Path-independence and closure operators with the anti-exchange property

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Abstract

Recently in Koshevoy (Math. Social Sci. 38 (1999) 35), it was established a connection between the theory of choice functions satisfying path-independence condition and closure operators with the anti-exchange property. Closure operators with the anti-exchange property are a combinatorial abstraction of usual convex hull closure in Euclidean spaces. Interest in these structures has its sources in different fields of mathematics. We demonstrate that path-independent choice functions provide another source for this structure. Specifically, we associate to a choice function \( f \) a collection of expanding maps. We prove that a function \( f \) is path-independent if and only if all the maps of this collection are coinciding anti-exchange closure operators. Consequences of such a characterization are demonstrated.

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1. Introduction

Among the various justifications considered by Arrow [2] for the condition of transitivity of a social preference is the argument that it ensures path-independence of the final choice from the path to it. Plott [11] developed this idea and considered the concept of “path-independence” of a choice function as a means of weakening the condition of rationality in a manner which preserves one of the key properties of rational choice, namely
that choice over any subset should be independent of the way the alternatives were initially divided up to consideration. More about path-independent choice functions may be found in a nice survey paper by Moulin [10], see also [8].

In [7] we have shown that extreme points under an anti-exchange closure operator form a path-independent choice function and any such function comes in such a form. Closure operators with the anti-exchange property are a combinatorial abstraction of usual convex hull closure in Euclidean spaces. Interest in these structures has its sources in different fields of mathematics.

Here, we provide another relation between path-independence and anti-exchange closure operators. Namely, we relate to a choice function $f$ a collection of expanding maps. We prove that a function $f$ is path-independent if and only if all the maps of this collection are coinciding anti-exchange closure operators.

2. Notations and definitions

A closure space $(E, \sigma)$ is a finite set $E$ endowed with a closure operator $\sigma$. A mapping $\sigma : 2^E \to 2^E$, $\sigma(\emptyset) = \emptyset$, is said to be a closure operator if the following holds:

CO1) $X \subseteq \sigma(X)$;
CO2) $\sigma(\sigma(X)) = \sigma(X)$;
CO3) $X \subseteq Y$ implies $\sigma(X) \subseteq \sigma(Y)$.

Subsets of $E$ of the form $X = \sigma(X)$ are said to be closed sets. Denote by $\mathcal{K}_\sigma$ the set of closed subsets of the closure space $(E, \sigma)$. Given a closure operator $\sigma$, a point $x \in A \subseteq E$ is said to be extreme to $A$ if $x \notin \sigma(A\setminus x)$. Denote by $ex_\sigma(A) = \{x \in A \mid x \notin \sigma(A\setminus x)\}$ the set of extreme points to $A$ with respect to $\sigma$. (Caution: for some closure operators the set of extreme points can be empty for some $A$.)

Any set system $(E, \mathcal{K})$ such that $\emptyset, E \in \mathcal{K}$ and $\mathcal{K}$ is closed under intersection is the set of closed sets of the closure operator $\sigma_{\mathcal{K}}(A) := \bigcap\{X : X \in \mathcal{K}, A \subseteq X\}$. Such set systems are called closure spaces. The set of closed sets of a closure operator (space) $\sigma$ forms a lattice $L(\sigma)$ with operations $A \land B = A \cap B$ and $A \lor B = \sigma(A \cup B)$.

On the other hand, any lattice $L$ forms a closure space $(J(L), \sigma)$, where $J(L)$ denotes the set of join-irreducible elements of $J$, and $\sigma(A) = \{j \in J \mid j \leq \bigvee_{a \in A} a, A \subseteq J\}$.

A closure operator $\sigma$ on a finite set $E$ is said to satisfy the anti-exchange property if for a $\sigma$-closed set $A$ and for any different $x, y \notin A$, if $x \in \sigma(A \cup y)$ holds, then $y \notin \sigma(A \cup x)$. This property is a combinatorial abstraction of the convex hull operator in usual Euclidean spaces.

If $x$ and $y$ are two points and $A$ is a set such that $x$ is not in the convex hull of $A$ but is in the convex hull of $A \cup \{y\}$, then $y$ is not in the convex hull of $A \cup \{x\}$.

For a closure operator $\sigma$ with the anti-exchange property, the set $ex_\sigma(A)$ is always non-empty with a non-empty $A$ and is the minimal spanning set of $A$, $\sigma(A) = \sigma(ex_\sigma(A))$.

The lattice of a closure operator with the anti-exchange property is meet-distributive and any meet-distributive lattice comes out to be the lattice of closed sets of an operator with the anti-exchange property [4].
The closure space with the anti-exchange closure operator is called an abstract convex geometry. Equivalently, an abstract convex geometry can be defined as a collection $\mathcal{K}$ of subsets of $E$ with the following properties: $\mathcal{K}$ is closed under intersection, $\emptyset, E \in \mathcal{K}$, and, for any $A \in \mathcal{K}$, there exists $x \in E \setminus A$ with $A \cup x \in \mathcal{K}$.

3. Choice functions and expanding maps

Let $f$ be a choice function on a finite set $E$, that is $f : 2^E \to 2^E$, $f(A) \subseteq A$, $f(A) = \emptyset \Rightarrow A = \emptyset$. In other words, $f$ is a shrinking operator sending a non-void set to a non-empty subset.

Given a choice function $f$, define the following increasing collection of expanding maps $\tau^i_f : 2^E \to 2^E$, $i = 1, \ldots, |E|$, by the rule:

$$\tau^i_f(A) = \bigcup \{A' : f(A') = f(A), |A' \Delta A| \leq i\}.$$ 

For any $A \subseteq E$, there holds $A \subseteq \tau^i_f(A)$ and $\tau^i_f(A) \subseteq \tau^{i+1}_f(A)$.

The meaning of $\tau^i_f(A)$ is the following. We take the union of all sets $A'$ that have the same choice set as $A$, $f(A') = f(A)$, and that have at most $i$ elements outside $A$. This union obviously contains $A$, therefore $\tau^i_f$ is an expanding map, $A \subseteq \tau^i_f(A)$. To simplify notations, denote $\kappa_f(A) := \tau^{|E|}_f(A)$.

For a path-independent choice function, the following properties hold ([7], see also [9]): $f(\kappa_f(A)) = f(A)$, $f(A') = f(A)$ for any $A'$ such that $A \subseteq A' \subseteq \kappa_f(A)$, and, moreover, $\kappa_f$ is a closure operator with the anti-exchange property.

4. Path-independent choice functions

Here, we will show that a choice function $f$ is path-independent if and only if all maps $\tau^i_f$ coincide and are anti-exchange closure operators.

Recall that a choice function $f$ satisfies the path-independence condition, if for any subsets $A$ and $B$ of $E$

$$f(A \cup B) = f(f(A) \cup B).$$

Here are some examples of path-independent choice functions.

Let $\leq$ be a linear order on $E$. The following choice function $f(A) = \{\max(A), \min(A)\}$ satisfies path-independence.

More generally, let $(\leq_i, i \in I)$ be a family of linear orders on $E$. Then a joint-extremal choice function $f$ is given by the union of choices made under maximization for each individual order $\leq_i$. That is

$$f(A) = \bigcup_{i \in I} \text{argmax}(\leq_i |A),$$

where $\text{argmax}(\leq |A) = \{a \in A : a' \leq a \forall a' \in A \setminus a\}$. It is easy to check that any joint-extremal choice function is path-independent (see, for example, [8]).
The following theorem provides a characterization of path-independent choice functions via the associated operators $\tau_f$.

**Theorem 1.** A choice function $f : 2^E \rightarrow 2^E$ is path-independent if and only if $\tau_f, i = 1, \ldots, |E|$ are coinciding closure operators with the anti-exchange property.

**Proof.** ($\Rightarrow$) Let $f$ be path-independent. Show that $\tau_1^f(A) = \kappa_f(A)$. Recall that we denoted $\kappa_f(A) := |E|^1(A)$.

By the definitions, the inclusion $\tau_1^f(A) \subseteq \kappa_f(A)$ holds. Assume, for some $A$, $x \in \kappa_f(A) \setminus \tau_1^f(A)$. Then $A \subset A \cup x \subseteq \kappa_f(A)$, hence, $f(A \cup x) = f(A)$, and thus $x \in \tau_1^f(A)$. Therefore, for any $A \subseteq E$, we have $\tau_1^f(A) = \kappa_f(A)$. Since $\kappa_f$ is a closure operator with the anti-exchange property and, for any $i$ and $A$, we have $\tau_i^f(A) \subseteq \tau_{i+1}^f(A)$, the implication $\Rightarrow$ is proven.

For the reverse implication, we will prove that there holds $f = \operatorname{ex}_{\tau_f}(A)$, where $\operatorname{ex}_{\tau_f}(A) := \{x \in \tau_1^f(A) : x \notin \tau_1^f(A \cup x)\}$. In [7] it was shown that extreme points of a closure operator with the anti-exchange property satisfy path-independence. Therefore, the identity $f = \operatorname{ex}_{\tau_f}$ will imply path-independence of $f$. The equality

$$\tau_1^f(A) = \kappa_f(A) \quad (1)$$

implies that if $f(A^I) = f(A)$ holds, then $f(A \cup A^I) = f(A)$ holds true. In fact, because $A^I \subseteq \kappa_f(A)$ and $\kappa_f(A) = \tau_1^f(A)$, we have $A^I \subseteq \tau_1^f(A)$. Therefore, for any $y \in E^I$, there holds $f(A \cup y) = f(A)$, and so $A^I \subseteq \kappa_f(A \cup y) = \tau_1^f(A \cup y)$. The identity $\kappa_f(A \cup y) = \tau_1^f(A \cup y)$ implies that for any $y^I \in A^I$ we have $f(A \cup y \cup y^I) = f(A \cup y) = f(A)$, and, because of monotonicity, $A^I \subseteq \kappa_f(A \cup y \cup y^I)$ holds. Continuing this procedure with other elements of $A^I$, we obtain $f(A \cup A^I) = f(A)$, and so $f(A) = f(\kappa_f(A))$ holds. Because of this and (1), we get $f(A) = f(\tau_1^f(A))$.

For the anti-exchange closure operator $\tau_1^f$, there holds $\tau_1^f(\operatorname{ex}_{\tau_f}(A)) = \tau_1^f(A)$, $A \subseteq E$. Therefore, for any $A \subseteq E$, we have

$$f(A) = f(\tau_1^f(A)) = f(\tau_1^f(\operatorname{ex}_{\tau_f}(A))) = f(\operatorname{ex}_{\tau_f}(A)). \quad (2)$$

Show that $f$ is idempotent, i.e. $f^2(A) := f(f(A)) = f(A)$. Assume not, then there exists a set $A$ such that $A \supset f(A) \supset f(f(A)) = f(f(f(A)))$, i.e. $A \neq f(A) \neq f(f(A))$. Then, because of (2), we have $f(f^2(A)) = f^2(A) = f(\operatorname{ex}_{\tau_f}(f^2(A)))$. Because both $f$ and $\operatorname{ex}_{\tau_f}$ send a set to a subset, we have $f^2(A) = \operatorname{ex}_{\tau_f}(f^2(A))$. Because of the identity (1), the inclusion $f(A) \subseteq \tau_1^f(f^2(A))$ holds. The latter, due to idempotence of the closure operator, implies the inclusion $\tau_1^f(f(A)) \subseteq \tau_1^f(f^2(A))$. Let $x \in f(A) \setminus f^2(A)$. Then $f^2(A) \subseteq \tau_1^f(f^2(A))$.

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1 For any $A \subseteq E$, let us consider a sequence of nested sets $A \subseteq f(A) \subseteq f(f(A)) \subseteq \cdots$. Because the set $E$ is finite, any such sequence has a finite number of distinct sets. If for any $A$, such distinct sets less or equal 2, then $f$ is idempotent. If there exists $A$ with at least 3 distinct sets in the corresponding sequence, we can consider a subsequence which contains exactly 3 distinct sets. Obviously these sets should be adjoint sets in the sequence.
Therefore, $f(A) \backslash x$ holds true. Because of $f^2(A) = \text{ex}_{\tau_f^1}(f^2(A))$, we have $\tau_f^1(f(A) \backslash x) \supseteq \tau_f^1(f^2(A))$, and due to monotonicity of the closure operator and $\tau_f^1(f(A) \subseteq \tau_f^1(f^2(A))$, we obtain the equality $\tau_f^1(f(A) \backslash x) = \tau_f^1(f^2(A))$. Therefore, the inclusion $x \in \tau_f^1(f(A) \backslash x)$ holds.

Because of the monotonicity of the closure operator, we have $x \in \tau_f^1(f(A) \backslash x) \subseteq \tau_f^1(A \backslash x)$. Therefore, $x \notin \text{ex}_{\tau_f^1}(A)$, that contradicts to that $x$ was chosen of $f(A)$. Because $f(A)$ is a subset of extreme points of $A$ (with respect to $\tau_f^1$), $f(A) = f(\text{ex}_{\tau_f^1}(A)) \subseteq \text{ex}_{\tau_f^1}(A)$. Thus, $f(f(A)) = f(A)$ holds for any $A \subseteq E$.

Because of (2) and $f(f(A)) = f(A)$, $\text{ex}_{\tau_f^1}(A) \subseteq \kappa_f(f(A))$ holds. Therefore, we have

$$\text{ex}_{\tau_f^1}(A) \subseteq \kappa_f(f(A)) \subseteq \kappa_f(A).$$

Because of monotonicity and idempotence of $\kappa_f$ and since $\kappa_f(A) = \kappa_f(\text{ex}_{\tau_f^1}(A))$, we conclude that $\kappa_f(A) = \kappa_f(f(A))$, and since $f(A) \subseteq \text{ex}_{\tau_f^1}(A)$, and $\text{ex}_{\tau_f^1}(A)$ is the minimal spanning set, we conclude, for any $A \subseteq E$ the equality $f(A) = \text{ex}_{\tau_f^1}(A)$. Thus, $f$ is path-independent. □

The following examples show that there exist choice functions which fail to be path-independent, but for which one of the operators $\tau_f^1$ and $\kappa_f$ is a closure operator with the anti-exchange property.

**Example 1.** Consider the following choice function $f$ on the set $E = \{a, b, c\}$, $f(A) = A$ for any $A \neq E$ and $f(E) = a$. The operator $\tau_f^1$ is an anti-exchange closure operator, but $\kappa_f$ fails to be closure operator. In fact, there holds $\tau_f^1(A) = A$ for all $A \subseteq E$ and, hence, $\mathcal{K}_{\tau_f^1} = 2^E$, while $\kappa_f(A) = A$ for $A \neq \{a\}$ and $\kappa_f(a) = E$ and, hence, $\mathcal{K}_{\kappa_f} = 2^E \backslash a$. The latter set is not closed under intersection, $\{ab\} \cap \{ac\} \notin \mathcal{K}_{\kappa_f}$, therefore, $\kappa_f$ fails to be a closure operator. $f$ fails to be path-independent.

**Example 2.** Consider the choice function $f$ of Example 2 [7]. $E = \{a, b, c\}$ and the choice function is given by $f(\{a, b\}) = a$, $f(\{a, c\}) = a$, $f(\{b, c\}) = b, f(\{a, b, c\}) = \{b, c\}$. $\kappa_f$ is a closure operator with the anti-exchange property, while $\tau_f^1$ fails to be a closure operator. In fact, there holds $\kappa_f(a) = \kappa_f(ab) = \kappa_f(ac) = \kappa_f(abc) = abc$, $\kappa_f(b) = \kappa_f(bc) = bc$, $\kappa_f(c) = c$.

While, we have $\tau_f^1(a) = \tau_f^1(ab) = abc$, and, because $f(abc) \neq a = f(ab) = f(ac)$, $\tau_f^1(ab) = ab$ and $\tau_f^1(ac) = ac$ hold. Therefore, $\tau_f^1(ab) \cap \tau_f^1(ac) = a \notin \mathcal{K}_{\tau_f^1}$ holds, and $\tau_f^1$ fails to be a closure operator. $f$ fails to satisfy path-independence, for instance $f(\{a, b, c\}) \neq f(f(a) \cup f(\{b, c\}))$.

5. **Path-independence, transitivity, choice lattices and shellability**

Because of Theorem 1, we can analyze path-independence in the framework of the pair of anti-exchange closure operators, $\tau_f^1$ and $\kappa_f$. 


5.1. Transitivity

Given a closure space \((E, \sigma)\), define a collection \(R_{\sigma, A} \subseteq E\) of preference relations. We specify the relation \(R_{\sigma, A}\) by its asymmetric part \(P_{\sigma, A}\) given by: for \(y \in E \setminus \sigma(A)\), \(x \neq y\) set

\[ xP_{\sigma, A}y \quad \text{iff} \quad x \in \sigma(A \cup y). \]

**Lemma 2.** A closure space \((E, \sigma)\) is an abstract convex geometry (that is \(\sigma\) is an anti-exchange closure) if and only if \(R_{\sigma, A}\) is quasi-transitive (\(xPy, yPz\) imply \(xPz\)) for any \(A \subseteq E\).

**Proof.** Because of monotonicity of closure operators, for \(x \neq z\), we have \(x \in \sigma(A \cup y) \subset \sigma(A \cup \sigma(A \cup z)) = \sigma(A \cup z)\). Anti-exchange property ensures that \(xP_{\sigma, A}y\) and \(yP_{\sigma, A}x\) do not hold simultaneously. Because of this, \(R_{\sigma, A}\) is quasi-transitive. On the other hand, quasi-transitivity implies that \(xP_{\sigma, A}y\) and \(yP_{\sigma, A}x\) do not hold simultaneously, i.e. \(\sigma\) is an anti-exchange closure operator. \(\square\)

As a consequence of Lemma 2 and Theorem 1, we obtain the following characterization of path-independence.

**Corollary 3.** A choice function \(f\) is path-independent if and only if, for any \(A \subseteq E\), the preferences \(R_{\kappa_f, A}\) and \(R_{\tau_f, A}\) coincide and are quasi-transitive.

5.2. Choice lattices

Let \(f : 2^E \to 2^E\) be a choice function. Define the following operations on the collection \(\mathcal{S}_f\) of idempotent sets with respect to \(f\), i.e. sets of the form \(A = f(A), A \subseteq E\):

\[
\begin{align*}
\sigma f(A) \land \kappa_f f(B) &= f(\kappa_f(A) \cap \kappa_f(B)), \\
\sigma f(A) \lor \kappa_f f(B) &= f(\kappa_f(A \cup B)), \\
\sigma f(A) \land \tau_f f(B) &= f(\tau_f(A) \cap \tau_f(B)), \\
\sigma f(A) \lor \tau_f f(B) &= f(\tau_f(A \cup B)).
\end{align*}
\]

We call the collection \(\mathcal{S}_f\) endowed with operations \(\land \kappa_f\) and \(\lor \kappa_f\) a choice lattice. \(\mathcal{S}_f\) endowed with operations \(\land \tau_f\) and \(\lor \tau_f\) also is another choice lattice.

Another consequence of Theorem 1 is the following:

**Corollary 4.** A choice function \(f\) is path-independent if and only if the collection \(\mathcal{S}_f\) endowed with operations \(\land \kappa_f\) and \(\lor \kappa_f\) is a meet-distributive lattice, \(\mathcal{S}_f\) endowed with operations \(\land \tau_f\) and \(\lor \tau_f\) also is a meet-distributive lattice, and the lattices coincide.

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2 This lemma is a variant of Theorem 2.3 in [3].
Remark. By methods of lattice theory, it was shown in [5], that, for a path-independent choice function \( f, \mathcal{S} \) endowed with the operations \( \bigwedge \kappa_f \) and \( f(A) \lor f(B) = f(A \cup B) \) is a meet-distributive lattice. For a path-independent \( f \), there holds \( f(A \cup B) = f(\kappa_f(A \cup B)) \).

5.3. Shellability

Let a set system \((E, \mathcal{K})\) be an abstract convex geometry, that is \((E, \mathcal{K})\) is a closure space with the anti-exchange property.

Consider a set \( A \in \mathcal{K}, A \neq E \). Let \( B \) cover \( A \) in the lattice \( L(\sigma_{\mathcal{K}}) \) of closed subsets, i.e. \( B \neq A \) and \( B \) is the minimal set of \( \mathcal{K} \) which contains \( A \) (because \( E \in \mathcal{K} \), such a set exists). Then there holds \(|B \setminus A| = 1\). In fact, assume two element \( x \neq y \in B \setminus A \). Then, because of monotonicity of closure operators, and closedness and minimality \( B \), we have \( \sigma_{\mathcal{K}}(A \cup x) = B \) and \( \sigma_{\mathcal{K}}(A \cup y) = B \). Thus, we have \( x, y \notin A, x \in B = \sigma_{\mathcal{K}}(A \cup y) \) and \( y \in B = \sigma_{\mathcal{K}}(A \cup x) \), that is not the case, because \( \sigma_{\mathcal{K}} \) satisfies the anti-exchange property.

If \( A \) is a minimal non-empty set of \( \mathcal{K} \). Then \(|A| = 1\). In fact, let \( x \) and \( y \in A \), then \( y \in \sigma(x \cup \emptyset) \) and \( x \notin \sigma(y \cup \emptyset) \), because \( \emptyset \) is a closed set, this is not the case.

Because of these two properties, an abstract convex geometry \((E, \mathcal{K})\) might be set as a collection of sets with the following properties:

CG1) \( \emptyset, E \in \mathcal{K} \); CG2) \( A \cap B \in \mathcal{K} \) with \( A, B \in \mathcal{K} \); CG3) for any \( S \in \mathcal{K} \) there exists a chain \( \emptyset = A_0 \subset A_1 \subset \ldots \subset A_{|S|} = S \) such that \( A_i \in \mathcal{K} \) and \( |A_i| = i \).

The property CG3 might be called the shellability property (see [6] for a case of antimatroids).

Lemma 5. Let \((E, \sigma)\) be an abstract convex geometry. Then, for a set \( S \subset E \), a point \( x \in S \) is an extreme point to \( S \), i.e., \( x \notin \sigma(S \setminus x) \), if and only if there exists a chain \( \emptyset = A_0 \subset A_1 \subset \ldots \subset A_{|\sigma(S)|} = \sigma(S) \) such that \( A_i \in \mathcal{K} \), \( |A_i| = i \), and \( x = \sigma(S) \setminus A_{|\sigma(S)|} \).

Proof. Let \( k = |\sigma(S)| \), and \( \emptyset = A_0 \subset A_1 \subset \ldots \subset A_{|\sigma(S)|} = \sigma(S) \) be a chain such that \( A_i \in \mathcal{K} \) and \( |A_i| = i \). Show that \( x = \sigma(S) \setminus A_{k-1} \) is an extreme point of \( S \). For check first that \( x \in S \) holds. Assume not, then there holds \( S \setminus x \subset A_{k-1} \). Because of monotonicity of \( \sigma \), we have \( \sigma(S) \subset A_{k-1} \) that is not the case. Thus, there holds \( x \notin S \). Then, again, \( S \setminus x \subset A_{k-1} \) holds, and monotonicity \( \sigma \) implies \( \sigma(S \setminus x) \subset A_{k-1} \), hence, \( x \notin \sigma(S \setminus x) \), i.e. \( x \in \text{ex}_{\sigma}(S) \).

Let \( x \) be an extreme point to \( S \), \( x \in \text{ex}_{\sigma}(S) \). Show that there holds

\[
\sigma(S) \setminus x = \sigma(S \setminus x). \tag{7}
\]

In fact, for any \( y \in \sigma(S) \setminus \sigma(S \setminus x), y \neq x \), we have \( y \in \sigma(S \setminus x) \cup x = \sigma(S) \). Because \( \sigma \) is a closure operator with the anti-exchange property, we have \( x \notin \sigma(S \setminus x) \cup y \). This implies that \( x \notin \sigma(S \setminus x) \cup (\sigma(S \setminus x)) \) and, hence, (7) holds. Because of property CG3, there exists a chain \( \emptyset = A_0 \subset A_1 \subset \ldots \subset A_{|\sigma(S \setminus x)|} = \sigma(S) \setminus x \) with \( A_i \in \mathcal{K} \) and \( |A_i| = i \). □
Given an abstract convex geometry \((E, \sigma)\), consider the set of full chains \(\emptyset = A_0 \subset A_1 \subset \cdots \subset A_{|E|} = E, A_i \in \mathcal{K}_\sigma \) and \(|A_i| = i\). Denote by \(\mathcal{P}_\sigma\) this set of chains. With a chain \(c := \emptyset = A_0 \subset A_1 \subset \cdots \subset A_{|E|} = E\) is associated a linear order on \(E\): \(A_1 \leq_c \{A_2 \setminus A_1\} \leq_c \cdots \leq_c \{E \setminus A_{|E| - 1}\}\). Because of Lemma 5, for any \(A \subset E\), there holds

\[
\text{ex}_\sigma(A) = \bigcup_{c \in \mathcal{P}_\sigma} \text{argmax}(\leq_c |A),
\]

where \(\text{argmax}(\leq |A)\) denotes the top element \(a \in A\) with respect to a linear order \(\leq\).

This establishes the following characterization of path-independence proposed in Aizerman and Malishevski [1] (see, also, [8]).

**Theorem 6.** A choice function \(f\) is path-independent if and only if

\[
f(A) = \bigcup_{\leq_i \in \mathcal{P}} \text{argmax}(\leq_i |A),
\]

with some collection \(\mathcal{P}\) of linear orders on \(E\).

**Proof.** Let \(f\) be path-independent. Then, because \(f = \text{ex}_{\kappa_f}\) and (8), (9) holds with \(\mathcal{P}_{\kappa_f}\), the set of linear orders corresponding the closure operator \(\kappa_f\) with the anti-exchange property.

On the other hand, let \(f\) be of the form (9) with some set \(\mathcal{P}\) of linear orders on \(E\). It is easy to check that such a choice function \(f\) is path-independent, see, for example, [8]. □

6. Conclusion

The main result of this paper is the characterization of a path-independent choice function via coincidence of two anti-exchange closure operators associated to a function. This allows to analyze path-independence in the framework of the pair of anti-exchange closure operators. Experiments show that the anti-exchange property of any pair of associated operators implies path-independence. We propose that the following conjecture is true.

**Conjecture.** A choice function \(f\) satisfies path independence if and only if the operators \(\tau^1_f\) and \(\kappa_f\) are anti-exchange closure operators.

It might be an interesting task to study choice functions which have at least one anti-exchange closure operator in the set of associated operators to the function. Example 1 shows that a choice function \(f\) with the anti-exchange closure operator \(\tau^1_f\) might fail to be path-independent. However, for example, such a function coincides with the path-independent choice function \(\text{ex}_{\tau^1_f}\) for subsets \(A \subseteq E\) which differ for at most one element of the indempotents of \(f\). \(f(A) = A\). Roughly speaking, such choice functions could make a “mistake” for choice sets being “far from obvious".
References