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Spectral radius of Hadamard product versus conventional product for non-negative matrices

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ABSTRACT

We prove an inequality for the spectral radius of products of non-negative matrices conjectured by Zhan. We show that for all $n \times n$ non-negative matrices A and B, $\rho(A \circ B) \leqslant \rho((A \circ A)(B \circ B))^{1/2} \leqslant \rho(AB)$, in which $\circ$ represents the Hadamard product.
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We denote by A_{ij} the entry of a matrix A in position (i,j), and by $\circ$ the Hadamard product of two matrices of the same size, that is, the entrywise product $(A \circ B)_{ij} = A_{ij}B_{ij}$. We are concerned with entrywise positive and entrywise non-negative matrices, which we refer to as positive and non-negative matrices, denoted by $A > 0$ and $A \geqslant 0$, respectively.

The spectral radius $\rho(A)$ of a square complex matrix A is the largest modulus of the eigenvalues of A. The spectral radius is not submultiplicative: $\rho(AB) \leqslant \rho(A)\rho(B)$ does not hold in general, not even for non-negative matrices. A counterexample is given by the pair (see, e.g. [2, Section 5.6, Problem 19])

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Of course, $\rho(A^m) = \rho(A)^m$ for all square complex A and all $m = 1, 2, \ldots$

On the other hand, for non-negative A and B, the spectral radius is submultiplicative with respect to the Hadamard product: $\rho(A \circ B) \leqslant \rho(A)\rho(B)$ ([3, Observation 5.7.4]). This can be generalised to Hadamard products (and powers) of several matrices as follows [1]: for non-negative matrices

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$A_1, A_2, \ldots, A_n$ and non-negative α_i such that $\sum_i \alpha_i \geqslant 1$, $\rho(A_1^{\circ\alpha_1} \circ \cdots \circ A_n^{\circ\alpha_n}) \leqslant \rho(A_1)^{\alpha_1} \cdots \rho(A_n)^{\alpha_n}$, in which $A^{\circ\alpha}$ denotes the entrywise power.

Zhan has conjectured [4] that for pairs of square non-negative matrices, the spectral radius of the Hadamard product is always bounded above by the spectral radius of the (conventional) matrix product.

The purpose of this note is to prove Zhan's conjecture, and a little more:

Theorem 1. *For $n \times n$ non-negative matrices A and B,*

$$\rho(A \circ B) \leqslant \rho((A \circ A)(B \circ B))^{1/2} \leqslant \rho(AB).$$

Before proving the theorem, we note that there is no reasonable lower bound on $\rho(A \circ B)$ in terms of $\rho(AB)$. The pair A and B in (1) shows that $\rho(A \circ B)$ can be zero while $\rho(AB)$ is not. In fact, this can happen even when A and B both have non-zero spectral radius. For example, with

$$A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix},$$

A, B and $A \circ B$ all have spectral radius 1, while $AB = \begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix}$. Since $\det(AB) = 1$ and $\mathrm{Tr}(AB) = 2 + xy$, the spectral radius of AB can be arbitrarily large.

Our proof of Theorem 1 is based on the following representation for the spectral radius of a positive matrix, which is essentially the power method:

Lemma 1. *For $A \in \mathbb{M}_n$ such that $A > 0$, $\rho(A) = \lim_{m\to\infty}(\mathrm{Tr}A^m)^{1/m}$.*

Proof. By Perron's theorem [2, Theorem 8.2.11], the spectral radius of a positive matrix A is a simple and strictly dominant eigenvalue of A; that is, $\rho = |\lambda_1| > |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$. Thus, $(\mathrm{Tr}A^m)^{1/m} = \rho(1 + (\lambda_2/\rho)^m + \cdots + (\lambda_n/\rho)^m)^{1/m}$, which tends to ρ as m goes to infinity. □

It suffices to prove Theorem 1 for positive A and B. Because of the continuity of the spectral radius, the theorem follows for non-negative A and B as well.

Proof of Theorem 1 for positive A and B. We prove that the following inequalities hold for any positive integer k:

$$\mathrm{Tr}((A \circ B)^{2k}) \leqslant \mathrm{Tr}\left(((A \circ A)(B \circ B))^k\right) \leqslant \mathrm{Tr}((AB)^{2k}). \tag{2}$$

Taking the $(2k)$th root, taking the limit $k \to \infty$, and invoking the lemma the theorem follows from (2).

The left-hand side $\mathrm{Tr}((A \circ B)^{2k})$ can be written as a $2k$-fold sum:

$$\begin{aligned}\mathrm{Tr}((A \circ B)^{2k}) &= \sum_{i_1,i_2,\ldots,i_{2k}} (A_{i_1 i_2} B_{i_1 i_2})\ (A_{i_2 i_3} B_{i_2 i_3}) \cdots (A_{i_{2k} i_1} B_{i_{2k} i_1}) \\ &= \sum_{i_1,i_2,\ldots,i_{2k}} \left(A_{i_1 i_2} B_{i_2 i_3} \cdots A_{i_{2k-1} i_{2k}} B_{i_{2k} i_1}\right) \\ &\quad \times \left(B_{i_1 i_2} A_{i_2 i_3} \cdots B_{i_{2k-1} i_{2k}} A_{i_{2k} i_1}\right). \end{aligned} \tag{3}$$

Alternation of the A and B factors in the last line is intentional.

The expression (3) is an inner product between two vectors in $\mathbb{R}_+^{n^{2k}}$, one with entries $A_{i_1 i_2} B_{i_2 i_3} \cdots A_{i_{2k-1} i_{2k}} B_{i_{2k} i_1}$, and the other with entries $B_{i_1 i_2} A_{i_2 i_3} \cdots B_{i_{2k-1} i_{2k}} A_{i_{2k} i_1}$. One sees that these two vectors have the same sets of entries (as can be seen by performing a cyclic permutation on the indices $i_1, i_2, \ldots, i_{2k}$). Thus, in particular, both vectors have the same Euclidean norm. Applying the Cauchy-Schwarz inequality then gives

$$\begin{aligned}\mathrm{Tr}((A\circ B)^{2k}) &\leqslant \sum_{i_1,i_2,\ldots,i_{2k}} \left(A_{i_1i_2}B_{i_2i_3}\cdots A_{i_{2k-1}i_{2k}}B_{i_{2k}i_1}\right)^2 \\ &= \sum_{i_1,i_2,\ldots,i_{2k}} A^2_{i_1i_2}B^2_{i_2i_3}\cdots A^2_{i_{2k-1}i_{2k}}B^2_{i_{2k}i_1} \\ &= \mathrm{Tr}(((A\circ A)(B\circ B))^k). \end{aligned} \tag{4}$$

This proves the first inequality of the theorem.

Now consider $\mathrm{Tr}((AB)^{2k})$, which can be written as a $4k$-fold summation:

$$\begin{aligned}\mathrm{Tr}((AB)^{2k}) = &\sum_{\substack{i_1,i_2,\ldots,i_{2k}\\ j_1,j_2,\ldots,j_{2k}}} (A_{i_1i_2}B_{i_2i_3}\cdots A_{i_{2k-1}i_{2k}}B_{i_{2k}j_1}) \\ &\times (A_{j_1j_2}B_{j_2j_3}\cdots A_{j_{2k-1}j_{2k}}B_{j_{2k}i_1}).\end{aligned}$$

The crucial observation is that if we take all terms of this sum for which $i_1=j_1, i_2=j_2,\ldots, i_{2k}=j_{2k}$, then we obtain the right-hand side of (4). The remaining terms are of course all positive. Therefore, we find that $\mathrm{Tr}((AB)^{2k})$ is an upper bound on (4). This proves the second inequality of the theorem. □

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