



# Algebraic $K$ -theory and abstract homotopy theory

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## Abstract

We decompose the  $K$ -theory space of a Waldhausen category in terms of its Dwyer–Kan simplicial localization. This leads to a criterion for functors to induce equivalences of  $K$ -theory spectra that generalizes and explains many of the criteria appearing in the literature. We show that under mild hypotheses, a weakly exact functor that induces an equivalence of homotopy categories induces an equivalence of  $K$ -theory spectra.

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## 1. Introduction

Quillen's higher algebraic  $K$ -theory provides a powerful and subtle invariant of rings and schemes. Waldhausen reformulated the definition and generalized the original input from algebra to homological algebra or homotopy theory; in place of exact categories, which are additive categories with a notion of exact sequence, Waldhausen's construction allows categories equipped with weak equivalences (quasi-isomorphisms) and a notion of cofibration sequence. Although

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designed to apply homotopy theory and  $K$ -theory to geometric topology, the added flexibility of Waldhausen  $K$ -theory turns out to be tremendously useful even when studying the original algebraic objects. For instance, the remarkable localization and Mayer–Vietoris theorems of Thomason and Trobaugh [18, 7.4, 8.1], which relate the  $K$ -theory of a scheme to the  $K$ -theories of open covers, depend on techniques possible only in Waldhausen’s framework. One of the most important of these techniques is the ability to change homological models, using different categories of complexes with equivalent  $K$ -theory. A central question then becomes when do different models yield the same  $K$ -theory [18, 1.9.9]? More generally, what is  $K$ -theory made of?

In terms of comparing different models, Waldhausen’s approximation theorem [20, 1.6.4] stands as the prototypical example of a  $K$ -theory equivalence criterion. Thomason and Trobaugh [18, 1.9.8] specialized Waldhausen’s approximation theorem to certain categories of complexes, where for appropriate complicial functors, an equivalence of derived categories implies an equivalence of  $K$ -theory. Based on this result and work of the Grothendieck school on  $K_0$ , they articulated the perspective that higher algebraic  $K$ -theory “essentially depends only on the derived category” [18, 1.9.9], with a caveat about choice of models. Indeed, Schlichting [16] subsequently constructed examples of Frobenius categories with abstractly equivalent derived categories but different algebraic  $K$ -theory groups. On the other hand, for the algebraic  $K$ -theory of rings, Dugger and Shipley [3] proved that an abstract equivalence of derived categories does imply a  $K$ -theory equivalence. Their argument relies on the folk theorem that a Quillen equivalence of model categories induces an equivalence of  $K$ -theory of appropriate Waldhausen subcategories. Toën and Vezzosi [19] generalized this from Quillen equivalences to equivalences on Dwyer–Kan simplicial localizations [4]. Other approaches have tried to construct higher algebraic  $K$ -theory directly from the derived category [14] (and sequels) or using the Heller–Grothendieck–Keller theory of “derivators” [9,8,10,11] in for example [12].

In this paper, we describe a precise relationship between the algebraic  $K$ -theory space and the Dwyer–Kan simplicial localization of the Waldhausen category. To any category with weak equivalences, the Dwyer–Kan simplicial localization associates simplicial mapping spaces that have the “correct” homotopy type [5] and that characterize the higher homotopy theory of the category [13, 5.7]. Our description elucidates the nature of the homotopical information encoded by  $K$ -theory, and leads to a very general criterion for functors to induce an equivalence of  $K$ -theory spectra, one that includes the approximation theorems above as special cases. We regard this decomposition as providing a conceptual explanation of the phenomena described in the preceding paragraphs.

**Theorem 1.1.** *Let  $\mathcal{C}$  be a Waldhausen category such that every map admits a factorization as a cofibration followed by a weak equivalence and assume that the weak equivalences satisfy the two out of three property. For  $n > 1$ , the nerve of  $\mathrm{wS}_n\mathcal{C}$  is weakly equivalent to the homotopy coend*

$$\mathrm{hocoend}_{(X_1, \dots, X_n) \in \mathrm{w}\mathcal{C}^n} LC(X_{n-1}, X_n) \times \cdots \times LC(X_1, X_2),$$

where  $LC$  denotes the Dwyer–Kan hammock localization.

The homotopy coend comes with a map to the classifying space  $B\mathrm{w}\mathcal{C}^n$ . We can identify this space and the homotopy fiber of the map intrinsically in terms of the Dwyer–Kan simplicial localization. For  $X$  an object in  $\mathcal{C}$ , let  $\mathrm{hAut} X$  denote the subspace of  $LC(X, X)$  consisting of the

components corresponding to the weak equivalences; precisely,  $\text{hAut } X$  consists of those components which have a vertex where all the forward maps are weak equivalences. Then  $\text{hAut}(X)$  is a grouplike monoid of homotopy automorphisms of  $X$  in  $LC$ .

**Theorem 1.2.** *Let  $\mathcal{C}$  be as in Theorem 1.1. For  $n \geq 1$ , the nerve of  $\text{wS}_n\mathcal{C}$  is weakly equivalent to the total space of a fibration where the base is the disjoint union of*

$$B \text{hAut } X_n \times \cdots \times B \text{hAut } X_1$$

*over  $n$ -tuples of weak equivalences classes of objects of  $\mathcal{C}$ , and the fiber is equivalent to*

$$LC(X_{n-1}, X_n) \times \cdots \times LC(X_1, X_2)$$

*for  $n > 1$  and contractible for  $n = 1$ .*

This description of the  $K$ -theory spaces may provide a replacement in the abstract setting for certain  $K$ -theory arguments that rely on the plus construction description, which is only available for the  $K$ -theory of rings or connective ring spectra. We expect this to apply to the study of Waldhausen’s chromatic convergence conjecture and related localization conjectures of Rognes. This is work in progress.

Currently, we can apply these theorems to the models question of Thomason and Trobaugh. We show that under mild hypotheses, a weakly exact functor that induces an equivalence of homotopy categories induces an equivalence of  $K$ -theory spectra. The hypotheses hold in particular in Waldhausen categories that come from model categories. The main hypothesis is that any map in  $\mathcal{C}$  admits a factorization as a cofibration followed by a weak equivalence; under this hypothesis, we say that  $\mathcal{C}$  admits factorization. Factorization generalizes Waldhausen’s notion of “cylinder functor satisfying the cylinder axiom”. The secondary hypothesis involves the relationship between the weak equivalences in the Waldhausen categories being compared. One version is the requirement (that often holds in practice) that the Waldhausen categories have their weak equivalences closed under retracts; we have included two alternative hypotheses for cases when this does not hold. We prove the following theorem in Section 3. This theorem can also be found in work of Cisinski [2], where it is proved by other techniques.

**Theorem 1.3.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be saturated Waldhausen categories that admit factorization. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a weakly exact functor that induces an equivalence on homotopy categories. If one of the following additional hypotheses holds:*

- (i) *the weak equivalences of  $\mathcal{C}$  and  $\mathcal{D}$  are closed under retracts,*
- (ii) *a map  $f$  in  $\mathcal{C}$  is a weak equivalence if and only if the map  $Ff$  in  $\mathcal{D}$  is a weak equivalence, or*
- (iii) *for any  $A, B \in \mathcal{C}$ , the image of  $\text{Ho}(\text{wC})(A, B)$  in  $\text{Ho } \mathcal{D}(FA, FB)$  coincides with the image of  $\text{Ho}(\text{wD})(FA, FB)$ ,*

*then  $F$  induces an equivalence of  $K$ -theory spectra.*

In the statement, a “weakly exact” functor is a homotopical generalization of an exact functor. An exact functor between Waldhausen categories preserves weak equivalences, cofibrations, and pushouts along cofibrations. A weakly exact functor preserves weak equivalences, but need

only preserve cofibrations and pushouts along cofibrations up to weak equivalence (see Definition 2.1 below). The “homotopy category” of a category  $\mathcal{C}$  with weak equivalence is the category  $\text{Ho}\mathcal{C}$  obtained by formally inverting the weak equivalences.  $\text{Ho}\mathcal{C}$  generalizes the derived category to this context; it is typically not a triangulated category without additional hypotheses on  $\mathcal{C}$ .

Following Waldhausen’s notation, we have used  $w\mathcal{C}$  and  $w\mathcal{D}$  to denote the subcategories of weak equivalences for  $\mathcal{C}$  and  $\mathcal{D}$ . The image of  $\text{Ho}(w\mathcal{C})$  in  $\text{Ho}\mathcal{C}$  consists of isomorphisms (by definition), but might not in general contain all isomorphisms of  $\text{Ho}\mathcal{C}$ . It does contain all the isomorphisms, however, under the hypotheses of Theorem 1.3 when the weak equivalences of  $\mathcal{C}$  are closed under retracts; see Section 6 for a complete discussion. Hypotheses (ii) and (iii) in Theorem 1.3 ensure that the weak equivalences of  $\mathcal{C}$  and  $\mathcal{D}$  and their formal inverses generate equivalent subcategories of  $\text{Ho}\mathcal{D}$  even when they do not necessarily generate all the isomorphisms of  $\text{Ho}\mathcal{D}$ .

In the proof of Theorem 1.3, we argue that a weakly exact functor that induces an equivalence of homotopy categories comes very near to being a DK-equivalence (a functor that induces a weak equivalence of Dwyer–Kan simplicial localizations); see Corollary 3.7. It remains an interesting question to determine when such a functor is a DK-equivalence. When we drop the weakly exact hypothesis and consider only functors that preserve weak equivalences, we can characterize DK-equivalences in terms of homotopy categories of undercategories. For an object  $A$  of  $\mathcal{C}$ , let  $\mathcal{C}\setminus A$  denote the category of objects in  $\mathcal{C}$  under  $A$ , i.e., an object consists of a map  $A \rightarrow X$  in  $\mathcal{C}$  and a map from  $A \rightarrow X$  to  $A \rightarrow Y$  consists of a map  $X \rightarrow Y$  in  $\mathcal{C}$  that commutes with the maps from  $A$ ; say that such a map is a weak equivalence when its underlying map  $X \rightarrow Y$  is a weak equivalence in  $\mathcal{C}$ . We can then form the homotopy category  $\text{Ho}(\mathcal{C}\setminus A)$  by formally inverting the weak equivalences. We prove in Section 8 the following theorem generalizing the main result of [13].

**Theorem 1.4.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be saturated Waldhausen categories that admit factorization, and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor that preserves weak equivalences. Then  $F$  is a DK-equivalence if and only if it induces an equivalence  $\text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{D})$  and an equivalence  $\text{Ho}(\mathcal{C}\setminus A) \rightarrow \text{Ho}(\mathcal{D}\setminus FA)$  for all objects  $A$  of  $\mathcal{C}$ .*

This interpretation relates to Waldhausen’s approximation theorem and provides a conceptual understanding of the role of Waldhausen’s approximation property [20, 1.6.4] in the more specialized approximation theorems. Recall that for Waldhausen categories  $\mathcal{C}$  and  $\mathcal{D}$ , an exact functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  satisfies the approximation property if

- (i) A map  $f : A \rightarrow B$  is a weak equivalence in  $\mathcal{C}$  if and only if the map  $F(f) : FA \rightarrow FB$  is a weak equivalence in  $\mathcal{D}$ .
- (ii) For every map  $FA \rightarrow X$  in  $\mathcal{D}$ , there exists a cofibration  $A \rightarrow B$  in  $\mathcal{C}$  and a weak equivalence  $FB \rightarrow X$  in  $\mathcal{D}$  such that the diagram

$$\begin{array}{ccc}
 FA & \longrightarrow & X \\
 \downarrow & \nearrow & \\
 FB & & \simeq
 \end{array}$$

commutes.

We prove the following theorem in Section 9.

**Theorem 1.5.** *Let  $\mathcal{C}$  be a saturated Waldhausen category that admits factorization. Let  $\mathcal{D}$  be a saturated Waldhausen category, and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor. If  $F$  satisfies Waldhausen’s approximation property, then  $\mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{D})$  is an equivalence and  $\mathrm{Ho}(\mathcal{C} \setminus A) \rightarrow \mathrm{Ho}(\mathcal{D} \setminus FA)$  is an equivalence for all objects  $A$  of  $\mathcal{C}$ .*

In all of the preceding theorems, we required the hypothesis that the Waldhausen categories be saturated, meaning that the weak equivalences satisfy the two out of three property: For composable maps  $f$  and  $g$ , if any two of  $f$ ,  $g$ , and  $f \circ g$  are weak equivalences then so is the third. This usage of the term “saturated” differs from the usage of the term in other sources such as [6]. Although much of the work in this paper could be adjusted to avoid this hypothesis, it is a hypothesis so common and pervasive in homotopy theory that to do so would lose more in the awkwardness it would engender than it would gain in the extra abstract generality it might achieve. Rather than continually repeating this hypothesis throughout the rest of the paper, we instead incorporate it by convention in the definition of weak equivalences.

**Convention.** In this paper we always understand a subcategory of weak equivalences to satisfy the two out of three property. In particular, all Waldhausen categories are assumed to be saturated in the sense of Waldhausen.

Finally, we should note that in virtually every example of interest, the factorizations hypothesized in the theorems above tend to be functorial. Assuming functorial factorizations simplifies many of the arguments; for these arguments, we assume functorial factorization in the body of the paper and treat the non-functorial case in Appendices A and B.

## 2. Weakly exact functors

This section defines weakly exact functor and shows that under mild technical hypotheses a weakly exact functor between Waldhausen categories induces a map between their  $K$ -theory spectra. Although we expect that the extra flexibility provided by stating Theorem 1.3 in terms of weakly exact functors rather than exact functors will increase its applicability, in fact, weakly exact functors play a vital technical role in its proof even in the case when the functor in question is exact. Specifically, the proof requires a version of Theorem 1.1 that is natural in weakly exact functors, which we state as Theorem 2.7 at the end of the section. We begin with the definition of weakly exact functor.

**Definition 2.1.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be Waldhausen categories. A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *weakly exact* if the initial map  $* \rightarrow F*$  in  $\mathcal{D}$  is a weak equivalence and  $F$  preserves weak equivalences, weak cofibrations, and homotopy cocartesian squares.

In the definition, a *weak cofibration* is a map that is weakly equivalent (by a zigzag) to a cofibration in the category  $\mathrm{Ar}\mathcal{C}$  of arrows in  $\mathcal{C}$ , and a *homotopy cocartesian square* is a square diagram that is weakly equivalent (by a zigzag) to a pushout square where one of the parallel sets of arrows consists of cofibrations. It follows that a functor that preserves weak equivalences will preserve weak cofibrations and homotopy cocartesian squares if and only if it takes cofibrations to weak cofibrations and takes pushouts along cofibrations to homotopy cocartesian squares.

Clearly the concept of weakly exact functor will only be useful when homotopy cocartesian squares have the usual expected properties. According to [1, §2], these properties hold when the Waldhausen category admits “functorial factorization of weak cofibrations”. (See Appendix A for a non-functorial generalization.)

Recall that a Waldhausen category  $\mathcal{C}$  admits *functorial factorization* when any map  $f : A \rightarrow B$  in  $\mathcal{C}$  factors as a cofibration followed by a weak equivalence

$$A \rightrightarrows Tf \xrightarrow{\cong} B,$$

$f$

functorially in  $f$  in the category  $\text{Ar}\mathcal{C}$  of arrows in  $\mathcal{C}$ . In other words, given the map  $\phi$  of arrows on the left (i.e., commuting diagram),

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a \downarrow & \phi & \downarrow b \\ A' & \xrightarrow{f'} & B' \end{array} \qquad \begin{array}{ccc} A & \rightrightarrows & Tf \xrightarrow{\cong} B \\ a \downarrow & T\phi \downarrow & \downarrow b \\ A' & \rightrightarrows & Tf' \xrightarrow{\cong} B' \end{array}$$

we have a map  $T\phi$  that makes the diagram on the right commute and that satisfies the usual identity and composition relations,  $T \text{id}_f = \text{id}_{Tf}$  and  $T(\phi' \circ \phi) = T\phi' \circ T\phi$ . A cylinder functor satisfying the cylinder axiom in the sense of Waldhausen [20, §1.6] is a factorization functor that in addition satisfies strong exactness properties.

In Waldhausen categories that admit functorial factorization, every map is weakly equivalent to a cofibration. This isn’t always the case in examples of interest, especially in “Waldhausen subcategories”. To get around this, in [1] we worked in terms of the technical hypothesis that  $\mathcal{C}$  admit *functorial factorization of weak cofibrations (FFWC)* [1, 2.2], which means that the weak cofibrations can be factored functorially (in  $\text{Ar}\mathcal{C}$ ) as above. Our interest in FFWC is the following theorem proved in this section.

**Theorem 2.2.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a weakly exact functor between Waldhausen categories and assume that  $\mathcal{D}$  admits FFWC. Then  $F$  induces a map of  $K$ -theory spectra.*

We prove Theorem 2.2 using the  $S'_\bullet$  construction of [1, §2]. To put this in context, we begin by reviewing the  $S_\bullet$  construction in detail. Recall that Waldhausen’s  $S_\bullet$  construction produces a simplicial Waldhausen category  $S_\bullet\mathcal{C}$  from a Waldhausen category  $\mathcal{C}$  and is defined as follows. Let  $\text{Ar}[n]$  denote the category with objects  $(i, j)$  for  $0 \leq i \leq j \leq n$  and a unique map  $(i, j) \rightarrow (i', j')$  for  $i \leq i'$  and  $j \leq j'$ .  $S_n\mathcal{C}$  is defined to be the full subcategory of the category of functors  $A : \text{Ar}[n] \rightarrow \mathcal{C}$  such that:

- $A_{i,i} = *$  for all  $i$ ,
- the map  $A_{i,j} \rightarrow A_{i,k}$  is a cofibration for all  $i \leq j \leq k$ , and
- the diagram

$$\begin{array}{ccc}
 A_{i,j} & \longrightarrow & A_{i,k} \\
 \downarrow & & \downarrow \\
 A_{j,j} & \longrightarrow & A_{j,k}
 \end{array}$$

is a pushout square for all  $i \leq j \leq k$ ,

where we write  $A_{i,j}$  for  $A(i, j)$ . The last two conditions can be simplified to the hypothesis that each map  $A_{0,j} \rightarrow A_{0,j+1}$  is a cofibration and the induced maps  $A_{0,j}/A_{0,i} \rightarrow A_{i,j}$  are isomorphisms. This becomes a Waldhausen category by defining a map  $A \rightarrow B$  to be a weak equivalence when each  $A_{i,j} \rightarrow B_{i,j}$  is a weak equivalence in  $\mathcal{C}$ , and to be a cofibration when each  $A_{i,j} \rightarrow B_{i,j}$  and each induced map  $A_{i,k} \cup_{A_{i,j}} B_{i,j} \rightarrow B_{i,k}$  is a cofibration in  $\mathcal{C}$ . The following definition gives a homotopical version of this construction for Waldhausen categories that admit FFWC.

**Definition 2.3.** Let  $\mathcal{C}$  be a Waldhausen category that admits FFWC. Define  $S'_n \mathcal{C}$  to be the full subcategory of functors  $A : \text{Ar}[n] \rightarrow \mathcal{C}$  such that:

- the initial map  $* \rightarrow A_{i,i}$  is a weak equivalence for all  $i$ ,
- the map  $A_{i,j} \rightarrow A_{i,k}$  is a weak cofibration for all  $i \leq j \leq k$ , and
- the diagram

$$\begin{array}{ccc}
 A_{i,j} & \longrightarrow & A_{i,k} \\
 \downarrow & & \downarrow \\
 A_{j,j} & \longrightarrow & A_{j,k}
 \end{array}$$

is a homotopy cocartesian square for all  $i \leq j \leq k$ .

We define a map  $A \rightarrow B$  to be a weak equivalence when each  $A_{i,j} \rightarrow B_{i,j}$  is a weak equivalence in  $\mathcal{C}$ , and to be a cofibration when each  $A_{i,j} \rightarrow B_{i,j}$  is a cofibration in  $\mathcal{C}$  and each induced map  $A_{i,k} \cup_{A_{i,j}} B_{i,j} \rightarrow B_{i,k}$  is a weak cofibration in  $\mathcal{C}$ .

Clearly  $S'_n \mathcal{C}$  assembles into a simplicial category with the usual face and degeneracy functors. Furthermore, we have the following comparison result [1, 2.8, 2.9].

**Proposition 2.4.** *Let  $\mathcal{C}$  be a Waldhausen category admitting FFWC.*

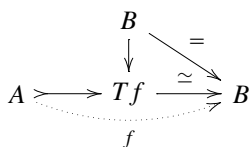
- (i)  $S'_n \mathcal{C}$  is a simplicial Waldhausen category admitting FFWC.
- (ii) The inclusion  $S_\bullet \mathcal{C} \rightarrow S'_n \mathcal{C}$  is a simplicial exact functor.
- (iii) For each  $n$ , the inclusion  $\text{w}S_n \mathcal{C} \rightarrow \text{w}S'_n \mathcal{C}$  induces a weak equivalence on nerves.

The following proposition is now clear from the definition of weakly exact. Theorem 2.2 then follows from this proposition and the previous proposition by iterating the  $S'_\bullet$  construction.

**Proposition 2.5.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  a weakly exact functor between Waldhausen categories and assume that  $\mathcal{D}$  admits FFWC. Then for each  $n$ ,  $F$  sends  $S_n\mathcal{C}$  into  $S'_n\mathcal{D}$  by a weakly exact functor.*

Finally, we can use the  $S'_n$  construction to express the full naturality of the weak equivalences in Theorem 1.1. Unfortunately, the hypothesis FFWC is not quite strong enough. We need a slight refinement of this hypothesis:

**Definition 2.6.** Let  $\mathcal{C}$  be a Waldhausen category. We say that  $\mathcal{C}$  has *functorial mapping cylinders for weak cofibrations (FMCWC)* when  $\mathcal{C}$  admits functorial factorization of weak cofibrations by a functor  $T$  together with a natural transformation  $B \rightarrow Tf$  splitting the natural weak equivalence  $Tf \rightarrow B$ , for weak cofibrations  $f : A \rightarrow B$ .



Functorial factorization of all maps implies functorial mapping cylinders: For a map  $f : A \rightarrow B$ , the factorization of the map  $f + \text{id}_B$  from the coproduct,  $A \vee B \rightarrow B$ , provides the functorial mapping cylinder. Thus, functorial factorization of all maps is equivalent to the conjunction of functorial mapping cylinders for weak cofibrations and all maps being weak cofibrations.

For a Waldhausen category  $\mathcal{C}$  that has functorial mapping cylinders for weak cofibrations, and  $A, B$  objects in  $\mathcal{C}$ , we use  $LC_{\text{co}}(A, B)$  to denote the components of the Dwyer–Kan hammock function complex  $LC(A, B)$  that correspond to the weak cofibrations; precisely,  $LC_{\text{co}}(A, B)$  consists of those components that contain as a vertex a zigzag where all the forward arrows are weak cofibrations. Likewise, we use  $LC_w(A, B)$  to denote the components of the Dwyer–Kan hammock function complex  $LC(A, B)$  that contain a zigzag where all the forward arrows are weak equivalences. Then  $LC_{\text{co}}$  and  $LC_w$  are simplicial subcategories of the Dwyer–Kan simplicial localization  $LC$ . We prove the following generalization of Theorems 1.1 and 1.2 in Section 7. (See Appendix B for the corresponding non-functorial statement.)

**Theorem 2.7.** *Let  $\mathcal{C}$  be a saturated Waldhausen category that has FMCWC.*

- (i) *For  $n > 1$ , the nerve of  $wS'_n\mathcal{C}$  is weakly equivalent to the homotopy coend*

$$\text{hocoend}_{(X_1, \dots, X_n) \in w\mathcal{C}^n} LC_{\text{co}}(X_{n-1}, X_n) \times \cdots \times LC_{\text{co}}(X_1, X_2),$$

*naturally in weakly exact functors.*

- (ii) *The nerve of  $w\mathcal{C}$  is weakly equivalent to the disjoint union of  $B \text{ hAut } X$  over the weak equivalence classes of objects of  $\mathcal{C}$ .*
- (iii) *For  $n \geq 1$ , the nerve of  $wS'_n\mathcal{C}$  is weakly equivalent to the total space of a fibration where the base is the disjoint union of*

$$B \text{ hAut } X_n \times \cdots \times B \text{ hAut } X_1$$



over  $n$ -tuples of weak equivalences classes of objects of  $\mathcal{C}$ , and the fiber is equivalent to

$$LC_{\text{co}}(X_{n-1}, X_n) \times \cdots \times LC_{\text{co}}(X_1, X_2),$$

for  $n > 1$  and contractible for  $n = 1$ .

### 3. Outline of the proof of Theorem 1.3

In this section we prove Theorem 1.3 from Theorem 2.7 above, Theorem 3.5 below, and Propositions 3.8 and 3.9 below, all of which are proved in later sections. Throughout this section (and this section only), we fix  $\mathcal{C}$ ,  $\mathcal{D}$ , and  $F : \mathcal{C} \rightarrow \mathcal{D}$ , satisfying the hypotheses of Theorem 1.3 and such that factorization is functorial. (See Appendix B for the proof in the non-functorial case.) Moreover, we fix factorization functors  $T$  on  $\mathcal{C}$  and  $\mathcal{D}$ ; we use the following terminology and notation.

**Definition 3.1.** For  $X$  in  $\mathcal{C}$  (resp.  $\mathcal{D}$ ), the *cone on  $X$* ,  $CX$ , is  $T(X \rightarrow *)$  and the *suspension of  $X$* ,  $\Sigma X$ , is  $CX/X$ . Let  $EX$  denote the cofiber sequence

$$X \twoheadrightarrow CX \longrightarrow \Sigma X$$

viewed as an object of  $S_2\mathcal{C}$  (resp.  $S_2\mathcal{D}$ ), with  $A_{0,1} = X$ ,  $A_{0,2} = CX$ , and  $A_{1,2} = \Sigma X$ .

It follows from the functoriality of  $T$  that  $CX$ ,  $\Sigma X$ , and  $EX$  assemble to functors in  $X$ . A straightforward application of factorization and [1, 2.5] (or the gluing axiom) shows that these functors preserve weak equivalences and homotopy cocartesian squares. This gives the following proposition; the corresponding result holds for  $\mathcal{D}$ .

**Proposition 3.2.** *The functors  $C$  and  $\Sigma$  are weakly exact functors  $\mathcal{C} \rightarrow \mathcal{C}$ , and  $E$  is a weakly exact functor  $\mathcal{C} \rightarrow S_2\mathcal{C}$ .*

The factorization functor for  $\mathcal{C}$  induces a factorization functor on  $S_2\mathcal{C}$ , and so  $E$  induces a map of  $K$ -theory spectra  $K\mathcal{C} \rightarrow KS_2\mathcal{C}$ . Applying the Additivity Theorem [20, 1.4.2, 1.3.2.(3)], we see that on  $K$ -theory, the sum in the stable category of the maps induced by the identity and suspension is the map induced by the cone. Since the cone induces the trivial map, it follows that  $\Sigma$  induces on  $K\mathcal{C}$  the map  $-1$  in the stable category. In particular, we obtain the following corollary; the corresponding result holds for  $\mathcal{D}$ .

**Corollary 3.3.**  *$\Sigma$  induces a weak equivalence on  $K$ -theory spectra  $K\mathcal{C} \rightarrow K\mathcal{C}$ .*

Although we do not assume any relationship between the factorization functors on  $\mathcal{C}$  and  $\mathcal{D}$ , nevertheless, we can relate the suspensions.

**Proposition 3.4.** *There is a functor  $\mathcal{E} : \mathcal{C} \rightarrow \mathcal{D}$  and natural weak equivalences*

$$F\Sigma X \xleftarrow{\simeq} \mathcal{E}X \xrightarrow{\simeq} \Sigma FX.$$

For example,  $\mathcal{E}X$  can be defined as the pushout

$$\begin{array}{ccc} FX & \xrightarrow{\quad} & T(F(X \rightarrow CX)) \\ \downarrow & & \downarrow \\ F* & \longrightarrow & \mathcal{E}X. \end{array}$$

The factorization weak equivalence  $T(F(X \rightarrow CX)) \rightarrow FCX$  and the universal property of the pushout induces the map  $\mathcal{E}X \rightarrow F\Sigma X$ , which is a weak equivalence [1, 2.5]. Functoriality of  $T$  in  $\text{Ar } \mathcal{D}$  then gives a map under  $FX$ ,

$$T(F(X \rightarrow CX)) \rightarrow T(FX \rightarrow *) = CFX,$$

which is a weak equivalence since the initial map to each is a weak equivalence. This map and the final map  $F* \rightarrow *$  induce the map  $\mathcal{E}X \rightarrow \Sigma FX$ , which is a weak equivalence by the gluing axiom.

To take advantage of the suspension functor, we need to relate it to the Dwyer–Kan function complexes. For this we use the following application of Theorem 6.2 from Section 6. Again, the corresponding theorem also holds for  $\mathcal{D}$ .

**Theorem 3.5.** *If the diagram on the left below is homotopy cocartesian in  $\mathcal{C}$ ,*

$$\begin{array}{ccc} A \longrightarrow B & & LC(D, Y) \longrightarrow LC(C, Y) \\ \downarrow & & \downarrow \\ C \longrightarrow D & & LC(B, Y) \longrightarrow LC(A, Y) \end{array}$$

*then for any object  $Y$  in  $\mathcal{C}$ , the diagram on the right is homotopy cartesian in the category of simplicial sets.*

Applying this theorem to the homotopy cocartesian square defining the suspension, we obtain the following corollary.

**Corollary 3.6.** *For any  $X, Y$  in  $\mathcal{C}$ ,  $LC(\Sigma X, Y)$  is weakly equivalent to the based loop space of  $LC(X, Y)$ , based at the trivial map  $X \rightarrow Y$ .*

Applying  $F$  to the homotopy cocartesian square defining the suspension, we see that likewise  $LD(F\Sigma X, FY)$  is weakly equivalent to the based loop space of  $LD(FX, FY)$ , based at  $F$  of the trivial map  $X \rightarrow Y$ , or equivalently, based at the trivial map  $FX \rightarrow FY$  since it is in the same component.

Iterating the suspension in the previous proposition, we see that  $\pi_n LC(X, Y)$  based at the trivial map is

$$\pi_0 LC(\Sigma^n X, Y) = \text{Ho } \mathcal{C}(\Sigma^n X, Y).$$

Since  $F$  induces an equivalence  $\text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$ , we obtain the following corollary.

**Corollary 3.7.** *For any  $X, Y$  in  $\mathcal{C}$ , the restriction of*

$$LF : LC(X, Y) \rightarrow LD(FX, FY)$$

*to the component of the trivial map is a weak equivalence; the map*

$$LF : LC(\Sigma X, Y) \rightarrow LD(F\Sigma X, FY)$$

*is a weak equivalence.*

To apply the previous corollary, we also need to know that the components of  $LC(\Sigma X, \Sigma X)$  in  $\text{hAut } \Sigma X$  correspond to the same components of  $LD(F\Sigma X, F\Sigma X)$  that are in  $\text{hAut } F\Sigma X$ . Since these components consist of exactly the components representing the image of  $\text{Ho } \text{w}\mathcal{C}(\Sigma X, \Sigma X)$  and  $\text{Ho } \text{w}\mathcal{D}(F\Sigma X, F\Sigma X)$ , respectively, this is clear when hypothesis (iii) in the statement of Theorem 1.3 holds. The other two cases are handled by the following propositions. The first is a special case of Theorem 6.4.

**Proposition 3.8.** *If hypothesis (i) in the statement of Theorem 1.3 holds, then a map in  $\mathcal{C}$  is a weak equivalence if and only if it represents an isomorphism in  $\text{Ho } \mathcal{C}$ , and likewise for  $\mathcal{D}$ .*

The second proposition is proved in Section 5 as Corollary 5.8.

**Proposition 3.9.** *If hypothesis (ii) in the statement of Theorem 1.3 holds, then so does hypothesis (iii).*

We have now assembled all we need to prove Theorem 1.3. By Proposition 2.4, we can use  $N(\text{w}S'_\bullet \mathcal{C})$ , the diagonal of the nerve of the simplicial category  $\text{w}S'_\bullet \mathcal{C}$ , as a model for the (de-looped)  $K$ -theory space of  $\mathcal{C}$ . Since suspension induces a weak equivalence on  $K$ -theory, the telescope under suspension

$$\text{Tel}_\Sigma N(\text{w}S'_\bullet \mathcal{C}) = \text{Tel}(N(\text{w}S'_\bullet \mathcal{C}) \xrightarrow{\Sigma} N(\text{w}S'_\bullet \mathcal{C}) \xrightarrow{\Sigma} N(\text{w}S'_\bullet \mathcal{C}) \xrightarrow{\Sigma} \dots)$$

is equivalent to  $N(\text{w}S'_\bullet \mathcal{C})$  via the inclusion. The same observations apply to  $\mathcal{D}$ , and Proposition 3.4 provides homotopies to construct a map of telescopes

$$\text{Tel}_\Sigma N(\text{w}S'_n \mathcal{C}) \rightarrow \text{Tel}_\Sigma N(\text{w}S'_n \mathcal{D}) \tag{3.10}$$

for all  $n$ . Now we use the models from Theorem 2.7. We write

$$LC(X_1, \dots, X_n) = LC(X_{n-1}, X_n) \times \dots \times LC(X_1, X_2)$$

and similarly for  $\mathcal{D}$  to save space. Then by Theorem 2.7(i), the square in the homotopy category formed by the  $S'_n$  constructions

$$\begin{array}{ccc}
 N(wS'_n\mathcal{C}) & \xrightarrow{F_*} & N(wS'_n\mathcal{D}) \\
 \Sigma_* \downarrow & & \downarrow \Sigma_* \\
 N(wS'_n\mathcal{C}) & \xrightarrow{F_*} & N(wS'_n\mathcal{D})
 \end{array}$$

is isomorphic in the homotopy category to the square in the homotopy category formed by the homotopy coends

$$\begin{array}{ccc}
 \text{hocoend } LC(X_1, \dots, X_n) & \xrightarrow{LF_*} & \text{hocoend } LD(Y_1, \dots, Y_n) \\
 L\Sigma_* \downarrow & & \downarrow L\Sigma_* \\
 \text{hocoend } LC(X_1, \dots, X_n) & \xrightarrow{LF_*} & \text{hocoend } LD(Y_1, \dots, Y_n),
 \end{array}$$

where the homotopy coends are over  $(X_1, \dots, X_n) \in w\mathcal{C}^n$  and  $(Y_1, \dots, Y_n) \in w\mathcal{D}^n$ . Writing  $\mathcal{C}_\Sigma$  and  $\mathcal{D}_\Sigma$  for the full subcategories of  $\mathcal{C}$  and  $\mathcal{D}$  consisting of objects that are weakly equivalent to suspensions, it is clear that the vertical map of arrows factors through the arrow

$$\text{hocoend}_{(X_1, \dots, X_n) \in w\mathcal{C}_\Sigma^n} LC(X_1, \dots, X_n) \xrightarrow{LF_*} \text{hocoend}_{(Y_1, \dots, Y_n) \in w\mathcal{D}_\Sigma^n} LD(Y_1, \dots, Y_n).$$

Corollary 3.7 and Theorem 2.7 imply this latter map is a weak equivalence, and we conclude that the map of telescopes (3.10) is a weak equivalence. This completes the proof of Theorem 1.3.

### 4. Universal simplicial quasifibrations

In this section, we introduce the first of two techniques which provide the foundation for our subsequent work in this paper. The proofs of the theorems in the previous sections depend on machinery for solving two related problems: The identification of certain squares of simplicial sets as homotopy cartesian squares and the identification of the homotopy fiber of certain maps of simplicial sets. Quillen’s Theorem B [15] and its simplicial variant [20, 1.4.B] provide a flexible tool for these purposes. We rely on a particular formulation of Theorem B in terms of a notion of “universal simplicial quasifibration”. Our exposition and viewpoint on the subject is heavily influenced by postings of Tom Goodwillie on Don Davis’ algebraic topology mailing list [7].

Recall that a map  $X \rightarrow Y$  of spaces is a quasifibration when for every point  $x$  of  $X$ , the map from the fiber to the homotopy fiber is a weak equivalence. We say that a map of simplicial sets  $X_\bullet \rightarrow Y_\bullet$  is a quasifibration when its geometric realization is a quasifibration of spaces.

**Definition 4.1.** A map of simplicial sets  $X_\bullet \rightarrow Y_\bullet$  is a *universal simplicial quasifibration* when for every map of simplicial sets  $Z_\bullet \rightarrow Y_\bullet$ , the induced map of the pullback  $X_\bullet \times_{Y_\bullet} Z_\bullet \rightarrow Z_\bullet$  is a quasifibration.

The definition specifies a class of maps for which it is easy to identify pullbacks as homotopy pullbacks. The following proposition implies that to verify that a map is a universal simplicial

quasifibration, it suffices to check the condition on the simplexes of  $X_\bullet$ ; for the simplices, checking that the pullback map to the simplex is a quasifibration then amounts to checking that the fiber over a vertex includes as a weak equivalence. The proof in one direction is the restriction of the universal simplicial quasifibration property to the standard simplices; the proof in the other direction is Waldhausen’s version of Theorem B [20, 1.4.B].

**Proposition 4.2.** *A map of simplicial sets  $X_\bullet \rightarrow Y_\bullet$  is a universal simplicial quasifibration if and only if for every  $n$  and every map  $\Delta[n] \rightarrow Y_\bullet$ , the induced map of the pullback  $X_\bullet \times_{Y_\bullet} \Delta[n] \rightarrow \Delta[n]$  is a quasifibration.*

In general, the simplicial sets we use arise as homotopy colimits. Thus, it is convenient to state the following proposition. For a small category  $\mathcal{C}$ , we write  $NC$  to denote the simplicial nerve.

**Proposition 4.3.** *Let  $\mathcal{C}$  be a small category and  $F$  a functor from  $\mathcal{C}$  to simplicial sets. Suppose that for every map  $f : x \rightarrow y$  in  $\mathcal{C}$ , the induced map  $F(f) : F(x) \rightarrow F(y)$  is a weak equivalence. Then the map  $\text{hocolim}_{\mathcal{C}} F \rightarrow NC$  is a universal simplicial quasifibration.*

**Proof.** We can identify an  $n$ -simplex of  $NC$  as a functor  $\sigma : \Delta[n] \rightarrow \mathcal{C}$ , where  $\Delta[n]$  is the poset of  $0, \dots, n$  under  $\leq$ . The pullback of  $\text{hocolim}_{\mathcal{C}} F$  over this simplex is  $\text{hocolim}_{\Delta[n]} F \circ \sigma$ . For any vertex  $i$ , the inclusion of the fiber,  $F(\sigma(i))$ , in  $\text{hocolim}_{\Delta[n]} F \circ \sigma$  is a weak equivalence.  $\square$

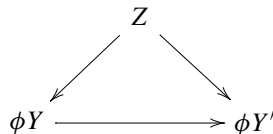
The same proof also gives the following proposition, which we apply directly in the proof of Theorem 2.7.

**Proposition 4.4.** *Let  $\mathcal{C}$  be a small category and  $F$  a functor from  $\mathcal{C}^{\text{op}} \times \mathcal{C}$  to simplicial sets. Suppose that for every  $z$  in  $\mathcal{C}$  and every map  $f : x \rightarrow y$  in  $\mathcal{C}$ , the induced maps  $F(f, z) : F(y, z) \rightarrow F(x, z)$  and  $F(z, f) : F(z, x) \rightarrow F(z, y)$  are weak equivalences. Then the map  $\text{hocoend}_{\mathcal{C}} F \rightarrow NC$  is a universal simplicial quasifibration.*

We also repeatedly use the following refinement of Quillen’s Theorem B [15]. Recall that for a functor  $\phi : \mathcal{D} \rightarrow \mathcal{C}$  and a fixed object  $Z$  of  $\mathcal{C}$ , the comma category  $Z \downarrow \phi$  has as objects the pairs  $(Y, Z \rightarrow \phi Y)$  consisting of an object  $Y$  of  $\mathcal{D}$  and a map in  $\mathcal{C}$  from  $Z$  to  $\phi Y$ . A morphism

$$(Y, Z \rightarrow \phi Y) \rightarrow (Y', Z \rightarrow \phi Y')$$

consists of a map  $Y \rightarrow Y'$  in  $\mathcal{D}$  such that the diagram



in  $\mathcal{D}$  commutes. The following theorem gives a useful sufficient condition for a commuting square of functors to induce a homotopy cartesian square of nerves.

**Theorem 4.5.** *Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$  be small categories, and let*

$$\begin{array}{ccc}
 \mathcal{D} & \xrightarrow{\delta} & \mathcal{C} \\
 \downarrow & & \downarrow \gamma \\
 \mathcal{B} & \xrightarrow{\beta} & \mathcal{A}
 \end{array}$$

*be a (strictly) commuting diagram of functors. If the following two conditions hold, then the induced square of nerves is homotopy cartesian:*

- (i) *for every map  $X \rightarrow X'$  in  $\mathcal{A}$ , the induced functor on comma categories  $X' \downarrow \beta \rightarrow X \downarrow \beta$  induces a weak equivalence of nerves, and*
- (ii) *for every object  $Z$  in  $\mathcal{C}$ , the functor  $Z \downarrow \delta \rightarrow \gamma Z \downarrow \beta$  induces a weak equivalence of nerves.*

**Proof.** As in Quillen’s proof of Theorem B, we have natural weak equivalences

$$\operatorname{hocolim}_{X \in \mathcal{A}} N(X \downarrow \beta) \rightarrow N\mathcal{B} \quad \text{and} \quad \operatorname{hocolim}_{Z \in \mathcal{C}} N(Z \downarrow \delta) \rightarrow N\mathcal{D}.$$

These then fit into the commutative diagram on the left

$$\begin{array}{ccc}
 \operatorname{hocolim}_{Z \in \mathcal{C}} N(Z \downarrow \delta) & \longrightarrow & N\mathcal{C} \\
 \downarrow & & \downarrow \\
 \operatorname{hocolim}_{X \in \mathcal{A}} N(X \downarrow \beta) & \longrightarrow & N\mathcal{A}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \operatorname{hocolim}_{Z \in \mathcal{C}} N(\gamma Z \downarrow \beta) & \longrightarrow & N\mathcal{C} \\
 \downarrow & & \downarrow \\
 \operatorname{hocolim}_{X \in \mathcal{A}} N(X \downarrow \beta) & \longrightarrow & N\mathcal{A}
 \end{array}$$

weakly equivalent to the diagram of nerves in question. By (ii), we have that the canonical map

$$\operatorname{hocolim}_{Z \in \mathcal{C}} N(Z \downarrow \delta) \rightarrow \operatorname{hocolim}_{Z \in \mathcal{C}} N(\gamma Z \downarrow \beta)$$

is a weak equivalence, and it follows that the square on the right above is weakly equivalent to the square on the left. The square on the right is a pullback square of simplicial sets, and by Proposition 4.3, (i) implies that the bottom horizontal map is a universal simplicial quasifibration. We conclude that the square is homotopy cartesian.  $\square$

### 5. Homotopy calculi of fractions and mapping cylinders

In this section, we describe the second technical device essential to the proof of the main theorems, the homotopy calculi of fraction introduced in [4]. When a category with weak equivalences admits a homotopy calculus of fractions, the Dwyer–Kan function complexes  $LC(A, B)$  admit significantly smaller models that are nerves of categories of words of a specified type. We begin the section with a concise review of this theory. We then prove that a Waldhausen category with FMCWC admits a homotopy calculus of left fractions. (See Appendix B for statements and proofs in the non-functorial case.)

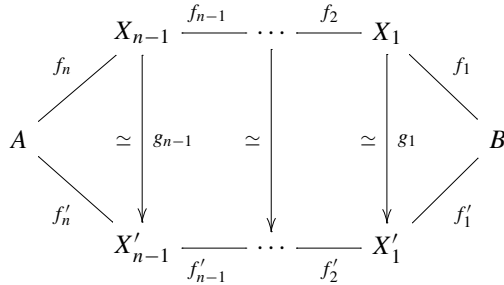
We begin with the notation for the categories of words of specified types. Let  $\mathcal{C}$  be a category with a subcategory  $w\mathcal{C}$  of weak equivalences. We consider the words on letters  $\mathbf{C}, \mathbf{W}$ , and

$W^{-1}$ : To every such word  $\Upsilon$  and pair of objects  $A, B$  in  $\mathcal{C}$ , we associate a category  $\Upsilon(A, B)$ , where the objects are roughly speaking words in  $\mathcal{C}$  of the type specified by the letters in  $\Upsilon$  and the morphisms are the weak equivalences. The precise definition is as follows.

**Definition 5.1.** Let  $\Upsilon$  be a word of length  $n$  on letters  $C, W$ , and  $W^{-1}$ , and let  $A, B$  be objects of  $\mathcal{C}$ . We define the  $\Upsilon(A, B)$  to be the following category. An object in  $\Upsilon(A, B)$  consists of:

- a collection of objects  $X_1, \dots, X_{n-1}$  of  $\mathcal{C}$ ,
- a map  $f_i : X_i \rightarrow X_{i-1}$  in  $\mathcal{C}$  whenever the  $i$ -th letter of  $\Upsilon$  is  $C$  for some  $1 \leq i \leq n$ ,
- a map  $f_i : X_i \rightarrow X_{i-1}$  in  $w\mathcal{C}$  whenever the  $i$ -th letter of  $\Upsilon$  is  $W$  for some  $1 \leq i \leq n$ , and
- a map  $f_i : X_{i-1} \rightarrow X_i$  in  $w\mathcal{C}$  whenever the  $i$ -th letter of  $\Upsilon$  is  $W^{-1}$  for some  $1 \leq i \leq n$ ,

where we interpret  $X_n$  as  $A$  and  $X_0$  as  $B$  in the conditions above. A morphism in  $\Upsilon(A, B)$  from  $\{X_i, f_i\}$  to  $\{X'_i, f'_i\}$  is a collection of maps  $g_i : X_i \rightarrow X'_i$  in  $w\mathcal{C}$  that are the identity on  $A$  and  $B$  and make the evident diagram commute.

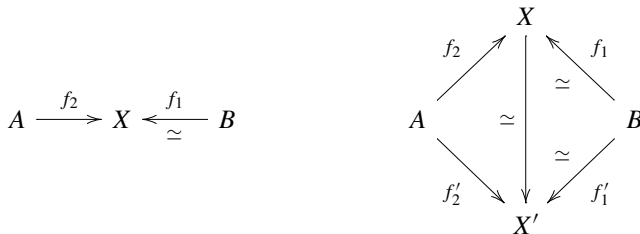


The numbering, which may seem unusual in the diagrams, is forced by the convention that the letters in the word follow composition order, where we think of the object  $\{X_i, f_i\}$  as representing a formal composition  $f_1 \circ \dots \circ f_n$  with the  $i$ -th map corresponding to the  $i$ -th letter (with  $f_i^{-1}$  in place of  $f_i$  in the formal composition when the  $i$ -th letter is  $W^{-1}$ ). Our words indicate the same categories and diagrams as those in [4], which are numbered slightly differently.

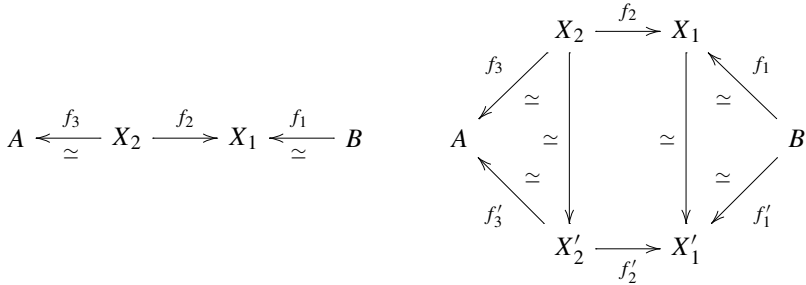
In our work below, the three most important words are  $W^{-1}C$ ,  $W^{-1}W$ , and  $W^{-1}CW^{-1}$ . For convenience, we spell out these categories explicitly.

**Example 5.2.** The categories of words  $W^{-1}C$ ,  $W^{-1}W$ , and  $W^{-1}CW^{-1}$ .

- (i)  $W^{-1}C$ : An object  $\{X, f_1, f_2\}$  is pictured on the left and a map from  $\{X, f_1, f_2\}$  to  $\{X', f'_1, f'_2\}$  is pictured on the right.



- (ii)  $\mathbf{W}^{-1}\mathbf{W}$ : Objects and maps look the same, but the map  $A \rightarrow X$  is required to be in  $w\mathcal{C}$ .
- (iii)  $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}$ : An object  $\{X_1, X_2, f_1, f_2, f_3\}$  is pictured on the left and a map from  $\{X_1, X_2, f_1, f_2, f_3\}$  to  $\{X'_1, X'_2, f'_1, f'_2, f'_3\}$  is pictured on the right.



As described in [4, 5.5], the function complex  $LC(A, B)$  is a colimit of the nerves of these categories of words. The hypothesis of a homotopy calculus of fractions is a homotopical requirement on how these nerves fit together. The following definition is [4, 6.1]. Although we use only homotopy calculus of left fractions and homotopy calculus of two-sided fractions in our work below, we include the definition of homotopy calculus of right fractions for completeness.

**Definition 5.3** (*Homotopy calculi of fractions*). Let  $\mathcal{C}$  be a category with a subcategory of weak equivalences  $w\mathcal{C}$ .

- (i)  $\mathcal{C}$  admits a *homotopy calculus of two-sided fractions (HC2F)* means that for every pair of integers  $i, j \geq 0$ , the functors given by inserting an identity morphism in the  $(i + 1)$ -st spot,

$$\begin{aligned} \mathbf{W}^{-1}\mathbf{C}^{i+j}\mathbf{W}^{-1} &\rightarrow \mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j\mathbf{W}^{-1}, \\ \mathbf{W}^{-1}\mathbf{W}^{i+j}\mathbf{W}^{-1} &\rightarrow \mathbf{W}^{-1}\mathbf{W}^i\mathbf{W}^{-1}\mathbf{W}^j\mathbf{W}^{-1}, \end{aligned}$$

induce weak equivalences on nerves for every pair of objects of  $\mathcal{C}$ .

- (ii)  $\mathcal{C}$  admits a *homotopy calculus of left fractions (HCLF)* means that for every pair of integers  $i, j \geq 0$ , the functors given by inserting an identity morphism in the  $(i + 1)$ -st spot,

$$\begin{aligned} \mathbf{W}^{-1}\mathbf{C}^{i+j} &\rightarrow \mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j, \\ \mathbf{W}^{-1}\mathbf{W}^{i+j} &\rightarrow \mathbf{W}^{-1}\mathbf{W}^i\mathbf{W}^{-1}\mathbf{W}^j, \end{aligned}$$

induce weak equivalences on nerves for every pair of objects of  $\mathcal{C}$ .

- (iii)  $\mathcal{C}$  admits a *homotopy calculus of right fractions (HCRF)* means that for every pair of integers  $i, j \geq 0$ , the functors given by inserting an identity morphism in the  $i$ -th spot,

$$\begin{aligned} \mathbf{C}^{i+j}\mathbf{W}^{-1} &\rightarrow \mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j\mathbf{W}^{-1}, \\ \mathbf{W}^{i+j}\mathbf{W}^{-1} &\rightarrow \mathbf{W}^i\mathbf{W}^{-1}\mathbf{W}^j\mathbf{W}^{-1}, \end{aligned}$$

induce weak equivalences on nerves for every pair of objects of  $\mathcal{C}$ .



Dwyer and Kan observe [4, 6.1, §9] that if  $\mathcal{C}$  admits a homotopy calculus of left or right fractions, then  $\mathcal{C}$  admits a homotopy calculus of two-sided fractions. The following proposition [4, 6.2] explains the utility and terminology of the definition.

**Proposition 5.4.** *Let  $\mathcal{C}$  be a category with a subcategory of weak equivalences  $w\mathcal{C}$ .*

(i) *If  $\mathcal{C}$  admits a homotopy calculus of two-sided fractions, then the maps*

$$NW^{-1}CW^{-1}(A, B) \rightarrow LC(A, B),$$

$$NW^{-1}WW^{-1}(A, B) \rightarrow L(w\mathcal{C})(A, B)$$

*are weak equivalences.*

(ii) *If  $\mathcal{C}$  admits a homotopy calculus of left fractions, then the maps*

$$NW^{-1}C(A, B) \rightarrow LC(A, B),$$

$$NW^{-1}W(A, B) \rightarrow L(w\mathcal{C})(A, B)$$

*are weak equivalences.*

(iii) *If  $\mathcal{C}$  admits a homotopy calculus of right fractions, then the maps*

$$NCW^{-1}(A, B) \rightarrow LC(A, B),$$

$$NWW^{-1}(A, B) \rightarrow L(w\mathcal{C})(A, B)$$

*are weak equivalences.*

The following theorem, the main theorem of this section, is proved below. In it,  $\tilde{co}\mathcal{C}$  denotes the category of weak cofibrations (q.v. [1, 2.4]), which by definition contains the weak equivalences  $w\mathcal{C}$  as a subcategory.

**Theorem 5.5.** *Let  $\mathcal{C}$  be a Waldhausen category with FMCWC. Then  $\mathcal{C}$ ,  $\tilde{co}\mathcal{C}$ , and  $w\mathcal{C}$  admit homotopy calculi of left fractions.*

Under the hypothesis of FMCWC, the previous theorem in particular allows us to model the function complex  $LC(A, B)$  in terms of the categories of words  $W^{-1}C(A, B)$  and  $W^{-1}CW^{-1}(A, B)$ . The following theorem identifies the subsets  $LC_{co}(A, B)$  and  $LC_w(A, B)$  in these terms. In it, for  $\mathcal{Y} = W^{-1}C$  and  $\mathcal{Y} = W^{-1}CW^{-1}$ , we write  $\mathcal{Y}(A, B)_{\tilde{co}}$  and  $\mathcal{Y}(A, B)_w$  for the full subcategory of  $\mathcal{Y}(A, B)$  of diagrams whose forward map is a weak cofibration and weak equivalence, respectively.

**Theorem 5.6.** *Let  $\mathcal{C}$  be a Waldhausen category with FMCWC. Then the maps*

$$W^{-1}C(A, B)_{\tilde{co}} \rightarrow LC_{co}(A, B), \quad W^{-1}CW^{-1}(A, B)_{\tilde{co}} \rightarrow LC_{co}(A, B),$$

$$W^{-1}C(A, B)_w \rightarrow LC_w(A, B), \quad \text{and} \quad W^{-1}CW^{-1}(A, B)_w \rightarrow LC_w(A, B)$$

*are weak equivalences*

**Proof.** We do the case for the weak cofibrations; the case for the weak equivalences is identical. Since  $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathcal{C}}}$  and  $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_{\tilde{\mathcal{C}}}$  are collections of components of the categories  $\mathbf{W}^{-1}\mathbf{C}(A, B)$  and  $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)$ , and  $LC_{\mathcal{C}o}(A, B)$  is a collection of components of the simplicial set  $LC(A, B)$ , we just need to check that these components coincide. For this, it suffices to show that every component of  $LC_{\mathcal{C}o}(A, B)$  contains as a vertex the image of an object of  $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathcal{C}}}$ . The key observation is that the category  $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathcal{C}}}$  is precisely the category of words  $\mathbf{W}^{-1}\mathbf{C}(A, B)$  in the category  $\tilde{\mathcal{C}}\mathcal{C}$ . By definition, a component of  $LC_{\mathcal{C}o}(A, B)$  contains a vertex represented by a zigzag in which all the forward arrows are weak cofibrations. Interpreting this zigzag as a vertex in  $L(\tilde{\mathcal{C}}\mathcal{C})(A, B)$ , we get a path in  $L(\tilde{\mathcal{C}}\mathcal{C})(A, B)$  to a vertex in the image of  $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathcal{C}}}$  since  $\tilde{\mathcal{C}}\mathcal{C}$  admits a homotopy calculus of left fractions. The inclusion of  $\tilde{\mathcal{C}}\mathcal{C}$  induces a map  $L(\tilde{\mathcal{C}}\mathcal{C})(A, B) \rightarrow LC_{\mathcal{C}o}(A, B)$  that gives us a path in  $LC_{\mathcal{C}o}(A, B)$  to a vertex in the image of  $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathcal{C}}}$ .  $\square$

Using the previous theorem and the observation in its proof that  $\mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{\mathcal{C}}}$  and  $\mathbf{W}^{-1}\mathbf{C}(A, B)_w$  coincide with the categories of words  $\mathbf{W}^{-1}\mathbf{C}(A, B)$  for  $\tilde{\mathcal{C}}\mathcal{C}$  and  $w\mathcal{C}$  (respectively), we obtain the following corollary.

**Corollary 5.7.** *For any  $A, B$  in  $\mathcal{C}$ , the inclusions  $L(\tilde{\mathcal{C}}\mathcal{C})(A, B) \rightarrow LC_{\mathcal{C}o}(A, B)$  and  $L(w\mathcal{C})(A, B) \rightarrow LC_w(A, B)$  are weak equivalences.*

Likewise, we obtain the proof of Proposition 3.9, which was used in the proof of Theorem 1.3:

**Corollary 5.8.** *In the context of Theorem 1.3, hypothesis (ii) implies hypothesis (iii).*

**Proof.** Suppose  $\phi$  is in the image of  $\text{Ho } w\mathcal{D}(FA, FB)$  in  $\text{Ho } \mathcal{D}(FA, FB)$ . Since  $F$  induces an equivalence  $\text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$ ,  $\phi$  is represented by  $F\phi'$  for some  $\phi'$  in  $\text{Ho } \mathcal{C}(A, B)$ . We can represent  $\phi'$  by a word

$$A \xrightarrow{f_2} X \xleftarrow[\simeq]{f_1} B$$

in  $\mathbf{W}^{-1}\mathbf{C}(A, B)$  for  $\mathcal{C}$ , and then  $\phi$  is represented by the word

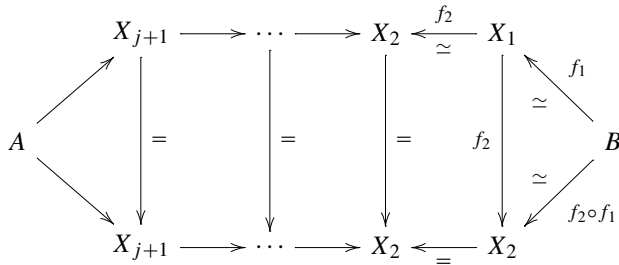
$$FA \xrightarrow{Ff_2} X \xleftarrow[\simeq]{Ff_1} B$$

in  $\mathbf{W}^{-1}\mathbf{C}(FA, FB)$  for  $\mathcal{D}$ . Theorem 5.6 implies that this word is in the components  $\mathbf{W}^{-1}\mathbf{C}(FA, FB)_w$  of  $\mathbf{W}^{-1}\mathbf{C}(FA, FB)$  and hence that  $Ff_2$  is a weak equivalence. Hypothesis (ii) implies that  $f_2$  is a weak equivalence in  $\mathcal{C}$ , and it follows that  $\phi$  is in the image of  $\text{Ho } w\mathcal{C}(A, B)$ .  $\square$

The remainder of the section is devoted to the proof of Theorem 5.5. Thus, assume that  $\mathcal{C}$  has FMCWC and fix  $A, B$  in  $\mathcal{C}$ ; we need to prove that for every pair of integers  $i, j \geq 0$ , the functor  $\mathbf{W}^{-1}\mathbf{C}^i\mathbf{C}^j(A, B) \rightarrow \mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(A, B)$  induces a weak equivalence of nerves in each of the three cases where the letter “C” in the words indicates the category  $\mathcal{C}$ ,  $\tilde{\mathcal{C}}\mathcal{C}$ , or  $w\mathcal{C}$ . The proof is the same in all three cases. The following lemma is the case  $i = 0$ .

**Lemma 5.9.** *The functor  $\mathbf{W}^{-1}\mathbf{C}^j(A, B) \rightarrow \mathbf{W}^{-1}\mathbf{W}^{-1}\mathbf{C}^j(A, B)$  induces a homotopy equivalence of nerves.*

**Proof.** Composition induces a functor back  $\mathbf{W}^{-1}\mathbf{W}^{-1}\mathbf{C}^j(A, B) \rightarrow \mathbf{W}^{-1}\mathbf{C}^j(A, B)$ . The composite functor on  $\mathbf{W}^{-1}\mathbf{C}^j(A, B)$  is the identity and the composite functor on  $\mathbf{W}^{-1}\mathbf{W}^{-1}\mathbf{C}^j(A, B)$  has a natural transformation to the identity.



These functors then induce inverse homotopy equivalences on nerves.  $\square$

The following lemma now completes the proof of Theorem 5.5.

**Lemma 5.10.** *For  $i > 0$  and  $j \geq 0$ , the functor*

$$\mathbf{W}^{-1}\mathbf{C}^i\mathbf{C}^j(A, B) \rightarrow \mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(A, B)$$

*induces a weak equivalence on nerves.*

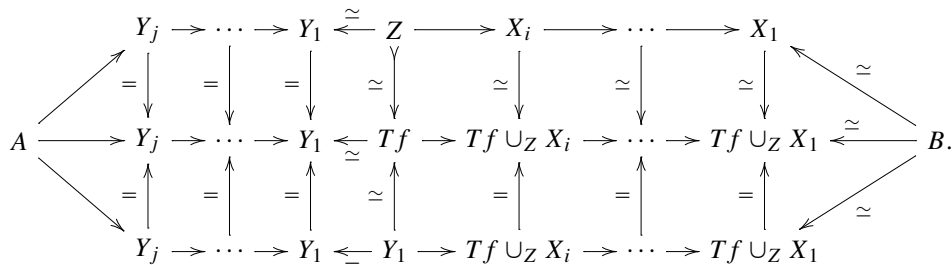
**Proof.** We obtain a functor  $\mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(A, B) \rightarrow \mathbf{W}^{-1}\mathbf{C}^i\mathbf{C}^j(A, B)$  by applying the mapping cylinder functor and taking pushouts: The object

$$A \longrightarrow Y_j \longrightarrow \cdots \longrightarrow Y_1 \xleftarrow[\simeq]{f} Z \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \xleftarrow[\simeq]{} B$$

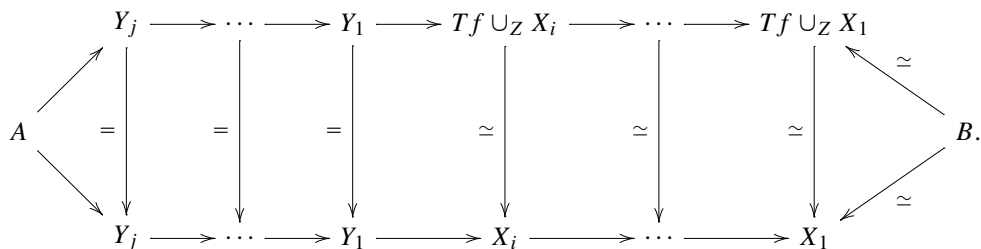
of  $\mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(A, B)$  is sent to the object

$$A \longrightarrow Y_j \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Tf \cup_Z X_i \longrightarrow Tf \cup_Z X_{i-1} \longrightarrow \cdots \longrightarrow Tf \cup_Z X_1 \xleftarrow[\simeq]{} B$$

of  $\mathbf{W}^{-1}\mathbf{C}^i\mathbf{C}^j(A, B)$ . The composite functor on  $\mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(A, B)$  has a zigzag of natural transformations relating it to the identity:



For the composite functor on  $\mathbf{W}^{-1}\mathbf{C}^i\mathbf{C}^j(A, B)$ ,  $f$  is the identity map  $Z = Y_1$ , and the map  $Tf \rightarrow Y_1 = Z$  composed with the map  $Z \rightarrow X_i$  induces a natural transformation from the composite functor to the identity



The induced map on nerves is therefore a generalized simplicial homotopy equivalence.  $\square$

### 6. Homotopy cocartesian squares in Waldhausen categories

Fundamentally, algebraic  $K$ -theory is about splitting “extensions”, and in Waldhausen’s framework, the category of cofibrations specifies the extensions to split. The key concept is the homotopy cocartesian square, i.e., a square diagram that is weakly equivalent (by a zigzag) to a pushout square where one of the parallel sets of arrows consists of cofibrations. In simplicial model categories, the homotopy cocartesian squares can be characterized in terms of mapping spaces. The Dwyer–Kan simplicial localization and function complexes extend this alternative definition of homotopy cocartesian to the context of Waldhausen categories. In this section we show that under mild hypotheses these two definitions are equivalent in Waldhausen categories. This equivalence reflects another aspect of the intrinsic relationship between the Dwyer–Kan localization and algebraic  $K$ -theory, and plays a key role in the proofs of Theorems 1.3 and 1.4. The following is the main theorem of the section.

**Theorem 6.1.** *Let  $\mathcal{C}$  be a Waldhausen category whose weak equivalences are closed under retracts. Suppose furthermore that  $\mathcal{C}$  admits a HCLF and every weak cofibration in  $\mathcal{C}$  has a mapping cylinder (e.g., when  $\mathcal{C}$  has FMCWC). Let  $A \rightarrow B$  be a weak cofibration and  $A \rightarrow C$ ,  $B \rightarrow D$ , and  $C \rightarrow D$  maps that make the square on the left commute.*

$$\begin{array}{ccc}
 A & \xrightarrow{wc} & B \\
 \downarrow & & \downarrow \\
 C & \longrightarrow & D
 \end{array}
 \qquad
 \begin{array}{ccc}
 LC(D, E) & \longrightarrow & LC(C, E) \\
 \downarrow & & \downarrow \\
 LC(B, E) & \longrightarrow & LC(A, E)
 \end{array}$$

Then the square on the left is homotopy cocartesian in  $\mathcal{C}$  if and only if for every  $E$  the square of simplicial sets on the right is homotopy cartesian.

In fact, the forward direction holds under slightly weaker hypotheses, and we state this as the following theorem, which implies Theorem 3.5 in Section 3. A similar result is the main theorem of [21].

**Theorem 6.2.** *Let  $\mathcal{C}$  be a Waldhausen category that admits a HCLF. For a cofibration  $A \rightarrow B$ , a map  $A \rightarrow C$ ,  $D = B \cup_A C$ , and any object  $E$ , the following square is homotopy cartesian:*

$$\begin{array}{ccc}
 LC(D, E) & \longrightarrow & LC(C, E) \\
 \downarrow & & \downarrow \\
 LC(B, E) & \longrightarrow & LC(A, E).
 \end{array}$$

The previous theorems make it easier to check in certain cases that functors are weakly exact. Applying Theorem 6.2 in  $\mathcal{C}$  and Theorem 6.1 in  $\mathcal{D}$  gives the following corollary.

**Corollary 6.3.** *Let  $\mathcal{C}$  be a Waldhausen category that admits a HCLF, and let  $\mathcal{D}$  be a Waldhausen category that admits a HCLF, whose weak equivalences are closed under retracts, and whose weak cofibrations have mapping cylinders. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor that preserves weak equivalences and weak cofibrations. If  $F$  is a DK-equivalence, then  $F$  preserves homotopy cocartesian squares and so is weakly exact.*

The hypothesis that weak equivalences are closed under retracts is familiar from the theory of model categories. Two other properties of weak equivalences in model categories are currently somewhat less well known but explored in [6]. The more subtle of these is the “two out of six” property [6, 7.3], which we abbreviate to DKHS-2/6. The subcategory  $w\mathcal{C}$  satisfies DKHS-2/6 when for any three composable maps

$$f : A \rightarrow B, \quad g : B \rightarrow C, \quad h : C \rightarrow D,$$

if the composites  $g \circ f$  and  $h \circ g$  are in  $w\mathcal{C}$ , then so are the original maps  $f$ ,  $g$ , and  $h$ . The proof of Theorem 6.1 depends more directly on the other property, which [6, 8.4] calls “saturated” and we call *DKHS-saturated* (to avoid confusion with Waldhausen’s terminology). By definition, the localization functor  $\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$  sends weak equivalences to isomorphisms. We say that the weak equivalences of  $\mathcal{C}$  are DKHS-saturated when a map is a weak equivalence if and only if its image in  $\text{Ho}\mathcal{C}$  is an isomorphism. Note that when  $\mathcal{C}$  admits a homotopy calculus of two-sided fractions, the weak equivalences of  $\mathcal{C}$  are DKHS-saturated if and only if the subcategory of  $\text{Ho}\mathcal{C}$  generated by the weak equivalences and their inverses consists of all the isomorphisms of  $\text{Ho}\mathcal{C}$ . We prove the following theorem at the end of the section.

**Theorem 6.4.** *Let  $\mathcal{C}$  be a Waldhausen category that admits a HCLF and assume that every weak cofibration in  $\mathcal{C}$  has a mapping cylinder. The following are equivalent:*

- (i) *The weak equivalences are closed under retracts.*
- (ii) *The weak equivalences satisfy the DKHS-2/6 property.*
- (iii) *The weak equivalences are DKHS-saturated.*

We can now prove Theorem 6.1, assuming Theorems 6.2 and 6.4.

**Proof of Theorem 6.1 from Theorems 6.2 and 6.4.** The “only if” direction follows from Theorem 6.2. For the “if” direction, by factoring the weak cofibration  $A \rightarrow B$  as a cofibration followed by a weak equivalence, it suffices to consider the case when  $A \rightarrow B$  is a cofibration. Consider the induced map  $B \cup_A C \rightarrow D$ ; in the commutative diagram of simplicial sets

$$\begin{array}{ccccc}
 LC(D, E) & \longrightarrow & LC(B \cup_A C, E) & \longrightarrow & LC(C, E) \\
 & \searrow & \downarrow & & \downarrow \\
 & & LC(B, E) & \longrightarrow & LC(A, E)
 \end{array}$$

both the outer “square” and the inner square are homotopy cartesian. It follows that the map  $LC(D, E) \rightarrow LC(B \cup_A C, E)$  is a weak equivalence for all objects  $E$ , and so in particular,  $B \cup_A C \rightarrow D$  is an isomorphism in  $\text{Ho}\mathcal{C}$ . Because  $\mathcal{C}$  is DKHS-saturated by Theorem 6.4,  $B \cup_A C \rightarrow D$  is a weak equivalence.  $\square$

The proof of Theorem 6.2 is slightly more complicated. We apply Quillen’s Theorem B as formulated in Theorem 4.5 to the short hammock version of  $LC(B, E)$ , the nerve of the category  $\mathbf{W}^{-1}\mathbf{C}(A, E)$ . Recall that  $\mathbf{W}^{-1}\mathbf{C}(A, E)$  is the category whose objects are the zigzags  $\overleftrightarrow{X}$

$$A \longrightarrow X \xleftarrow{\simeq} E$$

and whose maps are the maps  $X \rightarrow X'$  under  $A$  and  $E$ , as in Example 5.2(i). Composition with  $f : A \rightarrow B$  induces a functor  $f^* : \mathbf{W}^{-1}\mathbf{C}(B, E) \rightarrow \mathbf{W}^{-1}\mathbf{C}(A, E)$ . Theorem 6.2 is an immediate consequence of Theorem 4.5 and the following lemma.

**Lemma 6.5.** *Assume  $\mathcal{C}$  admits a HCLF and  $f : A \rightarrow B$  is a cofibration. For any map  $\overleftrightarrow{X} \rightarrow \overleftrightarrow{X}'$  in  $\mathbf{W}^{-1}\mathbf{C}(A, E)$  the induced map of comma categories from  $\overleftrightarrow{X}' \downarrow f^*$  to  $\overleftrightarrow{X} \downarrow f^*$  induces a weak equivalence of nerves.*

**Proof.** The argument is another application of Theorem 4.5. Let  $w\mathcal{C}_E$  denote the full subcategory of  $w\mathcal{C}$  consisting of those objects that are weakly equivalent to  $E$ . We have a functor  $G_{\overleftrightarrow{X}}$  from  $\overleftrightarrow{X} \downarrow f^*$  to  $w\mathcal{C}_E$  that sends the object  $\overleftrightarrow{Y}$

$$\begin{array}{ccc}
 A & \longrightarrow & X \xleftarrow{\simeq} E \\
 \downarrow & & \downarrow \simeq \\
 B & \longrightarrow & Y
 \end{array}$$

to  $Y$ . For a map  $\overleftarrow{X} \rightarrow \overleftarrow{X}'$  in  $\mathbf{W}^{-1}\mathbf{C}(A, E)$ , consider the following strictly commuting diagram of functors:

$$\begin{array}{ccc}
 \overleftarrow{X}' \downarrow f^* & \xrightarrow{G_{\overleftarrow{X}'}} & \mathbf{wC}_E \\
 \downarrow & & \downarrow = \\
 \overleftarrow{X} \downarrow f^* & \xrightarrow{G_{\overleftarrow{X}}} & \mathbf{wC}_E.
 \end{array} \tag{6.6}$$

We verify that this diagram satisfies the hypotheses of Theorem 4.5. For any object  $H$  in  $\mathbf{wC}_E$ , the comma category  $H \downarrow G_{\overleftarrow{X}}$  has as objects the diagrams of the form

$$\begin{array}{ccc}
 A & \longrightarrow & X < \xrightarrow{\cong} E \\
 \downarrow & & \downarrow \cong \\
 B & \longrightarrow & Y < \xrightarrow[\cong]{} H.
 \end{array}$$

Using the universal property of the pushout, we see that this category is equivalent to the full subcategory of  $\mathbf{W}^{-1}\mathbf{C}(B \cup_A X, H)$  of those zigzags

$$B \cup_A X \longrightarrow Y < \xrightarrow{\cong} H$$

for which the composite map  $E \rightarrow B \cup_A X \rightarrow Y$  is a weak equivalence. Thus,  $H \downarrow G_{\overleftarrow{X}}$  is equivalent to a disjoint union of certain components of  $\mathbf{W}^{-1}\mathbf{C}(B \cup_A X, H)$ . Hypotheses (i) and (ii) of Theorem 4.5 follow from the fact that  $N(\mathbf{W}^{-1}\mathbf{C}(B \cup_A X, H))$  preserves weak equivalences in  $H$  and  $X$ . We conclude that diagram (6.6) is a homotopy cartesian square. Since the vertical map on the right is a weak equivalence, so is the vertical map on the left.  $\square$

It remains to prove Theorem 6.4. Obviously (iii) implies both (i) and (ii); we show that (i) implies (ii) and (ii) implies (iii).

**Proof that (i) implies (ii).** Let  $A \rightarrow B \rightarrow C \rightarrow D$  be a sequence of composable maps, with  $A \rightarrow C$  and  $B \rightarrow D$  weak equivalences. Since  $A \rightarrow C$  is a weak equivalence, it is a weak cofibration, and so we can factor it as a cofibration  $A \rightarrow C'$  followed by a split weak equivalence  $C' \rightarrow C$ . Let  $B' = C' \cup_A B$ .

$$\begin{array}{ccc}
 A & \xrightarrow{\cong} & C \\
 \cong \downarrow & \nearrow & \uparrow = \\
 C' & \longleftarrow & C
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\cong} & C' \\
 \downarrow & & \downarrow \\
 B & \xrightarrow[\cong]{} & B'
 \end{array}$$

We have a composite map  $f : C \rightarrow C' \rightarrow B'$ , and the compatible maps  $C' \rightarrow C$  and  $B \rightarrow C$  induce a map  $g : B' \rightarrow C$  such that  $g \circ f$  is the identity on  $C$ . Since the composite map

$B \rightarrow C \rightarrow D$  is a weak equivalence, the composite of  $g$  with  $C \rightarrow D$  is a weak equivalence. We therefore obtain a commutative diagram

$$\begin{array}{ccccc}
 C & \xrightarrow{f} & B' & \xrightarrow{g} & C \\
 \downarrow & & \downarrow \simeq & & \downarrow \\
 D & \xrightarrow{=} & D & \xrightarrow{=} & D,
 \end{array}$$

where both horizontal composites are the identity and the middle vertical map is a weak equivalence. We conclude from (i) that the map  $C \rightarrow D$  is a weak equivalence, and it follows that  $B \rightarrow C$  and  $A \rightarrow B$  are weak equivalences.  $\square$

**Proof that (ii) implies (iii).** (Cf. [6, 36.4].) Let  $a : A \rightarrow B$  be a map in  $\mathcal{C}$  that becomes an isomorphism in  $\text{Ho}\mathcal{C}$ . Since  $\mathcal{C}$  admits a HCLF, the inverse isomorphism from  $\text{Ho}\mathcal{C}$  is represented by a zigzag (in  $\mathcal{C}$ ) of the form

$$B \xrightarrow{b} C \xleftarrow[\simeq]{c} A \tag{6.7}$$

for some  $C$ . Moreover, using a mapping cylinder, we can assume without loss of generality that  $c : A \rightarrow C$  is a cofibration as well as a weak equivalence. The composite zigzag

$$A \xrightarrow{b \circ a} C \xleftarrow[\simeq]{c} A$$

is in the component of the identity on  $A$ , and so  $b \circ a$  is a weak equivalence. Let  $B' = B \cup_A C$ , and let  $C' = C \cup_B B'$ .

$$\begin{array}{ccccc}
 A & \xrightarrow{a} & B & \xrightarrow{b} & C \\
 \downarrow c \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 B & \xrightarrow{b} & C & \longrightarrow & B' & \longrightarrow & C'
 \end{array} \tag{6.8}$$

The composite  $C \rightarrow C'$  is a weak equivalence because it is the pushout of the weak equivalence  $b \circ a$  over the cofibration  $c$ . The zigzag

$$B \xrightarrow{b} C \longrightarrow B' \xleftarrow[\simeq]{} B$$

is in the component representing the composite of the zigzag (6.7) with  $a$ , i.e., the component containing the identity of  $B$ . It follows that in the diagram (6.8), the horizontal composite map  $B \rightarrow B'$  is a weak equivalence. Applying DKHS-2/6 to the bottom horizontal sequence of maps in (6.8), we conclude that  $b$  is a weak equivalence, and hence that  $a$  is a weak equivalence.  $\square$



**7. Proof of Theorems 1.1, 1.2, and 2.7**

We begin with the proof of part (i) of Theorem 2.7. The first reduction is to replace  $wS'_n\mathcal{C}$  with a simpler category. Let  $F'_{n-1}\mathcal{C}$  denote the Waldhausen category whose objects are the sequences of  $n - 1$  composable weak cofibrations in  $\mathcal{C}$ . We have an exact forgetful functor  $S'_n\mathcal{C} \rightarrow F'_{n-1}\mathcal{C}$  that sends an object  $\{A_{i,j}\}$  of  $S'_n\mathcal{C}$  to the sequence

$$A_{0,1} \rightarrow A_{0,2} \rightarrow \cdots \rightarrow A_{0,n}.$$

The Waldhausen category  $F'_{n-1}\mathcal{C}$  is the analogue for  $S'_n\mathcal{C}$  of the Waldhausen category  $F_{n-1}\mathcal{C}$ , whose objects are the sequences of  $n - 1$  composable cofibrations in  $\mathcal{C}$ . The forgetful functor  $S_n\mathcal{C} \rightarrow F_{n-1}\mathcal{C}$  is exact and an equivalence of categories, whose inverse equivalence is also exact. Proposition 2.4(iii) and the analogous fact for  $F_{n-1}$  and  $F'_{n-1}$  then implies the following proposition. (See Appendix B for the statement in the non-functorial case.)

**Proposition 7.1.** *If  $\mathcal{C}$  admits FFWC, the forgetful functor  $wS'_n\mathcal{C} \rightarrow wF'_{n-1}\mathcal{C}$  induces a weak equivalence of nerves.*

We use Proposition 7.1 to simplify one side of the equivalence in part (i) of Theorem 2.7, and we use homotopy calculus of left fractions to simplify the other side. Although the categories  $\mathbf{W}^{-1}\mathbf{C}$  produce much more manageable simplicial sets than the hammock function complexes, they are contravariant in weak equivalences of each variable and so do not have the right functoriality to fit into a homotopy coend. The categories  $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}$  are covariant in weak equivalences of the source variable and contravariant in weak equivalences of the target variable, which is the opposite variance expected of a function complex. We do likewise have such an opposite variance on the hammock function complexes  $LC(X, Y)$  since the category  $LC$  contains “backward” copies of the weak equivalences. The following lemma compares the homotopy coend in Theorem 2.7 with the homotopy coend for the opposite variance.

**Lemma 7.2.** *When  $\mathcal{C}$  satisfies HCLF,*

$$\text{hocoend}_{(X_1, \dots, X_n) \in w\mathcal{C}^n} LC_{\text{co}}(X_{n-1}, X_n) \times \cdots \times LC_{\text{co}}(X_1, X_2)$$

and

$$\text{hocoend}_{(X'_1, \dots, X'_n) \in (w\mathcal{C}^{\text{op}})^n} LC_{\text{co}}(X'_{n-1}, X'_n) \times \cdots \times LC_{\text{co}}(X'_1, X'_2)$$

are weakly equivalent.

**Proof.** Write  $B$  and  $B'$  for these homotopy coends, and let  $D$  be the homotopy coend

$$D = \text{hocoend}_{(X_1, X'_1, \dots, X_n, X'_n) \in (w\mathcal{C} \times w\mathcal{C}^{\text{op}})^n} \mathbf{W}^{-1}\mathbf{W}(X'_n, X_n) \times LC_{\text{co}}(X'_{n-1}, X_n) \times \mathbf{W}^{-1}\mathbf{W}(X_{n-1}, X'_{n-1}) \\ \times \cdots \times \mathbf{W}^{-1}\mathbf{W}(X_2, X'_2) \times LC_{\text{co}}(X'_1, X_2) \times \mathbf{W}^{-1}\mathbf{W}(X_1, X'_1).$$

Composition then induces maps  $D \rightarrow B$  and  $D \rightarrow B'$ . Let  $C$  be the homotopy colimit

$$C = \operatorname{hocolim}_{(X_1, X'_1, \dots, X_n, X'_n) \in (\mathcal{wC} \times \mathcal{wC}^{\text{op}})^n} \mathbf{W}^{-1}\mathbf{W}(X'_n, X_n) \times \mathbf{W}^{-1}\mathbf{W}(X_{n-1}, X'_{n-1}) \\ \times \cdots \times \mathbf{W}^{-1}\mathbf{W}(X_1, X'_1).$$

We have an evident map  $D \rightarrow C$  obtained by dropping the  $LC_{\text{co}}$  factors, and we have maps  $C \rightarrow N\mathcal{wC}^n$  and  $C \rightarrow N(\mathcal{wC}^{\text{op}})^n$  obtained from the canonical map  $C \rightarrow N((\mathcal{wC} \times \mathcal{wC}^{\text{op}})^n)$  by dropping the  $X'_i$  or the  $X_i$  respectively. We then have the following commuting diagrams

$$\begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & N\mathcal{wC}^n \end{array} \qquad \begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow \\ B' & \longrightarrow & N(\mathcal{wC}^{\text{op}})^n \end{array}$$

that are easily seen to be pullback squares. By Proposition 4.3, the canonical map  $C \rightarrow N((\mathcal{wC} \times \mathcal{wC}^{\text{op}})^n)$  is a universal simplicial quasifibration as are projection maps, and so the right vertical maps above are universal simplicial quasifibrations. We will show that for each vertex of  $N\mathcal{wC}^n$  and of  $N(\mathcal{wC}^{\text{op}})^n$ , the fiber of the right vertical map is weakly contractible; it then follows that the vertical maps are weak equivalences, and this gives a zigzag of weak equivalences relating  $B$  and  $B'$ .

Thus, we are reduced to proving that the fibers of the right vertical maps are weakly contractible. We will treat the case of  $N\mathcal{wC}^n$ , the case of  $N(\mathcal{wC}^{\text{op}})^n$  being similar. The map  $C \rightarrow N\mathcal{wC}^n$  is the product of maps  $\operatorname{hocolim}_{X, X'} \mathbf{W}^{-1}\mathbf{W}(X, X') \rightarrow N\mathcal{wC}$ , and so it suffices to see that each of these maps has contractible fiber. Fixing a vertex  $X$  in  $N\mathcal{wC}$ , the fiber is the simplicial set with  $r$ -simplices the diagrams

$$\begin{array}{ccccccc} X & & X'_r & \xleftarrow{\cong} & \cdots & \xleftarrow{\cong} & X'_0 \\ & \searrow \cong & \downarrow \cong & & & & \\ & & A_0 & \xrightarrow{\cong} & \cdots & \xrightarrow{\cong} & A_r. \end{array}$$

We can regard this as the diagonal of the bisimplicial set  $F_{\bullet\bullet}$  with  $(q, r)$ -simplices the diagrams

$$\begin{array}{ccccccc} X & & X'_r & \xleftarrow{\cong} & \cdots & \xleftarrow{\cong} & X'_0 \\ & \searrow \cong & \downarrow \cong & & & & \\ & & A_0 & \xrightarrow{\cong} & \cdots & \xrightarrow{\cong} & A_q. \end{array}$$

We have a bisimplicial map to the bisimplicial set  $A_{\bullet\bullet}$  with  $(q, r)$ -simplices the diagrams

$$X \xrightarrow{\cong} A_0 \xrightarrow{\cong} \cdots \xrightarrow{\cong} A_q$$

(constant in the  $r$  direction) by forgetting the objects  $X'_0, \dots, X'_r$ . For each fixed  $q$ -simplex, this is a homotopy equivalence in the  $r$  direction using the usual simplicial contraction argument, i.e.,

using the contraction on  $N(wC^{op} \setminus A_0)$ . It follows that the map  $F_{\bullet\bullet} \rightarrow A_{\bullet\bullet}$  is a weak equivalence. On the other hand,  $A_{\bullet\bullet}$  is the constant bisimplicial set on  $N(wC \setminus X)$  and so is contractible. We conclude that  $F_{\bullet\bullet}$  and its diagonal are weakly contractible.  $\square$

We now prove part (i) of Theorem 2.7 by comparing  $N(wF'_{n-1})$  with the homotopy coend in Lemma 7.2. Thus, let  $\mathcal{C}$  be a Waldhausen category that has FMCWC and fix  $n \geq 2$ . We prove the comparison in a sequence of reductions  $\mathbb{A}, \mathbb{B}, \mathbb{C}$  obtained from applying simplicial homotopy theory.

**Lemma 7.3.**  *$NwF'_{n-1}\mathcal{C}$  is equivalent to the diagonal simplicial set of the bisimplicial set  $\mathbb{A}$  that has as its  $(q, r)$  simplices the commutative diagrams of the following form*

$$\begin{array}{ccccccc}
 X_{0,1} & \xrightarrow{wc} & \cdots & \xrightarrow{wc} & X_{0,n} & \xleftarrow{\simeq} & B_0 \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} B_q \\
 \simeq \downarrow & & & \simeq \downarrow & \simeq \downarrow & & \\
 \vdots & \xrightarrow{wc} & \cdots & \xrightarrow{wc} & \vdots & & \\
 \simeq \downarrow & & & \simeq \downarrow & \simeq \downarrow & & \\
 A_q & \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} A_0 & \xleftarrow{\simeq} X_{r,1} & \xrightarrow{wc} \cdots \xrightarrow{wc} & X_{r,n} & & 
 \end{array}$$

where the maps labeled “ $\simeq$ ” are weak equivalences and the maps labeled “ $wc$ ” are weak cofibrations.

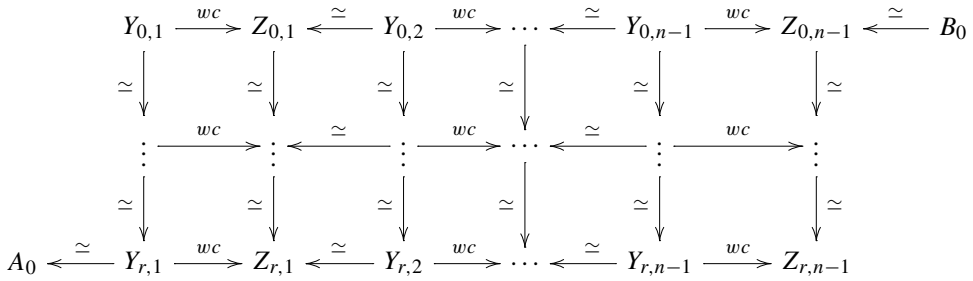
**Proof.** Each  $(q, r)$  simplex is specified by an  $r$ -simplex of  $NwF'_{n-1}\mathcal{C}$ , a  $q$ -simplex of  $Nw(\mathcal{C} \setminus X_{r,1})$  and a  $q$ -simplex of  $Nw(\mathcal{C} / X_{0,n})$ . Regarding  $NwF'_{n-1}\mathcal{C}$  as a bisimplicial set constant in the  $q$  direction, we get a bisimplicial map from  $\mathbb{A}$  to  $NwF'_{n-1}\mathcal{C}$ . Since for each fixed  $r$ -simplex of  $NwF'_{n-1}\mathcal{C}$ , the simplicial sets  $Nw(\mathcal{C} \setminus X_{r,1})$  and  $Nw(\mathcal{C} / X_{0,n})$  are contractible, the bisimplicial map induces a weak equivalence on diagonals.  $\square$

In the case  $n = 2$ , the diagonal of the bisimplicial set  $\mathbb{A}$  is

$$\text{hocoend}_{(A,B) \in (wC^{op})^2} N(\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_{\tilde{co}}),$$

where, as in Theorem 5.6,  $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_{\tilde{co}}$  denotes the full subcategory of  $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)$  of diagrams where the forward arrow is a weak cofibration. Lemma 7.2 then finishes the argument for the case  $n = 2$ . Now assume  $n \geq 3$ .

**Lemma 7.4.** *The diagonal of the bisimplicial set  $\mathbb{A}$  is weakly equivalent to the diagonal of the bisimplicial set  $\mathbb{B}$  that has as its  $(q, r)$  simplices the commutative diagrams of the following form*

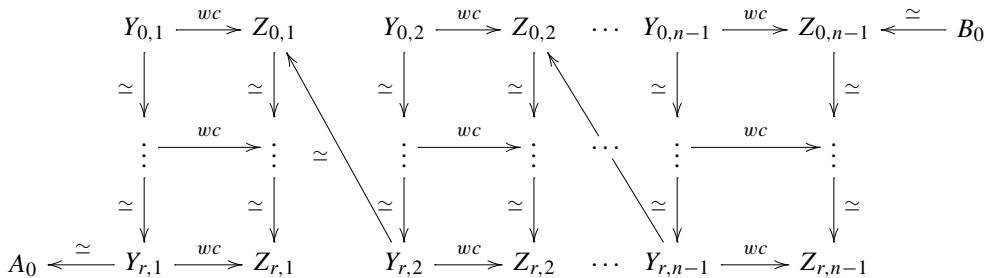


together with sequences of weak equivalences

$$A_q \xleftarrow{\cong} \cdots \xleftarrow{\cong} A_0, \quad B_0 \xleftarrow{\cong} \cdots \xleftarrow{\cong} B_q.$$

**Proof.** We have an inclusion of  $\mathbb{A}$  in  $\mathbb{B}$  by inserting identity maps in the appropriate columns. The lemma now follows from Theorem 5.5 and homotopy calculus of two-sided fractions [4, 9.4, 9.5].  $\square$

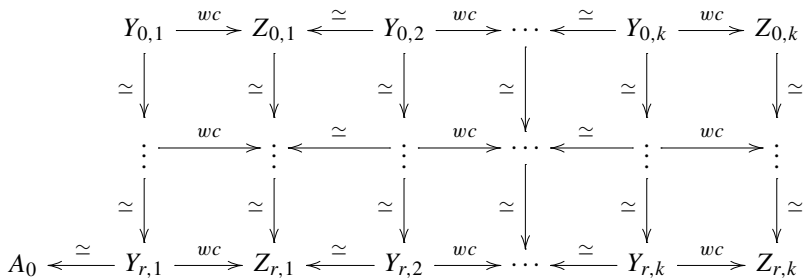
**Lemma 7.5.** *The diagonal of the bisimplicial set  $\mathbb{B}$  is weakly equivalent to the diagonal of the bisimplicial set  $\mathbb{C}$  that has as its  $(q, r)$  simplices the commutative diagrams of the following form*



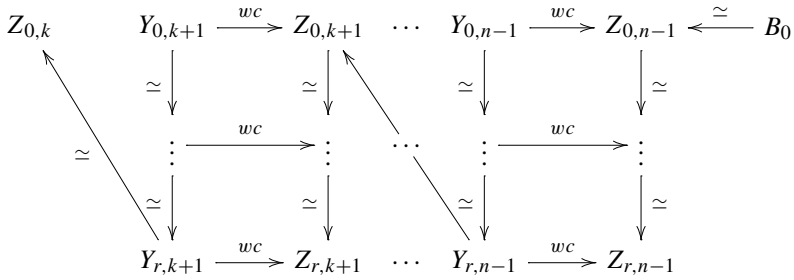
together with sequences of weak equivalences

$$A_q \xleftarrow{\cong} \cdots \xleftarrow{\cong} A_0, \quad B_0 \xleftarrow{\cong} \cdots \xleftarrow{\cong} B_q.$$

**Proof.** Fix  $A_0, B_0$  and consider the simplicial set  $\mathbb{C}_k$  that has its  $r$ -simplices the pairs of commutative diagrams



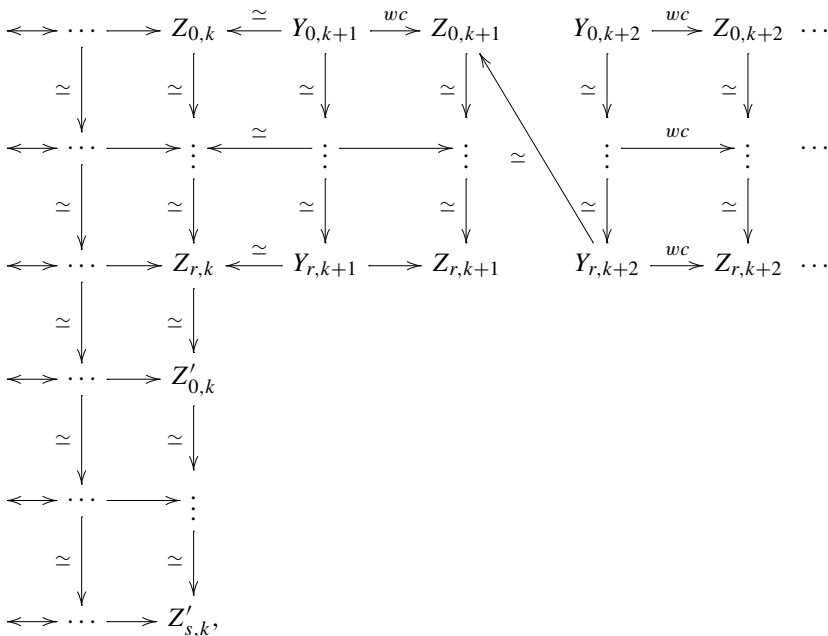
and



together with sequences of weak equivalences

$$A_q \xleftarrow{\cong} \cdots \xleftarrow{\cong} A_0, \quad B_0 \xleftarrow{\cong} \cdots \xleftarrow{\cong} B_q.$$

Then  $\mathbb{C}_1$  is  $\mathbb{C}$  and  $\mathbb{C}_{n-1}$  is  $\mathbb{B}$ . We construct a zigzag of weak equivalences between  $\mathbb{C}_{k+1}$  and  $\mathbb{C}_k$  with the diagonal of a bisimplicial set in the middle. Let  $\mathbb{D}_k$  be the bisimplicial set that has as its  $(r, s)$  simplices the commutative diagrams of the following form



where to the right the columns look like those in  $\mathbb{C}$  and to the left the columns look like those in  $\mathbb{B}$ . Regarding  $\mathbb{C}_{k+1}$  as a bisimplicial set constant in the  $s$ -direction, we obtain a bisimplicial map  $\mathbb{D}_k \rightarrow \mathbb{C}_{k+1}$  by forgetting the  $Y'_{i,j}$  and  $Z'_{i,j}$  parts of the diagram. It is easy to see that this map is a weak equivalence using the fact that the undercategory of

$$A_0 \xleftarrow{\simeq} Y_{r,k} \xrightarrow{wc} \cdots \longrightarrow Z_{r,k} \xleftarrow{\simeq} Y_{r,k+1}$$

in  $\mathbf{W}^{-1}\mathbf{C} \cdots \mathbf{W}^{-1}(A_0, Y_{r,k+1})$  has contractible nerve. To relate  $\mathbb{D}_k$  and  $\mathbb{C}_k$ , we regard  $\mathbb{C}_k$  as the diagonal of the bisimplicial set where  $Y_{i,j}, Z_{i,j}$  are indexed in  $i = 0, \dots, r$  for  $j > k$  and in  $i = 0, \dots, s$  for  $j \leq k$ ; to match the notation in  $\mathbb{D}_k$ , we will refer to these latter entries as  $Y'_{i,j}$  and  $Z'_{i,j}$  (for  $j \leq k$ ). We then get a bisimplicial map from  $\mathbb{D}_k$  to  $\mathbb{C}_k$  by forgetting the  $Y_{i,j}$  and  $Z_{i,j}$  parts of the  $\mathbb{D}_k$  diagram for  $j \leq k$ , and using the composite map  $Y_{r,k+1} \rightarrow Z'_{0,k}$ . For fixed  $s$ , this map is a simplicial homotopy equivalence: The inverse equivalence fills in the  $Y_{i,j}$  and  $Z_{i,j}$  entries for  $j \leq k$  with  $Y'_{i,0}$  and  $Z'_{i,0}$ . The composite on  $\mathbb{C}_k$  is the identity, and the composite on  $\mathbb{D}_k$  is homotopic to the identity by the usual argument. The map on diagonals from  $\mathbb{D}_k$  to  $\mathbb{C}_k$  is then a weak equivalence.  $\square$

Finally, to complete the argument, by Lemma 7.2 and Theorem 5.6, it suffices to see that the diagonal of the bisimplicial set  $\mathbb{C}$  is weakly equivalent to

$$\text{hocoend}_{(A, C_1, \dots, C_{n-2}, B) \in (\mathbf{wC}^{\text{op}})^n} N(\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(C_{n-2}, B)_{\tilde{\mathcal{C}}_0}) \times \cdots \times N(\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, C_1)_{\tilde{\mathcal{C}}_0}).$$

We can view the latter as the bisimplicial set with  $q$ -direction the nerve of  $(\mathbf{wC}^{\text{op}})^n$  and  $r$ -direction the nerve of the  $\mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(-, -)_{\tilde{\mathcal{C}}_0}$ . The  $(q, r)$ -simplices then look very similar to the diagrams that define  $\mathbb{C}$ , except that in place of the maps  $Z_{0,k} \leftarrow Y_{r,k+1}$ , we have sequences of maps of the form

$$Z_{0,k} \xleftarrow{\simeq} C_{q,k} \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} C_{0,k} \xleftarrow{\simeq} Y_{r,k+1}.$$

Composing induces a bisimplicial map from the homotopy coend to  $\mathbb{C}$  that is easily seen to be a weak equivalence.

Part (iii) of Theorem 2.7 follows from Proposition 4.4 and part (ii). Thus, it remains to prove part (ii), namely, that  $N\mathbf{wC}$  is weakly equivalent to the disjoint union of  $B \text{hAut}(X)$ . Fixing  $X$  in  $\mathcal{C}$ , the undercategory of  $X$  in  $\mathbf{wC}$  has contractible nerve. Then the (cartesian) commutative diagram of categories

$$\begin{array}{ccc} \mathbf{W}^{-1}\mathbf{W}(X, X) & \longrightarrow & \mathbf{wC} \setminus X \\ \downarrow & & \downarrow \\ \mathbf{wC} \setminus X & \longrightarrow & \mathbf{wC} \end{array}$$

satisfies the hypotheses of Theorem 4.5. Thus,  $\text{hAut}(X) \simeq \mathbf{W}^{-1}\mathbf{W}(X, X)$  is equivalent to the loop space of  $N\mathbf{wC}$  based at  $X$ .

This completes the proof of Theorem 2.7, which implies Theorems 1.1 and 1.2. We close the section with a remark on Theorem 1.1.

**Remark 7.6.** The decomposition of Theorem 1.1 does not fit into a simplicial structure to give a “construction” of the algebraic  $K$ -theory spectrum. An indirect construction of the algebraic  $K$ -theory spectrum for certain categories enriched in simplicial sets via the category of simplicial

functors can be found in [19, §4]. The Dwyer–Kan simplicial localization of a category that admits functorial factorization satisfies the hypotheses there, by Theorem 6.2.

A direct construction in terms of the Dwyer–Kan simplicial localization would include a description of face and degeneracy maps fitting the pieces together compatibly with the simplicial structure on the  $S_\bullet$  construction. Although we do not produce such a construction, we can reinterpret some of the simplicial structure maps of  $S_\bullet$  in terms of the spaces described above. The degeneracy maps, as in  $S_\bullet$ , are induced simply by repeating an object  $X_i$  and using the identity map in  $LC(X_i, X_i)$ , where we understand  $X_0$  as the distinguished zero object of  $\mathcal{C}$ . The face maps  $d_2, \dots, d_{n-1}$  are induced by composition

$$LC(X_i, X_{i+1}) \times LC(X_{i-1}, X_i) \rightarrow LC(X_{i-1}, X_{i+1}).$$

The face maps  $d_1$  and  $d_n$  essentially drop the first and  $n$ -th objects, respectively. The face map  $d_0$ , which in  $S_n$  corresponds to replacing the sequence of cofibrations  $X_1 \rightarrow \dots \rightarrow X_n$  with the quotient  $X_2/X_1 \rightarrow \dots \rightarrow X_n/X_1$ , is the impediment to making the spaces above into a simplicial object. Under the hypotheses of Theorem 2.7, Dwyer–Kan mapping complexes take pushouts along cofibrations to homotopy pullbacks, i.e., take homotopy cocartesian squares to homotopy cartesian squares. Roughly speaking, the face map  $d_0$  on the spaces above would involve composition and taking homotopy fibers.

**8. Proof of Theorem 1.4**

In this section we prove Theorem 1.4 following ideas in [13]. The first key step is relating the homotopy categories of undercategories to the higher homotopy data implicit in the Dwyer–Kan simplicial localization.

**Theorem 8.1.** *Let  $\mathcal{C}$  be a Waldhausen category that admits a HCLF and let  $A$  be an object of  $\mathcal{C}$ . Let  $\vec{B} = A \rightarrow B$  be a cofibration viewed as an object of  $\mathcal{C} \setminus A$ , let  $\vec{C} = A \rightarrow C$  be an object of  $\mathcal{C} \setminus A$ , and write  $\vec{A}$  for  $\text{id} : A \rightarrow A$ , viewed as an object of  $\mathcal{C} \setminus A$ . The following square is homotopy cartesian:*

$$\begin{CD} L(\mathcal{C} \setminus A)(\vec{B}, \vec{C}) @>>> L(\mathcal{C} \setminus A)(\vec{A}, \vec{C}) \\ @VVV @VVV \\ LC(B, C) @>>> LC(A, C). \end{CD}$$

**Proof.** Since  $\mathcal{C}$  admits a HCLF, it suffices to show that the square

$$\begin{CD} \mathbf{W}^{-1}\mathbf{C}_A(\vec{B}, \vec{C}) @>>> \mathbf{W}^{-1}\mathbf{C}_A(\vec{A}, \vec{C}) \\ @VVV @VVV \\ \mathbf{W}^{-1}\mathbf{C}(B, C) @>>> \mathbf{W}^{-1}\mathbf{C}(A, C) \end{CD}$$

is homotopy cartesian, where we have written  $\mathbf{W}^{-1}\mathbf{C}_A$  for the categories of words  $\mathbf{W}^{-1}\mathbf{C}$  in  $\mathcal{C} \setminus A$  to avoid confusion. We apply Theorem 4.5. An easy check of the definitions shows that this square satisfies the hypothesis (ii), and it satisfies hypothesis (i) by Lemma 6.5 (with  $E = C$ ).  $\square$

The previous theorem identifies the Dwyer–Kan function complexes in  $\mathcal{C}\backslash A$  for cofibrations. In the context of Theorem 1.4, factorization allows us to extend this to compute the Dwyer–Kan function complexes for arbitrary objects of  $\mathcal{C}\backslash A$ . The following proposition is immediately clear when  $\mathcal{C}$  admits functorial factorization; the non-functorial case is treated in Appendix B.

**Proposition 8.2.** *Let  $\mathcal{C}\backslash_{\text{co}}A$  denote the full subcategory of  $\mathcal{C}\backslash A$  consisting of the cofibrations. If  $\mathcal{C}$  admits factorization, then the inclusion  $\mathcal{C}\backslash_{\text{co}}A \rightarrow \mathcal{C}\backslash A$  is a DK-equivalence.*

For a map  $f : B \rightarrow C$  in  $\mathcal{C}$ , let  $\Omega_f LC(B, C)$  denote the space of based loops in the geometric realization of  $LC(B, C)$ , based at the vertex  $f$ . Thinking of  $\Omega_f LC(B, C)$  as the homotopy pullback of the diagram

$$\begin{array}{ccc} & LC(B, C) & \\ & \downarrow \Delta & \\ \{(f, f)\} & \longrightarrow & LC(B, C) \times LC(B, C), \end{array}$$

then up to weak equivalence, we can identify the lower right term as  $LC(B \vee B, C)$  by Theorem 6.2; we then get the following result as a corollary of the previous proposition and theorem.

**Corollary 8.3.** *Let  $\mathcal{C}$  be a Waldhausen category that admits factorization. Let  $f : B \rightarrow C$  be a map in  $\mathcal{C}$ . Viewing  $B$  as an object under  $B \vee B$  via the codiagonal map, and  $C$  as such via the composite with  $f$ , then the loop space  $\Omega_f LC(B, C)$  is weakly equivalent to  $L(\mathcal{C}\backslash(B \vee B))(B, C)$ .*

In comparing function complexes for  $\mathcal{C}$  to function complexes for  $\mathcal{D}$ , we need the following proposition, which is an easy consequence of Theorem 8.1 and Proposition 8.2.

**Proposition 8.4.** *Let  $A' \rightarrow A$  be a map in  $\mathcal{C}$  that is an isomorphism in  $\text{Ho}\mathcal{C}$ . If  $\mathcal{C}$  admits factorization, then the induced functor  $\mathcal{C}\backslash A \rightarrow \mathcal{C}\backslash A'$  is a DK-equivalence.*

Finally, we prove Theorem 1.4. Clearly, by Theorem 8.1 and Proposition 8.2, a DK-equivalence implies an equivalence of homotopy categories and homotopy categories of all under categories. For the converse, note that  $\mathcal{C}\backslash_{\text{co}}(B \vee B)$  inherits from  $\mathcal{C}$  the property of admitting factorization. Likewise, note that for an arbitrary component of  $LC(B, C)$ , we can find a vertex  $\phi$  of the form

$$B \xrightarrow{f} X \xleftarrow{\simeq} C$$

by HCLF. The loop space based at  $\phi$ ,  $\Omega_\phi LC(B, C)$ , is then homotopy equivalent to  $\Omega_f LC(B, X)$ , and so weakly equivalent to  $L(\mathcal{C}\backslash(B \vee B))(B, X)$ , as per Corollary 8.3. Thus, iterating Corollary 8.3 identifies the homotopy groups of  $LC(B, C)$  at arbitrary basepoints in terms of sets of maps in the homotopy categories of undercategories, as in [13, 5.4].

Specifically, we can identify  $\pi_n(LC(B, C))$  based at  $\phi$  as  $\text{Ho}(\mathcal{C}\backslash S^{n-1})(B, X)$ , for certain objects  $S^{n-1}$  formed inductively as follows: Starting with  $S^{-1} = *$  and  $B^0 = B$ ,  $S^n$  is formed as the coproduct  $B^n \cup_{S^{n-1}} B^n$  in  $\mathcal{C}\backslash_{\text{co}}S^{n-1}$  where  $B^n$  is an object of  $\mathcal{C}\backslash_{\text{co}}S^{n-1}$  with a weak equivalence  $B^n \rightarrow B$  in  $\mathcal{C}\backslash S^{n-1}$ . Now in  $\mathcal{D}$ , we can perform the analogous construction starting



with  $S_{\mathcal{D}}^{-1} = *$  and  $B_{\mathcal{D}}^0 = FB$  to form  $S_{\mathcal{D}}^{n-1}$  and  $B_{\mathcal{D}}^n$ . When inductively we choose the weak equivalence  $B_{\mathcal{D}}^n \rightarrow FB$  to factor through  $FB^n$  in  $\mathcal{D} \setminus S_{\mathcal{D}}^{n-1}$ , then  $S_{\mathcal{D}}^n \rightarrow FB$  factors through  $FS^n$ . Since we have not assumed that  $F$  is weakly exact, we cannot conclude that the map  $S_{\mathcal{D}}^n \rightarrow FS^n$  is a weak equivalence; however, we do have the following lemma, which then completes the proof of Theorem 1.4.

**Lemma 8.5.** *For all  $n$ , the restriction of  $F$  to a functor  $\mathcal{C} \setminus S^n \rightarrow \mathcal{D} \setminus S_{\mathcal{D}}^n$  induces an equivalence of homotopy categories  $\text{Ho}(\mathcal{C} \setminus S^n) \rightarrow \text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^n)$ .*

**Proof.** For  $n = -1$ ,  $S^{-1} = *$  and  $S_{\mathcal{D}}^{-1} = *$  are the initial object in each category; the equivalence  $\text{Ho}\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$  is part of the hypothesis on  $F$ . Now by induction, assume that  $\text{Ho}(\mathcal{C} \setminus S^{n-1}) \rightarrow \text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^{n-1})$  is an equivalence. By Theorem 6.2, we have that  $S^n = B^n \cup_{S^{n-1}} B^n$  is the coproduct of two copies of  $B$  in  $\text{Ho}(\mathcal{C} \setminus S^{n-1})$  and  $S_{\mathcal{D}}^n = B_{\mathcal{D}}^n \cup_{S_{\mathcal{D}}^{n-1}} B_{\mathcal{D}}^n$  is the coproduct of two copies of  $FB$  in  $\text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^{n-1})$ . It follows that the map  $S_{\mathcal{D}}^n \rightarrow FS^n$  is an isomorphism in  $\text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^{n-1})$  and hence in  $\text{Ho}\mathcal{D}$ . By Proposition 8.4, the map  $S_{\mathcal{D}}^n \rightarrow FS^n$  induces an equivalence  $\text{Ho}(\mathcal{D} \setminus FS^n) \rightarrow \text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^n)$ , and by hypothesis on  $F$ , the functor  $\text{Ho}(\mathcal{C} \setminus S^n) \rightarrow \text{Ho}(\mathcal{D} \setminus FS^n)$  is an equivalence. The functor  $\text{Ho}(\mathcal{C} \setminus S^n) \rightarrow \text{Ho}(\mathcal{D} \setminus S_{\mathcal{D}}^n)$  in question is the composite of these two equivalences.  $\square$

### 9. Proof of Theorem 1.5

This section proves Theorem 1.5, which relates the approximation property to the homotopy categories of the undercategories. It is convenient to prove the theorem in the following form.

**Theorem 9.1.** *Let  $\mathcal{C}$  be a Waldhausen category where every map factors as a cofibration followed by a weak equivalence. Let  $\mathcal{D}$  be a category with weak equivalences and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor that satisfies the approximation property, preserves finite coproducts, and preserves pushouts where one leg in  $\mathcal{C}$  is a cofibration. Then  $F$  induces an equivalence  $\text{Ho}\mathcal{C} \rightarrow \text{Ho}\mathcal{D}$ .*

Note that we do not assume that  $\mathcal{D}$  has all finite coproducts or pushouts; only the finite coproducts and pushouts required by the hypotheses are assumed to exist.

Theorem 9.1 implies Theorem 1.5: When  $\mathcal{C}$  has the property that every map can be factored as a cofibration followed by a weak equivalence, then for any object  $A$ , the inclusion of  $\mathcal{C} \setminus_{\text{co}} A$  in  $\mathcal{C} \setminus A$  (with notation as in Proposition 8.2 above) satisfies the hypothesis of Theorem 9.1 and so induces an equivalence of homotopy categories. Likewise,  $F$  regarded as a functor  $\mathcal{C} \setminus_{\text{co}} A \rightarrow \mathcal{D} \setminus FA$  satisfies the hypothesis of Theorem 9.1 and so induces an equivalence of homotopy categories. It follows that  $F$  induces an equivalence of homotopy categories  $\text{Ho}(\mathcal{C} \setminus A) \rightarrow \text{Ho}(\mathcal{D} \setminus FA)$ .

The remainder of the section is devoted to the proof of Theorem 9.1. We begin by constructing a functor  $R : \mathcal{D} \rightarrow \text{Ho}\mathcal{C}$ . For each object  $X$  in  $\mathcal{D}$ , apply the approximation property to the initial map  $F* \rightarrow X$  to choose an object  $RX$  in  $\mathcal{C}$  and a weak equivalence  $\epsilon_X : FRX \rightarrow X$  in  $\mathcal{D}$ . For each map in  $\mathcal{D}$ ,  $f : X \rightarrow Y$ , apply the approximation property to the map

$$F(RX \vee RY) \cong FRX \vee FRY \xrightarrow{f \circ \epsilon_X + \epsilon_Y} Y$$

to obtain an object  $Qf$  of  $\mathcal{C}$ , cofibrations  $RX \rightarrow Qf$  and  $RY \rightarrow Qf$ , and a commuting diagram in  $\mathcal{D}$ .

$$\begin{array}{ccc}
 FRX & \longrightarrow & FQf \longleftarrow FRY \\
 \simeq \downarrow \epsilon_X & & \searrow \simeq \downarrow \epsilon_Y \\
 X & \xrightarrow{f} & Y
 \end{array} \tag{9.2}$$

It follows from part (i) of the approximation property then that the map  $RY \rightarrow Qf$  is a weak equivalence, and so we obtain a zigzag in  $\mathcal{C}$ ,

$$RX \longrightarrow Qf \xleftarrow{\simeq} RY.$$

Let  $Rf$  be the map  $RX \rightarrow RY$  in  $\text{Ho}\mathcal{C}$  represented by this zigzag. The following lemma implies that  $Rf$  is independent of the choice of  $Qf$ .

**Lemma 9.3.** *Let  $f : X \rightarrow Y$  be a map in  $\mathcal{D}$ . Let*

$$RX \xrightarrow{\alpha} B \xleftarrow[\simeq]{\beta} RY$$

*be any zigzag in  $\mathcal{C}$ . If there exists a map  $\gamma : FB \rightarrow Y$  that makes the diagram*

$$\begin{array}{ccc}
 FRX & \longrightarrow & FB \xleftarrow[\simeq]{\beta} FRY \\
 \simeq \downarrow \epsilon_X & & \searrow \gamma \simeq \downarrow \epsilon_Y \\
 X & \xrightarrow{f} & Y
 \end{array}$$

*commutes in  $\mathcal{D}$ , then  $Rf = \beta^{-1} \circ \alpha$  in  $\text{Ho}\mathcal{C}$ .*

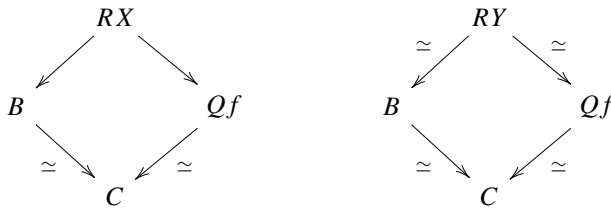
**Proof.** By construction, the map  $RX \vee RY \rightarrow Qf$  is a cofibration, and by factorization, we can assume without loss of generality that the map  $\alpha + \beta : RX \vee RY \rightarrow B$  is a cofibration. Since the maps  $FB \rightarrow Y$  and  $FQf \rightarrow Y$  both compose to the same maps  $FRX \rightarrow Y$  and the same maps  $FRY \rightarrow Y$ , we obtain a map

$$F(B \cup_{RX \vee RY} Qf) \cong FB \cup_{FRX \vee FRY} FQf \rightarrow Y.$$

Applying the approximation property to this map, we obtain an object  $C$  in  $\mathcal{C}$  and a map

$$B \cup_{RX \vee RY} Qf \rightarrow C$$

in  $\mathcal{C}$  such that the composites  $B \rightarrow C$  and  $Qf \rightarrow C$  are both weak equivalences and restrict to the same maps  $RX \rightarrow C$  and the same maps  $RY \rightarrow C$ . Thus, we have the following commutative diagrams in  $\mathcal{C}$ .



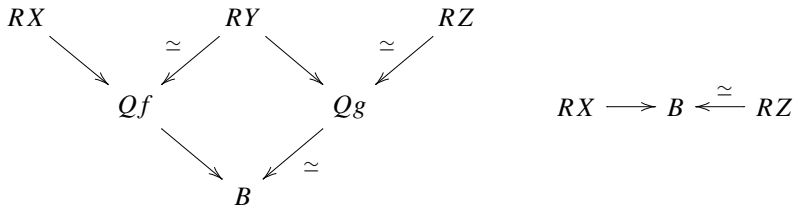
We see from these diagrams that the maps in  $\text{Ho}\mathcal{C}$  represented by the zigzags

$$RX \longrightarrow Qf \xleftarrow{\cong} RY, \quad RX \longrightarrow C \xleftarrow{\cong} RY, \quad RX \longrightarrow B \xleftarrow{\cong} RY,$$

coincide. The first is  $Rf$  and the third is  $\beta^{-1} \circ \alpha$ .  $\square$

**Theorem 9.4.**  $R$  is a functor  $\mathcal{D} \rightarrow \text{Ho}\mathcal{C}$ .

**Proof.** Applying Lemma 9.3 with  $\alpha = \text{id}_{RX} = \beta$  and  $\gamma = \epsilon_X$ , it follows that  $R \text{id}_X$  is  $\text{id}_{RX}$ . Now given maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  in  $\mathcal{D}$ , let  $B = Qf \cup_{RY} Qg$ . Then we see from the commuting diagram on the left



that  $Rg \circ Rf$  is represented by the zigzag on the right. Applying Lemma 9.3 to  $g \circ f$  with  $\alpha$  and  $\beta$  the maps  $RX \rightarrow B$  and  $RZ \rightarrow B$  above and  $\gamma : FB \rightarrow Z$  the map induced by the maps  $FQf \rightarrow Y \rightarrow Z$  and  $FQg \rightarrow Z$ , we see that  $R(f \circ g) = \beta^{-1} \circ \alpha = Rf \circ Rg$ .  $\square$

Clearly  $R$  takes weak equivalences in  $\mathcal{D}$  to isomorphisms in  $\text{Ho}\mathcal{C}$ , and so  $R$  factors through a functor  $\text{Ho}\mathcal{D} \rightarrow \text{Ho}\mathcal{C}$  that we also denote as  $R$ . It is clear from diagram (9.2) that  $\epsilon$  is a natural isomorphism from  $FR$  to the identity in  $\text{Ho}\mathcal{D}$ . For  $C$  an object of  $\mathcal{C}$ , applying the approximation property to the map

$$F(C \vee RFC) \cong FC \vee FRFC \rightarrow FC$$

constructs an object  $PC$  in  $\mathcal{C}$  with weak equivalences  $C \rightarrow PC$  and  $RFC \rightarrow PC$ . This then gives a zigzag in  $\mathcal{C}$  that represents an isomorphism in  $\text{Ho}\mathcal{C}$  from  $C$  to  $RFC$ . It is straightforward to verify using Lemma 9.3 that this isomorphism is natural.

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## Appendix A. USE of factorizations and mapping cylinders

In this appendix, we introduce the concept of a universal simplicial equivalence (USE) of a space of factorizations or a space of mapping cylinders for a Waldhausen category  $\mathcal{C}$ . A USE of factorizations or mapping cylinders is a way of encoding the data of a contractible space of choices of factorization or mapping cylinder for each morphism in  $\mathcal{C}$  (or a distinguished subcategory). Although our formulation is new (and relies on the definition of a universal quasifibration), this kind of approach to handling a lack of functoriality in factorizations derives from the work of Dwyer and Kan in [5].

The purpose of this appendix is to generalize the results of [1] on homotopy cocartesian squares and on the  $S'_\bullet$  construction from requiring functorial factorization of weak cofibrations to requiring a USE of factorizations of weak cofibrations. In addition, we show that the existence of (non-functorial) factorizations implies a USE of mapping cylinders. In the next appendix, we show how to modify the arguments of the body of the paper to remove functoriality hypotheses assuming just existence of factorization or a USE of mapping cylinders for weak cofibrations.

Throughout this section  $\mathcal{C}$  denotes a Waldhausen category whose weak equivalences satisfy the two out of three property (i.e., are saturated in the sense of Waldhausen).

### A.1. Categories of factorizations and mapping cylinders

We begin with the formal definition of the categories of factorizations and mapping cylinders for  $\mathcal{C}$ .

**Definition A.1.1.** The category  $\text{Fac } \mathcal{C}$  is the full subcategory of diagrams

$$A \twoheadrightarrow X \xrightarrow{\simeq} B$$

where the map  $A \rightarrow X$  is a cofibration and the map  $X \rightarrow B$  is a weak equivalence. The forgetful functor  $\text{Fac } \mathcal{C} \rightarrow \text{Ar } \mathcal{C}$  sends the diagram to the composite map  $A \rightarrow B$ .

The category  $\text{MCC}$  is the full subcategory of diagrams

$$\begin{array}{ccc}
 & B & \\
 & \downarrow & \searrow = \\
 A \twoheadrightarrow & X & \xrightarrow{\simeq} B,
 \end{array}$$

where the map  $A \rightarrow X$  is a cofibration, the map  $X \rightarrow B$  is a weak equivalence, and the composite map  $B \rightarrow B$  the identity. The forgetful functor  $\text{MCC} \rightarrow \text{Fac } \mathcal{C}$  forgets the map  $B \rightarrow X$ , and we obtain a composite forgetful functor  $\text{MCC} \rightarrow \text{Ar } \mathcal{C}$ .

In this terminology, the existence of factorizations is equivalent to the forgetful functor  $\text{Fac } \mathcal{C} \rightarrow \text{Ar } \mathcal{C}$  being surjective on objects. Likewise, functorial factorization as defined in Section 2 consists of a functor  $T : \text{Ar } \mathcal{C} \rightarrow \text{Fac } \mathcal{C}$  such that the composite with the forgetful functor  $\text{Fac } \mathcal{C} \rightarrow \text{Ar } \mathcal{C}$  is the identity on  $\text{Ar } \mathcal{C}$ . We write  $\text{Ar}_{\text{co}} \mathcal{C}$  for the full subcategory of  $\text{Ar } \mathcal{C}$

consisting of the weak cofibrations. Functorial factorization of weak cofibrations or functorial mapping cylinders for weak cofibrations then consists of a functor  $T : \text{Ar}_{\tilde{\infty}} \mathcal{C} \rightarrow \text{Fac} \mathcal{C}$  or  $T : \text{Ar}_{\tilde{\infty}} \mathcal{C} \rightarrow \text{MCC}$  such that the composite with the forgetful functor  $\text{Fac} \mathcal{C} \rightarrow \text{Ar} \mathcal{C}$  or  $\text{MCC} \rightarrow \text{Ar} \mathcal{C}$  is the inclusion  $\text{Ar}_{\tilde{\infty}} \mathcal{C} \rightarrow \text{Ar} \mathcal{C}$ .

## A.2. Universal simplicial equivalences

Building on the terminology of Definition 4.1, we call a map of simplicial sets  $X_{\bullet} \rightarrow Y_{\bullet}$  a *universal simplicial equivalence* when it is a universal simplicial quasifibration and a weak equivalence. Such a map is characterized by the property that for any simplicial map  $Z_{\bullet} \rightarrow Y_{\bullet}$ , the geometric realization of the pullback map  $|Z_{\bullet} \times_{Y_{\bullet}} X_{\bullet}| \rightarrow |Z_{\bullet}|$  has contractible fibers. For a subcategory  $\mathcal{S}$  of  $\text{Ar} \mathcal{C}$  (such as  $\text{Ar}_{\tilde{\infty}} \mathcal{C}$ ), we define a USE of factorizations and a USE of mapping cylinders as follows.

**Definition A.2.1.** Let  $\mathcal{S}$  be a subcategory of  $\text{Ar} \mathcal{C}$ . A *universal simplicial equivalence (USE) of factorizations for  $\mathcal{S}$*  consists of a simplicial set  $T_{\bullet}$  and a simplicial map  $T_{\bullet} \rightarrow N \text{Fac} \mathcal{C}$  such that the composite  $T_{\bullet} \rightarrow N \text{Ar} \mathcal{C}$  restricts to a universal simplicial equivalence

$$T_{\bullet} \times_{N \text{Ar} \mathcal{C}} N \mathcal{S} \rightarrow N \mathcal{S}.$$

A *universal simplicial equivalence (USE) of mapping cylinders for  $\mathcal{S}$*  consists of a simplicial set  $T_{\bullet}$  and a simplicial map  $T_{\bullet} \rightarrow N \text{Fac} \mathcal{C}$  such that the composite  $T_{\bullet} \rightarrow N \text{Ar} \mathcal{C}$  restricts to a universal simplicial equivalence

$$T_{\bullet} \times_{N \text{Ar} \mathcal{C}} N \mathcal{S} \rightarrow N \mathcal{S}.$$

When  $\mathcal{S}$  is the category of weak cofibrations  $\text{Ar}_{\tilde{\infty}}$ , we say that  $T_{\bullet}$  is a USE of factorizations or mapping cylinders for weak cofibrations.

This definition has several immediate consequences.

**Proposition A.2.2.** *Let  $\mathcal{S}$  be a subcategory of  $\text{Ar} \mathcal{C}$ .*

- (i) *Functorial factorization or functorial mapping cylinders for  $\mathcal{S}$  implies a USE of factorizations or mapping cylinders, respectively, with  $T_{\bullet} = N \mathcal{S}$  and with  $NT$  as the map  $T_{\bullet}$  to  $N \text{Fac} \mathcal{C}$  or  $N \text{MCC}$ .*
- (ii) *A USE of factorizations or mapping cylinders for  $\mathcal{S}$  implies a USE of factorizations or mapping cylinders, respectively, for any subcategory of  $\mathcal{S}$ .*
- (iii) *A USE of mapping cylinders for  $\mathcal{S}$  implies a USE of factorizations for  $\mathcal{S}$ .*

In particular, a USE of factorizations or mapping cylinders for  $\text{Ar} \mathcal{C}$  implies a USE of factorizations or mapping cylinders, respectively, for every subcategory of  $\text{Ar} \mathcal{C}$ . Moreover, a USE of factorizations or mapping cylinders for  $\text{Ar} \mathcal{C}$  implies existence of (non-functorial) factorizations or mapping cylinders, respectively. In fact, the following lemma, proved at the end of this appendix, shows that a USE of mapping cylinders for all arrows, a USE of factorizations for all arrows, and existence of factorizations for all arrows are all equivalent.

**Lemma A.2.3.** *If  $\mathcal{C}$  admits factorization (not necessarily functorially), then there is a USE of mapping cylinders for  $\text{Ar}\mathcal{C}$ .*

A.3. *Weak cofibrations and homotopy cocartesian squares*

We now generalize the results of Section 2 of [1] on homotopy cocartesian diagrams. In these results, we follow the notation of Definition A.2.1 and denote the domain of a USE as  $T_\bullet$ . For Proposition 2.3 of [1], recall that a full subcategory  $\mathcal{B}$  of  $\mathcal{C}$  is called a *Waldhausen subcategory* when it forms a Waldhausen category with weak equivalences the weak equivalences of  $\mathcal{C}$  and with cofibrations the cofibrations  $A \rightarrow B$  in  $\mathcal{C}$  (between objects  $A$  and  $B$  of  $\mathcal{B}$ ) whose quotient  $B/A = B \cup_A *$  is in  $\mathcal{B}$ . We say that the Waldhausen subcategory  $\mathcal{B}$  is *closed* if every object of  $\mathcal{C}$  weakly equivalent to an object of  $\mathcal{B}$  is an object of  $\mathcal{B}$ .

**Proposition A.3.1.** *(See [1, 2.3].) If  $\mathcal{B}$  is a closed Waldhausen subcategory of a Waldhausen category  $\mathcal{C}$  with a USE of factorizations for weak cofibrations, then  $\mathcal{B}$  has a USE of factorizations for weak cofibrations. Moreover, a weak cofibration  $f : A \rightarrow B$  in  $\mathcal{C}$  between objects in  $\mathcal{B}$  is a weak cofibration in  $\mathcal{B}$  if and only if there exists some factorization  $A \rightarrow X \rightarrow B$  in the image of  $T_\bullet$  such that the cofibration (in  $\mathcal{C}$ )  $A \rightarrow X$  is a cofibration in  $\mathcal{B}$ .*

**Proof.** Let  $f : A \rightarrow B$  be a weak cofibration in  $\mathcal{C}$  between objects in  $\mathcal{B}$ . Then  $f$  is weakly equivalent by a zigzag in  $\mathcal{B}$  to a cofibration  $f' : A' \rightarrow B'$  in  $\mathcal{C}$ ,

$$\begin{array}{ccccccc}
 A' & \xrightarrow{\cong} & A_1 & \xleftarrow{\cong} & \cdots & \xleftarrow{\cong} & A_n & \xrightarrow{\cong} & A \\
 \downarrow f' & & \downarrow f_1 & & & & \downarrow f_n & & \downarrow f \\
 B' & \xrightarrow{\cong} & B_1 & \xleftarrow{\cong} & \cdots & \xleftarrow{\cong} & B_n & \xrightarrow{\cong} & B
 \end{array}$$

This diagram specifies a generalized simplicial path in  $N \text{Ar}_{\infty} \mathcal{C}$  between the vertices  $A \rightarrow B$  and  $A' \rightarrow B'$ . It follows that there exists a generalized simplicial path between a lift of  $A \rightarrow B$  to  $T_0$  and some lift of  $A' \rightarrow B'$  to  $T_0$ . Without loss of generality (replacing the original generalized simplicial path if necessary), the image of this path in  $N \text{Fac} \mathcal{C}$  is a commutative diagram

$$\begin{array}{ccccccc}
 A' & \xrightarrow{\cong} & A_1 & \xleftarrow{\cong} & \cdots & \xleftarrow{\cong} & A_n & \xrightarrow{\cong} & A \\
 \downarrow f' & & \downarrow & & & & \downarrow & & \downarrow \\
 B' & \xrightarrow{\cong} & X_1 & \xleftarrow{\cong} & \cdots & \xleftarrow{\cong} & X_n & \xrightarrow{\cong} & X \\
 \downarrow = & & \downarrow & & & & \downarrow f_n & & \downarrow \\
 B' & \xrightarrow{\cong} & B_1 & \xleftarrow{\cong} & \cdots & \xleftarrow{\cong} & B_n & \xrightarrow{\cong} & B
 \end{array}$$

We then get weak equivalences,

$$B'/A' \xrightarrow{\cong} X_1/A_1 \xleftarrow{\cong} \cdots \xleftarrow{\cong} X_n/A_n \xrightarrow{\cong} X/A,$$

which imply that the map  $A \rightarrow X$  is a cofibration in  $\mathcal{B}$  if and only if  $f : A \rightarrow B$  is a weak cofibration in  $\mathcal{B}$ .  $\square$

The proof of Proposition 2.4 in [1] does not actually use the functoriality of the factorizations of weak cofibrations but just their existence. It therefore admits the following generalization.

**Proposition A.3.2.** (See [1, 2.4].) *Let  $\mathcal{C}$  be a Waldhausen category that admits factorization of weak cofibrations. If  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are weak cofibrations in  $\mathcal{C}$ , then  $g \circ f : A \rightarrow C$  is a weak cofibration in  $\mathcal{C}$ .*

The following proposition generalizes Proposition 2.5 in [1]. The proof is similar to that of Proposition A.3.1.

**Proposition A.3.3.** (See [1, 2.5].) *Let  $\mathcal{C}$  be a Waldhausen category with a USE of factorizations for weak cofibrations. A commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

with  $f$  a weak cofibration is homotopy cocartesian if and only if the map  $X \cup_A C \rightarrow D$  is a weak equivalence for some factorization  $A \rightarrow X \rightarrow B$  in the image of  $T_\bullet$ .

The previous proposition then implies the following proposition, which generalizes Proposition 2.6 of [1].

**Proposition A.3.4.** (See [1, 2.6].) *Let  $\mathcal{C}$  be a Waldhausen category with a USE of factorizations for weak cofibrations.*

(i) *Given a commutative cube*

$$\begin{array}{ccccc} A' & \longrightarrow & B' & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & A & \longrightarrow & B & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ C' & \longrightarrow & D' & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & C & \longrightarrow & D & \end{array}$$

with the  $(A, B, C, D)$ -face and  $(A', B', C', D')$ -face homotopy cocartesian, if the maps  $A' \rightarrow A$ ,  $B' \rightarrow B$ , and  $C' \rightarrow C$  are weak equivalences, then the map  $D' \rightarrow D$  is a weak equivalence.

(ii) Given a commutative diagram

$$\begin{array}{ccccc}
 A & \longrightarrow & B & \longrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow \\
 C & \longrightarrow & D & \longrightarrow & Y
 \end{array}$$

with the square  $(A, B, C, D)$  homotopy cocartesian, if either  $A \rightarrow C$  is a weak cofibration or both  $A \rightarrow B$  and  $B \rightarrow X$  are weak cofibrations, then the  $(A, X, C, Y)$  square is homotopy cocartesian if and only if the  $(B, X, D, Y)$  square is homotopy cocartesian.

#### A.4. Comparing $S_\bullet$ and $S'_\bullet$

Next we compare  $S_\bullet$  and  $S'_\bullet$  constructions. As a first step, we prove the following version of Theorems 2.8 and 2.9 of [1] for  $F_\bullet$  and  $F'_\bullet$ . Recall that  $F_n$  and  $F'_n$  are the Waldhausen categories with objects the sequences of  $n$  composable cofibrations and weak cofibrations respectively.

**Theorem A.4.1.** *Let  $C$  be a Waldhausen category with a USE of factorizations for weak cofibrations.*

- (i) *The Waldhausen category  $F'_n C$  has a USE of factorizations for weak cofibrations.*
- (ii) *The forgetful functor  $wF_n C \rightarrow wF'_n C$  induces a weak equivalence on nerves.*

**Proof.** The arguments are straightforward modifications of the usual pushout arguments. As above, let  $T_\bullet$  be the domain of the USE of factorizations of weak cofibrations in  $C$ . For part (i), we denote a typical object of  $\text{Ar } F'_n$  as

$$\begin{array}{ccccccc}
 A_0 & \longrightarrow & A_1 & \longrightarrow & \cdots & \longrightarrow & A_n \\
 \downarrow & & \downarrow & & & & \downarrow \\
 B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_n
 \end{array}$$

(where the maps  $A_i \rightarrow A_{i+1}$  and  $B_i \rightarrow B_{i+1}$  are weak cofibrations). Such an object is in  $\text{Ar}_{\text{co}} F'_n$  when the maps  $A_i \rightarrow B_i$  are weak cofibrations and for any factorization  $A_i \rightarrow X \rightarrow B_i$  in the image of  $T_\bullet$ , the map  $A_{i+1} \cup_{A_i} X \rightarrow B_{i+1}$  is a weak cofibration. For  $F'_n$ , let  $T^0_\bullet$  be the pullback

$$\begin{array}{ccc}
 T^0_\bullet & \longrightarrow & T_\bullet \\
 \simeq \downarrow & & \downarrow \simeq \\
 N_\bullet \text{Ar}_{\text{co}} F'_n & \longrightarrow & N_\bullet \text{Ar}_{\text{co}} C
 \end{array}$$

where the bottom map is induced by the zeroth object functor  $F'_n \rightarrow C$ . Using the map  $T_\bullet \rightarrow N \text{Fac } C$ , we get a simplicial map from  $T^0$  to the nerve of the category of diagrams of the form



$$\begin{array}{ccccccc}
 A_0 & \longrightarrow & A_1 & \longrightarrow & \cdots & \longrightarrow & A_n \\
 \downarrow & & \downarrow & & & & \downarrow \\
 X_0 & & & & & & \\
 \cong \downarrow & & \downarrow & & & & \downarrow \\
 B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_n.
 \end{array}$$

We have a functor from this category to  $\text{Ar}\mathcal{C}$  taking the object pictured above to the pushout map

$$X_0 \cup_{A_0} A_1 \rightarrow B_1,$$

and the composite map  $T_\bullet^0 \rightarrow \text{Ar}\mathcal{C}$  factors as a map  $p_0 : T_\bullet^0 \rightarrow \text{Ar}_{\tilde{\text{co}}}\mathcal{C}$ . Inductively, having constructed  $T_\bullet^i$  and  $p_i : T_\bullet^i \rightarrow \text{Ar}_{\tilde{\text{co}}}\mathcal{C}$ , define  $T_\bullet^{i+1}$  as the pullback

$$\begin{array}{ccc}
 T_\bullet^{i+1} & \longrightarrow & T_\bullet \\
 \cong \downarrow & & \downarrow \cong \\
 T_\bullet^i & \xrightarrow{p_i} & N_\bullet \text{Ar}_{\tilde{\text{co}}}\mathcal{C}
 \end{array}$$

and  $p_{i+1} : T_\bullet^{i+1} \rightarrow \text{Ar}_{\tilde{\text{co}}}\mathcal{C}$  by the analogous pushout. By construction,  $T_\bullet^n$  admits a map to the nerve of the category of diagrams of the form

$$\begin{array}{ccccccc}
 A_0 & \longrightarrow & A_1 & \longrightarrow & \cdots & \longrightarrow & A_n \\
 \downarrow & & \downarrow & & & & \downarrow \\
 X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_n \\
 \cong \downarrow & & \cong \downarrow & & & & \downarrow \cong \\
 B_0 & \longrightarrow & B_1 & \longrightarrow & \cdots & \longrightarrow & B_n
 \end{array}$$

such that each map  $X_i \cup_{A_i} A_{i+1} \rightarrow X_{i+1}$  is a weak cofibration, i.e., the category  $\text{Fac } F'_n$ . The composite functor  $T_\bullet^n \rightarrow \text{Ar}_{\tilde{\text{co}}} F'_n$  is the composite

$$T_\bullet^n \rightarrow T_\bullet^{n-1} \rightarrow \cdots \rightarrow T_\bullet^0 \rightarrow \text{Ar}_{\tilde{\text{co}}} F'_n$$

and so is a universal simplicial equivalence. This constructs the USE of factorizations of weak cofibrations for  $F'_n$  and proves part (i).

For part (ii), let  $U^1$  be the pullback

$$\begin{array}{ccc}
 U_\bullet^1 & \longrightarrow & T_\bullet \\
 \cong \downarrow & & \downarrow \cong \\
 NwF'_n\mathcal{C} & \longrightarrow & N \text{Ar}_{\tilde{\text{co}}}\mathcal{C}
 \end{array}$$

where the bottom map is induced by the functor  $f_1 : \mathbf{w}F'_n\mathcal{C} \rightarrow \mathbf{w}Ar_{\tilde{\infty}}\mathcal{C}$  that takes the object

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n,$$

to the arrow  $f_1$ . Using the map  $T_\bullet \rightarrow N\text{Fac}\mathcal{C}$ , we get a simplicial map from  $U^1$  to the nerve of (the subcategory of weak equivalences in) the category of commuting diagrams of the form

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f'_1} & A'_1 & & & & \\ & \searrow f_1 & \downarrow \simeq \phi_1 & \searrow g_2 & & & \\ & & A_1 & \xrightarrow{f_2} & A_2 & \longrightarrow \cdots & \xrightarrow{f_n} A_n \end{array}$$

where the map  $f'_1$  is a cofibration, the maps  $f_i$  are weak cofibrations, and the map  $\phi_1$  is a weak equivalence. Inductively constructing  $U^{i+1}$  as the pullback

$$\begin{array}{ccc} U^{i+1} & \longrightarrow & T_\bullet \\ \simeq \downarrow & & \downarrow \simeq \\ U^i & \xrightarrow{g_{i+1}} & NAr_{\tilde{\infty}}\mathcal{C}, \end{array}$$

we obtain a universal simplicial equivalence  $U^n \rightarrow N\mathbf{w}F'_n\mathcal{C}$  and a simplicial map from  $U^n$  to the nerve of the category of commuting diagrams of the form

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f'_1} & A'_1 & \longrightarrow & \cdots & \longrightarrow & A'_n \\ & \searrow f_1 & \downarrow \simeq \phi_1 & & & & \downarrow \simeq \phi_n \\ & & A_1 & \longrightarrow & \cdots & \longrightarrow & A_n. \end{array}$$

This constructs a map  $U^n \rightarrow N\mathbf{w}F_n\mathcal{C}$ . The maps  $\phi_i$  in the diagram above induce a homotopy between the composite map  $U^n \rightarrow N\mathbf{w}F_n\mathcal{C} \rightarrow N\mathbf{w}F'_n\mathcal{C}$  and the universal simplicial equivalence  $U^n \rightarrow N\mathbf{w}F'_n\mathcal{C}$ . It follows that the right-hand triangle in the diagram

$$\begin{array}{ccc} U^n \times_{N\mathbf{w}F'_n\mathcal{C}} N\mathbf{w}F_n\mathcal{C} & \longrightarrow & U^n \\ \simeq \downarrow & \swarrow \text{dotted} & \downarrow \simeq \\ N\mathbf{w}F_n\mathcal{C} & \longrightarrow & N\mathbf{w}F'_n\mathcal{C} \end{array}$$

commutes up to homotopy. Likewise, the maps  $\phi_i$  induce a homotopy for the left-hand triangle. This proves part (ii).  $\square$

A difficulty in generalizing the previous theorem to  $S'_n$  arises in part (i): One would like to continue the construction of the  $T_\bullet^*$  down the rows of  $S'_n$ , but the trick in the previous argument fails at this stage because the map from the pushout to the appropriate target object  $B_{1,2}$  need not be a weak cofibration. (In the case when all maps are weak cofibrations, Lemma A.2.3 immediately implies that  $S'_n$  has a USE of factorizations.) On the other hand, the analogue of part (ii) for  $S'_n$  follows from part (ii) for  $F_n$ .

**Corollary A.4.2.** (See [1, 2.9].) *Let  $\mathcal{C}$  be a Waldhausen category with a USE of factorizations for weak cofibrations. The forgetful functor  $wS_n\mathcal{C} \rightarrow wS'_n\mathcal{C}$  induces a weak equivalence on nerves.*

**Proof.** Let  $M = U^n \times_{NwF'_n\mathcal{C}} NwS'_n\mathcal{C}$ , where  $U^n \rightarrow NwF'_n\mathcal{C}$  is as in the proof of Theorem A.4.1. We have a universal simplicial equivalence  $M \rightarrow NwS'_n\mathcal{C}$ , and pushout induces a map  $\Phi : M \rightarrow NwS_n\mathcal{C}$  as in the proof of Theorem 2.9 in [1]. Looking at the diagram,

$$\begin{array}{ccc}
 M \times_{NwS'_n\mathcal{C}} NwS_n\mathcal{C} & \longrightarrow & M \\
 \cong \downarrow & \swarrow \text{dotted} & \downarrow \cong \\
 NwS_n\mathcal{C} & \longrightarrow & NwS'_n\mathcal{C}.
 \end{array}$$

The argument of [1, 2.9] generalizes to show that the left-hand triangle commutes up to simplicial homotopy and the right-hand triangle commutes up to generalized simplicial homotopy.  $\square$

A.5. Iterating  $F'_\bullet$  and  $S'_\bullet$

Corollary A.4.2 provides a space-level comparison of the (delooped)  $K$ -theory spaces provided by the  $S_\bullet$  and  $S'_\bullet$  constructions. We next extend this to a spectrum-level comparison by comparing the iterated  $S_\bullet$  construction with an  $S'_\bullet$  analogue. Because we cannot prove the analogue of part (i) of Theorem A.4.1 for  $S'_\bullet$  (the generalization of [1, 2.8]), we need to take a direct approach to the construction of iterated  $S'_\bullet$ . Again, we find it convenient to start by examining  $F'_\bullet$ .

**Definition A.5.1.** Let  $\mathcal{C}$  be a Waldhausen category with a USE of factorizations for weak cofibrations. For  $q, n_1, \dots, n_q \geq 0$ , let  $F_{n_1, \dots, n_q}^{(q)}\mathcal{C}$  be the Waldhausen category  $F_{n_1}F_{n_2} \cdots F_{n_q}\mathcal{C}$  and let  $F'_{n_1, \dots, n_q}{}^{(q)}\mathcal{C}$  be the Waldhausen category  $F'_{n_1}F'_{n_2} \cdots F'_{n_q}\mathcal{C}$ . Similarly, let  $S_{n_1, \dots, n_q}^{(q)}\mathcal{C}$  denote the Waldhausen category  $S_{n_1}S_{n_2} \cdots S_{n_q}\mathcal{C}$  given by iterating the  $S_\bullet$  construction.

Combining the construction of the universal simplicial equivalence  $U^n \rightarrow wF'_n\mathcal{C}$  in part (ii) of Theorem A.4.1 with the construction in part (i) of Theorem A.4.1 proves the following theorem.

**Theorem A.5.2.** *Let  $\mathcal{C}$  be a Waldhausen category with a USE of factorizations of weak cofibrations. There exist weak equivalences*

$$u : U_{n_1, \dots, n_q}^{(q)} \rightarrow NwF_{n_1, \dots, n_q}^{(q)}\mathcal{C} \quad \text{and} \quad p : U_{n_1, \dots, n_q}^{(q)} \rightarrow NwF'_{n_1, \dots, n_q}{}^{(q)}\mathcal{C}$$

and a simplicial homotopy from the composite map

$$U_{n_1, \dots, n_q}^{(q)} \rightarrow \text{Nw}F_{n_1, \dots, n_q}^{(q)} \mathcal{C} \rightarrow \text{Nw}F_{n_1, \dots, n_q}^{(q)} \mathcal{C}$$

to  $u$ ; moreover  $u$  is a universal simplicial quasifibration.

**Corollary A.5.3.** *Let  $\mathcal{C}$  be a Waldhausen category with a USE of factorizations of weak cofibrations. An  $n_1 \times \dots \times n_q$  rectangle of maps in  $\mathcal{C}$  is an object of  $F_{n_1, \dots, n_q}^{(q)} \mathcal{C}$  if and only there exists a weak equivalence (of rectangular diagrams) to it from an object of  $F_{n_1, \dots, n_q}^{(q)} \mathcal{C}$ .*

This corollary makes the categories  $F_{n_1, \dots, n_q}^{(q)} \mathcal{C}$  significantly more tractable. For example, it follows that the usual symmetric group action on  $F_{n_1, \dots, n_q}^{(q)} \mathcal{C}$  extends to  $F_{n_1, \dots, n_q}^{(q)} \mathcal{C}$ .

We now construct categories  $S_{\bullet, \dots, \bullet}^{(q)} \mathcal{C}$  that play the role of the iterated  $S'_\bullet$  construction. For this, recall that  $\text{Ar}[n]$  denotes the category with objects  $(i, j)$  for  $0 \leq i \leq j \leq n$  and a unique map  $(i, j) \rightarrow (i', j')$  for  $i \leq i'$  and  $j \leq j'$ . We write an object of  $\text{Ar}[n_1] \times \dots \times \text{Ar}[n_q]$  as  $(i_1, j_1; \dots; i_q, j_q)$ .

**Definition A.5.4.** Let  $\mathcal{C}$  be a Waldhausen category with a USE of factorizations of weak cofibrations. Let  $S_{n_1, \dots, n_q}^{(q)} \mathcal{C}$  be the full subcategory of functors

$$\text{Ar}[n_1] \times \dots \times \text{Ar}[n_q] \rightarrow \mathcal{C}$$

such that:

- The initial map  $* \rightarrow A_{i_1, j_1; \dots; i_q, j_q}$  is a weak equivalence whenever  $i_k = j_k$  for some  $k$ .
- The  $n_1 \times \dots \times n_q$  rectangular subdiagram  $A_{0, j_1; \dots; 0, j_q}$  is an object of  $F_{n_1, \dots, n_q}^{(q)} \mathcal{C}$ .
- For every object  $(i_1, j_1; \dots; i_q, j_q)$  in  $\text{Ar}[n_1] \times \dots \times \text{Ar}[n_q]$ , the square

$$\begin{array}{ccc} A_{0, i_1; \dots; 0, i_q} & \longrightarrow & A_{0, j_1; \dots; 0, j_q} \\ \downarrow & & \downarrow \\ A_{i_1, i_1; \dots; i_q, i_q} & \longrightarrow & A_{i_1, j_1; \dots; i_q, j_q} \end{array}$$

is homotopy cocartesian.

The subcategory  $\text{w}S_{n_1, \dots, n_q}^{(q)} \mathcal{C}$  consists of the maps in  $S'_{n_1, \dots, n_q}$  that are objectwise weak equivalences.

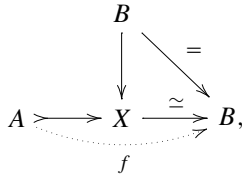
We understand  $S^{(0)} \mathcal{C}$  to be  $\mathcal{C}$  and we see that  $S_n^{(1)} \mathcal{C}$  is  $S'_n \mathcal{C}$ . As an easy consequence of Corollary A.5.3,  $S_{\bullet, \dots, \bullet}^{(q)} \mathcal{C}$  and  $\text{w}S_{\bullet, \dots, \bullet}^{(q)} \mathcal{C}$  assemble into simplicial categories. Likewise, it follows from Corollary A.5.3 that  $S_{\bullet, \dots, \bullet}^{(q)} \mathcal{C}$  and  $\text{w}S_{\bullet, \dots, \bullet}^{(q)} \mathcal{C}$  are simplicial subcategories of  $S_{\bullet, \dots, \bullet}^{(q)} \mathcal{C}$  and  $\text{w}S_{\bullet, \dots, \bullet}^{(q)} \mathcal{C}$ . The argument for Corollary A.4.2 generalizes to prove the following theorem, which provides the replacement for Theorems 2.8 and 2.9 of [1].

**Theorem A.5.5.** *The forgetful functor  $\text{w}S_{\bullet, \dots, \bullet}^{(q)} \mathcal{C} \rightarrow \text{w}S_{\bullet, \dots, \bullet}^{(q)} \mathcal{C}$  induces a weak equivalence on nerves.*

**Corollary A.5.6.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a weakly exact functor. If  $\mathcal{D}$  has a USE of factorizations of weak cofibrations, then  $F$  induces a map of  $K$ -theory spectra.*

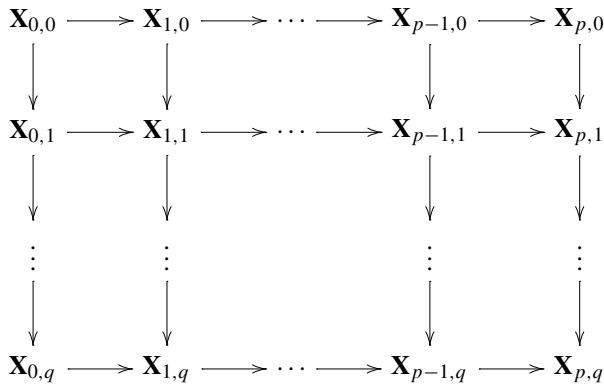
A.6. Proof of Lemma A.2.3

For a map  $f : A \rightarrow B$  in  $\mathcal{C}$ , say that a mapping cylinder



is a *strong* mapping cylinder when the map  $A \vee B \rightarrow X$  is a cofibration. Let  $\text{MC}^s \mathcal{C}$  be the full subcategory of the category of mapping cylinders  $\text{MCC}$  consisting of the strong mapping cylinders. For a fixed map  $f$  in  $\mathcal{C}$ , we write  $\text{MC}^s f$  for the category of strong mapping cylinders for  $f$ ; this is the subcategory of  $\text{MC}^s \mathcal{C}$  consisting of the objects that go to the object  $f$  of  $\text{Ar} \mathcal{C}$  and the maps that go to the identity map of  $f$  in  $\text{Ar} \mathcal{C}$  under the forgetful functor  $\text{MC}^s \mathcal{C} \rightarrow \text{Ar} \mathcal{C}$ .

Now assume that  $\mathcal{C}$  admits factorization. Consider the bisimplicial set  $T_{\bullet, \bullet}$  whose set of  $p, q$ -simplices  $T_{p,q}$  consists of the commuting diagrams in  $\text{MC}^s \mathcal{C}$



where each of the vertical maps  $\mathbf{X}_{i,j} \rightarrow \mathbf{X}_{i,j+1}$  forgets to an identity morphism  $\text{id}_f$  in the category  $\text{Ar} \mathcal{C}$ . We have a map from the diagonal simplicial set to  $N_{\bullet} \text{MC}$  that takes the diagram above (for  $p = q$ ) to the sequence

$$\mathbf{X}_{0,0} \rightarrow \mathbf{X}_{1,1} \rightarrow \cdots \rightarrow \mathbf{X}_{q,q}.$$

We show that the composite map to  $N_{\bullet} \text{Ar} \mathcal{C}$  is a universal simplicial equivalence.

The composite map  $T_{\bullet, \bullet} \rightarrow N_{\bullet} \text{Ar} \mathcal{C}$  is the diagonal of a map of the bisimplicial sets  $T_{p,q} \rightarrow N_p \text{Ar} \mathcal{C}$  induced by the forgetful functor  $\text{MC}^s \rightarrow \text{Ar} \mathcal{C}$ , where we regard  $N_p \text{Ar} \mathcal{C}$  as constant in the second simplicial direction. Since we can regard any simplicial map  $Z_{\bullet, \bullet} \rightarrow N_{\bullet} \text{Ar} \mathcal{C}$  as a bisimplicial map, constant in the second simplicial direction, and since the diagonal preserves pullbacks, it suffices to show that for each  $p$ , the map of simplicial sets from  $T_{p, \bullet}$  to the constant

simplicial set  $N_p \text{Ar} \mathcal{C}$  is a weak equivalence. Moreover, identifying the category  $N_p \text{Ar} \mathcal{C}$  with the category  $\text{Ar} N_p \mathcal{C}$ , this amounts to showing that for a map  $f$  between objects of  $N_p \mathcal{C}$ , the category of strong mapping cylinders for  $f$  has contractible nerve. Since the Waldhausen category  $N_p \mathcal{C}$  admits factorizations when  $\mathcal{C}$  does, it suffices to treat the case  $p = 0$ .

Thus, we need to show that for every  $f : A \rightarrow B$  in  $\mathcal{C}$ , the category  $\text{MC}^s f$  has contractible nerve. We view  $\text{MC}^s f$  as a subcategory of the category  $\mathcal{C} \mid f$  of diagrams

$$A \vee B \rightarrow X \rightarrow B$$

in  $\mathcal{C}$  such that the composite map  $A \vee B \rightarrow B$  is  $f + \text{id}_B$ . We apply Waldhausen’s argument for the Approximation Theorem as generalized in [17, A.2]. First, observe that factorization implies that  $\text{MC}^s f$  is nonempty. Since after suitable simplicial approximation and subdivision any homotopy class of maps from a sphere to the geometric realization of  $\text{MC}^s$  is represented by the geometric realization of a functor from a finite partially ordered set into  $\text{MC}^s$ , it suffices to show that any functor  $\alpha$  from a finite partially ordered set  $\mathcal{P}$  to  $\text{MC}^s$  admits a zigzag of natural transformations to a constant functor [17, A.10]. The key idea is to inductively apply factorization so that colimits over sub-posets exist as iterated pushouts over cofibrations; this approach constructs a functor  $\beta : \mathcal{P} \rightarrow \text{MC}^s f$  and natural transformation  $\beta \rightarrow \alpha$  such that the colimit of  $\beta : \mathcal{P} \rightarrow \mathcal{C} \mid f$  (exists and) can be constructed as an iterated pushout over cofibrations [17, A.6]. This colimit is not an element in  $\text{MC}^s f$ , but applying factorization in  $\mathcal{C}$ , gives an object  $\mathbf{X}$  in  $\text{MC}^s f$  and a map  $\text{colim}_{\mathcal{P}} \beta \rightarrow \mathbf{X}$  in  $\mathcal{C} \mid f$ . This then gives a natural transformation from  $\beta$  to the constant functor on  $\mathbf{X}$ .

**Appendix B. Generalizing to the non-functorial case**

In this appendix, we go section by section through the paper and indicate the changes in statements and proofs needed for the case when the required factorizations are not functorial.

*B.1. Introduction*

Statements are made in the non-functorial case.

*B.2. Weakly exact functors*

Corollary A.5.6 substitutes for Theorem 2.2 for categories that have a USE of factorizations for weak cofibrations in place of FFWC. Lemma A.2.3 implies that Waldhausen categories that admit factorization in particular have a USE of factorization for weak cofibrations.

The hypothesis of FMCWC in Theorem 2.7 generalizes to the hypothesis of a USE of mapping cylinders for weak cofibrations. Here is a full statement:

**Theorem B.2.1.** *Let  $\mathcal{C}$  be a saturated Waldhausen category that has a USE of mapping cylinders for weak cofibrations.*

- (i) *For  $n > 1$ , the nerve of  $\text{wS}'_n \mathcal{C}$  is weakly equivalent to the homotopy coend*

$$\text{hocoend}_{(X_1, \dots, X_n) \in \text{wC}^n} LC_{\text{co}}(X_{n-1}, X_n) \times \cdots \times LC_{\text{co}}(X_1, X_2),$$

*naturally in weakly exact functors.*

- (ii) *The nerve of  $w\mathcal{C}$  is weakly equivalent to the disjoint union of  $B \text{hAut } X$  over the weak equivalence classes of objects of  $\mathcal{C}$ .*
- (iii) *For  $n \geq 1$ , the nerve of  $wS'_n\mathcal{C}$  is weakly equivalent to the total space of a fibration where the base is the disjoint union of*

$$B \text{hAut } X_n \times \cdots \times B \text{hAut } X_1$$

*over  $n$ -tuples of weak equivalence classes of objects of  $\mathcal{C}$ , and the fiber is equivalent to*

$$LC_{\text{co}}(X_{n-1}, X_n) \times \cdots \times LC_{\text{co}}(X_1, X_2),$$

*for  $n > 1$  and contractible for  $n = 1$ .*

### B.3. Outline of the proof of Theorem 1.3

The outline proceeds somewhat differently without functorial factorizations. First, applying Lemma A.2.3, let  $T_\bullet \rightarrow N \text{Fac } \mathcal{C}$  be a USE of factorizations for  $\mathcal{C}$ . Next, in place of cone and suspension functors, we define a category of cone and suspension objects.

**Definition B.3.1.** Let  $EC$  be the Waldhausen subcategory of  $S_2\mathcal{C}$  of objects

$$X \rightrightarrows C \longrightarrow \Sigma$$

such that the initial map  $* \rightarrow C$  is a weak equivalence. Let  $E'\mathcal{C}$  be the Waldhausen subcategory of  $S'_2\mathcal{C}$  of objects  $\{A_{i,j}\}$  such that the initial map  $* \rightarrow A_{0,2}$  is a weak equivalence.

We have three exact functors  $EC \rightarrow \mathcal{C}$ , the *forgetful* functor, the *cone* functor, and the *suspension* functor defined by sending the object pictured above to  $X$ ,  $C$ , and  $\Sigma$  respectively. We refer to the corresponding functors  $E'\mathcal{C} \rightarrow \mathcal{C}$  by the same names; specifically, for an object  $\{A_{i,j}\}$  in  $E'\mathcal{C}$ , the forgetful functor sends it to  $A_{0,1}$ , the cone functor sends it to  $A_{0,2}$ , and the suspension functor sends it to  $A_{1,2}$ .

The Waldhausen categories  $EC$  and  $E'\mathcal{C}$  inherit factorizations from  $\mathcal{C}$ . The forgetful functors  $EC \rightarrow \mathcal{C}$  and  $E'\mathcal{C} \rightarrow \mathcal{C}$  satisfy Waldhausen’s approximation property. Using Schlichting’s extension of Waldhausen’s Approximation Theorem [17, A.2], Theorem 1.4, and Theorem 1.5, we obtain the following theorem.

**Theorem B.3.2.** *The forgetful functors  $EC \rightarrow \mathcal{C}$  and  $E'\mathcal{C} \rightarrow \mathcal{C}$  are DK-equivalences and induce weak equivalences on  $K$ -theory.*

Applying Waldhausen’s Additivity Theorem, we obtain the following result.

**Corollary B.3.3.** *The suspension functors  $EC \rightarrow \mathcal{C}$  and  $E'\mathcal{C} \rightarrow \mathcal{C}$  induce weak equivalences on  $K$ -theory.*

For the purposes of generalizing the arguments in Section 3, we say that an object of  $\mathcal{C}$  is a suspension object if it is in the image of the suspension functor  $E'\mathcal{C} \rightarrow \mathcal{C}$ . Then a weakly exact functor between Waldhausen categories that admit factorization sends suspension objects

to suspension objects. Theorem 3.5 (proved in Section 6) then implies the following result in this language.

**Corollary B.3.4.** *With hypotheses as in Theorem 1.3, the map  $LF : LC(X, Y) \rightarrow LD(FX, FY)$  is a weak equivalence when  $X$  is a suspension object.*

Define  $K'C$  as the homotopy colimit of the diagram

$$\begin{array}{ccccccc}
 & & N(wS'_\bullet E'C) & & N(wS'_\bullet E'C) & & \dots \\
 & \swarrow \simeq & & \swarrow \simeq & & \swarrow \simeq & \\
 N(wS'_\bullet C) & & & & N(wS'_\bullet C) & & N(wS'_\bullet C) \dots
 \end{array}$$

where the leftward arrows are induced by the forgetful functor and the rightward arrows are induced by the suspension functor. We have the corresponding construction  $K'D$  for  $\mathcal{D}$  and  $F$  induces a map  $K'C \rightarrow K'D$ . By Theorem B.3.2 and Corollary B.3.3, all the maps in the diagram are weak equivalences and it follows that  $K'C \rightarrow K'D$  models the induced map on delooped  $K$ -theory spaces. The proof of Theorem 1.3 is completed by showing that this map is a weak equivalence.

The argument in Section 3 generalizes as follows. According to Theorem 2.7, the commuting square of functors

$$\begin{array}{ccccc}
 wS'_n C & \xleftarrow{\simeq} & wS'_n E'C & \xrightarrow{\simeq} & wS'_n C \\
 F \downarrow & & F \downarrow & & F \downarrow \\
 wS'_n \mathcal{D} & \xleftarrow{\simeq} & wS'_n E'\mathcal{D} & \xrightarrow{\simeq} & wS'_n \mathcal{D}
 \end{array}$$

(where the left-hand functors are the forgetful functor and the righthand maps are the suspension functors) induces on nerves a map modeled by the diagram

$$\begin{array}{ccccc}
 \text{hocoend } LC(X_1, \dots, X_n) & \longleftarrow & \text{hocoend } LE'C(\mathbf{X}_1, \dots, \mathbf{X}_n) & \longrightarrow & \text{hocoend } LC(X_1, \dots, X_n) \\
 LF \downarrow & & LF \downarrow & & LF \downarrow \\
 \text{hocoend } LD(Y_1, \dots, Y_n) & \longleftarrow & \text{hocoend } LE'D(\mathbf{Y}_1, \dots, \mathbf{Y}_n) & \longrightarrow & \text{hocoend } LD(Y_1, \dots, Y_n)
 \end{array}$$

where the homotopy coends are over  $(X_1, \dots, X_n) \in wC^n$ ,  $(\mathbf{X}_1, \dots, \mathbf{X}_n) \in (wE'C)^n$ ,  $(Y_1, \dots, Y_n) \in wD^n$ , and  $(\mathbf{Y}_1, \dots, \mathbf{Y}_n) \in (wE'D)^n$ . The right-hand square factors as

$$\begin{array}{ccccc}
 \text{hocoend } LE'C(\mathbf{X}_1, \dots, \mathbf{X}_n) & \longrightarrow & \text{hocoend } LC(C_1, \dots, C_n) & \longrightarrow & \text{hocoend } LC(X_1, \dots, X_n) \\
 LF \downarrow & & LF \downarrow & & LF \downarrow \\
 \text{hocoend } LE'D(\mathbf{Y}_1, \dots, \mathbf{Y}_n) & \longrightarrow & \text{hocoend } LD(D_1, \dots, D_n) & \longrightarrow & \text{hocoend } LD(Y_1, \dots, Y_n)
 \end{array}$$



where the middle homotopy coends are over  $n$ -tuples of suspension objects in  $w\mathcal{C}$  and  $w\mathcal{D}$ . Corollary B.3.4 then implies that the middle vertical arrow is a weak equivalence. Going back to the homotopy colimit defining  $K'\mathcal{C}$  and  $K'\mathcal{D}$ , we see that the map  $K'\mathcal{C} \rightarrow K'\mathcal{D}$  is a weak equivalence. This completes the proof of Theorem 1.3.

*B.4. Universal simplicial quasifibrations*

No statements or arguments in this section involve factorization.

*B.5. Homotopy calculi of fractions and mapping cylinders*

The hypothesis of FMCWC in Theorems 5.5 and 5.6 (and implicitly Lemma 5.10, which shares the hypothesis of Theorem 5.5) generalizes to the hypothesis of USE of mapping cylinders for weak cofibrations. The statements become:

**Theorem B.5.1.** *Let  $\mathcal{C}$  be a Waldhausen category with a USE of mapping cylinders for weak cofibrations. Then  $\mathcal{C}$ ,  $\tilde{co}\mathcal{C}$ , and  $w\mathcal{C}$  admit homotopy calculi of left fractions.*

**Theorem B.5.2.** *Let  $\mathcal{C}$  be a Waldhausen category with a USE of mapping cylinders for weak cofibrations. Then the maps*

$$\begin{aligned} \mathbf{W}^{-1}\mathbf{C}(A, B)_{\tilde{co}} &\rightarrow LC_{co}(A, B), & \mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_{\tilde{co}} &\rightarrow LC_{co}(A, B), \\ \mathbf{W}^{-1}\mathbf{C}(A, B)_w &\rightarrow LC_w(A, B), & \text{and } \mathbf{W}^{-1}\mathbf{C}\mathbf{W}^{-1}(A, B)_w &\rightarrow LC_w(A, B) \end{aligned}$$

are weak equivalences.

The proof of Theorem 5.5 from Lemma 5.10 and Theorem 5.6 from Theorem 5.5 generalize immediately to Theorems B.5.1 and B.5.2. The proof of Lemma 5.10 is modified as follows:

**Proof of Lemma 5.10.** Let  $T_\bullet \rightarrow N \text{Fac } \mathcal{C}$  be a USE of mapping cylinders for weak cofibrations. Let  $T_\bullet^{i,j}(A, B)$  be the pullback

$$\begin{array}{ccc} T_\bullet^{i,j} & \longrightarrow & T_\bullet \\ \simeq \downarrow & & \downarrow \simeq \\ N\mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(A, B) & \xrightarrow{\phi} & N \text{Ar}_{\tilde{co}} \mathcal{C}, \end{array}$$

where the map  $\phi$  is induced by the functor that takes the object

$$A \longrightarrow Y_j \longrightarrow \cdots \longrightarrow Y_1 \xleftarrow[\simeq]{f} Z \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \xleftarrow[\simeq]{} B$$

of  $\mathbf{W}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(A, B)$  to the object  $f$  of  $\text{Ar}_{\tilde{co}} \mathcal{C}$ . Then the pushout construction in the proof of this lemma in Section 5 constructs a map

$$T_{\bullet}^{i,j}(A, B) \rightarrow \mathbf{NW}^{-1}\mathbf{C}^i\mathbf{C}^j(A, B)$$

and the natural transformations there give simplicial homotopies to make the diagram

$$\begin{array}{ccc}
 T_{\bullet}^{i,j}(A, B) \times_{\mathbf{NW}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(A,B)} \mathbf{NW}^{-1}\mathbf{C}^i\mathbf{C}^j(A, B) & \longrightarrow & T_{\bullet}^{i,j}(A, B) \\
 \downarrow \simeq & \swarrow \text{dotted} & \downarrow \simeq \\
 \mathbf{NW}^{-1}\mathbf{C}^i\mathbf{C}^j(A, B) & \longrightarrow & \mathbf{NW}^{-1}\mathbf{C}^i\mathbf{W}^{-1}\mathbf{C}^j(A, B)
 \end{array}$$

commute up to generalized simplicial homotopy.  $\square$

*B.6. Homotopy cocartesian squares in Waldhausen categories*

We note that the hypothesis of a USE of mapping cylinders for weak cofibrations also implies the existence of mapping cylinders for weak cofibrations and HCLF. The statements and proofs in this section are unchanged.

*B.7. Proof of Theorems 1.1, 1.2, and 2.7*

Proposition 7.1 generalizes to the case of a USE of factorizations for weak cofibrations by the work in the previous appendix. Combining Lemma A.2.3 as well, we have the following statement:

**Proposition B.7.1.** *If  $\mathcal{C}$  has a USE of factorizations of weak cofibrations or  $\mathcal{C}$  admits factorizations, then the forgetful functor  $\mathbf{wS}'_n\mathcal{C} \rightarrow \mathbf{wF}'_{n-1}\mathcal{C}$  induces a weak equivalence on nerves.*

*B.8. Proof of Theorem 1.4*

Most of the statements and proofs in this section are written in terms of non-functorial factorization. The only exception is Proposition 8.2, which we need to prove in the non-functorial case.

**Proof of Proposition 8.2.** Let  $A \rightarrow B$  and  $A \rightarrow C$  be objects in  $\mathcal{C}_{\text{co}}A$ . For each word  $\mathcal{Y}$  in  $\mathbf{C}$  and  $\mathbf{W}^{-1}$ , we have categories  $\mathcal{Y}_{\text{co}}(B, C)$  and  $\mathcal{Y}(B, C)$  of diagrams in  $\mathcal{C}_{\text{co}}A$  and  $\mathcal{C} \setminus A$ , respectively, as in Section 5; it suffices to show that the inclusion  $\mathcal{Y}_{\text{co}}(B, C) \rightarrow \mathcal{Y}(B, C)$  induces a weak equivalence on nerves.

Fix a word  $\mathcal{Y}$ ; the argument for Lemma 5.9 shows that it suffices to consider the case where  $\mathcal{Y}$  contains no subword of the form  $\mathbf{W}^{-1}\mathbf{W}^{-1}$ . Any two letter subword of  $\mathcal{Y}$  is then of the form  $\mathbf{CW}^{-1}$ ,  $\mathbf{CC}$ , or  $\mathbf{W}^{-1}\mathbf{C}$ :

$$\begin{array}{l}
 \mathbf{CW}^{-1}: \quad B \longrightarrow \cdots \longrightarrow X_{i+1} \xleftarrow{\simeq} X_i \longrightarrow X_{i-1} \longrightarrow \cdots \longrightarrow C, \\
 \mathbf{CC}: \quad B \longrightarrow \cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow X_{i-1} \longrightarrow \cdots \longrightarrow C, \\
 \mathbf{W}^{-1}\mathbf{C}: \quad B \longrightarrow \cdots \longrightarrow X_{i+1} \longrightarrow X_i \xleftarrow{\simeq} X_{i-1} \longrightarrow \cdots \longrightarrow C.
 \end{array}$$

In each of these cases we call  $X_i$  the *pivot* of the subword. Let  $\mathcal{Y}_{\text{co}(\mathbf{CW}^{-1})}(B, C)$  be the full subcategory of  $\mathcal{Y}(B, C)$  consisting of those objects where the structure maps  $A \rightarrow X_i$  is a cofibration whenever  $X_i$  is the pivot of a  $\mathbf{CW}^{-1}$  subword. Likewise, let  $\mathcal{Y}_{\text{co}(\mathbf{CW}^{-1}, \mathbf{CC})}(B, C)$  be the full subcategory where the structure map is a cofibration for the pivots of all  $\mathbf{CW}^{-1}$  and  $\mathbf{CC}$  subwords. We then have inclusions of full subcategories

$$\mathcal{Y}_{\text{co}}(B, C) \rightarrow \mathcal{Y}_{\text{co}(\mathbf{CW}^{-1}, \mathbf{CC})}(B, C) \rightarrow \mathcal{Y}_{\text{co}(\mathbf{CW}^{-1})}(B, C) \rightarrow \mathcal{Y}(B, C)$$

and it suffices to show that each of these induces weak equivalences on nerves.

We start with the inclusion  $\mathcal{Y}_{\text{co}(\mathbf{CW}^{-1})}(B, C) \rightarrow \mathcal{Y}(B, C)$ . By Lemma A.2.3, we have a USE of factorizations  $T_\bullet \rightarrow N \text{Fac} \mathcal{C}$ . For each subword  $\mathbf{CW}^{-1}$ , we have a functor  $\mathcal{Y}(B, C) \rightarrow \text{Ar} \mathcal{C}$  sending the object in  $\mathcal{Y}(B, C)$  pictured above to the object  $A \rightarrow X_i$  in  $\text{Ar} \mathcal{C}$ . Let  $U_\bullet$  be the pullback of the diagram below.

$$\begin{array}{ccc} U_\bullet & \longrightarrow & T_\bullet \times \cdots \times T_\bullet \\ \cong \downarrow & & \cong \downarrow \\ N\mathcal{Y}(B, C) & \longrightarrow & N \text{Ar} \mathcal{C} \times \cdots \times N \text{Ar} \mathcal{C} \end{array}$$

Then from the map  $T_\bullet \rightarrow N \text{Fac} \mathcal{C}$ , we get a map  $U_\bullet \rightarrow \mathcal{Y}_{\text{co}(\mathbf{CW}^{-1})}(B, C)$  and simplicial homotopies making the diagram

$$\begin{array}{ccc} U_\bullet \times_{N\mathcal{Y}(B, C)} N\mathcal{Y}_{\text{co}(\mathbf{CW}^{-1})}(B, C) & \longrightarrow & U_\bullet \\ \cong \downarrow & \swarrow \text{dotted} & \cong \downarrow \\ N\mathcal{Y}_{\text{co}(\mathbf{CW}^{-1})}(B, C) & \longrightarrow & N\mathcal{Y}(B, C) \end{array}$$

commute up to simplicial homotopy.

For the inclusion  $\mathcal{Y}_{\text{co}(\mathbf{CW}^{-1}, \mathbf{CC})}(B, C) \rightarrow \mathcal{Y}_{\text{co}(\mathbf{CW}^{-1})}(B, C)$ , we work by induction. Let  $\mathcal{Y}_i$  be the full subcategory of  $\mathcal{Y}_{\text{co}(\mathbf{CW}^{-1})}(B, C)$  consisting of the objects for which the structure maps are cofibrations for the pivots of the last  $i$  subwords of the form  $\mathbf{CC}$ . Then  $\mathcal{Y}_{\text{co}(\mathbf{CW}^{-1})}(B, C) = \mathcal{Y}_0(B, C)$  and  $\mathcal{Y}_{\text{co}(\mathbf{CW}^{-1}, \mathbf{CC})}(B, C)$  is  $\mathcal{Y}_n(B, C)$  for some  $n$ ; we show that the inclusions

$$\mathcal{Y}_{i+1}(B, C) \rightarrow \mathcal{Y}_i(B, C)$$

induce weak equivalences on nerves. In  $\mathcal{Y}_i(B, C)$ , consider the pivot of the  $(i + 1)$ -st from last subword of the form  $\mathbf{CC}$ :

$$B \text{ --- } \cdots \text{ --- } X_{j+1} \text{ --->} X_j \text{ --->} X_{j-1} \text{ --- } \cdots \text{ --- } C.$$

Either  $X_{j+1} = B$ , or  $X_{j+1}$  is a pivot of a subword  $\mathbf{CW}^{-1}$ , or  $X_{j+1}$  is the pivot of the  $i$ -th from last subword of the form  $\mathbf{CC}$ . In any of these cases, the structure map  $A \rightarrow X_{j+1}$  is a cofibration. Using the functor  $\mathcal{Y}_i(B, C) \rightarrow \text{Ar} \mathcal{C}$  sending the object in  $\mathcal{Y}_i(B, C)$  pictured above to the object  $X_{j+1} \rightarrow X_j$  of  $\text{Ar} \mathcal{C}$ , we get the solid arrow diagram below.

$$\begin{array}{ccc}
 T_{\bullet} \times_{N \text{Ar} \mathcal{C}} N \Upsilon_{i+1}(B, C) & \longrightarrow & T_{\bullet} \times_{N \text{Ar} \mathcal{C}} N \Upsilon_i(B, C) \\
 \cong \downarrow & \swarrow \text{dotted arrow} & \cong \downarrow \\
 N \Upsilon_{i+1}(B, C) & \longrightarrow & N \Upsilon_i(B, C)
 \end{array}$$

The map  $T_{\bullet} \rightarrow N \text{Fac} \mathcal{C}$  induces the dotted arrow and simplicial homotopies making both triangles commute up to simplicial homotopy.

For the inclusion  $\Upsilon_{\text{co}}(B, C) \rightarrow \Upsilon_{\text{co}(\mathbf{CW}^{-1}, \mathbf{CC})}(B, C)$ , consider the subwords of the form  $\mathbf{W}^{-1}\mathbf{C}$

$$B \text{ --- } \cdots \text{ --- } X_{k+1} \longrightarrow X_k \xleftarrow{\cong} X_{k-1} \text{ --- } \cdots \text{ --- } C.$$

If the subword is the final two symbols of  $\Upsilon$ , then  $X_{k+1}$  is  $B$ ; if not, then the next letter in  $\Upsilon$  is  $\mathbf{W}^{-1}$  or  $\mathbf{C}$ , and  $X_{k+1}$  is in the middle of a  $\mathbf{CW}^{-1}$  or  $\mathbf{CC}$  subword. In either case, the structure map  $A \rightarrow X_{k+1}$  is a cofibration. Likewise the structure map  $A \rightarrow X_{k-1}$  is a cofibration. We therefore obtain a functor  $\Upsilon_{\text{co}(\mathbf{CW}^{-1}, \mathbf{CC})}(B, C) \rightarrow \text{Ar} \mathcal{C}$  taking the object of  $\Upsilon_{\text{co}(\mathbf{CW}^{-1}, \mathbf{CC})}(B, C)$  pictured above to the object  $X_{k-1} \cup_A X_{k+1} \rightarrow X_k$  of  $\text{Ar} \mathcal{C}$ . The same argument as in the  $\mathbf{CW}^{-1}$  subword argument then shows that the inclusion of  $\Upsilon_{\text{co}}(B, C)$  in  $\Upsilon_{\text{co}(\mathbf{CW}^{-1}, \mathbf{CC})}(B, C)$  induces a weak equivalence on nerves, and completes the proof.  $\square$

### B.9. Proof of Theorem 1.5

No statements or arguments in this section involve functoriality of the factorizations.

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