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On symmetries in the theory of finite rank singular perturbations

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Abstract

For a nonnegative self-adjoint operator A_0 acting on a Hilbert space $\mathfrak{H}$ singular perturbations of the form $A_0 + V$, $V = \sum_1^n b_{ij}\langle\psi_j, \cdot\rangle\psi_i$ are studied under some additional requirements of symmetry imposed on the initial operator A_0 and the singular elements ψ_j. A concept of symmetry is defined by means of a one-parameter family of unitary operators $\mathfrak{U}$ that is motivated by results due to R.S. Phillips. The abstract framework to study singular perturbations with symmetries developed in the paper allows one to incorporate physically meaningful connections between singular potentials V and the corresponding self-adjoint realizations of $A_0 + V$. The results are applied for the investigation of singular perturbations of the Schrödinger operator in $L_2(\mathbb{R}^3)$ and for the study of a (fractional) p-adic Schrödinger type operator with point interactions.

Keywords: Self-adjoint operator; Singular perturbation with symmetries; Friedrichs and Krein–von Neumann extensions; Scaling transformation; p-Adic analysis

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1. Introduction

Let A_0 be an unbounded nonnegative self-adjoint operator acting on a Hilbert space $\mathfrak{H}$ and let $\mathfrak{H}_2(A_0) \subset \mathfrak{H}_1(A_0) \subset \mathfrak{H} \subset \mathfrak{H}_{-1}(A_0) \subset \mathfrak{H}_{-2}(A_0)$ be the standard scale of Hilbert spaces associated with A_0. More precisely,

$$\mathfrak{H}_k(A_0) = \mathcal{D}\big(A_0^{k/2}\big), \quad k = 1, 2, \tag{1.1}$$

equipped with the norm $\|u\|_k = \|(A_0 + I)^{k/2}u\|$. The dual spaces $\mathfrak{H}_{-k}(A_0)$ can be defined as the completions of $\mathfrak{H}$ with respect to the norms $\|u\|_{-k} = \|(A_0 + I)^{-k/2}u\|$ $(u \in \mathfrak{H})$. The resolvent operator $(A_0 + I)^{-1}$ can be continuously extended to an isometric mapping $(\mathbb{A}_0 + I)^{-1}$ from $\mathfrak{H}_{-2}(A_0)$ onto $\mathfrak{H}$ and the relation

$$\langle \psi, u\rangle = \big((A_0 + I)u, (\mathbb{A}_0 + I)^{-1}\psi\big), \quad u \in \mathfrak{H}_2(A_0), \tag{1.2}$$

enables one to identify the elements $\psi \in \mathfrak{H}_{-2}(A_0)$ as linear functionals on $\mathfrak{H}_2(A_0)$.

Consider the heuristic expression

$$A_0 + \sum_{i,j=1}^{n} b_{ij}\langle\psi_j, \cdot\rangle\psi_i, \quad b_{ij} \in \mathbb{C},\ n \in \mathbb{N}, \tag{1.3}$$

where elements ψ_j $(1 \leqslant j \leqslant n)$ form a linearly independent system in $\mathfrak{H}_{-2}(A_0)$. In what follows it is supposed that the linear span $\mathcal{X}$ of $\{\psi_j\}_{j=1}^n$ satisfies the condition $\mathcal{X} \cap \mathfrak{H} = \{0\}$, i.e., elements ψ_j are $\mathfrak{H}$-independent. In this case, the perturbation $V = \sum_{i,j=1}^n b_{ij}\langle\psi_j, \cdot\rangle\psi_i$ is said to be singular and the formula

$$A_{\mathrm{sym}} = A_0 \upharpoonright \mathcal{D}(A_{\mathrm{sym}}), \quad \mathcal{D}(A_{\mathrm{sym}}) = \big\{u \in \mathcal{D}(A_0)\colon \langle\psi_j, u\rangle = 0,\ 1 \leqslant j \leqslant n\big\} \tag{1.4}$$

determines a closed densely defined symmetric operator in $\mathfrak{H}$.

In the theory of singular perturbations, cf. e.g. [3,5,23], each intermediate extension A of A_{sym}, i.e., $A_{\mathrm{sym}} \subset A \subset A_{\mathrm{sym}}^*$, can be viewed to be singularly perturbed with respect to A_0 and, in general, such an extension can be regarded as an operator-realization of (1.3) in $\mathfrak{H}$. In this context, the natural question arises whether and how one could establish a physically meaningful correspondence between the parameters b_{ij} of the singular potential V and the intermediate extensions of A_{sym}. The investigation of this problem is one of goals of the present paper. In the approach developed by S. Albeverio and P. Kurasov in [4,5] one considers an operator realization A of (1.3) by setting

$$A = \mathbb{A}_{\mathbf{R}} \upharpoonright \mathcal{D}(A), \quad \mathcal{D}(A) = \big\{f \in \mathcal{D}\big(A_{\mathrm{sym}}^*\big)\colon \mathbb{A}_{\mathbf{R}} f \in \mathfrak{H}\big\}, \tag{1.5}$$

where

$$\mathbb{A}_{\mathbf{R}} = \mathbb{A}_0 + \sum_{i,j=1}^{n} b_{ij}\big\langle\psi_j^{\mathrm{ex}}, \cdot\big\rangle\psi_i \tag{1.6}$$

is seen as a regularization of (1.3).

Formula (1.6) involves a construction of the extended functionals $\langle\psi_j^{\mathrm{ex}},\cdot\rangle$ defined on $\mathcal{D}(A_{\mathrm{sym}}^*)$. These functionals are uniquely determined by the choice of a Hermitian matrix $\mathbf{R}=(r_{jp})_{j,p=1}^n$. Since for elements $\psi\in\mathcal{X}\cap\mathfrak{H}_{-1}(A_0)$ the functionals $\langle\psi,\cdot\rangle$ admit extensions by continuity onto $\mathfrak{H}_1(A_0)\cap\mathcal{D}(A_{\mathrm{sym}}^*)$, a lot of natural restrictions appears in the choice of $\mathbf{R}$. For their preservation the concept of admissible matrices $\mathbf{R}$ for the regularization of (1.3) has been introduced in [4, Definition 3.1.2]. However, this definition involves certain spectral measures and, in what follows, their calculation will be avoided. In fact, an equivalent operator concept of *admissible large coupling limits of* (1.3) is introduced in the form convenient for the further studies in the present paper.

If the singular potential V in (1.3) is not form-bounded (i.e., $\mathcal{X}\not\subset\mathfrak{H}_{-1}(A_0)$), then an admissible large coupling limit A_∞ cannot be determined uniquely and one needs to impose some extra assumptions to achieve the uniqueness. For instance, in many applications, the condition of extremality [9,10] allows one to select a unique operator A_∞ (see Theorem 3.12). It should be noted that the concept of extremality is physically reasonable. For example, extremal operators determine free evolutions in the Lax–Phillips scattering theory [31].

Another approach inspired by [4,5,30] deals with the preservation of initially existing symmetries of singular elements ψ_j in the definition of the extended functionals ψ_j^{ex}. To study this problem in an abstract framework, one needs to define the notion of symmetry for the unperturbed operator A_0 and for the singular elements ψ_j in (1.3). Generalizing the ideas suggested in [5,26,37], the required definitions will be formulated here as follows:

Let $\mathfrak{T}$ be a subset of the real line $\mathbb{R}$ and let $\mathfrak{U}=\{U_t\}_{t\in\mathfrak{T}}$ be a one-parameter family of unitary operators acting on $\mathfrak{H}$ with the following property:

$$U_t\in\mathfrak{U}\quad\Leftrightarrow\quad U_t^*\in\mathfrak{U}. \tag{1.7}$$

Definition 1.1. (See [20].) A linear operator A ($\neq 0$) acting in $\mathfrak{H}$ is said to be $p(t)$-homogeneous with respect to $\mathfrak{U}$ if there exists a real function $p(t)$ defined on $\mathfrak{T}$ such that

$$U_tA=p(t)AU_t,\quad\forall t\in\mathfrak{T}. \tag{1.8}$$

In other words, the set $\mathfrak{U}$ determines the structure of a symmetry and the property of A to be $p(t)$-homogeneous with respect to $\mathfrak{U}$ means that A possesses a certain symmetry with respect to $\mathfrak{U}$.

Definition 1.2. (See [20].) A singular element $\psi\in\mathfrak{H}_{-2}(A_0)\setminus\mathfrak{H}$ is said to be $\xi(t)$-invariant with respect to $\mathfrak{U}$ if there exists a real function $\xi(t)$ defined on $\mathfrak{T}$ such that

$$\mathbb{U}_t\psi=\xi(t)\psi,\quad\forall t\in\mathfrak{T}, \tag{1.9}$$

where $\mathbb{U}_t$ is the continuation of U_t onto $\mathfrak{H}_{-2}(A_0)$ (see Section 4 for details).

The main aim of the paper is to study (1.3) assuming that the initial operator A_0 is $p(t)$-homogeneous and the singular elements ψ_j are $\xi_j(t)$-invariant with respect to $\mathfrak{U}$. It appears that the preservation of $\xi_j(t)$-invariance for the extended functionals $\langle\psi_j^{\mathrm{ex}},\cdot\rangle$ is equivalent to the $p(t)$-homogeneity of the operator A_∞ which is used for the regularization of (1.3) (Theorem 4.8). Combining this result with the complete description of admissible large coupling limits (Theorem 3.6) allows one to select a unique admissible large coupling limit A_∞ by imposing the condition of $p(t)$-homogeneity (Theorems 4.13, 4.14). One of interesting properties discovered here is the possibility to get the Friedrichs and the Krein–von Neumann extension (and more generally, all $p(t)$-homogeneous self-adjoint extensions transversal to A_0) as solutions of a system of equations involving the functions $p(t)$ and $\xi_j(t)$ (Corollary 4.10, Proposition 4.16).

The choice of a $p(t)$-homogeneous admissible large coupling limit A_∞ for the regularization of (1.3) immediately gives a new specific relation for the corresponding Weyl function $\mathbf{M}(z)$ (Theorem 5.5) and enables one to establish simple relations involving the functions $p(t)$ and $\xi_j(t)$, and the properties of operator realizations of (1.3) (Theorem 5.1, Proposition 5.3).

It is well known, see e.g. [2,13,25,30] that the Schrödinger operators perturbed by potentials homogeneous with respect to a certain set $\mathfrak{U}$ of unitary operators might possess a lot of interesting properties. Obviously, such properties become even more meaningful if, in addition to (1.7), the set $\mathfrak{U}$ has further algebraic group properties. In particular, if $\mathfrak{U}$ is the set of scaling transformations, then the additional multiplicative property $U_{t_1}U_{t_2} = U_{t_2}U_{t_1} = U_{t_1t_2}$ of it elements enables one to get simple solutions of many problems (like description of nonnegative operator realizations, spectral properties, completeness of the wave operators, explicit form of the scattering matrix) for Schrödinger operators with singular potentials $\xi(t)$-invariant with respect to scaling transformations in $\mathbb{R}^3$ (Section 6).

The abstract approach to the notion of symmetry developed in the paper can be also useful for the study of supersingular perturbations [30], for applications in the non-Archimedean analysis (Example 5.6), and for the investigation of Weyl families of boundary relations [15].

In a very recent paper [36], K.A. Makarov and E. Tsekanovskii considered the so-called μ-scale invariant operators, which can be seen as a special case of $p(t)$-homogeneous operators in the present paper. The main result of [36] is intimately related to [20, Lemma 4.5], see also Section 4 below.

Throughout the paper $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker A$ denote the domain, the range, and the null-space of a linear operator A, respectively, while $A \upharpoonright \mathcal{D}$ stands for the restriction of A to the set $\mathcal{D}$.

2. Preliminaries on operator realizations

Following [4,5] an operator realization A of (1.3) in $\mathfrak{H}$ are defined by (1.5), (1.6). To clarify the meaning of $\mathbb{A}_0$ and ψ_j^{ex} in (1.6), observe that $\mathbb{A}_0$ stands for the continuation of A_0 as a bounded linear operator acting from $\mathfrak{H}$ into $\mathfrak{H}_{-2}(A_0)$. Using the extended resolvent $(\mathbb{A}_0 + I)^{-1}$ this continuation can be determined also by the formula

$$\mathbb{A}_0 f = \left[(\mathbb{A}_0 + I)^{-1}\right]^{-1} f - f, \quad \forall f \in \mathfrak{H}. \tag{2.1}$$

The linear functionals $\langle \psi_j^{\mathrm{ex}}, \cdot\rangle$ are extensions of $\langle \psi_j, \cdot\rangle$ onto $\mathcal{D}(A^*_{\mathrm{sym}})$. Using the well-known relation

$$\mathcal{D}\big(A^*_{\mathrm{sym}}\big) = \mathcal{D}(A_0) \dot{+} \mathcal{H}, \quad \text{where } \mathcal{H} = \ker\big(A^*_{\mathrm{sym}} + I\big), \tag{2.2}$$

one concludes that $\langle \psi_j, \cdot\rangle$ can be extended onto $\mathcal{D}(A^*_{\mathrm{sym}})$ by fixing their values on $\mathcal{H}$. It follows from (1.2) and (1.4) that the vectors

$$h_j = (\mathbb{A}_0 + I)^{-1}\psi_j, \quad j = 1, \dots, n, \tag{2.3}$$

form a basis of the defect subspace $\mathcal{H}=\ker(A_{\text{sym}}^*+I)$ of A_{sym}. Hence, the functionals $\langle\psi_j^{\text{ex}},\cdot\rangle$ are well defined by the formula

$$\langle\psi_j^{\text{ex}}, f\rangle = \langle\psi_j, u\rangle + \sum_{p=1}^{n}\alpha_p r_{jp} \tag{2.4}$$

for all elements $f = u + \sum_{p=1}^{n}\alpha_p h_p \in \mathcal{D}(A_{\text{sym}}^*)$ ($u \in \mathcal{D}(A_0)$, $\alpha_p \in \mathbb{C}$) if the entries $r_{jp} = \langle\psi_j, (\mathbb{A}_0+I)^{-1}\psi_p\rangle = \langle\psi_j, h_p\rangle$ of the matrix $\mathbf{R} = (r_{jp})_{j,p=1}^{n}$ are known.

If all $\psi_j \in \mathfrak{H}_{-1}(A_0)$, then r_{jp} are well defined and $\mathbf{R}$ is a Hermitian matrix [5]. Otherwise, the matrix $\mathbf{R}$ is not uniquely determined. In what follows, it is assumed that $\mathbf{R}$ is already chosen as a Hermitian matrix. The problem of an appropriate choice of $\mathbf{R}$ will be discussed in Section 3.

In order to describe an operator realization A of (1.3) in terms of parameters b_{ij} of the singular perturbation V, the method of boundary triplets (see [16,18] and the references therein) is now incorporated.

Definition 2.1. (See [18].) A triplet (N, Γ_0, Γ_1), where N is an auxiliary Hilbert space and Γ_0, Γ_1 are linear mappings of $\mathcal{D}(A_{\text{sym}}^*)$ into N, is called a boundary triplet of A_{sym}^* if $(A_{\text{sym}}^* f, g) - (f, A_{\text{sym}}^* g) = (\Gamma_1 f, \Gamma_0 g)_N - (\Gamma_0 f, \Gamma_1 g)_N$ for all $f, g \in \mathcal{D}(A_{\text{sym}}^*)$ and the mapping (Γ_0, Γ_1) : $\mathcal{D}(A_{\text{sym}}^*) \to N \oplus N$ is surjective.

The next two results (Lemma 2.2 and Theorem 2.3) are known (see e.g. [6,14]). For the convenience of the reader some principal steps of their proofs are repeated.

Lemma 2.2. *The triplet $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$, where the linear operators $\Gamma_i : \mathcal{D}(A_{\text{sym}}^*) \to \mathbb{C}^n$ are defined by the formulas*

$$\Gamma_0 f = \begin{pmatrix} \langle\psi_1^{\text{ex}}, f\rangle \\ \vdots \\ \langle\psi_n^{\text{ex}}, f\rangle \end{pmatrix}, \qquad \Gamma_1 f = -\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \tag{2.5}$$

where $f = u + \sum_{j=1}\alpha_j h_j \in \mathcal{D}(A_{\text{sym}}^*)$ ($u \in \mathcal{D}(A_0)$, $\alpha_j \in \mathbb{C}$) and $\langle\psi_j^{\text{ex}}, f\rangle$ is defined by (2.4), *forms a boundary triplet for A_{sym}^*.*

Proof. Using (1.2), (2.2), and (2.3) it is easy to verify that the mappings

$$\widehat{\Gamma}_0 f = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \qquad \widehat{\Gamma}_1 f = \begin{pmatrix} \langle\psi_1, u\rangle \\ \vdots \\ \langle\psi_n, u\rangle \end{pmatrix}, \quad f = u + \sum_{j=1}\alpha_j h_j \tag{2.6}$$

satisfy the conditions of Definition 2.1. Thus $(\mathbb{C}^n, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$ is a boundary triplet for A_{sym}^*. It follows from (2.4), (2.5), and (2.6) that

$$\Gamma_0 f = \widehat{\Gamma}_1 f + \mathbf{R}\widehat{\Gamma}_0 f, \qquad \Gamma_1 f = -\widehat{\Gamma}_0 f, \quad f \in \mathcal{D}\big(A_{\text{sym}}^*\big). \tag{2.7}$$

These relations between Γ_i and $\widehat{\Gamma}_i$ and the fact that $(\mathbb{C}^n, \widehat{\Gamma}_0, \widehat{\Gamma}_1)$ is a boundary triplet for A_{sym}^* imply that $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ also is a boundary triplet for A_{sym}^*. □

Theorem 2.3. *The operator realization A of* (1.3) *is an intermediate extension of A_{sym} which coincides with the operator*

$$A_{\mathbf{B}} = A^*_{\mathrm{sym}} \restriction \mathcal{D}(A_{\mathbf{B}}), \quad \mathcal{D}(A_{\mathbf{B}}) = \{ f \in \mathcal{D}(A^*_{\mathrm{sym}}) \colon \mathbf{B}\Gamma_0 f = \Gamma_1 f \}, \tag{2.8}$$

where Γ_i are defined by (2.5) *and $\mathbf{B} = (b_{ij})^n_{i,j=1}$ is the coefficient matrix of the singular perturbation $V = \sum^n_{i,j=1} b_{ij} \langle \psi_j, \cdot \rangle \psi_i$ in* (1.3).

If V is symmetric, i.e., $\langle Vu, v\rangle = \langle u, Vv\rangle$ $(u, v \in \mathfrak{H}_2(A_0))$, then the corresponding operator realization $A_{\mathbf{B}}$ becomes self-adjoint.

Proof. It follows from (2.1) that $\mathbb{A}_0 h_j = \psi_j - h_j$ for all h_j defined by (2.3). Rewriting $f \in \mathcal{D}(A^*_{\mathrm{sym}})$ in the form $f = u + \sum^n_{i=1} \alpha_i h_i$, where $u \in \mathcal{D}(A_0)$, $h_i \in \mathcal{H}$, $\alpha_i \in \mathbb{C}$, and using (1.6) and (2.5) leads to

$$\begin{aligned} \mathbb{A}_{\mathbf{R}} f &= A_0 u - \sum_{i=1}^n \alpha_i h_i + \sum_{i,j=1}^n b_{ij} \langle \psi_j^{\mathrm{ex}}, f \rangle \psi_i + \sum_{i=1}^n \alpha_i \psi_i \\ &= A^*_{\mathrm{sym}} f + (\psi_1, \ldots, \psi_n)[\mathbf{B}\Gamma_0 f - \Gamma_1 f]. \end{aligned}$$

This equality and (1.5) show that $f \in \mathcal{D}(A)$ if and only if $\mathbf{B}\Gamma_0 f - \Gamma_1 f = 0$. Therefore, the operator realization A of (1.3) is an intermediate extension of A_{sym} and A coincides with the operator $A_{\mathbf{B}}$ defined by (2.8).

To complete the proof it suffices to finally observe that V is symmetric if and only if the corresponding matrix of coefficients $\mathbf{B} = (b_{ij})^n_{i,j=1}$ is Hermitian. In this case (2.8) immediately implies the self-adjointness of $A_{\mathbf{B}}$. □

Corollary 2.4. *The operator realization $A_{\mathbf{B}}$ of* (1.3) *in Theorem* 2.3 *determined by the boundary condition $\mathbf{B}\Gamma_0 f = \Gamma_1 f$ in* (2.8) *takes the form*

$$A_{\mathbf{B}} f = \mathbb{A}_0 f + \sum_{i,j=1}^n b_{ij} \langle \psi_j^{\mathrm{ex}}, f \rangle \psi_i, \quad f \in \mathcal{D}(A_{\mathbf{B}}), \tag{2.9}$$

where the extended functionals $\langle \psi_j^{\mathrm{ex}}, \cdot \rangle$, $j = 1, \ldots, n$, are determined by (2.4).

Proof. Since the vectors h_j in (2.3) span the defect subspace $\mathcal{H} = \ker(A^*_{\mathrm{sym}} + I)$ of A_{sym}, one has $\mathbb{A}_0 h_j = \psi_j - h_j = \psi_j + A^*_{\mathrm{sym}} h_j$ and hence

$$A^*_{\mathrm{sym}} f = A_0 u + \sum_{i=1}^n \alpha_i (\mathbb{A}_0 h_i - \psi_i) = \mathbb{A}_0 f - \sum_{i=1}^n \alpha_i \psi_i \tag{2.10}$$

for $f = u + \sum^n_{i=1} \alpha_i h_i \in \mathcal{D}(A^*_{\mathrm{sym}})$. By substituting the boundary condition $\mathbf{B}\Gamma_0 f = \Gamma_1 f$ in (2.10) yields the desired perturbation formula for $A_{\mathbf{B}}$ in (2.9). □

Remark 2.5. Another approach, also involving the use of boundary triplets, to determine self-adjoint operator realizations of finite rank singular perturbations of the form $A_0 + G\alpha G^*$, where G is an injective linear mapping from $\mathbb{C}^n$ to $\mathfrak{H}_{-k}(A_0)$ was presented in [14, Section 4].

3. Admissible matrices and admissible large coupling limits

There are certain natural requirements for the determination of the entries r_{jp} of the matrix $\mathbf{R}$ in (2.4). Indeed, if the linear span $\mathcal{X}$ of $\{\psi_j\}_{j=1}^n$ has a nonzero intersection with $\mathfrak{H}_{-1}(A_0)$, then for any $\psi \in \mathcal{X} \cap \mathfrak{H}_{-1}(A_0)$, the corresponding element $h = (\mathbb{A}_0 + I)^{-1}\psi$ belongs to $\mathfrak{H}_1(A_0)$ and, hence, the functional $\langle \psi, \cdot \rangle$ defined by (1.2) admits the following extension by continuity onto $\mathfrak{H}_1(A_0)$:

$$\langle \psi, f \rangle = \left((A_0 + I)^{1/2} f, (A_0 + I)^{1/2} h\right), \quad \forall f \in \mathfrak{H}_1(A_0). \tag{3.1}$$

To preserve such natural extensions of $\langle \psi, \cdot \rangle$ onto $\mathcal{D}(A_{\mathrm{sym}}^*) \cap \mathfrak{H}_1(A_0)$ in the definition (2.4), the concept of admissible matrices $\mathbf{R}$ as introduced in [4] is used.

Definition 3.1. A Hermitian matrix $\mathbf{R} = (r_{jp})_{j,p=1}^n$ is called admissible for the regularization $\mathbb{A}_{\mathbf{R}}$ of (1.3) if its entries r_{jp} are chosen in such a way that if a singular element $\psi = c_1\psi_1 + \cdots + c_n\psi_n$ belongs to $\mathfrak{H}_{-1}(A_0)$, then for all $f \in \mathcal{D}(A_{\mathrm{sym}}^*) \cap \mathfrak{H}_1(A_0)$

$$\langle \psi^{\mathrm{ex}}, f \rangle = \left((A_0 + I)^{1/2} f, (A_0 + I)^{1/2} h\right) = \sum_{j=1}^{n} c_j \langle \psi_j^{\mathrm{ex}}, f \rangle, \tag{3.2}$$

where $\langle \psi_j^{\mathrm{ex}}, f \rangle$ are defined by (2.4) and $h = (\mathbb{A}_0 + I)^{-1}\psi$.

It is convenient to describe the set of admissible matrices in terms of a certain associated operators. It follows from the relations in (2.7) that the choice of a matrix $\mathbf{R}$ in (2.4) is equivalent to the choice of an operator A_∞ defined by

$$A_\infty = A_{\mathrm{sym}}^* \restriction \mathcal{D}(A_\infty), \quad \mathcal{D}(A_\infty) = \ker \Gamma_0 = \left\{ f \in \mathcal{D}\left(A_{\mathrm{sym}}^*\right) \colon -\mathbf{R}\widehat{\Gamma}_0 f = \widehat{\Gamma}_1 f \right\}. \tag{3.3}$$

Since $\mathbf{R}$ is Hermitian, the general theory of boundary triplets [16] implies that A_∞ is a self-adjoint extension of A_{sym}. By the construction, A_∞ and A_0, $\mathcal{D}(A_0) = \ker \Gamma_1$ $(= \ker \widehat{\Gamma}_0)$, are *transversal extensions* of A_{sym}, i.e., $\mathcal{D}(A_0) + \mathcal{D}(A_\infty) = \mathcal{D}(A_{\mathrm{sym}}^*)$. Furthermore, it follows from Theorem 2.3 that A_∞ and the operator realization $A_{\mathbf{B}}$ of (1.3) determined by the boundary condition $\mathbf{B}\Gamma_0 f = \Gamma_1 f$, $f \in \mathcal{D}(A_{\mathrm{sym}}^*)$ are also transversal extensions of A_{sym} for every coefficient matrix $\mathbf{B}$ in (1.3), i.e., *the operator* A_∞ *determined by* (3.3) *is always transversal to the singular perturbations* $A_{\mathbf{B}}$ *in* (2.9). The operator A_∞ corresponds formally to the matrix $\mathbf{B}$ with infinite entries in (2.9) (such an extension of A_{sym} need not be unique). In this sense, A_∞ can be considered as a large coupling limit of operator realizations $A_{\mathbf{B}}$ of (1.3) with finite entries of $\mathbf{B}$.

Definition 3.2. An operator A_∞ is called admissible large coupling limit of (1.3) if A_∞ is defined by (3.3) with an admissible matrix $\mathbf{R}$.

So, the choice of an admissible large coupling limit A_∞ of (1.3) *is equivalent* to the choice of an admissible matrix $\mathbf{R}$ for the regularization $\mathbb{A}_{\mathbf{R}}$ of (1.3).

The next lemma contains some useful facts concerning the (unperturbed) nonnegative self-adjoint operator A_0 and its relation to the Friedrichs extension A_F of A_{sym}. They can be considered to be well known from the extension theory of nonnegative operators, therefore details for the present formulations with their proofs are left to the reader; see e.g. [8,17,21,22,29,32].

Lemma 3.3. *Let* $C=(A_0+I)^{-1}-(A_F+I)^{-1}$ *and let* $S_0=A_0\cap A_F$. *Moreover, denote* $\mathcal{H}=\ker(A^*_{\mathrm{sym}}+I)$ *and* $\mathcal{H}'=\ker(S_0^*+I)$. *Then*:

(i) $\overline{\mathcal{R}(C)}=\mathcal{H}'$;
(ii) $\ker C=\mathcal{R}(S_0+I)=\mathcal{R}(A_{\mathrm{sym}}+I)\oplus\mathcal{H}''$, *where* $\mathcal{H}''=\mathcal{H}\ominus\mathcal{H}'$;
(iii) $\mathcal{R}(C^{1/2})=\mathcal{D}(A_0^{1/2})\cap\mathcal{H}=\mathcal{H}'$;
(iv) $\mathcal{D}(A_0^{1/2})=\mathcal{D}(A_F^{1/2})\dotplus\mathcal{R}(C^{1/2})$.

Using the spaces introduced in (1.1) and (iii) in Lemma 3.3 one can rewrite the decomposition in part (iv) of Lemma 3.3 as follows:

$$\mathfrak{H}_1(A_0)=\mathcal{D}\oplus_1\mathcal{H}',\quad \mathcal{H}'=\mathcal{H}\cap\mathfrak{H}_1(A_0)=(\mathbb{A}_0+I)^{-1}\big[\mathcal{X}\cap\mathfrak{H}_{-1}(A_0)\big],\tag{3.4}$$

where $\mathcal{D}$ $(=\mathcal{D}(A_F^{1/2}))$ stands for the completion of $\mathcal{D}(A_{\mathrm{sym}})$ in $\mathfrak{H}_1(A_0)$, $\oplus_1$ denotes the orthogonal sum in $\mathfrak{H}_1(A_0)$, and $\mathcal{X}$ is the linear span of $\{\psi_j\}_{j=1}^n$.

The set of admissible large coupling limits of (1.3) can now be characterized in 'coordinate free' manner as follows.

Theorem 3.4. *A self-adjoint extension* $\widetilde{A}$ *of* A_{sym} *is an admissible large coupling limit of* (1.3) *if and only if* $\widetilde{A}$ *is transversal to* A_0 (*i.e.,* $\mathcal{D}(A_0)+\mathcal{D}(\widetilde{A})=\mathcal{D}(A^*_{\mathrm{sym}})$) *and*

$$\mathcal{D}(\widetilde{A})\cap\mathfrak{H}_1(A_0)\subset\mathcal{D}(A_F),\tag{3.5}$$

where A_F *is the Friedrichs extension of* A_{sym}.

Proof. Assume that the self-adjoint extension $\widetilde{A}$ of A_{sym} is transversal to A_0 and it satisfies the condition (3.5). In view of (2.6), $\mathcal{D}(A_0)=\ker\widehat{\Gamma}_0$. Therefore, the transversality of $\widetilde{A}$ and A_0 is equivalent to the representation of $\mathcal{D}(\widetilde{A})$ in the form (3.3) with an $n\times n$ Hermitian matrix $\mathbf{R}$ (here A_{sym} has finite defect numbers (n,n)), cf. [17, Proposition 1.4].

Since

$$\mathcal{D}(A_F)=\mathcal{D}\cap\mathcal{D}\big(A^*_{\mathrm{sym}}\big),\tag{3.6}$$

the decomposition (3.4) shows that the condition (3.5) is equivalent to the relation

$$\big((A_0+I)^{1/2}\tilde{f},(A_0+I)^{1/2}h\big)=0,\quad \forall\tilde{f}\in\mathcal{D}(\widetilde{A})\cap\mathfrak{H}_1(A_0),\ \forall h\in\mathcal{H}'.\tag{3.7}$$

Now it is shown that $\mathbf{R}$ is an admissible matrix in the sense of Definition 3.1 by verifying (3.2) for all $\psi\in\mathcal{X}\cap\mathfrak{H}_{-1}(A_0)$. Observe, that the mapping Γ_0 defined in Lemma 2.2, see also (2.7), determines the extended functionals $\langle\psi_j^{\mathrm{ex}},f\rangle$ in (2.4).

The transversality of $\widetilde{A}$ and A_0 yields the following decomposition for the elements $f\in\mathcal{D}(A^*_{\mathrm{sym}})$:

$$f=\tilde{f}+u,\tag{3.8}$$

where $\tilde{f} \in \mathcal{D}(\widetilde{A})$ and $u \in \mathcal{D}(A_0)$ are uniquely determined modulo $\mathcal{D}(A_{\mathrm{sym}})$. If $\psi = \sum_{j=1}^{n} c_j \psi_j \in \mathfrak{H}_{-1}(A_0)$, then by (3.4) $h = (\mathbb{A}_0 + I)^{-1}\psi \in \mathcal{H}'$. Now with $f \in \mathcal{D}(A_{\mathrm{sym}}^*) \cap \mathfrak{H}_1(A_0)$ decomposed as in (3.8) one obtains:

$$\begin{aligned}\langle \psi^{\mathrm{ex}}, f \rangle &= \sum_{j=1}^{n} c_j \langle \psi_j^{\mathrm{ex}}, f \rangle = \mathbf{c}\Gamma_0 f \overset{(3.8)}{=} \mathbf{c}\Gamma_0(\tilde{f} + u) \\ &\overset{(2.7)}{=} \mathbf{c}(\widehat{\Gamma}_1 + \mathbf{R}\widehat{\Gamma}_0)u = \mathbf{c}\widehat{\Gamma}_1 u \overset{(2.6)}{=} \langle \psi, u \rangle \overset{(1.2)}{=} \big((A_0 + I)u, h\big) \end{aligned} \tag{3.9}$$

where $\mathbf{c} := (c_1, \dots, c_n)$. On the other hand, it follows from (3.7) that

$$\big((A_0 + I)^{1/2} f, (A_0 + I)^{1/2} h\big) = \big((A_0 + I)^{1/2}(\tilde{f} + u), (A_0 + I)^{1/2} h\big) = \big((A_0 + I)u, h\big),$$

which combined with (3.9) proves (3.2). Thus, $\mathbf{R}$ is an admissible matrix and $\widetilde{A}$ $(= A_\infty)$ is an admissible large coupling limit of (1.3).

Conversely, assume that $\widetilde{A} = A_\infty$ satisfies the condition of Definition 3.2. Then (3.3) ensures the transversality of $\widetilde{A}$ and A_0 and $\mathbf{R}$ determines the extended functionals $\langle \psi_j^{\mathrm{ex}}, \cdot \rangle$ via (2.4). Reasoning as in (3.9) it is seen that (3.2) implies

$$0 = \big((A_0 + I)^{1/2} f, (A_0 + I)^{1/2} h\big) - \langle \psi^{\mathrm{ex}}, f \rangle = \big((A_0 + I)^{1/2} \tilde{f}, (A_0 + I)^{1/2} h\big)$$

for all $f \in \mathcal{D}(A_{\mathrm{sym}}^*) \cap \mathfrak{H}_1(A_0)$ and $h \in \mathcal{H}'$. Thus, the relation (3.7) and, equivalently, the relation (3.5) is satisfied. Theorem 3.4 is proved. □

For some further study of admissible large coupling limits the following lemma is needed.

Lemma 3.5. *Let $\widetilde{\mathcal{H}}$ be a subspace of $\mathcal{H} = \ker(A_{\mathrm{sym}}^* + I)$. Then the symmetric operator*

$$S = A_F \restriction_{\mathcal{D}(S)}, \quad \mathcal{D}(S) = (A_F + I)^{-1}\big[\mathcal{R}(A_{\mathrm{sym}} + I) \oplus \widetilde{\mathcal{H}}\big] \tag{3.10}$$

satisfies the relations

$$\mathcal{D}(S) \cap \mathcal{D}(A_0) = \mathcal{D}(A_{\mathrm{sym}}) \quad \textit{and} \quad \mathcal{D}(S) + \mathcal{D}(A_0) = \mathcal{D}(A_F) \dot{+} \mathcal{H}' \tag{3.11}$$

if and only if

$$\dim \widetilde{\mathcal{H}} = \dim \mathcal{H}' \quad \textit{and} \quad \widetilde{\mathcal{H}} \cap \mathcal{H}'' = \{0\}, \tag{3.12}$$

where $\mathcal{H}' = \mathcal{H} \cap \mathfrak{H}_1(A_0)$ and $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$. In this case, the domain of S admits the description

$$\mathcal{D}(S) = \mathcal{D}(A_{\mathrm{sym}}) \dot{+} \big\{h' + u\colon\ h' \in \mathcal{H}',\ u = u(h')\big\}, \tag{3.13}$$

where $u = u(h') \in \mathcal{D}(A_0)$ is (uniquely) determined by $h' \in \mathcal{H}'$ and satisfies the relation

$$\big((A_0 + I)u, \widetilde{h}^{\perp}\big) = \langle \psi, u \rangle = 0, \quad \forall \widetilde{h}^{\perp} \in \mathcal{H} \ominus \widetilde{\mathcal{H}},\ \psi = (\mathbb{A}_0 + I)\widetilde{h}^{\perp}. \tag{3.14}$$

Proof. Denote $S_0 = A_F \cap A_0$. By Lemma 3.3

$$\mathcal{D}(S_0) = (A_0 + I)^{-1}\bigl[\mathcal{R}(A_{\text{sym}} + I) \oplus \mathcal{H}''\bigr] = (A_F + I)^{-1}\bigl[\mathcal{R}(A_{\text{sym}} + I) \oplus \mathcal{H}''\bigr], \quad (3.15)$$

where $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}'$. Comparing (3.10) and (3.15), one concludes that

$$\mathcal{D}(S) \cap \mathcal{D}(A_0) = \mathcal{D}(S) \cap \mathcal{D}(S_0) = (A_F + I)^{-1}\bigl[\mathcal{R}(A_{\text{sym}} + I) \oplus (\widetilde{\mathcal{H}} \cap \mathcal{H}'')\bigr].$$

Thus,

$$\mathcal{D}(S) \cap \mathcal{D}(A_0) = \mathcal{D}(A_{\text{sym}}) \quad \Leftrightarrow \quad \widetilde{\mathcal{H}} \cap \mathcal{H}'' = \{0\}.$$

The relations (3.10) and (3.15) also show that

$$\mathcal{D}(S) + \mathcal{D}(A_0) = (A_F + I)^{-1}\bigl[\mathcal{R}(A_{\text{sym}} + I) \oplus (\widetilde{\mathcal{H}} \dot{+} \mathcal{H}'')\bigr] + (A_0 + I)^{-1}\mathcal{H}'. \quad (3.16)$$

Here $(A_0 + I)^{-1}\mathcal{H}'$ can be represented as

$$(A_0 + I)^{-1}\mathcal{H}' = \bigl\{(A_F + I)^{-1}h' + Ch' \colon h' \in \mathcal{H}'\bigr\}, \quad (3.17)$$

where $C = (A_0 + I)^{-1} - (A_F + I)^{-1}$. It follows from Lemma 3.3 that

$$\mathcal{R}(C) = \mathcal{H}', \qquad \ker C = \operatorname{ran}(A_{\text{sym}} + I) \oplus \mathcal{H}''. \quad (3.18)$$

Relations (3.16)–(3.18) show that the second identity in (3.11) holds if and only if $\widetilde{\mathcal{H}} \dot{+} \mathcal{H}'' = \mathcal{H}$. Obviously, this representation is possible only in the case where $\dim \widetilde{\mathcal{H}} = \dim \mathcal{H}'$.

The definition (3.10) shows that $\mathcal{D}(S) = \mathcal{D}(A_{\text{sym}}) \dot{+} (A_F + I)^{-1}\widetilde{\mathcal{H}}$, where

$$(A_F + I)^{-1}\widetilde{\mathcal{H}} = \bigl\{(A_0 + I)^{-1}\widetilde{h} - C\widetilde{h} \colon \widetilde{h} \in \widetilde{\mathcal{H}}\bigr\}.$$

Since $\widetilde{\mathcal{H}}$ satisfies (3.12), it follows from (3.18) that $C\widetilde{\mathcal{H}} = \mathcal{H}'$. Now, setting $u = (A_0 + I)^{-1}\widetilde{h}$ and $h' = -C\widetilde{h}$, one obtains (3.13) and (3.14). Note that the preimage $\widetilde{h} = C^{-1}h' \in \widetilde{\mathcal{H}}$, and therefore also u, is uniquely determined by $h' \in \mathcal{H}'$. □

The next theorem gives a description of all admissible large coupling limits.

Theorem 3.6. *Let $\widetilde{A}$ be a self-adjoint extension of A_{sym} and let the symmetric operator $S = \widetilde{A} \cap A_F$ be represented as in* (3.10) *with some subspace $\widetilde{\mathcal{H}}$ of $\mathcal{H}$. Then the following statements are equivalent*:

(i) *$\widetilde{A}$ $(= A_\infty)$ is an admissible large coupling limit of* (1.3);
(ii) *$\widetilde{A}$ is a self-adjoint extension of S transversal to the Friedrichs extension S_F of S and the subspace $\widetilde{\mathcal{H}}$ satisfies the conditions in* (3.12).

Proof. Let $\widetilde{A}$ be an admissible large coupling limit. Since $\widetilde{A}$ and A_0 are transversal, one has

$$\mathcal{D}(\widetilde{A}) \cap \mathcal{D}(A_0) = \mathcal{D}(A_{\text{sym}}), \qquad \mathcal{D}(\widetilde{A}) + \mathcal{D}(A_0) = \mathcal{D}(A_F) \dot{+} \mathcal{H} = \mathcal{D}\bigl(A^*_{\text{sym}}\bigr). \quad (3.19)$$

The condition (3.5) is equivalent to

$$\mathcal{D}(\widetilde{A}) \cap \mathfrak{H}_1(A_0) = \mathcal{D}(\widetilde{A}) \cap \mathcal{D}(A_F) = \mathcal{D}(\widetilde{A} \cap A_F).$$

Thus, intersecting all parts of (3.19) with $\mathfrak{H}_1(A_0)$ one concludes that the relations (3.11) are true for $S = \widetilde{A} \cap A_F$. By Lemma 3.5, the subspace $\widetilde{\mathcal{H}}$ satisfies (3.12). Furthermore, since the Friedrichs extension S_F of S coincides with A_F, one gets $\mathcal{D}(S_F) \cap \mathcal{D}(\widetilde{A}) = \mathcal{D}(A_F) \cap \mathcal{D}(\widetilde{A}) = \mathcal{D}(S)$. This implies the transversality of S_F and $\widetilde{A}$. The implication (i) $\Rightarrow$ (ii) is proved.

Now, assume that (ii) is satisfied. Since $S \supset A_{\text{sym}}$, the operator $\widetilde{A}$ is a self-adjoint extension of A_{sym}. It follows from (3.10) that $\ker(S^* + I) = \mathcal{H} \ominus \widetilde{\mathcal{H}}$ and hence, $\mathcal{D}(S^*) = \mathcal{D}(S_F) + \ker(S^* + I) = \mathcal{D}(A_F) \dot{+} (\mathcal{H} \ominus \widetilde{\mathcal{H}})$. On the other hand, the transversality of S_F and $\widetilde{A}$ gives $\mathcal{D}(S^*) = \mathcal{D}(A_F) + \mathcal{D}(\widetilde{A})$. Therefore, $\mathcal{D}(A_F) + \mathcal{D}(\widetilde{A}) = \mathcal{D}(A_F) \dot{+} (\mathcal{H} \ominus \widetilde{\mathcal{H}})$. This equality and the second relation in (3.11) yield

$$\begin{aligned}\mathcal{D}(A_0) + \mathcal{D}(\widetilde{A}) &= \mathcal{D}(S) + \mathcal{D}(A_0) + \mathcal{D}(\widetilde{A}) \\ &= \big(\mathcal{D}(A_F) \dot{+} \mathcal{H}'\big) + \mathcal{D}(\widetilde{A}) = \mathcal{D}(A_F) \dot{+} \mathcal{H}' \dot{+} (\mathcal{H} \ominus \widetilde{\mathcal{H}}). \qquad (3.20)\end{aligned}$$

The conditions (3.12) imply that $\mathcal{H}' \dot{+} (\mathcal{H} \ominus \widetilde{\mathcal{H}}) = \mathcal{H}$. Hence, (3.20) shows that $\mathcal{D}(A_0) + \mathcal{D}(\widetilde{A}) = \mathcal{D}(A_F) \dot{+} \mathcal{H} = \mathcal{D}(A^*_{\text{sym}})$, i.e., $\widetilde{A}$ and A_0 are transversal. Furthermore, by Lemma 3.3, see also (3.6), $\mathcal{D}(A_F) \dot{+} \mathcal{H}' = \mathfrak{H}_1(A_0) \cap \mathcal{D}(A^*_{\text{sym}})$. Now, employing the second relation in (3.11) one obtains

$$\mathcal{D}(\widetilde{A}) \cap \mathfrak{H}_1(A_0) = \mathcal{D}(\widetilde{A}) \cap \big(\mathcal{D}(S) + \mathcal{D}(A_0)\big) = \mathcal{D}(S) + \mathcal{D}(A_{\text{sym}}) = \mathcal{D}(S) \subset \mathcal{D}(A_F).$$

According to Theorem 3.4 this means that $\widetilde{A}$ is an admissible large coupling limit of (1.3). The implication (ii) $\Rightarrow$ (i) is proved. □

It follows from Theorem 3.6 that there is at least one admissible large coupling limit of (1.3). Some further specifications are given in the following two corollaries.

Corollary 3.7. *If all the elements ψ_j in* (1.3) *belong to $\mathfrak{H}_{-1}(A_0)$, then there exists a unique admissible large coupling limit A_∞ and it coincides with the Friedrichs extension A_F of A_{sym}.*

Proof. Assume that $\psi_j \in \mathfrak{H}_{-1}(A_0)$ for all $j = 1, \ldots, n$. Then $\mathcal{D}(A^*_{\text{sym}}) \subset \mathfrak{H}_1(A_0)$ and $\mathcal{H}' = \mathcal{H}$. Let $\widetilde{A} = A_\infty$ be an admissible large coupling limit of (1.3) and let $S = \widetilde{A} \cap A_F$. By Theorem 3.6 the corresponding subspace $\widetilde{\mathcal{H}}$ satisfies (3.12) in Lemma 3.5, so that $\widetilde{\mathcal{H}} = \mathcal{H}$. Now (3.10) gives $S = A_F$ and since $S = \widetilde{A} \cap A_F$, one concludes that $\widetilde{A} = A_F$. This completes the proof. □

Corollary 3.8. *If all the elements ψ_j in* (1.3) *are $\mathfrak{H}_{-1}(A_0)$-independent* (*i.e.* $\mathcal{X} \cap \mathfrak{H}_{-1}(A_0) = \{0\}$), *then every self-adjoint extension $\widetilde{A}$ of A_{sym} transversal to A_0 is an admissible large coupling limit of* (1.3). *The Friedrichs extension of A_{sym} coincides with A_0.*

Proof. The condition of $\mathfrak{H}_{-1}(A_0)$-independency means that $\mathcal{H}' = \{0\}$. In this case, only the zero subspace $\widetilde{\mathcal{H}} = \{0\}$ can satisfy (3.12). The corresponding operator S coincides with A_{sym}. Moreover, since $\mathcal{H}' = \{0\}$, Lemma 3.3 shows that $S_F = A_F = A_0$. Thus, by Theorem 3.6, $\widetilde{A}$ is an admissible large coupling limit if and only if $\widetilde{A}$ is transversal to A_0. □

Observe, that the condition $\mathcal{X} \subset \mathfrak{H}_{-1}(A_0)$ in Corollary 3.7 is equivalent to $\mathcal{D}(A_{\mathrm{sym}}^*) \subset \mathfrak{H}_1(A_0)$; see (2.2). Since in this case all the elements $\psi_j \in \mathfrak{H}_{-1}(A_0)$ admit their natural extension by continuity onto $\mathfrak{H}_1(A_0)$ via (3.1), the matrix $\mathbf{R} = (r_{jp})_{j,p=1}^n$ in (2.4) is uniquely determined and the extended functionals $\langle \psi_j^{\mathrm{ex}}, \cdot \rangle$ in (2.4) are obtained by restricting their natural continuations to the subset $\mathcal{D}(A_{\mathrm{sym}}^*)$ of $\mathfrak{H}_1(A_0)$. It follows that if $\mathcal{X} \subset \mathfrak{H}_{-1}(A_0)$, then the operator realizations of (1.3) described in Theorem 2.3 reduce to the so-called form bounded perturbations of A_0:

Corollary 3.9. *If all the elements ψ_j in* (1.3) *belong to $\mathfrak{H}_{-1}(A_0)$, then the operator realization $A_{\mathbf{B}}$ of* (1.3) *in Theorem* 2.3 *determined by the boundary condition $\mathbf{B}\Gamma_0 f = \Gamma_1 f$ in* (2.8) *takes the form*

$$A_{\mathbf{B}} f = \mathbb{A}_0 f + \sum_{i,j=1}^n b_{ij} \langle \psi_j^{\mathrm{ex}}, f \rangle \psi_i, \quad f \in \mathcal{D}(A_{\mathbf{B}}),$$

where the extended functionals $\langle \psi_j^{\mathrm{ex}}, \cdot \rangle$, $j = 1, \ldots, n$, are determined by their continuations onto $\mathfrak{H}_1(A_0)$ via (3.1) *and $\mathbb{A}_0$ as defined by* (2.1) *can be considered as a bounded operator acting from $\mathfrak{H}_{-1}(A_0)$ into $\mathfrak{H}_1(A_0)$.*

Proof. The statement is immediate from Corollary 2.4 and the fact that in this case $\mathcal{D}(A_{\mathrm{sym}}^*) \subset \mathfrak{H}_1(A_0)$. Note that $\mathbb{A}_0$ defined by (2.1) satisfies $\mathbb{A}_0(\mathfrak{H}_1(A_0)) \subset \mathfrak{H}_{-1}(A_0)$ and its restriction to $\mathfrak{H}_1(A_0)$ coincides with the continuation of A_0 as a bounded operator from $\mathfrak{H}_{-1}(A_0)$ into $\mathfrak{H}_1(A_0)$. □

The properties of admissible large coupling limits are closely related to the transversality of the Friedrichs and the Krein–von Neumann extensions of A_{sym}.

Theorem 3.10. *There exists a nonnegative admissible large coupling limit of* (1.3) *if and only if the Friedrichs extension A_F and the Krein–von Neumann extension A_N of A_{sym} are transversal.*

Proof. Let $\widetilde{A}$ be a nonnegative admissible large coupling limit. Then $\widetilde{A}$ is a nonnegative extension of A_{sym} and therefore

$$(A_F + I)^{-1} \leqslant (\widetilde{A} + I)^{-1} \leqslant (A_N + I)^{-1}, \tag{3.21}$$

where A_F is the Friedrichs extension and A_N is the Krein–von Neumann extension of A_{sym} (see e.g. [21] and the references therein).

Recall that transversality of self-adjoint extensions $\widetilde{A}_1$ and $\widetilde{A}_2$ of A_{sym} is equivalent to

$$\big[(\widetilde{A}_1 + I)^{-1} - (\widetilde{A}_2 + I)^{-1}\big]\mathcal{H} = \mathcal{H} \tag{3.22}$$

(see e.g. [16]). Hence, if A_F and A_N are not transversal then $(A_F + I)^{-1} h = (A_N + I)^{-1} h$ for some nonzero $h \in \mathcal{H}$. Then nonnegativity of $\widetilde{A}$ and A_0 yields $(\widetilde{A} + I)^{-1} h = (A_0 + I)^{-1} h$ due to (3.21) (with similar inequalities for A_0), so that

$$\big[(\widetilde{A} + I)^{-1} - (A_0 + I)^{-1}\big]\mathcal{H} \subset \mathcal{H} \ominus \langle h \rangle$$

and by (3.22) $\widetilde{A}$ and A_0 cannot be transversal. This is a contradiction to the admissibility of $\widetilde{A}$. Thus A_F and A_N are transversal.

To prove the converse statement assume that A_F and A_N are transversal. Let $\widetilde{\mathcal{H}}$ be a subspace of $\mathcal{H}$, which satisfies (3.12) and let the symmetric operator S be defined by (3.10) in Lemma 3.5. Moreover, let $\widetilde{A}$ be the Krein–von Neumann extension of S. Clearly, $\widetilde{A}$ is a nonnegative self-adjoint extension of A_{sym}. It remains to prove that $\widetilde{A}$ is an admissible large coupling limit of (1.3). To see this, observe that the Friedrichs extension of S coincides with A_F. Then it follows from [10, Proposition 7.2] that the Friedrichs extension $S_F = A_F$ and the Krein–von Neumann extension $\widetilde{A}$ of S are transversal with respect to S. Therefore, by Theorem 3.6, $\widetilde{A}$ is an admissible large coupling limit. □

Observe that S in Theorem 3.10 is a restriction of the Friedrichs extension A_F of A_{sym}. Since the admissible large coupling limit $\widetilde{A}$ constructed in Theorem 3.10 is the Krein–von Neumann extension of S it is a consequence of [10, Theorem 6.4] that $\widetilde{A}$ is an extremal extension of A_{sym} in the sense of the following definition.

Definition 3.11. (See [9,10].) A self-adjoint extension $\widetilde{A}$ of A_{sym} is called extremal if it is nonnegative and satisfies the condition

$$\inf_{u\in\mathcal{D}(A_{\mathrm{sym}})} \big(\widetilde{A}(f-u), f-u\big) = 0 \quad \text{for all } f \in \mathcal{D}(\widetilde{A}).$$

Theorem 3.12. *Let the Friedrichs extension A_F and the Krein–von Neumann extension A_N of A_{sym} be transversal, and let S be defined by* (3.10) *and* (3.12). *Then among all self-adjoint extensions of S there exists a unique extremal admissible large coupling limit $\widetilde{A}$ of* (1.3).

Proof. By Theorem 3.10, it suffices to show that the Krein–von Neumann extension $\widetilde{A}$ of S is the only extremal extension of A_{sym} which coincides with admissible large coupling limit of (1.3).

To prove this assume that $\widehat{A}$ is extremal and admissible in the sense of Definition 3.2. Then by [10, Theorem 6.4] $\widehat{A}$ as an extremal extension of A_{sym} is the Krein–von Neumann extension of the symmetric operator $\widehat{S} = \widehat{A} \cap A_F$. Moreover, by Theorem 3.6 the admissibility of $\widehat{A}$ means that $\widehat{S}$ is determined via (3.10) where the corresponding subspace $\widehat{\mathcal{H}}$ satisfies (3.12).

Since $\widehat{A}$ is an extension of S, one has $S \subseteq \widehat{S}$ or, equivalently, $\widetilde{\mathcal{H}} \subseteq \widehat{\mathcal{H}}$, where the subspaces $\widetilde{\mathcal{H}}$ and $\widehat{\mathcal{H}}$ correspond to S and $\widehat{S}$ in (3.10). Now the first equality in (3.12) forces that $\widetilde{\mathcal{H}} = \widehat{\mathcal{H}}$ and hence $S = \widehat{S}$. Therefore, $\widehat{A} = \widetilde{A}$ and this completes the proof. □

Remark 3.13. The selection of a self-adjoint operator $\widetilde{A}$ transversal to the initial one A_0 (but without the condition (3.5)) is also a key point of the approach used in [11] to the determination of self-adjoint realizations of a formal expression $A_0 + V$, where a singular perturbation V is assumed to be (in general) an unbounded self-adjoint operator $V : \mathfrak{H}_2(A_0) \to \mathfrak{H}_{-2}(A_0)$ such that $\ker V$ is dense in $\mathfrak{H}$. In this case, the regularization of $A_0 + V$ takes the form $A_{\mathcal{P},V} = \mathbb{A}_0 + V\mathcal{P}$ and it is well defined on the domain $\mathcal{D}(A_{\mathcal{P},V}) = \{f \in \mathcal{D}(A^*_{\mathrm{sym}}): \mathcal{P}f \in \mathcal{D}(V)\}$, where $\mathcal{P}$ is the skew projection onto $\mathfrak{H}_2(A_0)$ in $\mathcal{D}(A^*_{\mathrm{sym}})$ that is uniquely determined by $\widetilde{A}$.

4. Singular perturbations with symmetries and uniqueness of admissible large coupling limits

According to (2.4) and (3.3) the regularization $\mathbb{A}_{\mathbf{R}}$ of (1.3) depends on the choice of an admissible large coupling limit A_∞. Apart from the case of form bounded singular perturbations, admissible large coupling limits are not determined uniquely, cf. Theorem 3.6. However, in many cases (see e.g. [4,5]), the uniqueness can be attained by imposing extra assumptions of symmetry motivated by the specific nature of the underlying physical problem. In this section, we study this problem in an abstract framework.

4.1. Preliminaries

First some general facts concerning $p(t)$-homogeneous operators are given. Let an operator A in $\mathfrak{H}$ be $p(t)$-homogeneous with respect to a one-parameter family $\mathfrak{U}=\{U_t\}_{t\in\mathfrak{T}}$ of unitary operators acting on $\mathfrak{H}$, cf. Definition 1.1. It follows from (1.7) and (1.8) that

$$p(t)p(g(t))=1, \quad \forall t\in\mathfrak{T}, \tag{4.1}$$

where the function of conjugation $g(t):\mathfrak{T}\to\mathfrak{T}$ is determined by the formula

$$U_{g(t)}=U_t^*, \quad \forall t\in\mathfrak{T}. \tag{4.2}$$

Lemma 4.1. *Let A be a $p(t)$-homogeneous operator with respect to a family $\mathfrak{U}=\{U_t\}_{t\in\mathfrak{T}}$. Then for all $t\in\mathfrak{T}$ and all $z\in\mathbb{C}$,*

$$U_t\big(\ker(A-zI)\big)=\ker\big(p(t)A-zI\big). \tag{4.3}$$

In particular, $\ker A$ *is a reducing subspace for every U_t, $t\in\mathfrak{T}$. Furthermore, $z\in\sigma_a(A) \Leftrightarrow zp(t)^n\in\sigma_a(A)$, $n\in\mathbb{Z}$, $t\in\mathfrak{T}$, $a\in\{p,r,c\}$.*

If $p(t)\neq 1$ at least for one point $t\in\mathfrak{T}$, then the essential spectrum of A contains the point $z=0$.

Proof. In view of (4.1), $p(t)\neq 0$ for all $t\in\mathfrak{T}$. Using (1.8) one gets

$$U_t(A-zI)=\big(p(t)A-zI\big)U_t=p(t)\left(A-\frac{z}{p(t)}I\right)U_t \tag{4.4}$$

that gives $U_t(\ker(A-zI))\subset\ker(p(t)A-zI)$. The reverse inclusion is obtained by using (4.1). The property of $\ker A$ to be a reducing subspace for every U_t follows from (4.3) with $z=0$ if one takes into account that $p(t)\neq 0$.

The remaining assertions of the lemma immediately follow from (4.4). □

Lemma 4.2. *Let A be a closed densely defined $p(t)$-homogeneous operator with respect to a family $\mathfrak{U}=\{U_t\}_{t\in\mathfrak{T}}$. Then also its adjoint A^* is $p(t)$-homogeneous with respect to $\mathfrak{U}$.*

Proof. Since A is $p(t)$-homogeneous one has $U_tA=p(t)AU_t$ for all $t\in\mathfrak{T}$. As a unitary operator U_t is bounded with bounded inverse, and therefore, the previous equality is equivalent to

$A^*U_t^* = p(t)U_t^*A^* \Leftrightarrow U_tA^* = p(t)A^*U_t$, $\forall t \in \mathfrak{T}$, which means that A^* is $p(t)$-homogeneous with respect to $\mathfrak{U}$. □

In the case that A is symmetric the formula (4.3) in Lemma 4.1 shows how the unitary operators U_t, $t \in \mathfrak{T}$, transform the defect subspaces $\ker(A^* - zI)$ of A.

Corollary 4.3. *Let A in Lemma 4.2 be nonnegative and $p(t)$-homogeneous with respect to $\mathfrak{U} = \{U_t\}_{t\in\mathfrak{T}}$ and let A_0 be a nonnegative selfadjoint extension of A. Then $(p(t)A_0 + I)(A_0 + I)^{-1}U_t(\ker(A^* + I)) = \ker(A^* + I)$.*

Proof. By Lemma 4.2 the adjoint A^* of A is also $p(t)$-homogeneous and (4.3) implies that $U_t(\ker(A^* + I)) = \ker(A^* + 1/p(t)I)$. Moreover, the equality

$$\bigl(p(t)A_0 + I\bigr)(A_0 + I)^{-1}\ker\left(A^* + \frac{1}{p(t)}I\right) = \ker(A^* + I)$$

is always satisfied for a nonnegative self-adjoint extension A_0 of A. □

For the next result recall that if A is a nonnegative operator (or in general a nonnegative relation) in a Hilbert space $\mathfrak{H}$, then the Friedrichs extension A_F and the Krein–von Neumann extension A_N of A can be characterized as follows (see [8] for the densely defined case and [19,21,22] for the general case):

If $\{f, f'\} \in A^*$, then $\{f, f'\} \in A_F$ if and only if

$$\inf\bigl\{\|f - h\|^2 + (f' - h', f - h)\colon \{h, h'\} \in A\bigr\} = 0. \tag{4.5}$$

If $\{f, f'\} \in A^*$, then $\{f, f'\} \in A_N$ if and only if

$$\inf\bigl\{\|f' - h'\|^2 + (f' - h', f - h)\colon \{h, h'\} \in A\bigr\} = 0. \tag{4.6}$$

Lemma 4.4. *Let A be a nonnegative densely defined $p(t)$-homogeneous operator with respect to $\mathfrak{U}$. Then the Friedrichs extension A_F and the Krein–von Neumann extension A_N of A are also $p(t)$-homogeneous with respect to $\mathfrak{U}$. Moreover, $U_t(\mathcal{D}(A_F^{1/2})) \subset \mathcal{D}(A_F^{1/2})$ and $U_t(\mathcal{R}(A_N^{1/2})) \subset \mathcal{R}(A_N^{1/2})$ for all $t \in \mathfrak{T}$.*

Proof. By Lemma 4.2 A^* is $p(t)$-homogeneous with respect to $\mathfrak{U}$. Hence, in view of (1.7) and (1.8), an intermediate extension $\widetilde{A}$ of A is $p(t)$-homogeneous with respect to $\mathfrak{U}$ if and only if

$$U_t : \mathcal{D}(\widetilde{A}) \to \mathcal{D}(\widetilde{A}), \quad \forall t \in \mathfrak{T}. \tag{4.7}$$

To prove that A_F is $p(t)$-homogeneous with respect to $\mathfrak{U}$, assume that $f \in \mathcal{D}(A_F)$. Then $g = U_t f \in \mathcal{D}(A^*)$ and there is a sequence $h_n \in \mathcal{D}(A)$ attaining the infimum in (4.5). Then $U_t h_n \in \mathcal{D}(A)$, $U_t h_n \to U_t f = g$, and

$$(A^*U_t f - AU_t h_n, U_t f - U_t h_n) = p\bigl(g(t)\bigr)(A^* f - Ah_n, f - h_n) \to 0, \tag{4.8}$$

so that $g \in \mathcal{D}(A_F)$ by (4.5). Therefore, $U_t(\mathcal{D}(A_F)) \subset \mathcal{D}(A_F)$ and A_F is $p(t)$-homogeneous with respect to $\mathfrak{U}$.

To prove the $p(t)$-homogeneity of A_N assume that $f \in \mathcal{D}(A_N)$. Then again $g = U_t f \in \mathcal{D}(A^*)$ and there is a sequence $h_n \in \mathcal{D}(A)$ attaining the infimum in (4.6). In particular, $Ah_n \to A^* f$, $U_t h_n \in \mathcal{D}(A)$, and

$$AU_t h_n = p\big(g(t)\big)U_t A h_n \to p\big(g(t)\big)U_t A^* f = A^* U_t f = A^* g.$$

Moreover, (4.8) is satisfied. Therefore, (4.6) shows that $g \in \mathcal{D}(A_N)$. This proves that $U_t(\mathcal{D}(A_N)) \subset \mathcal{D}(A_N)$ and thus A_N is $p(t)$-homogeneous with respect to $\mathfrak{U}$.

Finally, recall that the domain $\mathcal{D} = \mathcal{D}(A_F^{1/2})$, see (3.4), can be characterized as the set of vectors $f \in \mathfrak{H}$ satisfying

$$h_n \to f, \quad \big(A(h_n - h_m), h_n - h_m\big) \to 0, \quad m, n \to \infty,$$

and the range $\mathcal{R}(A_N^{1/2})$ as the set of vectors $g \in \mathfrak{H}$ satisfying

$$Ah_n \to g, \quad \big(A(h_n - h_m), h_n - h_m\big) \to 0, \quad m, n \to \infty,$$

with $h_n \in \mathcal{D}(A)$, see (4.5) and (4.6). The last statement is clear from these characterizations using similar arguments as above with the sequence h_n. This completes the proof. □

Let the operator A_0 in (1.3) be $p(t)$-homogeneous with respect to $\mathfrak{U} = \{U_t\}_{t \in \mathfrak{T}}$. Define a family of self-adjoint operators on $\mathfrak{H}$ by

$$G_t = \big(p(t)A_0 + I\big)(A_0 + I)^{-1}, \quad t \in \mathfrak{T}. \tag{4.9}$$

Clearly, G_t is positive and bounded with bounded inverse for all $t \in \mathfrak{T}$. Moreover, it follows from (1.8) and (4.1) that $(A_0 + I)^{-1}U_t = U_t(p(g(t))A_0 + I)^{-1}$ and

$$G_t U_t = U_t G_{g(t)}^{-1} = (G_{g(t)} U_{g(t)})^{-1}. \tag{4.10}$$

Since $\|u\|_{-2} = \|(A_0 + I)^{-1}u\|$, the identity $(A_0 + I)^{-1}U_t = G_t U_t (A_0 + I)^{-1}$ implies that $\|U_t u\|_{-2} \leqslant \|G_t\| \, \|u\|_{-2}$ for all $u \in \mathfrak{H}$. Hence, the operators U_t can be continuously extended to bounded operators $\mathbb{U}_t$ in $\mathfrak{H}_{-2}(A_0)$ and, furthermore,

$$(\mathbb{A}_0 + I)^{-1}\mathbb{U}_t \psi = G_t U_t (\mathbb{A}_0 + I)^{-1}\psi \tag{4.11}$$

for all $\psi \in \mathfrak{H}_{-2}(A_0)$ and $t \in \mathfrak{T}$. The equality (4.2) shows that $\mathbb{U}_t$ has a bounded inverse which satisfies $\mathbb{U}_t^{-1} = \mathbb{U}_{g(t)}$. The operator $\mathbb{U}_t$ can be characterized also as the dual mapping (adjoint) of $U_{g(t)}$ with respect to the form defined in (1.2). In fact, using (1.2), (1.8), (4.2), and (4.11), it is seen that the action of the functional $\langle \mathbb{U}_t \psi, \cdot \rangle$ on the elements $u \in \mathfrak{H}_2(A_0)$ is determined by the formula

$$\begin{aligned} \langle \mathbb{U}_t \psi, u \rangle &= \big((A_0 + I)u, G_t U_t h\big) = \big(U_{g(t)}\big(p(t)A_0 + I\big)u, h\big) \\ &= \big((A_0 + I)U_{g(t)}u, h\big) = \langle \psi, U_{g(t)}u \rangle, \end{aligned} \tag{4.12}$$

where $h = (\mathbb{A}_0 + I)^{-1}\psi$.

Now consider a singular element $\psi \in \mathfrak{H}_{-2}(A_0)$, cf. (1.3). The assumption that ψ is $\xi(t)$-invariant with respect to $\mathfrak{U}$, i.e. $\mathbb{U}_t\psi = \xi(t)\psi$ for all $t \in \mathfrak{T}$ (see Definition 1.2), implies some relations between $\xi(t)$, $p(t)$, and $g(t)$.

Proposition 4.5. *Let the operator A_0 in* (1.3) *be $p(t)$-homogeneous with respect to the family $\mathfrak{U}$ and let $\psi \in \mathfrak{H}_{-2}(A_0) \setminus \mathfrak{H}$ be $\xi(t)$-invariant with respect to $\mathfrak{U}$. Then for all $t \in \mathfrak{T}$ one has*

$$\xi(t)\xi\big(g(t)\big) = 1 \tag{4.13}$$

and, moreover, $|\xi(t)| = 1$ *if* $p(t) = 1$ *and* $\min\{1, p(t)\} < |\xi(t)| < \max\{1, p(t)\}$ *if* $p(t) \neq 1$.

Proof. It follows from (1.9) and (4.11) that $\psi \in \mathfrak{H}_{-2}(A_0) \setminus \mathfrak{H}$ is $\xi(t)$-invariant with respect to $\mathfrak{U}$ if and only if

$$G_t U_t h = \xi(t)h, \quad \forall t \in \mathfrak{T}, \tag{4.14}$$

where $h = (\mathbb{A}_0 + I)^{-1}\psi$. This together with (4.10) implies that

$$h = (G_{g(t)}U_{g(t)})(G_t U_t)h = \xi(t)G_{g(t)}U_{g(t)}h = \xi(t)\xi\big(g(t)\big)h,$$

which proves (4.13). Moreover, (4.14) shows that $|\xi(t)|\|h\| = \|G_t U_t h\|$. In particular, if $p(t) = 1$, then $G_t = I$ and $|\xi(t)|\|h\| = \|U_t h\| = \|h\|$ that gives $|\xi(t)| = 1$.

In the case where $p(t) \neq 1$ the formula for G_t in (4.9) with an evident reasoning leads to the estimates $\alpha(t)\|h\| = \alpha(t)\|U_t h\| < \|G_t U_t h\| < \beta(t)\|U_t h\| = \beta(t)\|h\|$, where $\alpha(t) = \min\{1, p(t)\}$ and $\beta(t) = \max\{1, p(t)\}$. This completes the proof. □

4.2. $p(t)$-homogeneous self-adjoint extensions of A_{sym}

Let A_{sym} be defined by (1.4). This means that A_{sym} is a nonnegative symmetric operator with finite defect numbers.

Lemma 4.6. *If $p(t) \neq 1$ at least for one point $t \in \mathfrak{T}$, then an arbitrary $p(t)$-homogeneous self-adjoint extension of the symmetric operator A_{sym} is nonnegative.*

Proof. Assume that z is a negative eigenvalue of a $p(t)$-homogeneous self-adjoint extension A of A_{sym} and that $p(t) \neq 1$ for $t \in \mathfrak{T}$. Then, according to Lemma 4.1, there exists infinite series of negative eigenvalues $zp(t)^n$ ($n \in \mathbb{Z}$) of A that contradicts to the assumption of finite defect numbers of A_{sym}. Hence, A is a nonnegative extension of A_{sym}. □

Lemma 4.7. *Let A_0 be $p(t)$-homogeneous and let ψ_j be $\xi_j(t)$-invariant with respect to $\mathfrak{U}$, $j = 1, \ldots, n$. Then the symmetric operator A_{sym} defined by* (1.4) *and its adjoint A^*_{sym} are also $p(t)$-homogeneous with respect to $\mathfrak{U}$.*

Proof. It follows from (1.4) and (4.12) that

$$\langle\psi_j, U_t u\rangle = \langle\mathbb{U}_{g(t)}\psi_j, u\rangle = \xi_j\big(g(t)\big)\langle\psi_j, u\rangle = 0$$

for every $u \in \mathcal{D}(A_{\mathrm{sym}})$. Thus $U_t : \mathcal{D}(A_{\mathrm{sym}}) \to \mathcal{D}(A_{\mathrm{sym}})$ and hence by (1.8) A_{sym} is $p(t)$-homogeneous: $U_t A_{\mathrm{sym}} = p(t) A_{\mathrm{sym}} U_t$. By Lemma 4.2 also the adjoint A^*_{sym} is $p(t)$-homogeneous with respect to $\mathfrak{U}$. □

In view of (1.9) and (4.12) the $\xi_j(t)$-invariance of ψ_j is equivalent to the relation

$$\xi_j(t)\langle\psi_j, u\rangle = \langle\psi_j, U_{g(t)}u\rangle, \quad \forall u \in \mathfrak{H}_2(A_0),\ \forall t \in \mathfrak{T}, \tag{4.15}$$

where the linear functionals $\langle\psi_j, \cdot\rangle$ are defined by (1.2). The next theorem shows that the preservation of (4.15) for the extended functionals $\langle\psi_j^{\mathrm{ex}}, \cdot\rangle$ is closely related to the existence of $p(t)$-homogeneous self-adjoint extensions of A_{sym} transversal to A_0.

Theorem 4.8. *Let A_0 be $p(t)$-homogeneous, let $\psi_1, \ldots, \psi_n$ be $\xi_j(t)$-invariant with respect to $\mathfrak{U}$, and let $\langle\psi_j^{\mathrm{ex}}, f\rangle$ be defined by* (2.4). *Then the relations*

$$\xi_j(t)\bigl\langle\psi_j^{\mathrm{ex}}, f\bigr\rangle = \bigl\langle\psi_j^{\mathrm{ex}}, U_{g(t)}f\bigr\rangle, \quad 1 \leqslant j \leqslant n,\ \forall t \in \mathfrak{T}, \tag{4.16}$$

are satisfied for all $f \in \mathcal{D}(A^*_{\mathrm{sym}})$ if and only if the corresponding self-adjoint operator A_∞ defined by (3.3) *is $p(t)$-homogeneous with respect to $\mathfrak{U}$.*

Proof. Denote

$$\boldsymbol{\Xi}(t) = \begin{pmatrix} \xi_1(t) & 0 & \ldots & 0 \\ 0 & \xi_2(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \xi_n(t) \end{pmatrix}. \tag{4.17}$$

Then $\det \boldsymbol{\Xi}(t) \neq 0$, $t \in \mathfrak{T}$, by Proposition 4.5, since ψ_i is $\xi_j(t)$-invariant with respect to $\mathfrak{U}$. By using (2.5) in Lemma 2.2 the relations (4.16) can be rewritten as follows:

$$\boldsymbol{\Xi}(t)\Gamma_0 f = \Gamma_0 U_{g(t)} f, \quad \forall f \in \mathcal{D}\bigl(A^*_{\mathrm{sym}}\bigr),\ \forall t \in \mathfrak{T}. \tag{4.18}$$

Since $\mathcal{D}(A_\infty) = \ker \Gamma_0$, (4.18) immediately implies that $U_t(\mathcal{D}(A_\infty)) \subset \mathcal{D}(A_\infty)$, cf. (4.2). Thus the equalities (4.16) ensure $p(t)$-homogeneity of A_∞ with respect to $\mathfrak{U}$.

Conversely, assume that A_∞ is $p(t)$-homogeneous with respect to $\mathfrak{U}$. According to (3.3), (4.2), and (4.7) this is equivalent to

$$-\mathbf{R}\widehat{\Gamma}_0 U_{g(t)} f = \widehat{\Gamma}_1 U_{g(t)} f, \quad \forall f \in \mathcal{D}(A_\infty),\ \forall t \in \mathfrak{T}. \tag{4.19}$$

Using (4.9), (4.13), and (4.14) it is seen that

$$\begin{aligned} U_{g(t)}h_j &= p(t)G_{g(t)}U_{g(t)}h_j + \bigl(I - p(t)G_{g(t)}\bigr)U_{g(t)}h_j \\ &= \frac{p(t)}{\xi_j(t)}h_j + \bigl(1 - p(t)\bigr)(A_0 + I)^{-1}U_{g(t)}h_j, \end{aligned} \tag{4.20}$$

where $h_j = (\mathbb{A}_0 + I)^{-1}\psi_j$, $j = 1, \dots, n$. This expression and relations (2.6), (4.12) yield the following equalities for all $f = u + \sum_{j=1} \alpha_j h_j \in \mathcal{D}(A^*_{\mathrm{sym}})$ and $t \in \mathfrak{T}$:

$$\widehat{\Gamma}_0 U_{g(t)} f = p(t)\boldsymbol{\Xi}^{-1}(t)\widehat{\Gamma}_0 f, \qquad \widehat{\Gamma}_1 U_{g(t)} f = \boldsymbol{\Xi}(t)\widehat{\Gamma}_1 f + \big(1 - p(t)\big)\mathbf{G}^{\top}(t)\widehat{\Gamma}_0 f, \tag{4.21}$$

where $\mathbf{G}^{\top}(t)$ is the transpose of the matrix $\mathbf{G}(t) = ((h_i, U_t h_j))^n_{i,j=1}$. Now with $f \in \mathcal{D}(A_\infty)$ substituting these expressions into (4.19), using (3.3), and taking into account that $\widehat{\Gamma}_0(\mathcal{D}(A_\infty)) = \mathbb{C}^n$, one concludes that the $p(t)$-homogeneity of A_∞ is equivalent to the matrix equality

$$\boldsymbol{\Xi}(t)\mathbf{R} - p(t)\mathbf{R}\boldsymbol{\Xi}^{-1}(t) = \big(1 - p(t)\big)\mathbf{G}^{\top}(t), \quad \forall t \in \mathfrak{T}. \tag{4.22}$$

Finally, employing (2.7) and (4.21) it is easy to see that equality (4.22) is equivalent to (4.18). Therefore, the extended functionals $\langle \psi_j^{\mathrm{ex}}, \cdot \rangle$ satisfy the relations (4.16). Theorem 4.8 is proved. □

Remark 4.9. In the particular case where $p(t) = t^\beta$ and $\xi(t) = t^\theta$ with $\beta, \theta \in \mathbb{R}$, another condition for the preservation of $\xi(t)$-invariance for $\langle \psi_j^{\mathrm{ex}}, \cdot \rangle$ has been obtained in [5, Lemma 1.3.2].

Corollary 4.10. *A self-adjoint extension $\widetilde{A}$ of A_{sym} transversal to A_0 is $p(t)$-homogeneous if and only if $\widetilde{A}$ is defined by* (3.3) *and the entries r_{ij} of* $\mathbf{R}$ *in* (3.3) *satisfy the following system of equations for all $t \in \mathfrak{T}$:*

$$\beta_{ij}(t) r_{ij} = \big(1 - p(t)\big)(h_j, U_t h_i), \qquad \beta_{ij}(t) = \left(\xi_i(t) - \frac{p(t)}{\xi_j(t)}\right), \quad 1 \leqslant i, j \leqslant n. \tag{4.23}$$

Proof. Since $\ker \widehat{\Gamma}_0 = \mathcal{D}(A_0)$, formula (3.3) describe all self-adjoint extensions of A_{sym} transversal to A_0 when the parameter $\mathbf{R} = (r_{ij})^n_{i,j=1}$ runs the set of all Hermitian matrices. Hence, $\widetilde{A} = A_\infty$ for some choice of $\mathbf{R}$ in (3.3). The proof of Theorem 4.8 shows that A_∞ is $p(t)$-homogeneous if and only if $\mathbf{R}$ is a solution of (4.22) that does not depend on $t \in \mathfrak{T}$. Rewriting (4.22) componentwise one gets (4.23). □

Remark 4.11. In the case that $p(x) \equiv 1$, the right-hand side of (4.23) vanishes and (4.23) reduces to $\beta_{ij}(t) r_{ij} = 0$, $1 \leqslant i, j \leqslant n$. Moreover, by Proposition 4.5 $\beta_{ii}(t) \equiv 0$ and, therefore, the entries r_{ii} cannot be uniquely determined from (4.23). This implies the existence of infinitely many 1-homogeneous self-adjoint extensions of A_{sym} transversal to A_0.

Example 4.12. Let $\alpha > 0$ and let $\widetilde{A}$ be defined by

$$\widetilde{A}_\alpha = A^*_{\mathrm{sym}} \restriction \mathcal{D}(\widetilde{A}_\alpha), \quad \mathcal{D}(\widetilde{A}_\alpha) = \mathcal{D}(A_{\mathrm{sym}}) \dotplus \ker\big(A^*_{\mathrm{sym}} + \alpha I\big).$$

Then for all $\alpha > 0$, $\widetilde{A}_\alpha$ is a 1-homogeneous self-adjoint extensions of A_{sym} transversal to A_0.

4.3. Uniqueness of $p(t)$-homogeneous admissible large coupling limits of (1.3)

Let the operator A_0 be $p(t)$-homogeneous and let the singular elements ψ_j appearing in (1.3) be $\xi_j(t)$-invariant with respect to $\mathfrak{U}$.

If all ψ_j belong to $\mathfrak{H}_{-1}(A_0)$, then the extended functionals $\langle\psi_j^{\mathrm{ex}},\cdot\rangle$ are determined by continuity onto $\mathcal{D}(A_{\mathrm{sym}}^*)$ and they automatically possess the property of $\xi_j(t)$-invariance (4.16), since $U_t \upharpoonright_{\mathcal{D}(A_0)}$ can be extended by continuity onto $\mathfrak{H}_1(A_0)$. In this case, the set of admissible large coupling limits consists of a unique element (the Friedrichs extension A_F, see Corollary 3.7) and this operator is $p(t)$-homogeneous.

If $\mathfrak{H}_{-1}(A_0)$ does not contain all ψ_j, then admissible large coupling limits A_∞ of (1.3) are not determined uniquely. In this case, the natural assumption of $\xi_j(t)$-invariance for the extended functionals $\langle\psi_j^{\mathrm{ex}},\cdot\rangle$ can be used to select a unique operator A_∞. By Theorem 4.8 the $\xi_j(t)$-invariance of $\langle\psi_j^{\mathrm{ex}},\cdot\rangle$ is equivalent to the $p(t)$-homogeneity of the corresponding operator A_∞ defined by (3.3). Therefore, instead of assumption of $\xi_j(t)$-invariance one can use the requirement of $p(t)$-homogeneity imposed on the set of admissible large coupling limits A_∞ of (1.3) to achieve their uniqueness.

Theorem 4.13. *Assume that the singular elements ψ_j in* (1.3) *are $\mathfrak{H}_{-1}(A_0)$-independent and the system of equations* (4.23) *has a unique solution $\mathbf{R}=(r_{ij})_{i,j=1}^n$ that does not depend on $t\in\mathfrak{T}$. Then there exists a unique $p(t)$-homogeneous admissible large coupling limit A_∞ of* (1.3) *and it coincides with the Krein–von Neumann extension A_N of A_{sym}.*

Proof. Let $\mathbf{R}=(r_{ij})_{i,j=1}^n$ be a unique solution of (4.23) and let A_∞ be the corresponding self-adjoint extension of A_{sym} determined by (3.3).

Since (4.23) has a unique solution, $p(t)\neq 1$ for at least one point $t\in\mathfrak{T}$ (see Remark 4.11). In this case, Lemma 4.6 and relation (3.3) imply that A_∞ is a nonnegative extension of A_{sym} transversal to A_0. Then also A_F and A_N are transversal extensions of A_{sym}; cf. the proof of Theorem 3.10. These extensions are also $p(t)$-homogeneous (see Lemmas 4.7, 4.4).

Since elements ψ_j in (1.3) form an $\mathfrak{H}_{-1}(A_0)$-independent system, Corollary 3.8 gives that any self-adjoint extension of A_{sym} transversal to A_0 is an admissible large coupling limit of (1.3) and $A_0=A_F$. The unique solution of (4.23) allows one to select a unique $p(t)$-homogeneous self-adjoint extension A_∞ of A_{sym} transversal to $A_0=A_F$. Obviously, it coincides with the Krein–von Neumann extension A_N. □

The next statement concerns to the general case.

Theorem 4.14. *Let A_F and A_N be transversal, let the operator S defined in* (3.10) *be $p(t)$-homogeneous for some choice of $\widetilde{\mathcal{H}}$ satisfying conditions* (3.12)*, and assume that for every $\beta_{ij}(t)$ in* (4.23) *there exists at least one point $t_{ij}\in\mathfrak{T}$ such that $\beta_{ij}(t_{ij})\neq 0$. Then there exists a unique $p(t)$-homogeneous admissible large coupling limit of* (1.3).

Proof. Let $\widetilde{A}$ be the Krein–von Neumann extension of S. The second part of the proof of Theorem 3.10 shows that $\widetilde{A}$ is an admissible large coupling limit of (1.3). By Lemma 4.4, $\widetilde{A}$ is $p(t)$-homogeneous. Its uniqueness follows from the fact that condition $\beta_{ij}(t_{ij})\neq 0$ ensures in view of (4.23) the uniqueness of $p(t)$-homogeneous self-adjoint extensions of A_{sym} transversal to A_0. □

The next statement contains conditions for the $p(t)$-homogeneity of the symmetric operator S defined by (3.10) in Lemma 3.5 which appear to be useful in applications.

Proposition 4.15. *Let A_0 be $p(t)$-homogeneous, let the singular elements ψ_j in* (1.3) *be $\xi_j(t)$-invariant with respect to $\mathfrak{U}$, and let $\mathcal{Y} = (\mathbb{A}_0 + I)(\mathcal{H} \ominus \widetilde{\mathcal{H}})$. Then*:

(i) *S is $p(t)$-homogeneous if and only if $\mathcal{Y}$ is invariant under $\mathbb{U}_t$, $t \in \mathfrak{T}$, and*

$$\big(h', U_t\widetilde{h}^{\perp}\big) = 0, \quad \forall h' \in \mathcal{H}',\ \forall \widetilde{h}^{\perp} \in \mathcal{H} \ominus \widetilde{\mathcal{H}},\ \forall t \in \mathfrak{T}_0 = \big\{t \in \mathfrak{T}\colon\ p(t) \neq 1\big\}. \tag{4.24}$$

(ii) *If G_tU_t, $t \in \mathfrak{T}$, is self-adjoint, then S with $\widetilde{\mathcal{H}} = \mathcal{H}'$ is $p(t)$-homogeneous if and only if* (4.24) *holds.*

(iii) *If $\mathcal{Y}$ is a linear span of some singular elements ψ_j in* (1.3)*, then S is $p(t)$-homogeneous if and only if* (4.24) *holds.*

Proof. (i) The definition (3.10) shows that $\ker(S^* + I) = \mathcal{H} \ominus \widetilde{\mathcal{H}}$. Hence, if S is $p(t)$-homogeneous with respect to $\mathfrak{U}$ then $G_tU_t(\mathcal{H} \ominus \widetilde{\mathcal{H}}) = \mathcal{H} \ominus \widetilde{\mathcal{H}}$ by Corollary 4.3. According to (4.11) the subspace $\mathcal{H} \ominus \widetilde{\mathcal{H}}$ is invariant under G_tU_t if and only if $\mathcal{Y} = (\mathbb{A}_0 + I)(\mathcal{H} \ominus \widetilde{\mathcal{H}})$ is invariant under the operator $\mathbb{U}_t$, $t \in \mathfrak{T}$. Thus, if S is $p(t)$-homogeneous with respect to $\mathfrak{U}$ then $\mathcal{Y}$ is invariant under $\mathbb{U}_t$, $t \in \mathfrak{T}$.

By Lemma 4.7, A^*_{sym} is $p(t)$-homogeneous with respect to $\mathfrak{U}$. Since S is an intermediate extension of A_{sym} its $p(t)$-homogeneity is equivalent to the relation $U_{g(t)}(\mathcal{D}(S)) \subset \mathcal{D}(S)$, $t \in \mathfrak{T}$, see (4.7).

The definition of S in (3.10) implies that

$$U_{g(t)}f \in \mathcal{D}(S) \quad \Leftrightarrow \quad \big((A_F + I)U_{g(t)}f, \widetilde{h}^{\perp}\big) = 0, \quad \forall \widetilde{h}^{\perp} \in \mathcal{H} \ominus \widetilde{\mathcal{H}}. \tag{4.25}$$

Now let $f = h' + u \in \mathcal{D}(S)$ be decomposed as in Lemma 3.5, see (3.13), (3.14). It follows from (4.20) that

$$(A_F + I)U_{g(t)}f = \big(A^*_{\mathrm{sym}} + I\big)U_{g(t)}f = \big(1 - p(t)\big)U_{g(t)}h' + (A_0 + I)U_{g(t)}u.$$

By taking (4.12) into account one obtains

$$\begin{aligned}\big((A_F + I)U_{g(t)}f, \widetilde{h}^{\perp}\big) &= \big(1 - p(t)\big)\big(U_{g(t)}h', \widetilde{h}^{\perp}\big) + \big((A_0 + I)U_{g(t)}u, \widetilde{h}^{\perp}\big)\\ &= \big(1 - p(t)\big)\big(h', U_t\widetilde{h}^{\perp}\big) + \langle \mathbb{U}_t\psi, u\rangle.\end{aligned} \tag{4.26}$$

If $\mathcal{Y}$ is invariant under $\mathbb{U}_t$, $t \in \mathfrak{T}$, then $\langle \mathbb{U}_t\psi, u\rangle = 0$ for all $f = h' + u \in \mathcal{D}(S)$. Now (4.25) and (4.26) show that S is $p(t)$-homogeneous if and only if $\mathcal{Y}$ is invariant under $\mathbb{U}_t$ and (4.24) holds.

(ii) Since A_0 and A_F are $p(t)$-homogeneous, the symmetric restriction $S_0 := A_F \cap A_0$ and its adjoint S_0^* are also $p(t)$-homogeneous, see Lemma 4.2. It follows from (3.15) that $f \in \mathcal{D}(S_0)$ if and only if $f \in \mathcal{D}(A_0)$ and

$$\big((A_0 + I)f, h'\big) = 0, \quad \forall h' \in \mathcal{H}' = \mathcal{H} \cap \mathfrak{H}_1(A_0).$$

Hence, $\ker(S_0^* + I) = \mathcal{H}'$ and $G_tU_t\mathcal{H}' = \mathcal{H}'$ for all $t \in \mathfrak{T}$ by Corollary 4.3. Similarly $G_tU_t\mathcal{H} = \mathcal{H}$ for all $t \in \mathfrak{T}$, since A_{sym} is $p(t)$-homogeneous. Therefore, if G_tU_t is self-adjoint, then $\mathcal{H}$ and $\mathcal{H}'$ are reducing subspaces for the operators G_tU_t and consequently $G_tU_t\mathcal{H}'' \subset \mathcal{H}''$ is satisfied for all $t \in \mathfrak{T}$. Then, according to (4.11), $\mathcal{Y} = (\mathbb{A}_0 + I)\mathcal{H}''$ is invariant under $\mathbb{U}_t$. Now the claim follows from part (i) with $\widetilde{\mathcal{H}} = \mathcal{H}'$ and $\mathcal{H} \ominus \widetilde{\mathcal{H}} = \mathcal{H}''$.

(iii) If $\mathcal{Y}$ has a basis formed by some $\xi_j(t)$-invariant singular elements ψ_j, then $\mathcal{Y}$ is invariant under $\mathbb{U}_t$, see (1.9). So, the statement is reduced to (i). □

4.4. The case of rank one singular perturbations

In the case of rank one singular perturbations $A_0 + b\langle\psi,\cdot\rangle\psi$, where A_0 is $p(t)$-homogeneous and ψ is $\xi(t)$-invariant, the system (4.23) takes the form

$$\big(\xi^2(t) - p(t)\big)r = \xi(t)\big(1 - p(t)\big)(h, U_t h) \quad \big(h = (\mathbb{A}_0 + I)^{-1}\psi\big),\ \forall t \in \mathfrak{T}. \tag{4.27}$$

Proposition 4.16.

(1) *If* (4.27) *has no solutions $r \in \mathbb{R}$, then there is only one $p(t)$-homogeneous extension $A_0 = A_F = A_N$ and any self-adjoint extension of A_{sym} different from A_0 has a negative eigenvalue.*
(2) *If* (4.27) *has at least two solutions $r_1, r_2 \in \mathbb{R}$, then all self-adjoint extensions of A_{sym} are $p(t)$-homogeneous.*
(3) *If* (4.27) *has a unique solution $r \in \mathbb{R}$, then the symmetric operator A_{sym} associated with $A_0 + b\langle\psi,\cdot\rangle\psi$ possesses exactly two $p(t)$-homogeneous extensions: the Friedrichs A_F and the Krein–von Neumann A_N extensions. One of them coincides with A_0, another one is the unique $p(t)$-homogeneous admissible large coupling limit A_∞ of $A_0 + b\langle\psi,\cdot\rangle\psi$. More precisely, $A_0 = A_F$ and $A_\infty = A_N$ if $\psi \in \mathfrak{H}_{-2}(A_0) \setminus \mathfrak{H}_{-1}(A_0)$; $A_0 = A_N$ and $A_\infty = A_F$ if $\psi \in \mathfrak{H}_{-1}(A_0)$.*

Proof. In the case of rank one perturbations, an arbitrary self-adjoint extension $A(\neq A_0)$ of the symmetric operator $A_{\mathrm{sym}} = A_0 \upharpoonright \{u \in \mathcal{D}(A_0)\colon \langle\psi, u\rangle = 0\}$ is transversal to A_0. This means that there is a one-to-one correspondence between the set of solutions $r \in \mathbb{R}$ of (4.27) and the set of $p(t)$-homogeneous self-adjoint extensions $A(\neq A_0)$ of A_{sym}.

By Lemmas 4.4, 4.7 the symmetric operator A_{sym} and its Friedrichs A_F and Krein–von Neumann A_N extensions are $p(t)$-homogeneous. Therefore, if (4.27) has no solutions, then $A_N = A_F = A_0$ that justifies assertion (1).

Two different solutions of (4.27) may appear only in the case where $\xi^2(t) = p(t)$ and $(1 - p(t))(h, U_t h) = 0$ for all $t \in \mathfrak{T}$. But these equalities are equivalent to the fact that any $r \in \mathbb{R}$ is a solution of (4.27). Therefore, an arbitrary self-adjoint extension of A_{sym} is $p(t)$-homogeneous. Assertion (2) is proved.

Finally, assume that (4.27) has a unique solution. It follows from Corollary 4.10 that the set of all $p(t)$-homogeneous extensions of A_{sym} is exhausted by the Friedrichs A_F and the Krein–von Neumann A_N extensions. One of them coincides with A_0, another one is the unique $p(t)$-homogeneous admissible large coupling limit A_∞. To complete the proof it suffices to use Theorem 4.13 for $\psi \in \mathfrak{H}_{-2}(A_0) \setminus \mathfrak{H}_{-1}(A_0)$ and Corollary 3.7 for $\psi \in \mathfrak{H}_{-1}(A_0)$. □

Example 4.17. *One point interaction in $\mathbb{R}^n$* ($n = 1, 2, 3$). Consider the singular rank one perturbation $-\Delta + b\langle\delta,\cdot\rangle\delta(x)$, where $A_0 = -\Delta$ ($\mathcal{D}(A_0) = W_2^2(\mathbb{R}^n)$) is the Laplace operator in $\mathfrak{H} = L_2(\mathbb{R}^n)$ and the associated symmetric operator $A_{\mathrm{sym}} = -\Delta \upharpoonright \{u(x) \in W_2^2(\mathbb{R}^n)\colon u(0) = 0\}$.

The operator A_0 is t^{-2}-homogeneous with respect to the set of scaling transformations $\mathfrak{U} = \{U_t\}_{t\in(0,\infty)}$ in $L_2(\mathbb{R}^n)$, where $U_t f(x) = t^{n/2} f(tx)$. Furthermore, the singular element $\psi = \delta$ is $t^{-n/2}$-invariant (cf. [5]).

If $n = 1$, then $\delta(x) \in \mathfrak{H}_{-1}(A_0) = W_2^{-1}(\mathbb{R})$, Eq. (4.27) has a unique solution and by Proposition 4.16 the free Laplace operator $-\Delta$ coincides with the Krein–von Neumann extension A_N of A_{sym}. The Friedrichs extension A_F has the form $A_F = -d^2/dx^2 \restriction \{u(x) \in W_2^2(\mathbb{R} \setminus \{0\}) \cap W_2^1(\mathbb{R})\colon u(0) = 0\}$.

If $n = 2$, then (4.27) has no solutions and there exists the unique nonnegative self-adjoint extension $-\Delta = A_N = A_F$ of A_{sym}.

If $n = 3$, then $\delta(x) \in W_2^{-2}(\mathbb{R}^3) \setminus W_2^{-1}(\mathbb{R}^3)$, Eq. (4.27) has a unique solution and $-\Delta = A_F$. The Krein–von Neumann extension A_N has the form

$$A_N f(x) = -\Delta u(x) - u(0)\frac{e^{-|x|}}{|x|}, \quad \mathcal{D}(A_N) = \left\{ f = u(x) + u(0)\frac{e^{-|x|}}{|x|}\colon u \in W_2^2(\mathbb{R}^3) \right\}.$$

Another description of the Krein–von Neumann extension of A_{sym} obtained with the aid of the Fourier transformation can be founded in [12].

5. Operator realizations in the case of singular perturbations with symmetries

In this section, operator realizations $A_{\mathbf{B}}$ of (1.3) given by formulas (2.8), (2.9) are studied under the condition that the unperturbed operator A_0 and the singular elements ψ_j in (1.3) are, respectively, $p(t)$-homogeneous and $\xi_j(t)$-invariant with respect to $\mathfrak{U}$.

5.1. $p(t)$-Homogeneous operator realizations

Theorem 5.1. *Let an admissible large coupling limit A_∞ of* (1.3) *be chosen to be $p(t)$-homogeneous. Then the operator $A_{\mathbf{B}}$ defined by* (2.8) *is $p(t)$-homogeneous if and only if the relations*

$$\xi_i(t)\xi_j(t) = p(t), \quad \forall t \in \mathfrak{T},$$

hold for all indices $1 \leqslant i, j \leqslant n$ corresponding to non-zero entries b_{ij} of $\mathbf{B}$.

Proof. By Lemma 4.7, the operator A_{sym}^* is $p(t)$-homogeneous. Hence, in view of (4.7), $A_{\mathbf{B}}$ is $p(t)$-homogeneous if and only if $U_{g(t)} : \mathcal{D}(A_{\mathbf{B}}) \to \mathcal{D}(A_{\mathbf{B}})$, $\forall t \in \mathfrak{T}$. By (2.8), this relation can be rewritten as

$$\mathbf{B}\Gamma_0 U_{g(t)} f = \Gamma_1 U_{g(t)} f, \quad \forall t \in \mathfrak{T},\ \forall f \in \mathcal{D}(A_{\mathbf{B}}). \tag{5.1}$$

Since the admissible large coupling limit A_∞ is $p(t)$-homogeneous, the boundary operator Γ_0 satisfies (4.18). Therefore, $\mathbf{B}\Gamma_0 U_{g(t)} f = \mathbf{B}\boldsymbol{\Xi}(t)\Gamma_0 f$. On the other hand, relations (2.7) and (4.21) lead to the equality

$$\Gamma_1 U_{g(t)} f = p(t)\boldsymbol{\Xi}^{-1}(t)\Gamma_1 f, \quad \forall f \in \mathcal{D}\big(A_{\mathrm{sym}}^*\big). \tag{5.2}$$

The last two equalities and (2.8) show that the relation (5.1) is equivalent to the matrix equality $\boldsymbol{\Xi}(t)\mathbf{B}\boldsymbol{\Xi}(t) = p(t)\mathbf{B}$, $t \in \mathfrak{T}$. Rewriting this componentwise, one obtains the equalities $\xi_i(t)\xi_j(t)b_{ij} = p(t)b_{ij}$, $1 \leqslant i, j \leqslant n$. $\square$

Corollary 5.2. *If there exists a point $t_0 \in \mathfrak{T}$ such that $p(t_0) \neq 1$ and relations $\xi_i(t_0)\xi_j(t_0) = p(t_0)$ hold for all indices $1 \leqslant i, j \leqslant n$ corresponding to non-zero entries b_{ij} of* **B**, *then*: (i) *the point $\lambda = 0$ belongs to the essential spectrum of $A_{\mathbf{B}}$ and $\lambda \in \sigma(A_{\mathbf{B}}) \Leftrightarrow \lambda p(t_0)^n \in \sigma(A_{\mathbf{B}})$, $n \in \mathbb{Z}$*; (ii) *the operator $A_{\mathbf{B}}$ is nonnegative if and only if the matrix* **B** *is Hermitian.*

Proof. If the matrix **B** satisfies the conditions above, then $A_{\mathbf{B}}$ is $p(t)$-homogeneous with respect to the family $\mathfrak{U}_0 := \{U_t \in \mathfrak{U}\colon t \in \{t_0, g(t_0)\}\}$. Now, to establish (i), it suffices to use Lemma 4.1 with $A = A_{\mathbf{B}}$.

Obviously, the matrix **B** is Hermitian if and only if the operator $A_{\mathbf{B}}$ defined by (2.8) is self-adjoint. Using Lemma 4.6 and Theorem 5.1 one derives (ii). □

Proposition 5.3. *Assume that the singular elements ψ_j in* (1.3) *form a $\mathfrak{H}_{-1}(A_0)$-independent orthonormal system in $\mathfrak{H}_{-2}(A_0)$, the system* (4.23) *has a unique solution* **R**, *and a $p(t)$-homogeneous admissible large coupling limit A_∞ of* (1.3) *is chosen. Then a self-adjoint operator realization $A_{\mathbf{B}}$ of* (1.3) *is nonnegative if and only if* $\det(\mathbf{BR} + \mathbf{E}) \neq 0$ *and* $0 \leqslant -(\mathbf{BR} + \mathbf{E})^{-1}\mathbf{B} \leqslant -\mathbf{R}^{-1}$, *where* **E** *stands for the identity matrix.*

Proof. By Theorem 4.13, the Krein–von Neumann extension A_N of A_{sym} coincides with a $p(t)$-homogeneous admissible large coupling limit A_∞ and it is defined by (3.3), where **R** is the solution of (4.23). Furthermore, the Friedrichs extension A_F coincides with A_0. Combining these observations with [33, Theorem 3] the statement follows. For completeness some of the details are repeated here.

By (3.21) a self-adjoint operator $A_{\mathbf{B}}$ is nonnegative if and only if $-1 \in \rho(A_{\mathbf{B}})$ and

$$0 \leqslant C_{\mathbf{B}} \leqslant C_N, \tag{5.3}$$

where $C_{\mathbf{B}} = (A_{\mathbf{B}} + I)^{-1} - (A_0 + I)^{-1}$ and $C_N = (A_N + I)^{-1} - (A_0 + I)^{-1}$ are self-adjoint operators in $\mathcal{H} = \ker(A^*_{\mathrm{sym}} + I)$.

It follows from (2.7) and (2.8) that

$$\mathcal{D}(A_{\mathbf{B}}) = \bigl\{ f \in \mathcal{D}\bigl(A^*_{\mathrm{sym}}\bigr)\colon\ \mathbf{B}\widehat{\Gamma}_1 f = -(\mathbf{BR} + \mathbf{E})\widehat{\Gamma}_0 f \bigr\}. \tag{5.4}$$

Relations (2.5) and (5.4) imply $-1 \in \rho(A_{\mathbf{B}}) \Leftrightarrow \mathcal{D}(A_{\mathbf{B}}) \cap \mathcal{H} = \{0\} \Leftrightarrow \det(\mathbf{BR} + \mathbf{E}) \neq 0$. Since the elements ψ_j are orthonormal in $\mathfrak{H}_{-2}(A_0)$, the corresponding vectors h_j in (2.3) form an orthonormal basis of $\mathcal{H}$. In that case, the domain $\mathcal{D}(A_{\mathbf{B}})$ can be also presented as $\mathcal{D}(A_{\mathbf{B}}) = \{f \in \mathcal{D}(-\Delta^*_{\mathrm{sym}})\colon\ \mathbf{C}_{\mathbf{B}}\widehat{\Gamma}_1 f = \widehat{\Gamma}_0 f\}$, where $\mathbf{C}_{\mathbf{B}}$ is the matrix representation of $C_{\mathbf{B}}$ with respect to the basis $\{h_j\}_1^n$. Comparing this with (5.4) one gets $\mathbf{C}_{\mathbf{B}} = -(\mathbf{BR} + \mathbf{E})^{-1}\mathbf{B}$.

Similar reasonings for the operator A_N defined by (3.3) give $\det \mathbf{R} \neq 0$ (since $-1 \in \rho(A_N)$) and $\mathbf{C}_N = -\mathbf{R}^{-1}$. By substituting the obtained expressions for $\mathbf{C}_{\mathbf{B}}$ and $\mathbf{C}_N$ into (5.3) one completes the proof. □

Remark 5.4. A description of nonnegative self-adjoint operator realizations of (1.3) given above is based on the specific form of boundary operators Γ_i. A general approach to the description of nonnegative self-adjoint extensions of a symmetric operator has been proposed recently in [12].

5.2. The Weyl function and the resolvent formula

Let $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ be the boundary triplet of A_{sym}^* constructed in Lemma 2.2 and let A_∞ be a self-adjoint extension of A_{sym} defined by (3.3).

The γ-field $\gamma(z)$ and the Weyl function $\mathbf{M}(z)$ associated with the boundary triplet $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ are defined by

$$\gamma(z) = (\Gamma_0 \restriction \mathcal{H}_z)^{-1}, \qquad \mathbf{M}(z) = \Gamma_1 \gamma(z), \quad z \in \rho(A_\infty), \tag{5.5}$$

see [16,17]. Here $\mathcal{H}_z = \ker(A_{\mathrm{sym}}^* - zI)$, $z \in \mathbb{C}$, denote the defect subspaces of A_{sym}. The mappings Γ_i are defined by (2.5) and $\mathbf{M}(z)$ is an $n \times n$-matrix function.

Theorem 5.5. *The operator A_∞ is $p(t)$-homogeneous if and only if for at least one point $z = z_0 \in \mathbb{C} \setminus \mathbb{R}$ (and then for all non-real points z) the Weyl function $\mathbf{M}(z)$ satisfies the relation*

$$p(t)\mathbf{M}(z) = \boldsymbol{\Xi}(t)\mathbf{M}\big(p(t)z\big)\boldsymbol{\Xi}(t), \quad \forall t \in \mathfrak{T}, \tag{5.6}$$

where $\boldsymbol{\Xi}(t)$ is defined by (4.17).

Proof. Let $f_z \in \mathcal{H}_z$, $z \in \mathbb{C}$. Then Lemma 4.1 and relation (4.1) imply

$$U_{g(t)} f_z \in \ker\left(A_{\mathrm{sym}}^* - \frac{z}{p(g(t))} I\right) = \ker\big(A_{\mathrm{sym}}^* - p(t)zI\big) = \mathcal{H}_{p(t)z}. \tag{5.7}$$

Putting $f = f_z \in \mathcal{H}_z$ in (5.2), using (5.7), and observing that $\mathbf{M}(z)\Gamma_0 f_z = \Gamma_1 f_z$, $z \in \mathbb{C}$ (see (5.5)), one can rewrite (5.2) as follows:

$$\mathbf{M}\big(p(t)z\big)\Gamma_0 U_{g(t)} f_z = p(t)\boldsymbol{\Xi}^{-1}(t)\mathbf{M}(z)\Gamma_0 f_z. \tag{5.8}$$

If the identity (5.6) holds for some non-real $z = z_0$, then (5.8) implies that

$$\Gamma_0 U_{g(t)} f = \boldsymbol{\Xi}(t)\Gamma_0 f \tag{5.9}$$

for all $f = f_{z_0} \in \mathcal{H}_{z_0}$. Since $\mathbf{M}^*(z) = \mathbf{M}(\bar{z})$ [16] and hence, (5.6) holds for $\bar{z}_0$, the relation (5.9) is also true for $f = f_{\bar{z}_0} \in \mathcal{H}_{\bar{z}_0}$. Moreover, (5.9) holds for all $f \in \mathcal{D}(A_{\mathrm{sym}})$ since $\Gamma_0 f = \Gamma_0 U_{g(t)} f = 0$ by (1.4). Consequently, (5.9) is true on the domain $\mathcal{D}(A_{\mathrm{sym}}^*) = \mathcal{D}(A_{\mathrm{sym}}) \dot{+} \mathcal{H}_{z_0} \dot{+} \mathcal{H}_{\bar{z}_0}$. By Theorem 4.8 this provides the $p(t)$-homogeneity of A_∞.

Conversely, assume that A_∞ is $p(t)$-homogeneous. In this case, (5.9) holds for all $f \in \mathcal{D}(A_{\mathrm{sym}}^*)$ (see (4.18)). But then, for all non-real z and all $f_z \in \mathcal{H}_z$,

$$\begin{aligned}\mathbf{M}\big(p(t)z\big)\boldsymbol{\Xi}(t)\Gamma_0 f_z &\overset{(5.9)}{=} \mathbf{M}\big(p(t)z\big)\Gamma_0 U_{g(t)} f_z \overset{(5.7)}{=} \Gamma_1 U_{g(t)} f_z \\ &\overset{(5.2)}{=} p(t)\boldsymbol{\Xi}^{-1}(t)\Gamma_1 f_z = p(t)\boldsymbol{\Xi}^{-1}(t)\mathbf{M}(z)\Gamma_0 f_z\end{aligned}$$

that justifies (5.6). Theorem 5.5 is proved. □

Let $A_{\mathbf{B}}$ be a self-adjoint realization of (1.3) defined by (2.8). Then the resolvents of $A_{\mathbf{B}}$ and A_∞ are connected via Krein's formula

$$(A_{\mathbf{B}} - zI)^{-1} = (A_\infty - zI)^{-1} + \gamma(z)\big(\mathbf{B} - \mathbf{M}(z)\big)^{-1}\gamma(\bar{z})^*, \quad z \in \rho(A_{\mathbf{B}}) \cap \rho(A_\infty). \tag{5.10}$$

The explicit form of $\mathbf{M}(z)$ can be found as follows. By (2.7) it is easy to see that the Weyl functions $\mathbf{M}(z)$ and $\widehat{\mathbf{M}}(z)$ associated with the boundary triplets (2.5) and (2.6), respectively, are connected via the linear fractional transform

$$\mathbf{M}(z) = -\big(\mathbf{R} + \widehat{\mathbf{M}}(z)\big)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{5.11}$$

The boundary triplet (2.6) is one of the most used boundary triplets and the corresponding Weyl function $\widehat{\mathbf{M}}(z)$ is studied well. In particular, if the singular elements ψ_j in (1.3) form an orthonormal system in $\mathfrak{H}_{-2}$, then (see [16, Remark 4])

$$\widehat{\mathbf{M}}(z) = (z+1)P_{\mathcal{H}}\big[I + (z+1)(A_0 - zI)^{-1}\big]P_{\mathcal{H}}.$$

By combining this relation with (5.11) one gets an explicit form for $\mathbf{M}(z)$.

Example 5.6. *A point interaction for* p*-adic Schrödinger type operator*. Let p be a fixed prime number and let $\mathbb{Q}_{\mathsf{p}}$ be the field of p-adic numbers. The operation of differentiation is not defined in the p-adic analysis of complex-valued functions defined on $\mathbb{Q}_{\mathsf{p}}$ and the Vladimirov operator of the fractional p-adic differentiation

$$D^\alpha f(x) = \frac{\mathsf{p}^\alpha - 1}{1 - \mathsf{p}^{-1-\alpha}} \int_{\mathbb{Q}_{\mathsf{p}}} \frac{f(x) - f(y)}{|x-y|_{\mathsf{p}}^{1+\alpha}}\, d\mu(y), \quad \alpha > 0,$$

is used as an analog of it (see [27] for details). Here $|\cdot|_{\mathsf{p}}$ and $d\mu(y)$ are, respectively, the p-adic norm and the Haar measure on $\mathbb{Q}_{\mathsf{p}}$. The operator D^α is positive and self-adjoint in the Hilbert space $L_2(\mathbb{Q}_{\mathsf{p}})$ of complex-valued square integrable functions on $\mathbb{Q}_{\mathsf{p}}$. p-Adic Schrödinger-type operators with potentials $V(x): \mathbb{Q}_{\mathsf{p}} \to \mathbb{C}$ are defined as $D^\alpha + V(x)$.

Denote $\mathfrak{T} = \{t = \mathsf{p}^n\colon\ n \in \mathbb{Z}\}$ and consider a family $\mathfrak{U} = \{U_t\}_{t\in\mathfrak{T}}$ of unitary operators $U_t f(x) = t^{-1/2} f(tx)$ acting in $L_2(\mathbb{Q}_{\mathsf{p}})$. Obviously, U_t satisfies (1.7) with the function of conjugation $g(t) = 1/t$, cf. (4.2). It follows from [28] that $U_t D^\alpha = t^\alpha D^\alpha U_t$, $t \in \mathfrak{T}$. Hence, D^α is t^α-homogeneous with respect to $\mathfrak{U}$.

Since D^α is a p-adic pseudo-differential operator its domain of definition $\mathcal{D}(D^\alpha)$ need not contain functions continuous on $\mathbb{Q}_{\mathsf{p}}$ and, in general, it may happen that the formal expression

$$D^\alpha + b\langle\delta, \cdot\rangle\delta(x), \quad b \in \mathbb{R}, \tag{5.12}$$

and the associated symmetric operator $A_{\mathrm{sym}} = D^\alpha \restriction \{u(x) \in \mathcal{D}(D^\alpha)\colon\ u(0) = 0\}$ are not defined on $\mathcal{D}(D^\alpha)$. It is known [35] that the domain $\mathcal{D}(D^\alpha)$ consists of continuous functions on $\mathbb{Q}_{\mathsf{p}}$ and the Dirac delta function $\delta(x)$ is well defined on $\mathfrak{H}_2(D^\alpha) = \mathcal{D}(D^\alpha)$ if and only if $\alpha > 1/2$. Furthermore, $\delta(x)$ is $\sqrt{t}$-invariant with respect to $\mathfrak{U}$ and $\delta(x) \in \mathfrak{H}_{-2}(D^\alpha) \setminus \mathfrak{H}_{-1}(D^\alpha)$ if $1/2 < \alpha \leqslant 1$, while $\delta(x) \in \mathfrak{H}_{-1}(D^\alpha)$ if $\alpha > 1$.

It follows from [27, Lemma 3.7] and [35, Lemma 2.1] that

$$h(x) = \left(D^\alpha + I\right)^{-1}\delta = \sum_{N=-\infty}^{\infty}\sum_{j=1}^{\mathsf{p}-1}\mathsf{p}^{-N/2}\left[\mathsf{p}^{\alpha(1-N)}+1\right]^{-1}\psi_{Nj0}(x),$$

where the functions $\psi_{Nj0}(x)$ ($N \in \mathbb{Z}$, $j = 1, \dots, \mathsf{p}-1$) form a part of the p-adic wavelet basis $\{\psi_{Nj\epsilon}(x)\}$ recently constructed in [28].

Eq. (4.27) takes the form

$$\left(t - t^\alpha\right)r = \sqrt{t}\left(1 - t^\alpha\right)(h, U_t h), \quad \forall t \in \mathfrak{T}. \tag{5.13}$$

A simple analysis shows that (5.13) has no solutions $r \in \mathbb{R}$ for $\alpha = 1$. In that case the initial operator D^1 is a unique nonnegative self-adjoint extension of A_{sym}, see Proposition 4.16. If $\alpha \neq 1$ ($\alpha > 1/2$), then (5.13) has a unique solution $r \in \mathbb{R}$ that determines a unique t^α-homogeneous admissible large coupling limit A_∞ of (5.12) by the formula (cf. (3.3))

$$A_\infty f(x) = D^\alpha u(x) + \frac{u(0)}{r}h(x), \quad \mathcal{D}(A_\infty) = \left\{f = u(x) - \frac{u(0)}{r}h(x):\ u \in \mathcal{D}\left(D^\alpha\right)\right\}.$$

In view of Proposition 4.16, the operator A_∞ coincides with the Krein–von Neumann (Friedrichs) extension of A_{sym} for $1/2 < \alpha < 1$ (respectively for $\alpha > 1$).

Let $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ be the boundary triplet of A^*_{sym} constructed in Lemma 2.2 so that $\ker \Gamma_0 = \mathcal{D}(A_\infty)$. By Theorem 2.3, self-adjoint operator realizations of (5.12) in $L_2(\mathbb{Q}_\mathsf{p})$ have the form $A_b f = A_b(u + ch) = D^\alpha u - ch$, $\forall u \in \mathcal{D}(D^\alpha)$, where the parameter $c = c(u, b) \in \mathbb{C}$ is uniquely determined by the relation $bu(0) = -c[1 + br]$. Since $\xi^2(t) = t \neq t^\alpha = p(t)$ ($\alpha \neq 1$), Theorem 5.1 shows that A_b is t^α-homogeneous if and only if $b = 0$ or $b = \infty$.

Let $\alpha > 1$. It follows from [7] that the Weyl function associated with $(\mathbb{C}^n, \Gamma_0, \Gamma_1)$ has the form

$$\mathbf{M}(z) = -\frac{1}{(\mathsf{p}-1)\sum_{N=-\infty}^{\infty}\frac{\mathsf{p}^{-N}}{\mathsf{p}^{\alpha(1-N)}-z}}.$$

By virtue of Theorem 5.5, $\mathbf{M}(z)$ satisfies the relation $t^{\alpha-1}\mathbf{M}(z) = \mathbf{M}(t^\alpha z)$, $\forall t \in \mathfrak{T}$. This simplifies the spectral analysis of A_b, see [7] for details.

Example 5.7. *A general zero-range potential in* $\mathbb{R}$. A one-dimensional Schrödinger operator corresponding to a general zero-range potential at the point $x = 0$ can be given by the expression

$$A_0 + b_{11}\langle\delta, \cdot\rangle\delta(x) + b_{12}\langle\delta', \cdot\rangle\delta(x) + b_{21}\langle\delta, \cdot\rangle\delta'(x) + b_{22}\langle\delta', \cdot\rangle\delta'(x),$$

where $A_0 = -d^2/dx^2$ ($\mathcal{D}(A_0) = W_2^2(\mathbb{R})$) acts in $\mathfrak{H} = L_2(\mathbb{R})$, $\delta'(x)$ is the derivative of the Dirac δ-function (with support at 0).

In this case, $A_{\mathrm{sym}} = -d^2/dx^2 \restriction \{u(x) \in W_2^2(\mathbb{R}):\ u(0) = u'(0) = 0\}$ and the corresponding Friedrichs and Krein–von Neumann extensions are transversal (see, e.g., [10]). The functions

$$h'(x) = (\mathbb{A}_0 + I)^{-1}\psi_1 = \frac{1}{2}\begin{cases} e^{-x}, & x > 0, \\ e^{x}, & x < 0, \end{cases} \qquad h''(x) = (\mathbb{A}_0 + I)^{-1}\psi_2 = -(\operatorname{sign} x)h'(x),$$

where $\psi_1 = \delta(x)$ and $\psi_2 = \delta'(x)$, form an orthogonal basis of $\mathcal{H} = \ker(A^*_{\mathrm{sym}} + I)$ such that $\mathcal{H}' = \mathcal{H} \cap \mathfrak{H}_1(A_0) = \langle h'(x)\rangle$ and $\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}' = \langle h''(x)\rangle$.

Define $\mathfrak{U} = \{U_t\}_{t\in[0,\infty)}$ as a collection of *the space parity operator* $U_0 f(x) = f(-x)$ $(f(x) \in L_2(\mathbb{R}))$ and the set of *scaling transformations* $U_t f(x) = \sqrt{t} f(tx)$, $t > 0$. In this case, A_0 is $p(t)$-homogeneous with respect to $\mathfrak{U}$, where $p(0) = 1$ and $p(t) = t^{-2}$ if $t > 0$. The elements ψ_j $(j = 1, 2)$ are $\xi_j(t)$-invariant, where $\xi_1(0) = 1$, $\xi_1(t) = t^{-1/2}$ $(t > 0)$ and $\xi_2(0) = -1$, $\xi_2(t) = t^{-3/2}$ $(t > 0)$. Furthermore, for such a choice of $\mathfrak{U}$, $\mathfrak{T}_0 = \{t \in [0,\infty)\colon\ p(t) \neq 1\} = (0,\infty)$ and

$$(h', U_t h'') = t^{1/2} \int_{-\infty}^{\infty} h'(x)\overline{h''(tx)}\, dx = 0, \qquad \forall t \in \mathfrak{T}_0.$$

Let us put $\widetilde{\mathcal{H}} = \mathcal{H}'$. Then $\mathcal{Y} = (\mathbb{A}_0 + I)\mathcal{H}'' = \langle\psi_2\rangle$ and part (iii) of Proposition 4.15 implies that the corresponding operator S defined by (3.10) is $p(t)$-homogeneous. Calculating $\beta_{ij}(t)$ in (4.23) for $\xi_1(t)$, $\xi_2(t)$, and $p(t)$ as given above, it is easy to see that $\beta_{ij}(0) \neq 0$ if $i \neq j$ and $\beta_{ii}(t) \neq 0$ for all $t > 0$. In this case, by Theorem 4.14 there exists a unique $p(t)$-homogeneous admissible large coupling limit A_∞.

To identify A_∞ it suffices to determine the entries r_{ij} of $\mathbf{R}$ in (3.3) with the aid of (4.23): for $t = 0$, (4.23) takes the form $\begin{pmatrix} 0 & 2r_{12} \\ -2r_{21} & 0 \end{pmatrix} = 0$ and, hence, $r_{12} = r_{21} = 0$; on the other hand, for $t > 0$ calculating both sides of (4.23) leads to

$$t^{-3/2}(t-1)\begin{pmatrix} r_{11} & 0 \\ 0 & -r_{22} \end{pmatrix} = \left(1 - t^{-2}\right)\begin{pmatrix} \frac{\sqrt{t}}{2(1+t)} & 0 \\ 0 & \frac{\sqrt{t}}{2(1+t)} \end{pmatrix}$$

and thus $r_{11} = 1/2, r_{22} = -1/2$. Substituting the coefficients r_{ij} in (2.4) results in the well-known extensions of $\delta(x)$ and $\delta'(x)$ onto $\mathcal{D}(A^*_{\mathrm{sym}}) = W_2^2(\mathbb{R} \setminus \{0\})$ (see [5]):

$$\langle\delta_{\mathrm{ex}}, f\rangle = \frac{f(+0) + f(-0)}{2}, \qquad \langle\delta'_{\mathrm{ex}}, f\rangle = -\frac{f'(+0) + f'(-0)}{2}.$$

The corresponding operator A_∞ is the restriction of $-d^2/dx^2$ to $\mathcal{D}(A_\infty) = \{f(x) \in W_2^2(\mathbb{R} \setminus \{0\})\colon\ -f(-0) = f(+0),\ -f'(-0) = f'(+0)\}$ and A_∞ is transversal to the singular perturbations $A_{\mathbf{B}}$ of A_0 that are determined by (2.9).

It follows from Theorem 5.1 that $A_{\mathbf{B}}$ is t^{-2}-homogeneous with respect to the scaling transformations U_t $(t > 0)$ if and only if $\mathbf{B} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}$. In that case $A_{\mathbf{B}} = A^*_{\mathbf{B}}$ (i.e., $b_{21} = \overline{b_{12}}$) $\Leftrightarrow A_{\mathbf{B}} \geqslant 0$ (by Corollary 5.2).

6. Schrödinger operators with singular perturbations $\xi(t)$-invariant with respect to scaling transformations in $\mathbb{R}^3$

It is well known (see, e.g., [5,13]) that the Schrödinger operator $A_0 = -\Delta$ $(D(\Delta) = W_2^2(\mathbb{R}^3))$, is t^{-2}-homogeneous with respect to the set of scaling transformations $\mathfrak{U} = \{U_t\}_{t\in(0,\infty)}$ $(U_t f(x) = t^{3/2} f(tx))$ in $L_2(\mathbb{R}^3)$. It is clear that U_t satisfies (1.7) with the function of conjugation $g(t) = 1/t$.

The elements U_t of $\mathfrak{U}$ possess the additional multiplicative property $U_{t_1}U_{t_2} = U_{t_2}U_{t_1} = U_{t_1t_2}$ that enables one to describe all measurable functions $\xi(t)$ for which there exist $\xi(t)$-invariant singular elements $\psi \in W_2^{-2}(\mathbb{R}^3)$.

Theorem 6.1. *Let $\xi(t)$ be a real measurable function defined on $(0,\infty)$. Then $\xi(t)$-invariant singular elements $\psi\in W_2^{-2}(\mathbb{R}^3)\setminus L_2(\mathbb{R}^3)$ exist if and only if $\xi(t)=t^{-\alpha}$, where $0<\alpha<2$.*

Proof. Let $\psi\in W_2^{-2}(\mathbb{R}^3)\setminus L_2(\mathbb{R}^3)$ be $\xi(t)$-invariant with respect to $\mathfrak{U}$. Since $U_{t_1}U_{t_2}=U_{t_2}U_{t_1}=U_{t_1t_2}$, equality (1.9) gives $\xi(t_1)\xi(t_2)=\xi(t_1t_2)$ $(t_i>0)$ that is possible only if $\xi(t)=0$ or $\xi(t)=t^{-\alpha}$ $(\alpha\in\mathbb{R})$ [24, Chapter IV]. Furthermore, Proposition 4.5 enables one to restrict the set of possible functions $\xi(t)$ as follows: $\xi(t)=t^{-\alpha}$, where $0<\alpha<2$.

To complete the proof of Theorem 6.1 it suffices to construct $t^{-\alpha}$-invariant singular elements for $0<\alpha<2$.

Fix $m(w)\in L_2(S^2)$, where $L_2(S^2)$ is the Hilbert space of square-integrable functions on the unit sphere S^2 in $\mathbb{R}^3$, and determine the functional $\psi(m,\alpha)\in W_2^{-2}(\mathbb{R}^3)$ by the formula

$$\langle\psi(m,\alpha),u\rangle=\int_{\mathbb{R}^3}\frac{m(w)}{|y|^{3/2-\alpha}(|y|^2+1)}\big(|y|^2+1\big)\widehat{u}(y)\,dy\quad\big(y=|y|w\in\mathbb{R}^3\big),\tag{6.1}$$

where $\widehat{u}(y)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}e^{ix\cdot y}u(x)\,dx$ is the Fourier transformation of $u(\cdot)\in W_2^2(\mathbb{R}^3)$.

It is easy to verify that

$$(\widehat{U_{g(t)}u})(y)=(\widehat{U_{1/t}u})(y)=\frac{1}{(2\pi t)^{3/2}}\int_{\mathbb{R}^3}e^{iy\cdot x}u(x/t)\,dx=U_t\widehat{u}(y)=t^{3/2}\widehat{u}(ty).\tag{6.2}$$

Using (6.1) and (6.2), one obtains $\langle\psi(m,\alpha),U_{g(t)}u\rangle=t^{-\alpha}\langle\psi(m,\alpha),u\rangle$ for all $u\in W_2^2(\mathbb{R}^3)$. By (4.15) this means that the functional $\psi(m,\alpha)$ is $t^{-\alpha}$-invariant with respect to $\mathfrak{U}$. Theorem 6.1 is proved. □

A more detailed study of functionals that are $t^{-\alpha}$-invariant with respect to scaling transformations and the results of [38] lead to the conclusion that the collection $\mathcal{L}_\alpha$ of all $t^{-\alpha}$-invariant singular elements $\psi\in W_2^{-2}(\mathbb{R}^3)\setminus L_2(\mathbb{R}^3)$ can be described as follows: $\mathcal{L}_\alpha=\{\psi=\psi(m,\alpha)\colon m(w)\in L_2(S^2),\ m(w)\neq 0\}$.

Let us consider the formal expression

$$-\Delta+\sum_{i,j=1}^{n}b_{ij}\langle\psi_j,\cdot\rangle\psi_i,\quad b_{ij}\in\mathbb{C},\ n\in\mathbb{N},\tag{6.3}$$

where all singular elements ψ_j are assumed to be $t^{-\alpha}$-invariant with respect to scaling transformations for a fixed α, i.e., $\psi_j=\psi(m_j,\alpha)$. The symmetric operator $A_{\mathrm{sym}}=-\Delta_{\mathrm{sym}}$ associated with (6.3) takes the form

$$-\Delta_{\mathrm{sym}}=-\Delta\restriction_{\mathcal{D}(\Delta_{\mathrm{sym}})},\quad\mathcal{D}(\Delta_{\mathrm{sym}})=\big\{u(x)\in W_2^2\big(\mathbb{R}^3\big)\colon\langle\psi_j,u\rangle=0,\ 1\leqslant j\leqslant n\big\},\tag{6.4}$$

where $\langle\psi_j,u\rangle$ are defined by (6.1).

Comparing (1.2) and (6.1), one sees that the functions $h_j=(\mathbb{A}_0+I)^{-1}\psi(m_j,\alpha)$ in (2.3) have the form

$$h_j(x)=\left(\frac{\overline{m_j(w)}}{|y|^{3/2-\alpha}(|y|^2+1)}\right)^{\vee}(x)=\overline{\left(\frac{m_j(w)}{|y|^{3/2-\alpha}(|y|^2+1)}\right)^{\wedge}(x)},\tag{6.5}$$

where the symbol $^\vee$ denotes the inverse Fourier transformation.

A simple analysis of (6.5) shows that $h_j \in L_2(\mathbb{R}^3) \setminus W_2^1(\mathbb{R}^3)$ for $1 \leqslant \alpha < 2$ and $h_j \in W_2^1(\mathbb{R}^3)$ for $0 < \alpha < 1$. In the latter case, Corollary 3.7 and Lemma 4.4 imply that the Friedrichs extension $-\Delta_F$ is a unique t^{-2}-homogeneous admissible large coupling limit of (6.3).

Proposition 6.2. *Let $1 < \alpha < 2$. Then the Krein–von Neumann extension $-\Delta_N$ of $-\Delta_{\mathrm{sym}}$ is a unique t^{-2}-homogeneous admissible large coupling limit of* (6.3).

Proof. If $1 < \alpha < 2$, then all the elements ψ_j in (6.3) are $W_2^{-1}(\mathbb{R}^3)$-independent. Let us show that the system (4.23) has a unique solution $\mathbf{R} = (r_{ij})_{i,j=1}^n$ that does not depend on $t > 0$. Since the both parts of (4.23) are equal to zero for $t = 1$, one can suppose that $t > 0$ and $t \neq 1$.

It follows from (6.2) and (6.5) that

$$\overline{U_t h_i(x)} = U_t\left(\frac{m_i(w)}{|y|^{3/2-\alpha}(|y|^2+1)}\right)^\wedge(x) = \left(U_{1/t}\frac{m_i(w)}{|y|^{3/2-\alpha}(|y|^2+1)}\right)^\wedge(x)$$
$$= t^{2-\alpha}\left(\frac{m_i(w)}{|y|^{3/2-\alpha}(|y|^2+t^2)}\right)^\wedge(x).$$

Hence,

$$(h_j, U_t h_i) = t^{2-\alpha}\int_{\mathbb{R}^3}\frac{m_i(w)\overline{m_j(w)}}{|y|^{3-2\alpha}(|y|^2+t^2)(|y|^2+1)}\,dy$$
$$= (m_i, m_j)_{L_2}\int_0^\infty \frac{t^{2-\alpha}}{|y|^{1-2\alpha}(|y|^2+t^2)(|y|^2+1)}\,d|y|$$
$$= c_\alpha \frac{t^\alpha - t^{2-\alpha}}{t^2-1}(m_i, m_j)_{L_2},$$

where $c_\alpha = \int_0^\infty \frac{|y|^{3-2\alpha}}{|y|^2+1}\,d|y|$ and $(m_i, m_j)_{L_2} = \int_{S^2} m_i(w)\overline{m_j(w)}\,dw$ is the scalar product in $L_2(S^2)$. Substituting the expression for $(h_j, U_t h_i)$ into (4.23) one gets a unique solution $\mathbf{R} = (r_{ij})_{i,j=1}^n$, where $r_{ij} = -c_\alpha(m_i, m_j)_{L_2}$. By Theorem 4.13, the obtained solution determines a unique t^{-2}-homogeneous admissible large coupling limit A_∞ of (6.3) that coincides with $-\Delta_N$. □

Remark 6.3. If $\alpha = 1$, then (4.23) has no solution, there are no t^{-2}-homogeneous admissible large coupling limits of (6.3), and the Friedrichs $-\Delta = -\Delta_F$ and the Krein–von Neumann $-\Delta_N$ extensions of $-\Delta_{\mathrm{sym}}$ are not transversal.

Corollary 6.4. *For a fixed $1 < \alpha < 2$ assume that $\psi_j = \psi(m_j, \alpha)$ in* (6.3) *form an orthonormal system in $W_2^{-2}(\mathbb{R}^3)$ and self-adjoint operator realizations $A_{\mathbf{B}} = -\Delta_{\mathbf{B}}$ of* (6.3) *are defined by*

(2.8) *with* $\ker\Gamma_0=\mathcal{D}(-\Delta_N)$. *Then* $-\Delta_{\mathbf{B}}$ *is nonnegative if and only if* $\det(\beta_\alpha\mathbf{B}-\mathbf{E})\neq 0$ *and* $0\leqslant\beta_\alpha\mathbf{B}[\beta_\alpha\mathbf{B}-\mathbf{E}]^{-1}\leqslant\mathbf{E}$, *where*

$$\beta_\alpha=\left[\int_0^\infty\frac{|y|^{3-2\alpha}}{|y|^2+1}\,d|y|\right]\left[\int_0^\infty\frac{1}{|y|^{1-2\alpha}(|y|^2+1)^2}\,d|y|\right]^{-1}. \tag{6.6}$$

Proof. Since $\psi(m_j,\alpha)$ are orthonormal in $W_2^{-2}(\mathbb{R}^3)$ the functions $h_j(x)$ determined by (6.5) are orthonormal in $L_2(\mathbb{R}^3)$. This means that

$$(m_i,m_j)_{L_2}=0\quad(i\neq j)\quad\text{and}\quad(m_i,m_i)_{L_2}\int_0^\infty\frac{1}{|y|^{1-2\alpha}(|y|^2+1)^2}\,d|y|=1.$$

The obtained relations allow one to rewrite the unique solution $\mathbf{R}=-c_\alpha((m_i,m_j)_{L_2})_{i,j=1}^n$ of (4.23) in a more explicit form: $\mathbf{R}=-\beta_\alpha\mathbf{E}$, where β_α is defined by (6.6). Using Proposition 5.3 one completes the proof. □

Note that the delta function $\delta(\cdot)$ belongs to $\mathcal{L}_{3/2}$. For this reason, the expression (6.3) where all $\psi_j\in\mathcal{L}_{3/2}$ can be considered as a generalization of the classical one-point interaction $-\Delta+b\langle\delta,\cdot\rangle\delta$. In that case the parameter β_α in Corollary 6.4 can be easily calculated: $\beta_{3/2}=2$.

Theorem 6.5. *Let* $\alpha=3/2$. *Then for any self-adjoint operator realization* $A_{\mathbf{B}}=-\Delta_{\mathbf{B}}$ *of* (6.3) *defined by* (2.8)*, the following statements are true*:

(i) *if* $-\Delta_{\mathbf{B}}$ *is nonnegative, then the wave operators* $W_\pm=\lim_{t\to\pm\infty}e^{-it\Delta_{\mathbf{B}}}e^{i\Delta t}$ *exist and are unitary operators in* $L_2(\mathbb{R}^3)$;
(ii) *if* $-\Delta_{\mathbf{B}}$ *is nonnegative and the singular elements* $\psi_j=\psi(m_j,3/2)$ *in* (6.3) *form an orthonormal system in* $W_2^{-2}(\mathbb{R}^3)$, *then the S-matrix*

$$\mathbb{S}_{(-\Delta_{\mathbf{B}},-\Delta)}=FW_+^*W_-F^{-1}$$

(*F is the Fourier transformation in* $L_2(\mathbb{R}^3)$) *of the Schrödinger equation* $iu_t=-\Delta_{\mathbf{B}}u$ *coincides with the boundary value* $\mathbb{S}_{(-\Delta_{\mathbf{B}},-\Delta)}(\delta)$ ($\delta\in\mathbb{R}$) *of the contractive operator-valued function*

$$\mathbb{S}_{(-\Delta_{\mathbf{B}},-\Delta)}(z)=(\mathbf{E}-2iz\mathbf{B})(\mathbf{E}+2iz\mathbf{B})^{-1},\quad z\in\mathbb{C}_+, \tag{6.7}$$

analytic in the upper half-plane $\mathbb{C}_+$.

Proof. The statements follow from [34, Theorem 3.3] and [33, Section 4].

Remark 6.6. In [33] the expression (6.7) was obtained by using the Lax–Phillips scattering scheme. Another description of $\mathbb{S}_{(-\Delta_{\mathbf{B}},-\Delta)}(z)$ in terms of the Krein's resolvent formula was obtained in [1]. In that paper, the stationary scattering theory approach has been used.

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