(h)\, v)^* \otimes (F_{I_0}(h)\, v_{n+1})^*)\, F_I v \\
&\quad = \prod_{j=1}^{n} z_j^{(\omega_h^\vee)^{(j)}} \sum_{I, J \in P(v_0, n)} \sum_{I_0 \in \mathscr{A}(h)} \langle (F_I(h)\, v)^*, A(F_{I_0}(h)) \\
&\quad\quad \times \tau(P_{I_0}(F(h), \Lambda_{n+1})(F_J(h))\, v\rangle\, D((F_J(h)\, v)^*)\, F_I(h)\, v \\
&\quad = \prod_{j=1}^{n} z_j^{(\omega_h^\vee)^{(j)}} \sum_{J \in P(v_0, n)} \sum_{I_0 \in \mathscr{A}(h)} D((F_J(h)\, v)^*)\, A(F_{I_0}(h)) \\
&\quad\quad \times \tau(P_{I_0}(F(h), \Lambda_{n+1})(F_J(h)\, v) \\
&\quad = \prod_{j=1}^{n} z_j^{(\omega_h^\vee)^{(j)}} \sum_{J \in P(v_0, n)} D((F_J(h)\, v)^*)(\mathbb{B}_{\omega[h]}(\Lambda_{n+1} + \rho + \tfrac{1}{2}\nu)(F_J(h)\, v)).
\end{aligned}
$$

We have applied equality (42), Lemma 9.7, Lemma 9.8, contraction of the summation by I, and Theorem 7.1 consecutively.

Lemma 9.1 asserts that $D((F_J(h)\,v)^*) = \phi(J, h)$. Since $\lambda = \Lambda_{n+1} + \rho + \frac{1}{2}\nu$ and $K_h(z, \lambda) = \prod_{j=1}^{n} z_j^{(\omega_h^\vee)^{(j)}} \mathbb{B}_{\omega[h]}(\lambda)$ we have

$$(45) \qquad \left[\prod_{j=1}^{n} z_j^{(\Lambda_j, \omega_h^\vee)} \prod_{d=1}^{m_h} (t_h^{(d)})^{-1}\right] \phi(z, t) \backsim K_h(z, \lambda)\,\phi(z, t),$$

which is equivalent to the statement of the theorem for a hypergeometric solution $v(z, \lambda)$ with values in a tensor product of Verma modules. The statement of the theorem for a hypergeometric solution with values in a tensor product of any highest weight sl_N-modules follows from the functorial properties of the operator K_h. ▮

10. APPENDIX A

Matsuo's Hypergeometric Solutions of the KZ Equations for sl_N

A construction in [Ma] gives hypergeometric solutions of the rational KZ equations in a tensor product of lowest weight sl_N modules. We modify that procedure to a construction of solutions of the rational KZ equations in a tensor product of highest weight sl_N modules; cf. [SV].

Adopt the notation from Section 4. We have defined a function $\Phi'\colon \mathbb{C}_{z'}^{n+1} \times \mathbb{C}_t^m \to \mathbb{C}$, integration cycles $\gamma(z')$ and rational functions $\{\phi_I\}_{I \in P(v_0, n+1)}$.

Order the basis $\{e_{N-1, N}, ..., e_{1, 2}\}$ of the Lie subalgebra $\mathbf{n}_+$ by $e_{k, l} \succ e_{k', l'}$ if and only if $\alpha_{k, l} \succ \alpha_{k', l'}$. The corresponding PBW-basis of $U(\mathbf{n}_+)$ is

$$\left\{ E_{I_0} = \frac{e_{N-1, N}^{i_{N, N-1}}}{i_{N, N-1}!} \cdots \frac{e_{1, 2}^{i_{2, 1}}}{i_{2, 1}!} \right\}.$$

The index $I_0 = (i_{l, k})_{l > k}$ runs over all sequences of non-negative integers.

Let $W' = W_1 \otimes \cdots \otimes W_{n+1}$, where W_j is a lowest weight sl_N module of lowest weights $-\Lambda_j$ with lowest weight vector w_j. Fix a weight subspace $W'[-v'] \subset W'$, where $v' = \sum_{j=1}^{n+1} \Lambda_j - \sum_{i=1}^{N-1} m_i \alpha_i$ as in Section 4.

To every $I \in P(v_0, n+1)$, associate a vector $E_I w = E_{I_1} w_1 \otimes \cdots \otimes E_{I_{n+1}} w_{n+1}$.

THEOREM 10.1 (Theorem 2.4 in [Ma]). *The function*

$$w(z') = \int_{\gamma(z')} \Phi'(z', t)^{1/\kappa} \left(\sum_{I \in P(v_0, n+1)} \phi_I E_I w \right) dt$$

takes values in the subspace of singular vectors of $W'[-v']$ and satisfies the rational KZ equations with parameter $\kappa \in \mathbb{C}$.

From lowest weight modules to highest weight modules. Recall that τ denotes the Chevalley involution. We have $\tau(e_{k,l}) = -e_{l,k}$ for any $1 \leqslant l \neq k \leqslant N$.

Let $\lambda \in \mathbf{h}$, and let V_λ be the highest weight Verma module of highest weight λ with highest weight vector v_λ, and let $W_{-\lambda}$ be the associated lowest weight Verma module of lowest weight $-\lambda$ with lowest weight vector $w_{-\lambda}$. The following proposition is well known.

PROPOSITION 10.2. *The Chevalley involution defines an isomorphism of Verma modules $W_{-\lambda} \to V_\lambda$ for generic $\lambda \in \mathbf{h}$. Namely, for all $x \in U(\mathbf{n}_+)$ we have $xw_{-\lambda} \mapsto \tau(x)\, v_\lambda$. This isomorphism induces an isomorphism from any lowest weight sl_N module to the corresponding highest weight sl_N module.*

Assume that the $(\Lambda_j)_{j=1}^{n+1}$ are generic. Let $V' = V_1 \otimes \cdots \otimes V_{n+1}$, where V_j is a highest weight module with highest weight Λ_j and highest weight vector v_j, corresponding to W_j under the isomorphism of Proposition 10.2. Notice that $\tau(E_J) = F_J$ for every set of positive integers $J = (j_{l,k})_{k<l}$. Thus, $\sum_{I \in P(v_0, n+1)} \phi_I E_I w \mapsto \phi'$. Theorem 10.1 and Proposition 10.2 imply the following corollary.

COROLLARY 10.3. *The function*

$$u(z') = \int_{\gamma(z')} \Phi'(z', t)^{1/\kappa}\, \phi'(z', t)\, dt$$

takes values in the subspace of all singular vectors of $V'[v']$ and satisfies the rational KZ equations with parameter $\kappa \in \mathbb{C}$.

11. APPENDIX B

Proof of Proposition

Recall the statement of Proposition 2.1.

Fix a weight subspace $V'[v'] \subset V'$, $v' = \sum_{j=1}^{n+1} \Lambda_j - v_0$, where $v_0 \in Q_+$. Let $u\colon \mathbb{C}^{n+1} \to V'$ be a solution of the rational KZ equations with parameter $\kappa \in \mathbb{C}$ taking values in the subspace of $V'[v']$ consisting of all singular vectors. Set $v = \sum_{j=1}^{n} \Lambda_j - v_0$.

Then $v(z_1, ..., z_n) = u(z_1, ..., z_n, 0)\, |v_{n+1}^*\rangle \prod_{i=1}^{n} z_i^{(\Lambda_i, \Lambda_i + 2\rho)/2\kappa}$ is a solution of the trigonometric KZ equations with values in the weight subspace $V[v] \subset V$ with parameter $\lambda = \Lambda_{n+1} + \rho + \frac{1}{2} v \in \mathbf{h}$ and the same parameter $\kappa \in \mathbb{C}$.

Proof. Set $u_0 = u(z_1, ..., z_n, 0)$. The function u_0 satisfies the system of equations $\nabla_i(\kappa, \lambda)\, u_0 = 0$, $i = 1, ..., n$. Rewrite this equations separating the input of the $(n+1)^{st}$ point in the sum:

$$\kappa \frac{\partial u_0}{\partial z_i} = \left(\sum_{j=1, j\neq i}^{n} \frac{\Omega^{(ij)}}{z_i - z_j} + \frac{\Omega^{(i, n+1)}}{z_i} \right) u_0 . \tag{46}$$

Multiply Eq. (46) by z_i and rearrange it using $z_i \Omega^{(ij)}/z_i - z_j = r(z_i/z_j)^{(ij)} + (\Omega^-)^{(ij)}$ to obtain

$$\kappa z_i \frac{\partial u_0}{\partial z_i} = \left(\sum_{j=1, j\neq i}^{n} (r(z_i/z_j)^{(ij)} + (\Omega^-)^{(ij)}) + \Omega^{(i, n+1)} \right) u_0 .$$

Set $v' = u_0\, |v^*_{n+1}\rangle$. For all $\alpha \in \Sigma_+$, $i = 1, ..., n$, we have $e^{(i)}_{\alpha} v' = (e^{(i)}_{\alpha} u_0)\, |v^*_{n+1}\rangle$. $e^{(i)}_{-\alpha} v' = (e^{(i)}_{-\alpha} u_0)\, |v^*_{n+1}\rangle$. This implies

(47)
$$\kappa z_i \frac{\partial v'}{\partial z_i} = \left(\sum_{j=1, j\neq i}^{n} r(z_i/z_j)^{(ij)} \right) v' + \left(\sum_{j=1, j\neq i}^{n} (\Omega^-)^{(ij)}\, v' + (\Omega^{(i, n+1)})\, u_0\, |v^*_{n+1}\rangle \right).$$

Now we claim that

(48)
$$(\Lambda_{n+1} + \rho + \tfrac{1}{2}\, \nu - \tfrac{1}{2}\, (\Lambda_i, \Lambda_i + 2\rho))^{(i)}\, v' = \sum_{j=1, j\neq i}^{n} (\Omega^-)^{(ij)}\, v' + (\Omega^{(i, n+1)})\, u_0\, |v^*_{n+1}\rangle).$$

First write

(49)
$$(\Omega^{(i, n+1)})\, u_0\, |v^*_{n+1}\rangle$$
$$= \left(\sum_k x^{(i)}_k \otimes x^{(n+1)}_k u_0 \right) |v^*_{n+1}\rangle + \sum_{\alpha \in \Sigma_+} (e^{(i)}_{\alpha} \otimes e^{(n+1)}_{-\alpha} + e^{(i)}_{-\alpha} \otimes e^{(n+1)}_{\alpha})\, u_0\, |v^*_{n+1}\rangle.$$

The coupling $|v^*_{n+1}\rangle$ allows us to compute the first summand explicitly

$$\left(\sum_k x^{(i)}_k \otimes x^{(n+1)}_k u_0 \right) |v^*_{n+1}\rangle = \sum_k (x_k, \Lambda_{n+1})\, x^{(i)}_k u_0)\, |v^*_{n+1}\rangle \tag{50}$$
$$= \Lambda^{(i)}_{n+1} u_0\, |v^*_{n+1}\rangle = \Lambda^{(i)}_{n+1} v'.$$

Obviously $(e^{(i)}_{\alpha} \otimes e^{(n+1)}_{-\alpha})\, u_0\, |v^*_{n+1}\rangle = 0$.

Since u_0 is a singular vector we have $e_\alpha u_0 = 0$. Equivalently, $e_\alpha^{(n+1)} u_0 = -\sum_{j=1}^n e_\alpha^{(j)} u_0$. Therefore

(51)

$$\sum_{\alpha\in\Sigma_+} (e_{-\alpha}^{(i)} \otimes e_\alpha^{(n+1)})\, u_0\, |v_{n+1}^*\rangle$$

$$= \sum_{\alpha\in\Sigma_+} (e_{-\alpha}^{(i)} e_\alpha^{(n+1)})\, u_0\, |v_{n+1}^*\rangle = -\sum_{\alpha\in\Sigma_+} e_{-\alpha}^{(i)} \left(\sum_{j=1}^n e_\alpha^{(j)}\right) u_0\, |v_{n+1}^*\rangle$$

$$= -\sum_{\alpha\in\Sigma_+} \left(\sum_{j=1,\, j\neq i}^n e_{-\alpha}^{(i)} \otimes e_\alpha^{(j)}\right) u_0\, |v_{n+1}^*\rangle - \sum_{\alpha\in\Sigma_+} (e_{-\alpha}^{(i)} e_\alpha^{(i)})\, u_0\, |v_{n+1}^*\rangle$$

$$= -\sum_{j=1,\, j\neq i}^n (\Omega^-)^{(ij)}\, u_0\, |v_{n+1}^*\rangle$$

$$-\left(\tfrac{1}{2}\sum_k x_k^{(i)} x_k^{(i)} + \sum_{\alpha\in\Sigma_+} e_{-\alpha}^{(i)} e_\alpha^{(i)}\right) u_0\, |v_{n+1}^*\rangle + \tfrac{1}{2}\sum_{j=1}^n \sum_k x_k^{(i)} x_k^{(j)} u_0\, |v_{n+1}^*\rangle.$$

Since $u_0 \in V[\nu]$ we have

$$\sum_{j=1}^n \sum_k x_k^{(i)} x_k^{(j)} u_0\, |v_{n+1}^*\rangle = \sum_k (x_k, \nu)\, x_k^{(i)} u_0\, |v_{n+1}^*\rangle = \nu^{(i)} u_0\, |v_{n+1}^*\rangle = \nu^{(i)} v'.$$

Let $C \in U(\mathbf{g})$ be the Casimir element. Since $e_{-\alpha} e_\alpha = -h_\alpha + e_\alpha e_{-\alpha}$ we have

$$-\left(\tfrac{1}{2}\sum_k x_k^{(i)} x_k^{(i)} + \sum_{\alpha\in\Sigma_+} e_{-\alpha}^{(i)} e_\alpha^{(i)}\right) u_0\, |v_{n+1}^*\rangle$$

$$= -\tfrac{1}{2} C^{(i)} v' + \tfrac{1}{2}\sum_{\alpha\in\Sigma_+} h_\alpha^{(i)} v' = -\tfrac{1}{2} C^{(i)} v' + \rho^{(i)} v'.$$

We have $C^{(i)} v' = (\Lambda_i, \Lambda_i + 2\rho)\, v'$. Rewrite (51) as

(52)

$$\sum_{\alpha\in\Sigma_+} (e_{-\alpha}^{(i)} \otimes e_\alpha^{(n+1)})\, u_0\, |v_{n+1}^*\rangle = \left(-\sum_{j=1,\, j\neq i}^n (\Omega^-)^{(ij)} - \tfrac{1}{2} C^{(i)} + \rho^{(i)} + \tfrac{1}{2}\nu^{(i)}\right) v'.$$

Combine (52), (50) and (49) to obtain claim (48). Equation (47) for v' becomes

(53)

$$\kappa z_i \frac{\partial v'}{\partial z_i} = \left(\sum_{j=1,\, j\neq i}^n r(z_i/z_j)^{(ij)} + \left(\Lambda_{n+1} + \rho + \frac{1}{2}\nu\right)^{(i)} - \frac{1}{2}\sum_{i=1}^n (\Lambda_i, \Lambda_i + 2\rho)\right) v'.$$

Finally, $v = v' \prod_{i=1}^n z_i^{(\Lambda_i, \Lambda_i + 2\rho)/2\kappa}$, and equation (53) implies the proposition. ∎

12. APPENDIX C

Formulae for sl_3.

Let M_λ be the sl_3 Verma module with highest weight $\lambda \in \mathbf{h}$ and highest weight vector v_λ. Let V be a tensor product of highest weight sl_3-modules.

Set $(\lambda, \alpha_j) = \lambda_j$, for $j = 1, 2$, and $p_k(t) = t(t-1)\cdots(t-k+1) \in \mathbb{C}[t]$ for any $k \in \mathbb{N}$.

Formulae for the dynamical operators $\mathbb{B}_{\omega[1], V}(\lambda) = \mathbb{B}^{\alpha_1+\alpha_2}(\lambda)\, \mathbb{B}^{\alpha_1}(\lambda)$ and $\mathbb{B}_{\omega[2], V}(\lambda) = \mathbb{B}^{\alpha_1+\alpha_2}(\lambda)\, \mathbb{B}^{\alpha_2}(\lambda)$ are given in Section 7.

Let ω_0 be the longest element of the Weyl group. It has two reduced presentations, $\omega_0 = s_1 s_2 s_1 = s_2 s_1 s_2$, which imply two presentations for the operator $\mathbb{B}_{\omega_0, V}$,

$\mathbb{B}_{\omega_0, V}(\lambda) = \mathbb{B}^{\alpha_1}(\lambda)\, \mathbb{B}^{\alpha_1+\alpha_2}(\lambda)\, \mathbb{B}^{\alpha_2}(\lambda) = \mathbb{B}^{\alpha_2}(\lambda)\, \mathbb{B}^{\alpha_1+\alpha_2}(\lambda)\, \mathbb{B}^{\alpha_1}(\lambda)$. For any $v \in V[\nu]$, direct computation gives

$$\mathbb{B}_{\omega_0, V}(\lambda+\rho+\tfrac{1}{2}\nu)\, v = \sum_{a,b=0}^{\infty} \sum_{m,k=0}^{\min(a,b)} (-1)^m B^{a,b}_{m,k}(\lambda)\, e_{21}^{a-m} e_{31}^{m} e_{32}^{b-m} e_{12}^{a-k} e_{13}^{k} e_{23}^{b-k} v$$

$$= \sum_{a,b=0}^{\infty} \sum_{m,k=0}^{\min(a,b)} (-1)^m B^{a,b}_{m,k}(\lambda)\, e_{32}^{b-k} e_{31}^{k} e_{21}^{a-k} e_{23}^{b-m} e_{13}^{m} e_{12}^{a-m} v,$$

where, for any four non-negative integers a, b, m, k, such that $\max(m,k) \leqslant \min(a,b)$, $B^{a,b}_{m,k}(\lambda)$ is a complex-valued function depending on $\lambda \in \mathbf{h}$ defined by

$$B^{a,b}_{m,k}(\lambda) = \frac{1}{m!\,k!\,p_{a-m}(\lambda_1)\, p_{b-k}(\lambda_2)} \times \sum_{l=\max(m,k)}^{\min(a,b)} \frac{(-1)^l\, l!}{(a-l)!\,(b-l)!\,(l-k)!\,(l-m)!\,p_l(\lambda_1+\lambda_1+1)}.$$

The shifted universal fusion matrix is

$$J_+(\lambda) = \sum_{a,b=0}^{\infty} \sum_{m,k=0}^{\min(a,b)} (-1)^{a+b} B^{a,b}_{m,k}(\lambda)\, e_{32}^{b-m} e_{31}^{m} e_{21}^{a-m} \otimes e_{12}^{a-k} e_{13}^{k} e_{23}^{b-k}$$

$$= \sum_{a,b=0}^{\infty} \sum_{m,k=0}^{\min(a,b)} (-1)^{a+b+m+k} B^{a,b}_{m,k}(\lambda)\, e_{21}^{a-k} e_{31}^{k} e_{32}^{b-k} \otimes e_{23}^{b-m} e_{13}^{m} e_{12}^{a-m}.$$

The inverse to the Shapovalov form as an element of $M_\lambda \hat{\otimes} M_\lambda$ is

$$S_\lambda^{-1} = \sum_{a,b=0}^{\infty} \sum_{m,k=0}^{\min(a,b)} (-1)^k B_{m,k}^{a,b}(\lambda)\, e_{32}^{b-m} e_{31}^{m} e_{21}^{a-m} v_\lambda \otimes e_{21}^{a-k} e_{31}^{k} e_{32}^{b-k} v_\lambda$$
$$= \sum_{a,b=0}^{\infty} \sum_{m,k=0}^{\min(a,b)} (-1)^k B_{m,k}^{a,b}(\lambda)\, e_{21}^{a-k} e_{31}^{k} e_{32}^{b-k} v_\lambda \otimes e_{32}^{b-m} e_{31}^{m} e_{21}^{a-m} v_\lambda .$$

Finaly, given $v \in V$, there is a unique singular vector, $sing(v_\lambda \otimes v)$, in $M_\lambda \otimes V$ of the form $sing(v_\lambda \otimes v) = v_\lambda \otimes v + \{\text{lower order terms}\}$. In [ESt] the vector $sing(v_\lambda \otimes v)$ is given in terms of the inverse of the Shapovalov form. As a corollary we get

$$sing(v_\lambda \otimes v) = \sum_{a,b=0}^{\infty} \sum_{m,k=0}^{\min(a,b)} (-1)^{a+b} B_{m,k}^{a,b}(\lambda)\, e_{32}^{b-m} e_{31}^{m} e_{21}^{a-m} v_\lambda \otimes e_{12}^{a-k} e_{13}^{k} e_{23}^{b-k} v$$
$$= \sum_{a,b=0}^{\infty} \sum_{m,k=0}^{\min(a,b)} (-1)^{a+b+m+k} B_{m,k}^{a,b}(\lambda)\, e_{21}^{a-k} e_{31}^{k} e_{32}^{b-k} v_\lambda \otimes e_{23}^{b-m} e_{13}^{m} e_{12}^{a-m} v.$$

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