Generalized Multivalued
Variational Inequalities II

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Abstract—In this paper, we prove that the generalized multivalued variational inequalities are equivalent to the generalized multivalued Wiener-Hopf equations. This equivalence is used to suggest a number of iterative algorithms for solving generalized multivalued variational inequalities. We also discuss the convergence of these algorithms.

Keywords—Variational inequalities, Wiener-Hopf equations, Iterative algorithms.

1. INTRODUCTION

Variational inequality theory, introduced by Stampacchia [1], has emerged an interesting branch of applicable mathematics. This theory provides us a simple, natural, general and unified framework to study a wide class of linear and nonlinear problems arising in elasticity, fluid flow through porous media, economics, oceanography, transportation, optimization, pure and applied sciences (see [2–20] and the references therein). Variational inequality theory has been extended and generalized in several directions using novel and innovative techniques. Inspired and motivated by the recent research going on in this field, Noor [16] introduced and studied a new class of variational inequalities known as generalized multivalued variational inequalities. This class includes many known classes of variational inequalities as special cases. In this paper, we prove the equivalence between the generalized multivalued variational inequalities and the generalized multivalued Wiener-Hopf equations. This equivalence is used to suggest a number of iterative methods for solving generalized multivalued variational inequalities.

In Section 2, we formulate the problem and review some basic facts. The main results are discussed in Section 3.

2. PRELIMINARIES

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively. Let $K$ be a nonempty closed convex set in $H$.

Given singlevalued operators $A, g : H \to H$ and multivalued operators $T, V : H \to 2^H$, we consider the problem of finding $u \in H$, $w \in (Tu)$, $y \in V(u)$ such that $g(u) \in K$ and

$$\langle w + Ay, v - g(u) \rangle \geq 0, \quad \text{for all } v \in K. \quad (2.1)$$

The inequality of the type (2.1) is called the generalized multivalued variational inequality. This problem is due to Noor [16], where he used the projection technique to suggest an iterative algorithm.
**Special Cases**

**CASE I.** If \( A \equiv 0 \), then problem \( (2.1) \) is equivalent to finding \( u \in H, w \in (Tu) \) such that
\[
\langle w, v - g(u) \rangle \geq 0, \quad \text{for all } v \in K, \tag{2.2}
\]
which is called the generalized variational inequality and appears to be a new one.

**CASE II.** If \( V : H \to 2^H \) is the identity operator, then problem \( (2.1) \) is equivalent to finding \( u \in H, w \in (Tu) \) such that \( g(u) \in K \) and
\[
\langle w + Au, v - g(u) \rangle \geq 0, \quad \text{for all } u \in K. \tag{2.3}
\]
The problem \( (2.3) \) is called the multivalued strongly nonlinear variational inequality, and appears to be a new one.

**CASE III.** If \( T : H \to H \) is the single-valued operator and \( g, V : H \to H \) are the identity operators, then problem \( (2.1) \) is equivalent to finding \( u \in K \) such that
\[
\langle Tu + Au, v - u \rangle \geq 0, \quad \text{for all } v \in K, \tag{2.4}
\]
which is known as the strongly nonlinear variational inequality. The problem \( (2.4) \) is mainly due to Noor [17]. For the recent applications, generalizations and numerical methods for the problem \( (2.4) \), see [4,5,7,15-19] and the references therein.

In brief, for suitable and appropriate choice of the operators \( T, A, g, V \) and the convex set \( K \), one can obtain a large number of various classes of variational inequalities and complementarity problems as special cases. It is clear that the problem \( (2.1) \) is the most general and unifying one.

Let \( P_K \) be the projection of \( H \) into the convex set \( K \) and \( Q_K = I - P_K \), where \( I \) is the identity operator. Given multivalued operators \( T, V : H \to 2^H \) and nonlinear operators \( A, g : H \to H \), consider the problem of finding \( z \in H, u \in H, w \in (Tu), y \in V(u) \) such that
\[
w + \rho^{-1}Q_Kz = -Ay. \tag{2.5}
\]
The equations of the type \( (2.5) \) are called the generalized multivalued Wiener-Hopf equations. For the general treatment and applications of the Wiener-Hopf equations, see Speck [21].

**Lemma 2.1.** [2] Let \( K \) be a closed convex set in \( H \). Then, given \( z \in H, u = P_Kz \) if and only if \( u \in K \) satisfies
\[
\langle u - z, v - u \rangle \geq 0, \quad \text{for all } v \in K.
\]

**Definition 2.1.** For all \( u_1, u_2 \in H \), the operator \( T : H \to 2^H \) is said to be:

(a) *Strongly monotone* if there exists a constant \( \alpha > 0 \) such that
\[
\langle w_1 - w_2, u_1 - u_2 \rangle \geq \alpha \|u_1 - u_2\|^2, \quad \text{for all } w_1 \in (Tu_1), w_2 \in (Tu_2).
\]

(b) *Lipschitz continuous*, if there exists a constant \( \beta > 0 \) such that
\[
\|w_1 - w_2\| \leq \beta \|u_1 - u_2\|, \quad \text{for all } w_1 \in (Ty_1), w_2 \in (Ty_2).
\]

From (1) and (b), it follows that \( \alpha \leq \beta \).

**Definition 2.2.** The multivalued operator \( V : H \to C(H) \) is said to be *\( M \)-Lipschitz continuous* if there exists a constant \( \eta > 0 \) such that
\[
M(V(u), V(v)) \leq \eta \|u - v\|, \quad \text{for all } u, v \in H,
\]
where \( C(H) \) is the family of all nonempty compact subsets of \( H \) and \( M(.,.) \) is the Hausdorff metric on \( C(H) \).
3. MAIN RESULTS

In this section, we show that the variational inequalities (2.1) are equivalent to the Wiener-Hopf equation. This formulation is used to suggest iterative algorithms for solving the variational inequalities.

**Theorem 3.1.** The generalized multivalued variational inequality (2.1) has a solution \( u \in H \), \( w \in (Tu) \), \( y \in V(u) \) if and only if the generalized multivalued Wiener-Hopf equation (2.5) has a solution \( z \in H \), \( u \in H \), \( w \in (Tu) \), \( y \in V(u) \), where

\[
\begin{align*}
g(u) &= P_K z, \\
z &= g(u) - \rho(w + Ay),
\end{align*}
\]

where \( \rho > 0 \) is a constant and \( g^{-1} \) is the inverse of \( g \).

**Proof.** The method of proof is similar to that of Noor [15] and Shi [22,23]. Let \( u \in H \), \( w \in (Tu) \), \( y \in V(u) \) such that \( g(u) \in K \) be a solution of the variational inequality (2.1). Then by Lemma 2.1, we have

\[
g(u) = P_K [g(u) - \rho(w + Ay)].
\]

Now using the fact \( Q_K = I - P_K \) and the equation (3.3), we obtain

\[
Q_K [g(u) - \rho(w + Ay)] = g(u) - \rho(w + Ay) - P_K [g(u) - \rho(w + Ay)] = -\rho(w + Ay),
\]

from which and (3.2) it follows that

\[
w + \rho^{-1}Q_K z = -Ay.
\]

Conversely, let \( z \in H \), \( u \in H \), \( w \in (Tu) \) and \( y \in V(u) \) be solution of (2.5), then

\[
\rho(w + Ay) = -Q_K z = P_K z - z.
\]

Now from Lemma 2.1 and (3.4), for all \( v \in K \), we have

\[
0 \leq \langle P_K z - z, v - P_K z \rangle = \rho \langle w + Ay, v - P_K z \rangle.
\]

Thus, \((u, w, y)\), where \( u = g^{-1}P_K z \) is a solution of (2.1).

Theorem 3.1 establishes the equivalence between the variational inequality (2.1) and the Wiener-Hopf equation (2.5). This equivalence is quite general and flexible. For the appropriate rearrangement of the Wiener-Hopf equation (2.5), we can suggest a number of iterative algorithms for solving variational inequalities.

**Case I.** The equation (2.5) can be written as

\[
Q_K z = -\rho(w + Ay),
\]

from which it follows that

\[
\begin{align*}
z &= P_K z - \rho(w + Ay) \\
&= g(u) - \rho(w + Ay), \text{ using (3.1)}.
\end{align*}
\]

This fixed point formulation enables us to suggest the iterative algorithm for solving the variational inequality (2.1).
ALGORITHM 3.1. Assume that $K$ is a closed convex set in $H$, $T : H \to 2^H$, a multivalued operator, $g, A : H \to H$, nonlinear single-valued operators and $V : H \to C(H)$. For given $z_0 \in H, u_0 \in H$, let us take $g(u_0) = P_K z_0, w_0 \in (Tu_0), y_0 \in V(u_0)$ and

$$z_1 = g(u_0) - \rho(w_0 + Ay_0).$$

Since $y_0 \in V(u_0) \in C(H)$, by [24], there exists a $y_1 \in V(u_1)$ such that

$$\|y_1 - y_0\| \leq M(V(u_1), V(u_0)),$$

where $M(., .)$ is the Hausdorff metric on $C(H)$. Let $g(u_1) = P_K z_1, w_1 \in T(u_1), y_1 \in V(u_1)$ and

$$u_2 = g(u_1) - \rho(w_1 + Ay_1).$$

By induction, we can obtain sequences $\{z_n\}, \{u_n\}, \{w_n\}$ and $\{y_n\}$ such that

$$g(u_n) = P_K z_n,$$  \hspace{1cm} (3.6)

$$w_n \in T(u_n),$$  \hspace{1cm} (3.7)

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)),$$  \hspace{1cm} (3.8)

$$z_{n+1} = g(u_n) - \rho(w_n + Ay_n), \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (3.9)

CASE II. The equation (2.5) may be written as

$$Q_K z = (1 - \rho^{-1})Q_K z - (w + Ay),$$

which implies that

$$z = P_K z - (w + Ay) + (1 - \rho^{-1})Q_K z$$

$$= g(u) - (w + Ay) + (1 - \rho^{-1})Q_K z,$$

using (3.1).

Using this fixed point formulation, we can suggest the following iterative scheme.

ALGORITHM 3.2. For given $z_0 \in H, u_0 \in H, w_0 \in (Tu_0), y_0 \in V(u_0)$, compute $\{z_n\}, \{u_n\}, \{w_n\}$ and $\{y_n\}$ from the iterative schemes.

$$g(u_n) = P_K z_n, \quad w_n \in (Tu_n),$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)),$$

$$z_{n+1} = g(u_n) - \rho(w_n + Ay_n) + (1 - \rho^{-1})Q_K z_n, \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (3.10)

For suitable and appropriate choice of the operators $T, A, g, V$ and the convex set $K$, one can obtain a number of known and new iterative algorithms for solving various classes of variational inequalities and complementarity problems.

We now study the convergence criteria of the Algorithm 3.1. In a similar way, one may study the convergence of the approximate solutions obtained from Algorithm 3.2.

THEOREM 3.2. Let the operators $T : H \to 2^H$ and $g : H \to H$ be both strongly monotone with constants $\alpha > 0$ and $\sigma > 0$, respectively, and Lipschitz continuous with Lipschitz constants $\beta > 0, \delta > 0$, respectively. Let $A : H \to H$ be Lipschitz continuous with constant $\gamma > 0$ and $V : H \to C(H)$ be $M$-Lipschitz continuous with constant $\eta > 0$. Assume that

$$\rho - \frac{(\alpha - (1 - k)\gamma \eta)}{\beta^2 - \gamma^2 \eta^2} \leq \left\{ \frac{\sqrt{\alpha - (1 - k)\gamma \eta}^2 - (\beta^2 - \eta^2 \gamma^2)(2k - k^2)}{\beta^2 - \gamma^2 \eta^2} \right\},$$  \hspace{1cm} (3.10)

$$\alpha > (1 - k)\gamma + \sqrt{(\beta^2 - \gamma^2 \eta^2)(2k - k^2)},$$  \hspace{1cm} (3.11)

$$\rho \eta \gamma < 1 - k, \quad \text{where} \quad k = 2 \left( \sqrt{1 - 2\sigma + \delta^2} \right).$$  \hspace{1cm} (3.12)
Then there exist \( z \in H, u \in H, w \in (Tu), y \in V(u) \), which are the solutions of the generalized multivalued Wiener-Hopf equations (2.5) and the sequences \( \{z_n\}, \{u_n\}, \{w_n\} \) and \( \{y_n\} \) generated by Algorithm 3.1 converge to \( z, u, w, y \) strongly in \( H \), respectively.

**Proof.** From Algorithm 3.1, we have

\[
\|z_{n+1} - z_n\| = \|g(u_n) - g(u_{n-1}) - \rho(w_n - w_{n-1}) - \rho(Ay_n - Ay_{n-1})\|
\]

\[
\leq \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| + \|u_n - u_{n-1} - \rho(w_n - w_{n-1})\|
\]

\[
+ \rho\|Ay_n - Ay_{n-1}\|. \tag{3.13}
\]

Since \( T : H \to 2^H \) is a strongly monotone and Lipschitz continuous operator, so

\[
\|u_n - u_{n-1} - \rho(w_n - w_{n-1})\|^2 = \|u_n - u_{n-1}\|^2 - \rho (w_n - w_{n-1}u_n - u_{n-1})
\]

\[
+ \rho^2\|w_n - w_{n-1}\|^2
\]

\[
\leq (1 - 2\rho \alpha + \beta^2\rho^2)\|u_n - u_{n-1}\|^2. \tag{3.14}
\]

Now using the strong monotonicity and Lipschitz continuity of the operator \( g : H \to H \), we have

\[
\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\|^2 \leq (1 - 2\sigma + \delta^2)\|u_n - u_{n-1}\|^2. \tag{3.15}
\]

From the Lipschitz continuity of \( A : H \to H \) and the \( M \)-Lipschitz continuity of \( V : H \to C(H) \), we have

\[
\|Ay_n - Ay_{n-1}\| \leq \gamma\|y_n - y_{n-1}\| \leq \gamma M(V(u_n), V(u_{n-1})) \leq \gamma\|u_n - u_{n-1}\|. \tag{3.16}
\]

From (3.13), (3.14), (3.15) and (3.16), we have

\[
\|z_{n+1} - z_n\| \leq \left\{ \sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\rho \alpha + \beta^2\rho^2 + \rho \eta} \right\}\|u_n - u_{n-1}\|
\]

\[
= \left\{ \frac{k}{2} + \left( \sqrt{1 - 2\rho \alpha + \beta^2\rho^2 + \rho \eta} \right) \right\}\|u_n - u_{n-1}\|, \tag{3.17}
\]

where \( k = 2 \left( \sqrt{1 - 2\sigma + \delta^2} \right) \). \( \tag{3.18} \)

And from (3.6) and (3.15) we have

\[
\|u_n - u_{n-1}\| = \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) + P_K z_n - P_K z_{n-1}\|
\]

\[
\leq \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| + \|P_K z_n - P_K z_{n-1}\|
\]

\[
\leq \left( \sqrt{(1 - 2\sigma + \delta^2)} \right)\|z_n - z_{n-1}\| + \|z_n - z_{n-1}\|,
\]

from which along with (3.18) it follows that

\[
\|u_n - u_{n-1}\| \leq \frac{1}{1 - (1/2)k} \|z_n - z_{n-1}\|. \tag{3.19}
\]

Combining (3.18) and (3.19), we have

\[
\|z_{n+1} - z_n\| \leq \left\{ \frac{(1/2)k + \sqrt{1 - 2\rho \alpha + \beta^2\rho^2 + \eta \gamma \rho}}{1 - (1/2)k} \right\}\|z_n - z_{n-1}\|
\]

\[
= \theta\|z_n - z_{n-1}\|, \tag{3.20}
\]

where

\[
\theta = \left\{ \frac{(1/2)k + \sqrt{1 - 2\rho \alpha + \beta^2\rho^2 + \eta \gamma \rho}}{1 - (1/2)k} \right\} \quad \text{and} \quad k = 2 \left( \sqrt{1 - 2\sigma + \delta^2} \right).
From (3.10), (3.11) and (3.12), it follows that \( \theta < 1 \). Hence, from (3.20), we know that the sequence \( \{z_n\} \) is a Cauchy sequence in \( H \) so that there exists \( z \in H \) with \( z_n \rightarrow z \) as \( n \rightarrow \infty \). Also from (3.8) and (3.19), we have
\[
\|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) \leq \eta \|u_{n+1} - u_n\|
\]
\[
\leq \frac{\eta}{1 - (1/2)k} \|z_{n+1} - z_n\|
\]
from which it follows that the sequence \( \{y_n\} \) is also a Cauchy sequence in \( H \). Thus, there exists \( y \in H \) such that \( y_n \rightarrow y \) as \( n \rightarrow \infty \).

Now by using the continuity of the operators \( T, A, g, V, P_K \) and Algorithm 3.1, we have
\[
g(u) = P_K z,
\]
\[
z = g(u) - \rho(w + Ay) \in H.
\]
Now we shall prove that \( y \in V(u) \). In fact, using (3.13),
\[
d(y, V(u)) \leq \|y - y_n\| + d(y_n, V(u)) \leq \|y - y_n\| + M(V(u_n), V(u))
\]
\[
\leq \|y - y_n\| + \eta \|u_n - u\| \leq \|y - y_n\| + \frac{\eta}{1 - k} \|z_n - z\|,
\]
where \( d(y, V(u)) = \inf \{\|y - v\| : v \in V(u)\} \). Since \( y_n \rightarrow y \) and \( z_n \rightarrow z \) as \( n \rightarrow \infty \), it follows that \( d(y, V(u)) = 0 \). Thus, we conclude that \( y \in V(u) \), since \( V(u) \in C(H) \).

By Theorem 3.1, it follows that \( z \in H, u \in H, w \in (Tu), y \in V(u) \) are solutions of the Wiener-Hopf equation (2.5) and consequently \( z_n \rightarrow z, u_n \rightarrow u, w_n \rightarrow w \) and \( y_n \rightarrow y \) strongly in \( H \), the required result.

REMARK 3.1. We would like to point out that if the convex set \( K \) also depends upon the solution implicitly or explicitly, then the generalized multivalued variational inequality (2.1) is called the generalized multivalued quasi-variational inequality. To be more precise, given a point-to-set mapping \( K : u \rightarrow K(u) \), which associates a closed convex set \( K(u) \) with any element \( u \) of \( H \), consider the problem of finding \( u \in H, \omega \in T(u), y \in V(u) \) such that \( g(u) \in K(u) \) and
\[
(\omega + Ay, v - g(u)) \geq 0, \quad \text{for all } v \in K(u).
\]
The inequality (3.21) is known as the generalized multivalued quasi-variational inequality. Now using essentially the techniques and ideas of this paper, one can suggest and analyse analogue iterative methods for the generalized multivalued quasi-variational inequality (3.21). This will be discussed in a subsequent paper.

REFERENCES