CrossMark

Selberg's orthonormality conjecture and joint universality of L-functions

Yoonbok Lee[1,2] · **Takashi Nakamura[3]** ·
Łukasz Pańkowski[4,5]

Abstract In the paper we introduce a new method how to use only an orthonormality relation of coefficients of Dirichlet series defining given L-functions from the Selberg class to prove joint universality.

1 Introduction

In 1975, Voronin [22] discovered the so-called universality property, which is one of the most remarkable result concerning the value-distribution of $\zeta(s)$. The modern version states that

✉ Łukasz Pańkowski
   lpan@amu.edu.pl

   Yoonbok Lee
   leeyb131@gmail.com; leeyb@inu.ac.kr

   Takashi Nakamura
   nakamuratakashi@rs.tus.ac.jp

1 Department of Mathematics, Incheon National University, Incheon, Korea

2 Research Institute of Natural Sciences, Incheon National University, Incheon, Korea

3 Department of Liberal Arts, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda-shi, Chiba-ken 278-8510, Japan

4 Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznan, Poland

5 Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan

🌀 Springer

for any continuous non-vanishing function $f(s)$ on a compact set with connected complement $K \subset \{s \in \mathbb{C} : 1/2 < \mathrm{Re}(s) < 1\}$, analytic in the interior of K, we have

$$\forall_{\varepsilon > 0} \liminf_{T \to \infty} \frac{1}{T} \, \mathrm{meas} \left\{ \tau \in [0, T] : \max_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0,$$

where meas$\{\cdot\}$ denotes the Lebesgue real measure.

Voronin's universality theorem has been generalized for many zeta and L-functions from number theory. For example, a universality theorem is known for: Dirichlet L-functions [22], Dedekind zeta functions (Reich 1980), Artin L-functions (Bauer 2003), L-functions associated with newforms (Laurinčikas et al. 2003), and many others. A quite general class of universal L-functions with polynomial Euler product was introduced by Steuding [21], and recently, his result was generalized by Nagoshi and Steuding [17] to all L-functions from the Selberg class with coefficients $a(n)$ of Dirichlet series representation satisfying

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a(p)|^2 = \kappa, \tag{1}$$

for some positive constant κ depending on L; here $\pi(x)$, as usual, counts the number of primes not exceeding x.

Let us recall that the Selberg class $\mathcal{S}$ consists of functions $L(s)$ defined by a Dirichlet series $\sum_{n=1}^{\infty} a_L(n) n^{-s}$ in the half-plane $\sigma := \mathrm{Re}(s) > 1$ satisfying the following axioms:

 (i) *Ramanujan hypothesis:* $a_L(n) \ll_\varepsilon n^\varepsilon$ for every $\varepsilon > 0$;
 (ii) *analytic continuation:* there exists a non-negative integer m_L such that $(s - 1)^{m_L} L(s)$ is an entire function of finite order;
(iii) *functional equation:* $L(s)$ satisfies the following functional equation

$$\Lambda(s) = \theta \, \overline{\Lambda(1 - \bar{s})},$$

where

$$\Lambda(s) := L(s) Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j),$$

$|\theta| = 1$, $Q, \lambda_j \in \mathbb{R}$, and $\mu_j \in \mathbb{C}$ with $\mathrm{Re}(\mu_j) \geq 0$;
(iv) *Euler product:* for $\sigma > 1$ we have

$$\log L(s) = \sum_{p} \sum_{k=1}^{\infty} \frac{b_L(p^k)}{p^{ks}},$$

where $b_L(p^k)$ are complex numbers satisfying $b_L(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$.

Note that almost all known proofs of universality requires existing of the mean-square, which is rather difficult problem in the general setting of Selberg class. For example, the best known result (see [18] or [21, Corollary 6.11]) says that, for $L \in \mathcal{S}$, we have

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |L(\sigma + it)|^2 dt = \sum_{n=1}^{\infty} \frac{|a_L(n)|^2}{n^{2\sigma}} < \infty, \qquad \sigma > \max \left\{ \frac{1}{2}, 1 - \frac{1}{d_L} \right\},$$

where d_L denotes the degree of L defined by $2 \sum_{j=1}^{k} \lambda_j$, where λ_j's are given by the functional equation of L. Therefore, it is natural that Nagoshi's and Steuding's universality theorem of

L-function from the Selberg class was proved only in the strip $\{s \in \mathbb{C} : \sigma_{\mathrm{m}}(L) < \mathrm{Re}(s) < 1\}$, where $\sigma_{\mathrm{m}}(L)$ denotes the abscissa of the mean-square half-plane for L.

Voronin [23] (see also [9, Chapter VII, Theorem 3.2.1]) proved also the so-called joint universality theorem for Dirichlet L-functions associated with pairwise non-equivalent Dirichlet characters. Roughly speaking, he proved that any collection of analytic non-vanishing functions $f_1, f_2, \ldots, f_n$ can be approximated, in the Voronin sense, by the shift $L(s + i\tau; \chi_1), L(s + i\tau; \chi_2), \ldots, L(s + i\tau; \chi_n)$, where $\chi_1, \ldots, \chi_n$ are pairwise non-equivalent Dirichlet characters. Joint universality was also proved for many other zeta and L-functions from number theory. However, it is still open problem put forward by Steuding [21], whether a collection of L-functions from Selberg class is jointly universal under the assumption of the following so-called Selberg's orthonormality conjecture, which is widely believed to be true in the Selberg class.

Conjecture 1.1 (Selberg) *For any function $1 \neq L \in \mathcal{S}$ there is a positive integer κ_L such that*

$$\sum_{p \leq x} \frac{|a_L(p)|^2}{p} = \kappa_L \log \log x + R(x) \tag{2}$$

and, for any distinct primitive functions $L_1, L_2 \in \mathcal{S}$, we have

$$\sum_{p \leq x} \frac{a_{L_1}(p)\overline{a_{L_2}(p)}}{p} = R(x), \tag{3}$$

where $R(x) \ll 1$.

The last equation can be called the orthonormality relation.

Obviously, to expect joint universality for at least two functions L_1 and L_2 we need some kind of their independence, so Selberg's conjecture seems to be the most natural assumption of this kind in the Selberg class. Interesting evidence for the truth of this conjecture was given by Bombieri and Hejhal [3], where they showed the statistical independence of any collection of L-functions under a stronger version of Selberg's conjecture. Moreover, it is known that Selberg's conjecture with $R(x) \ll 1$ is not sufficient to prove joint universality. The second author in [15, Example 7.5] observed that, for non-principle Dirichlet character χ, the Dirichlet L-functions $L(s, \chi)$ and $L(s - i, \chi)$ cannot be jointly universal, whereas it is easy to observe that

$$\sum_{p \leq x} \frac{\chi(p)\overline{\chi(p)p^{-i}}}{p} = \sum_{\substack{p \leq x \\ \chi(p) \neq 0}} p^{i-1} \ll 1,$$

so Selberg's conjecture with $R(x) \ll 1$ holds.

The main purpose of this paper is to introduce a new method how to use only orthonormality to prove joint universality of L-functions with Euler product. In order to illustrate this idea we prove a general joint universality theorem for any collection of L-functions $L_1, \ldots, L_m$ from the Selberg class satisfying some stronger analogue of Selberg's conjecture, namely

$$\sum_{p \leq x} |a_{L_k}(p)|^2 = \sum_{j=1}^{2m+1} \frac{c_j^{(k)} x}{(\log x)^j} + O\left(\frac{x}{(\log x)^{2m+2}}\right) \quad (1 \leq k \leq m), \tag{4}$$

and

$$\sum_{p \leq x} a_{L_k}(p)\overline{a_{L_l}(p)} = \sum_{j=2}^{2m+1} \frac{c_j^{(k,l)}x}{(\log x)^j} + O\left(\frac{x}{(\log x)^{2m+2}}\right) \quad (1 \leq k \neq l \leq m), \quad (5)$$

where $c_j^{(k)}$, $c_j^{(k,l)}$ are some constants and $c_1^{(k)} > 0$. It is easy to observe, by partial summation, that it is equivalent to the Selberg's conjecture (2) and (3), where

$$R(x) = \sum_{j=0}^{2m+2} \frac{c_j}{(\log x)^j} + O\left(\frac{1}{(\log x)^{2m+3}}\right)$$

for suitable c_j depending on given L-functions.

Although the above formulas are obviously stronger than the original Selberg's conjecture, it is quite likely that they are fulfilled by all L-functions. We refer to Sect. 4 for a detailed discussion of this matter, where several unconditional joint universality theorems for automorphic L-functions are deduced from our method. Here we only mention that the evidence for the truth of this conjecture is the fact that there is a grand hypothesis that each L-function from Selberg class can be defined as a suitable automorphic L-function and, so far, all automorphic L-functions satisfying Selberg's conjecture fulfill in fact (4) and (5).

Theorem 1.2 *Let $L_1, \ldots, L_m$ be elements of $\mathcal{S}$, $K_1, \ldots, K_m \subset \{s \in \mathbb{C} : \max_{j=1,2,\ldots,m} \sigma_m (L_j) < \mathrm{Re}\, s < 1\}$ be compact sets with connected complements and g_j, $j = 1, \ldots, m$ be continuous non-vanishing function on K_j, and analytic in the interior of K_j. Then, if (4) and (5) hold, we have, for every $\varepsilon > 0$, that*

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \max_{j=1,\ldots,m} \max_{s \in K_j} |L_j(s + i\tau) - g_j(s)| < \varepsilon \right\} > 0.$$

Noteworthy is the fact that most of proofs of universality rely on periodicity and orthonormality property of coefficients of L-functions. Recently, Mishou [14] invented a new approach to prove joint universality without periodicity, which works for a pair of L-functions with real coefficients under the assumption of some analogue of (4) and (5). The purpose of this paper is to introduce another new approach how to use only orthonormality relation to prove joint universality for any collection of L-functions with complex coefficients. It should be noted here that Mishou's result was formulated in the language of automorphic L-functions, which belong to the Selberg class only conjecturally. Nevertheless, one can easily compare his result to Theorem 1.2 and notice that the essential difference is that our approach works even for more than two L-functions and, moreover, it does not require the assumption that coefficients of Dirichlet series defining given L-functions are real.

This method can be easily generalized to other zeta and L-functions, which joint universality property relies on some independence of coefficients of Dirichlet series representation. For example, in [11] the authors proved joint universality for a collection of Lerch zeta functions $L(s; \alpha, \lambda_j) = \sum_{n=0}^{\infty} \frac{\exp(2\pi i \lambda_j)}{(n+\alpha)^s}$, $j = 1, 2, \ldots, m$, associated to transcendental $\alpha \in (0, 1]$ and distinct λ_j's with $\lambda_j \in (0, 1]$.

As standard consequences of universality, one can easily prove the following corollaries. For the proofs we refer, for example, to [14, Section 8], where Mishou showed similar results for a pair of L-functions. However, the modifications needed are straightforward and can be left to the reader (see [21, Section 10]).

Corollary 1.3 *Let $m \geq 2$, $0 \neq a_1, \ldots, a_m \in \mathbb{C}$ and $L_1, \ldots, L_m \in \mathcal{S}$ satisfy (4) and (5). Then the function*

$$L(s) = \sum_{j=1}^{m} a_j L_j(s)$$

is strongly universal in the strip $\max_{1 \leq j \leq m} \sigma_m(L_j) =: \sigma_L < \sigma < 1$, which means that Theorem 1.2 holds also for functions g_j having zeros on K_j.

Moreover, the function $L(s)$ has infinitely many zeros in the strip $\sigma_L < \sigma < 1$, namely for any σ_1, σ_2 with $\sigma_L < \sigma_1 < \sigma_2 < 1$ and sufficiently large T there exist $\gg T$ zeros $\rho = \beta + i\gamma$ of $L(s)$ in the rectangle $\sigma_1 \leq \beta \leq \sigma_2$, $0 \leq \gamma \leq T$.

Corollary 1.4 *Let $N \in \mathbb{N}$, $L_1, \ldots, L_m \in \mathcal{S}$ satisfy (4) and (5) and σ_0 be a real number satisfying $\max_{1 \leq j \leq m} \sigma_m(L_j) < \sigma_0 < 1$. Then the set*

$$\left\{ \left(L_1(\sigma_0 + it), \ldots, L_m(\sigma_0 + it), \ldots, L_1^{(N-1)}(\sigma_0 + it), \ldots, L_m^{(N-1)}(\sigma_0 + it) \right) : t \in \mathbb{R} \right\}$$

is dense in $\mathbb{C}^{mN}$.

Corollary 1.5 *Let $N \in \mathbb{N}$ and $L_1, \ldots, L_m \in \mathcal{S}$ satisfy (4) and (5). If continuous functions $f_l : \mathbb{C}^{mN} \to \mathbb{C}$, $l = 0, 1, \ldots, L$ satisfy*

$$\sum_{l=0}^{L} s^l f_l \left(L_1(s), \ldots, L_m(s), \ldots, L_1^{(N-1)}(s), \ldots, L_m^{(N-1)}(s) \right) \equiv 0$$

for all $s \in \mathbb{C}$, then $f_l \equiv 0$ for all $0 \leq l \leq L$.

2 A denseness lemma

Let us fix L-functions $L_1, \ldots, L_m \in \mathcal{S}$ and compact sets $K_1, \ldots, K_m \subset \{s \in \mathbb{C} : \max_{j=1,2,\ldots,m} \sigma_m(L_j) < \mathrm{Re}\, s < 1\}$. Take $\sigma_1 > \max_{j=1,2,\ldots,m} \{\sigma_m(L_j)\}$ and $\sigma_2 < 1$ such that K_j, $j = 1, 2, \ldots, m$, are the subset of the strip $D := \{s \in \mathbb{C} : \sigma_1 < \mathrm{Re}\, s < \sigma_2\}$ and denote the space of analytic functions on D equipped with the topology of uniform convergence on compacta by $H(D)$.

Then, the main purpose of this section is to prove the so-called denseness lemma in the space $H(D)^m$, which plays a crucial role in the proof of universality and says that any collection of analytic functions from $H(D)^m$ can be approximated by given L-functions $L_1, \ldots, L_m$ twisted by certain sequence of complex numbers with absolute value 1.

In order to show it, let $\gamma := \{s \in \mathbb{C} : |s| = 1\}$ and $\Omega := \prod_p \gamma_p$ be an infinite-dimensional torus with product topology and pointwise multiplication, where $\gamma_p = \gamma$ for each prime p. It is well known that Ω is a compact topological abelian group, so there is a normalized Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(\Omega)$ denotes the class of Borel sets of Ω.

Let $\omega(p)$ denote the projection of $\omega \in \Omega$ to the coordinate space γ_p and $\omega : \mathbb{N} \to \mathbb{C}$ be a unimodular completely multiplicative extansion of ω. Then for any $L \in \mathcal{S}$ defined for $\sigma > 1$ by the series $\sum_{n=1}^{\infty} a_L(n) n^{-s}$ we put

$$L(s, \omega) = \sum_{n=1}^{\infty} \frac{a_L(n) \omega(n)}{n^s}, \qquad s \in D.$$

It turned out (see for example [21, Lemma 4.1]) that $L(s, \omega)$ is a random element on the probabilistic space $(\Omega, \mathcal{B}(\Omega), m_H)$ and for almost all $\omega \in \Omega$ we have (see [17, Eq. (3.17)])

$$\log L(s, \omega) = \sum_p \sum_{k=1}^{\infty} \frac{b_L(p^k)\omega(p)^k}{p^{ks}}, \qquad (s \in D).$$

Thus, for $L_j \in \mathcal{S}$, $j = 1, 2, \ldots, m$, let us put

$$g_{p,j}(s, \omega(p)) = \sum_{k=1}^{\infty} \frac{b_{L_j}(p^k)\omega(p)^k}{p^{ks}}, \qquad \omega \in \Omega$$

and

$$\underline{g}_p(s, \omega(p)) = (g_{p,1}(s, \omega(p)), \ldots, g_{p,m}(s, \omega(p))).$$

Therefore, the main result of this section is the following proposition, which strongly relies on Selberg's conjecture.

Proposition 2.1 *If we assume the truth of* (4) *and* (5)*, then the set of convergent series*

$$\left\{ \sum_p \underline{g}_p(s, \omega(p)) : \omega \in \Omega \right\}$$

is dense in the space $H(D)^m$.

Let U be a bounded simply connected smooth Jordan domain satisfying $\overline{U} \subset D$ and $K_j \subset U$ for every $j = 1, 2, \ldots, m$. Let $L^2(U)$ be the complex Hilbert space of all square integrable complex functions on U with the inner product

$$\langle f, g \rangle = \iint_U f(s)\overline{g(s)}d\sigma dt.$$

Define the Bergman space H_1 as the closure of $H(D)$ in $L^2(U)$. Then H_1^m is the complex Hilbert space with the inner product given, for $\underline{f} = (f_1, \ldots, f_m)$ and $\underline{g} = (g_1, \ldots, g_m)$, by

$$\langle \underline{f}, \underline{g} \rangle = \sum_{j=1}^{m} \iint_U f_j(s)\overline{g_j(s)}d\sigma dt.$$

Now, define

$$\underline{h}_p(s) = (h_{p,1}(s), \ldots, h_{p,m}(s)) := \left(\frac{a_{L_1}(p)}{p^s}, \ldots, \frac{a_{L_m}(p)}{p^s} \right).$$

Then, by the fact that $b_L(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$, one can easily prove that

$$\sum_p r_{p,j}(s, \omega) := \sum_p \left(g_{p,j}(s, \omega(p)) - \omega(p)h_{p,j}(s) \right), \qquad (j = 1, \ldots, n, \ |\omega(p)| = 1)$$

is absolutely convergent on $\overline{U}$.

Hence, in order to prove Proposition 2.1 it suffices to prove that the set of all convergent series

$$\left\{ \sum_{p > v} \omega(p)\underline{h}_p(s) : \omega \in \Omega \right\} \tag{6}$$

is dense in H_1^m for an arbitrary given $v > 0$. Indeed, let v be a sufficiently large number such that

$$\sum_{j=1}^{m} \max_{s \in \overline{U}} \left| \sum_{p > v} r_{p,j}(s, \omega) \right| < \frac{\varepsilon}{2} \qquad \text{for all } \omega \in \Omega.$$

The fact that for every $f \in H_1$ with the norm $||f||$ and $s \in U$ we have $|f(s)| < \frac{||f||}{\sqrt{\pi}\,\mathrm{dist}(s, \partial U)}$ (see for example [6, Chapter I, Section 1, Lemma 1]) clearly implies that the approximation with respect to the norm $|| \cdot ||$ in H_1 gives the uniform approximation on every compact subset K of U. Hence, from the fact that the set (6) is dense in H_1^m, we obtain that, for every $\underline{f} = (f_1, \ldots, f_m) \in H(D)^m$, there exists a sequence $\omega'(p)$ such that

$$\max_{1 \le j \le m} \max_{s \in K_j} \left| \sum_{p > v} \omega'(p) h_{p,j}(s) - f_j(s) + \sum_{p \le v} g_{p,j}(s, 1) \right| < \frac{\varepsilon}{2}.$$

Therefore, putting

$$\omega(p) = \begin{cases} 1 & \text{if } p \le v, \\ \omega'(p) & \text{if } p > v \end{cases}$$

gives that

$$\max_{1 \le j \le m} \max_{s \in K_j} \left| \sum_{p} g_{p,j}(s, \omega(p)) - f_j(s) \right| < \varepsilon. \tag{7}$$

In order to prove that the set (6) is dense in H_1^m we shall use the following lemma for the sequence $\underline{h}_p(s)$ and the Hilbert space H_1^m.

Lemma 2.2 *Let H be a complex Hilbert space. Assume that a sequence $u_n \in H$, $n \in \mathbb{N}$, is such that*

(i) the series $\sum_n ||u_n||^2 < \infty$;
(ii) for any element $0 \ne e \in H$ the series $\sum_n |\langle u_n, e \rangle|$ is divergent.

Then the set of convergent series

$$\left\{ \sum_n a_n u_n \in H : |a_n| = 1 \right\}$$

is dense in H.

Proof This is [21, Theorem 5.4].

Since $\mathrm{Re}\, s > \sigma_1 > 1/2$ for all $s \in U$, one can easily show that

$$\sum_{p} ||\underline{h}_p(s)||^2 < \infty$$

and the condition (i) holds.

Now let $\underline{g} = (g_1, \ldots, g_m) \in H_1^m$ be a non-zero element. Then

$$\langle \underline{h}_p(s), \underline{g}(s) \rangle = \sum_{j=1}^{m} a_{L_j}(p) \Delta_j(\log p),$$

where $\Delta_j(z) = \iint_U e^{-sz} \overline{g_j(s)} \, d\sigma \, dt$. Then, in order to complete the proof of Propositon 2.1 it suffices to prove the following lemma.

Lemma 2.3 *Let $\underline{g}(s) = (g_1(s), \ldots, g_m(s)) \in H_1^m$ be a non-zero element and $\Delta_j(z) = \iint_U e^{-sz} \overline{g_j(s)} d\sigma dt$. Then, assuming Selberg's conjecture (4) and (5) for $L_1, \ldots, L_m \in \mathcal{S}$ gives that the series*

$$\sum_p \left| a_{L_1}(p) \Delta_1(\log p) + \cdots + a_{L_m}(p) \Delta_m(\log p) \right|$$

is divergent.

Before we prove the above lemma, we need to obtain good estimation for $\Delta(\log p) = \iint_U p^{-s} \overline{g(s)} d\sigma dt$, where $g(s)$ is a given non-zero element of H_1. In order to prove it we use Markov's inequality.

Lemma 2.4 (Markov's inequality) *Suppose that $P(t)$ is a polynomial of degree n with real coefficients, which satisfies*

$$\max_{t \in [-1,1]} |P(t)| \leq 1.$$

Then for every $t \in [-1, 1]$ we have

$$|P'(t)| \leq n^2.$$

Proof For a proof see for example [1]. □

Corollary 2.5 *Let $P(s)$ be polynomial of degree n with complex coefficients. Then for every a, b with $a < b$ and every real $t \in [a, b]$ we have*

$$|P'(t)| \leq \frac{2n^2}{b-a} \max_{t \in [a,b]} |P(t)|.$$

Proof Let $t_0 \in [a, b]$ be such that $|P(t_0)| = \max_{t \in [a,b]} |P(t)|$. Then let us define

$$P_1(t) = \frac{P\left(\frac{b-a}{2}t + \frac{a+b}{2}\right)}{|P(t_0)|}.$$

Now, let us take an arbitrary $t \in [-1, 1]$ and let $c \in \mathbb{C}$ with $|c| = 1$ be such that $c P_1'(t)$ be real. Then applying Markov's inequality for $P_2(t) := \mathrm{Re}(c P_1(t))$ gives

$$|P_1'(t)| = |c P_1'(t)| = |P_2'(t)| \leq n^2,$$

so

$$\max_{t \in [-1,1]} |P_1'(t)| \leq n^2.$$

On the other hand, we can easily observe that

$$\max_{t \in [-1,1]} |P_1'(t)| = \frac{b-a}{2|P(t_0)|} \max_{t \in [a,b]} |P'(t)|$$

and the proof is complete. □

Lemma 2.6 *Let $U \subset \mathbb{C}$ be an open and bounded set and g be a Lebesgue square integrable function on U. For $z \in \mathbb{C}$ we put*

$$\Delta(z) = \iint_U e^{-sz} \overline{g(s)} d\sigma dt.$$

Then for every $A > 0$ and every interval $I = [x, x + \frac{B}{x^M}] \subset [x, x + 1]$ with $B > 0$, $M \geq 0$, $x > 2$ there exist an interval $I' \subset I$ of length $|I'| \geq \frac{B'}{x^{M+2}}$ with $B' := B'(B, A) > 0$ and $x_0 \in I'$ such that for all $\xi \in I'$ we have

$$\frac{1}{2}|\Delta(x)| + O\left(e^{-Ax}\right) \leq \frac{1}{2}|\Delta(x_0)| + O\left(e^{-Ax}\right) \leq |\Delta(\xi)| \leq |\Delta(x_0)| + O\left(e^{-Ax}\right).$$

Moreover, for every $\xi \in I$ we have

$$|\Delta'(\xi)| \ll x^{M+2}|\Delta(x_0)| + O(x^{M+2}e^{-Ax}).$$

Proof Let $c_0 > 0$, $K = [c_0 x]$ and $C > 0$ be such that $\max_{s \in \overline{U}} |s| \leq C$. Then, for every $\xi \in [x, x + 1]$, by Stirling's formula we get

$$
\begin{aligned}
e^{-s\xi} &= \sum_{l=0}^{K} \frac{(-s\xi)^l}{l!} + O\left(\sum_{l=0}^{\infty} \frac{(xC)^{l+K+1}}{(l+K+1)!}\right) \\
&= \sum_{l=0}^{K} \frac{(-s\xi)^l}{l!} + O\left(\frac{(xC)^{K+1}}{(K+1)!} \sum_{l=0}^{\infty} \frac{(xC)^l (K+1)!}{(l+K+1)!}\right) \\
&= \sum_{l=0}^{K} \frac{(-s\xi)^l}{l!} + O\left(e^{xC} \exp\left(-(K+1)\log\left(\frac{K+1}{xC}\right)\right)\right) \\
&= \sum_{l=0}^{K} \frac{(-s\xi)^l}{l!} + O\left(e^{xC} \exp\left(-c_0 x \log\left(\frac{c_0}{C}\right)\right)\right).
\end{aligned}
$$

Similarly,

$$
(-s)e^{-s\xi} = \sum_{l=0}^{K-1} \frac{(-s)^{l+1}\xi^l}{l!} + O\left(e^{xC} \exp\left(-(c_0 x - 1)\log\left(\frac{c_0}{2C}\right)\right)\right).
$$

Hence, for every $A > 0$ there exists sufficiently large $c_0 = c_0(A, C)$ such that

$$
e^{-s\xi} = \sum_{l=0}^{K} \frac{(-s\xi)^l}{l!} + O\left(e^{-x(A+C)}\right)
$$

and

$$
(-s)e^{-s\xi} = \sum_{l=0}^{K-1} \frac{(-s)^{l+1}\xi^l}{l!} + O\left(e^{-x(A+C)}\right)
$$

for every $\xi \in [x, x + 1]$.

Therefore, for $\xi \in [x, x + 1]$ we have

$$\Delta(\xi) = P(\xi) + O(e^{-Ax}) \quad \text{and} \quad \Delta'(\xi) = P'(\xi) + O(e^{-Ax}), \tag{8}$$

where $P(\xi) = \sum_{l=0}^{K} \frac{\xi^l}{l!} \iint_U (-s)^l \overline{g(s)} d\sigma dt$ is a polynomial of degree $\ll x$.

Let $x_0 \in I$ be such that $|P(x_0)| = \max_{\xi \in I} |P(\xi)|$. Then by Corollary 2.5 we get

$$\max_{\xi \in I} |P'(\xi)| \ll x^{M+2}|P(x_0)|$$

and hence

$$|\Delta'(\xi)| = |P'(\xi)| + O(e^{-Ax}) \ll x^{M+2}|\Delta(x_0)| + O(x^{M+2}e^{-Ax}).$$

Therefore, for $\xi \in I$ satisfying $|\xi - x_0| \leq \frac{B'}{x_0^{M+2}}$ with sufficiently small $B' > 0$ we have

$$|P(x_0)| - |P(\xi)| \leq |P(\xi) - P(x_0)| \leq |\xi - x_0| \max_{\xi \in I} |P'(\xi)| \leq \frac{1}{2}|P(x_0)|. \tag{9}$$

Therefore, for $\xi \in I' := I \cap \left[x_0 - \frac{B'}{x_0^{M+2}}, x_0 + \frac{B'}{x_0^{M+2}} \right]$ it holds

$$\frac{1}{2}|P(x)| \leq \frac{1}{2}|P(x_0)| \leq |P(\xi)| \leq |P(x_0)|,$$

and hence, by (8), the proof is complete. $\square$

Corollary 2.7 *Let $U \subset \mathbb{C}$ be open and bounded and g_j, $j = 1, 2, \ldots, m$, be Lebesgue square integrable functions on U. For $z \in \mathbb{C}$ we put*

$$\Delta_j(z) = \iint_U e^{-sz}\overline{g_j(s)}d\sigma dt.$$

Then for every $A > 0$ and every $x > 1$ there exist $B_1 > \cdots > B_m > 0$, $x_0^{(0)} = x, x_0^{(1)}, \ldots, x_0^{(m)}$ and intervals $I_j \subset [x, x+1]$ of length $|I_j| \geq \frac{B_j}{x^{2j}}$ such that $x_0^{(j)} \in I_j$, $I_{j+1} \subset I_j$, and for all $\xi \in I_j$ we have

$$\frac{1}{2}|\Delta_j(x_0^{(j-1)})| + O\left(e^{-Ax}\right) \leq \frac{1}{2}|\Delta_j(x_0^{(j)})| + O\left(e^{-Ax}\right)$$
$$\leq |\Delta_j(\xi)| \leq |\Delta_j(x_0^{(j)})| + O\left(e^{-Ax}\right).$$

Moreover, for every $t \in I_j$ we have

$$|\Delta_j'(\xi)| \ll x^{2j}|\Delta_j(x_0^{(j)})| + O(x^{2j}e^{-Ax}).$$

Proof Firstly, let us apply the last lemma for $\Delta_1(z)$ and the interval $I_0 := [x, x+1]$. Then there is an interval $I_1 \subset I_0$ of length $|I_1| \geq \frac{B_1}{x^2}$ and $x_0^{(1)} \in I_1$ such that for $\xi \in I_1$ we have

$$\frac{1}{2}|\Delta_1(x)| + O\left(e^{-Ax}\right) \leq \frac{1}{2}|\Delta_1(x_0^{(1)})| + O\left(e^{-Ax}\right) \leq |\Delta_1(\xi)| \leq |\Delta_1(x_0^{(1)})| + O\left(e^{-Ax}\right)$$

and

$$|\Delta_1'(\xi)| \ll x^2|\Delta_1(x_0^{(1)})| + O(x^2e^{-Ax}).$$

Next, we apply again the last lemma for $\Delta_2(z)$ and the interval $I_1 = [x', x' + \frac{B_1}{x^2}] \subset [x, x+1]$. Thus there is an interval $I_2 \subset I_1$ of length $|I_2| \geq \frac{B_2'}{x'^4} \geq \frac{B_2}{x^4}$ and $x_0^{(2)} \in I_2$ such that

$$\frac{1}{2}|\Delta_2(x_0^{(1)})| + O\left(e^{-Ax}\right) \leq \frac{1}{2}|\Delta_2(x_0^{(2)})| + O\left(e^{-Ax}\right) \leq |\Delta_2(\xi)| \leq |\Delta_2(x_0^{(2)})| + O\left(e^{-Ax}\right)$$

and

$$|\Delta_2'(\xi)| \ll x^4|\Delta_2(x_0^{(2)})| + O(x^4e^{-Ax}).$$

Next, repeating the application of the last lemma for each function Δ_j, $3 \leq j \leq m$, completes the proof. $\square$

Proof of Lemma 2.3 Without loss of generality we can assume that g_1 is a non-zero element, since the fact that $\underline{g} \neq 0$ implies that at least one of g_j's is a non-zero element.

Obviously, $\Delta_1(z) \ll e^{C|z|}$ for some positive constant C depending on U. Let us recall that for all $s \in U$ we have $1/2 < \sigma_1 < \operatorname{Re} s < \sigma_2 < 1$. Then for sufficiently small $\eta = \eta(U) > 0$ and for all complex z with $|\arg(-z)| \leq \eta$ we have

$$|e^{\sigma_2 z} \Delta_1(z)| \ll 1.$$

Moreover, $\Delta_1 \not\equiv 0$, since otherwise for every positive integer k we have $0 = \Delta_1^{(k)}(0) = \iint_U (-s)^k \overline{g_1(s)} d\sigma dt$, which means that g_1 is orthogonal to all polynomials in $L_2(U)$ and we get contradiction to the fact that g_1 is a non-zero element and the linear space of polynomials is dense in the Bergman space H_1 (see for example [19, Theorem 7.2.2]). Hence, by [8, Lemma 3], which proof based on the Phragmén-Lindelöf theorem, there is a real sequence x_k tending to ∞ such that

$$|\Delta_1(x_k)| \gg e^{-\sigma_2 x_k}.$$

Let us fix k and put $x = x_k$. Hence, using Corollary 2.7, for every $A > 0$ and $x = x_k$ there exist $B_1 > \cdots > B_m > 0$, $x_0^{(0)} = x, x_0^{(1)}, \ldots, x_0^{(m)}$ and intervals $I_j \subset [x, x+1]$ of length $|I_j| \geq \frac{B_j}{x^{2j}}$ such that $x_0^{(j)} \in I_j$, $I_{j+1} \subset I_j$, and for all $\xi \in I_j$ we have

$$\frac{1}{2}|\Delta_j(x_0^{(j-1)})| + O\left(e^{-Ax}\right) \leq \frac{1}{2}|\Delta_j(x_0^{(j)})| + O\left(e^{-Ax}\right)$$
$$\leq |\Delta_j(\xi)| \leq |\Delta_j(x_0^{(j)})| + O\left(e^{-Ax}\right) \tag{10}$$

and

$$|\Delta_j'(\xi)| \ll x^{2j}|\Delta_j(x_0^{(j)})| + O(x^{2j}e^{-Ax}). \tag{11}$$

Now let $I := I_m = \left[x', x' + \frac{B_m}{x'^{2m}}\right] \subset [x, x+1]$. Since $I \subset I_j$ for every $j = 1, 2, \ldots, m$, the above inequalities hold also for all $\xi \in I$.

In particular, since $x_0^{(0)} = x$, for $\xi \in I$ we have

$$|\Delta_1(\xi)| \geq \frac{1}{2}|\Delta_1(x_0^{(0)})| \gg e^{-\sigma_2 x}.$$

Moreover, for every $j = 1, 2, \ldots, m$ we have

$$|\Delta_j(\xi)| \ll e^{-\sigma_1 x} \qquad (\xi \in [x, x+1]).$$

Now, let $\sum_p^*$ denote the sum over primes $p \in \left[e^{x'}, e^{x' + \frac{B_m}{x'^{2m}}}\right]$. Then for these p we have $\log p \in I$.

It is easy to notice that

$$S(x) := \sum_p^* \left|a_{L_1}(p)\Delta_1(\log p) + \cdots + a_{L_m}(p)\Delta_m(\log p)\right|^2$$
$$= \sum_{j=1}^m \sum_p^* |a_{L_j}(p)|^2 |\Delta_j(\log p)|^2$$
$$+ \sum_{1 \leq k \neq l \leq m} \sum_p^* a_{L_k}(p)\overline{a_{L_l}(p)}\Delta_k(\log p)\overline{\Delta_l(\log p)}.$$

Using (5) it is easy to prove that for any $1 \leq k \neq l < m$ we have

$$\phi_{k,l}(u) := \sum_{p \leq u} a_{L_k}(p)\overline{a_{L_l}(p)} = \sum_{j=2}^{2m+1} \frac{c_j^{(k,l)}u}{(\log u)^j} + O\left(\frac{u}{(\log u)^{2m+2}}\right).$$

For $\log u \in I$, by (11), we get

$$\frac{d}{du}\Delta_j(\log u) = \frac{1}{u}\Delta_j'(\log u) \ll \frac{x^{2m}}{u}|\Delta_j(x_0^{(j)})| + O(x^{2m}e^{-Ax})$$

and, since $\overline{\Delta_j(\log u)} = \overline{\langle u^{-s}, g_j(s)\rangle} = \langle u^{-\bar{s}}, \overline{g_j(s)}\rangle$, we have

$$\frac{d}{du}\overline{\Delta_j(\log u)} = \frac{1}{u}\iint_U -\bar{s}u^{-\bar{s}}\overline{g_j(s)}d\sigma\,dt = \frac{1}{u}\overline{\Delta_j'(\log u)}$$

$$\ll \frac{x^{2m}}{u}|\Delta_j(x_0^{(j)})| + O(x^{2m}e^{-Ax}).$$

Hence, using partial summation and (10), gives

$$\sum_{1\leq k\neq l\leq m}\sum_p{}^* a_{L_k}(p)\overline{a_{L_l}(p)}\Delta_k(\log p)\overline{\Delta_l(\log p)}$$

$$= \sum_{1\leq l\neq k\leq m}\int_{X_1}^{X_2}\Delta_k(\log u)\overline{\Delta_l(\log u)}d\phi_{k,l}(u)$$

$$\ll \frac{e^x}{x^{2m+2}}\sum_{1\leq k\neq l\leq m}|\Delta_k(x_0^{(k)})||\Delta_l(x_0^{(l)})| + O(e^{(-A+1-\sigma_1)x})$$

$$+ \sum_{1\leq k\neq l\leq m}\int_{X_1}^{X_2}\frac{u}{(\log u)^{2m+2}}\left|\left(\Delta_k(\log u)\overline{\Delta_l(\log u)}\right)'\right|du$$

$$\ll \frac{e^x}{x^{2m+2}}\sum_{1\leq j\leq m}|\Delta_j(x_0^{(j)})|^2 + O(e^{(-A+1-\sigma_1)x})$$

$$+ x^{2m}\sum_{1\leq j\leq m}|\Delta_j(x_0^{(j)})|^2\int_{X_1}^{X_2}\frac{1}{(\log u)^{2m+2}}du$$

$$\ll \frac{e^x}{x^{2m+2}}\sum_{1\leq j\leq m}|\Delta_j(x_0^{(j)})|^2 + O(e^{(-A+1-\sigma_1)x}) := E(x),$$

where $X_1 = e^{x'}$, $X_2 = e^{x'+\frac{Bm}{x'^{2m}}}$.

Therefore, by (4), we get

$$S(x) = \sum_{p}^{*} \sum_{j=1}^{m} |a_{L_j}(p)|^2 |\Delta_j(\log p)|^2 + E(x)$$

$$\gg \sum_{j=1}^{m} \left(|\Delta_j(x_0^{(j)})|^2 + |\Delta_j(x_0^{(j)})| O(e^{-Ax}) + O(e^{-2Ax}) \right) \sum_{p}^{*} |a_{L_j}(p)|^2 + E(x)$$

$$\gg \frac{e^x}{x^{2m+1}} \sum_{j=1}^{m} |\Delta_j(x_0^{(j)})|^2 + O(e^{(-A+1-\sigma_1)x}) + E(x)$$

$$\gg \frac{e^x}{x^{2m+1}} \sum_{j=1}^{m} |\Delta_j(x_0^{(j)})|^2 + O(e^{(-A+1-\sigma_1)x}) + O\left(\frac{e^x}{x^{2m+2}} \sum_{j=1}^{m} |\Delta_j(x_0^{(j)})|^2 \right)$$

$$\gg \frac{e^x}{x^{2m+1}} \left(\sum_{j=1}^{m} |\Delta_j(x_0^{(j)})| \right)^2 + O(e^{(-A+1-\sigma_1)x})$$

$$\gg \frac{e^{(1-\sigma_2)x}}{x^{2m+1}} \sum_{j=1}^{m} |\Delta_j(x_0^{(j)})| + O(e^{(-A+1-\sigma_1)x}).$$

On the other hand, since $a_{L_j}(p) \ll p^\varepsilon$ for every $\varepsilon > 0$, we have

$$S(x) \ll e^{\varepsilon x} \sum_{p}^{*} \left| \sum_{j=1}^{m} a_{L_j}(p) \Delta_j(\log p) \right| \sum_{j=1}^{m} |\Delta_j(\log p)|$$

$$\ll e^{\varepsilon x} \sum_{p}^{*} \left| \sum_{j=1}^{m} a_{L_j}(p) \Delta_j(\log p) \right| \sum_{j=1}^{m} |\Delta_j(x_0^{(j)})| + O(e^{(-A+1+\varepsilon-\sigma_1)x}).$$

Finally, dividing the last inequalities by $\sum_{j=1}^{m} |\Delta_j(x_0^{(j)})|$ and taking sufficiently large $A > 0$ gives

$$\sum_{p}^{*} \left| \sum_{j=1}^{m} a_{L_j}(p) \Delta_j(\log p) \right| \gg \frac{e^{x(1-\sigma_2-\varepsilon)}}{x^{2m+1}}$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3 Proof of Theorem 1.2

Now we shall use the denseness lemma proved above, to give the proof of joint universality for a collection of L-functions $L_1, \ldots, L_m$ from the Selberg class. In order to do it we need a joint limit theorem for the following probabilistic measure on $(H(D)^m, \mathcal{B}(H(D)^m))$, where $\mathcal{B}(H(D)^m)$ denotes the class of Borel sets of $H(D)^m$. Basically, the proof of the joint limit theorem and the remaining steps of the proof of Theorem 1.2 are based on [21, Chapter 12], where Steuding proved conditional joint universality (see [21, Theorem 12.5]) for a slightly different class of L-functions. The modification needed are easy and straightforward. Nevertheless, we give a sketch of the proof for sake of completeness.

For

$$\underline{L}(s) = (L_1(s), \ldots, L_m(s))$$

define a probabilistic measure $\mathbf{P}_T^{\underline{L}}$ by

$$\mathbf{P}_T^{\underline{L}}(A) = \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \underline{L}(s + i\tau) \in A \right\}, \qquad \text{for } A \in \mathcal{B}(H(D)^m).$$

Moreover, it is known that

$$\underline{L}(s, \omega) := (L_1(s, \omega), \ldots, L_m(s, \omega)), \qquad (\omega \in \Omega),$$

is an $H(D)^m$-valued random element on $(\Omega, \mathcal{B}(\Omega), m_H)$. Therefore, denoting the distribution of $\underline{L}(s, \omega)$ by $\mathbf{P}^{\underline{L}}$ on $(H(D)^m, \mathcal{B}(H(D)^m))$, gives the following joint limit theorem.

Theorem 3.1 [21, Theorem 12.1] *For $L_1, \ldots, L_m \in \mathcal{S}$ the probability measure $\mathbf{P}_T^{\underline{L}}$ converges weakly to $\mathbf{P}^{\underline{L}}$, as $T \to \infty$.*

The immediate consequence of the above theorem is the following result.

Corollary 3.2 *Let $L_1, \ldots, L_m \in \mathcal{S}$ and $D_M := \{s \in \mathbb{C} : \sigma_1 < \operatorname{Re}(s) < \sigma_2, \ |t| < M\}$ for any $M > 0$. Then the probability measure*

$$\mathbf{Q}_T^{\underline{L}}(A) := \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \underline{L}(s + i\tau) \in A \right\},$$

for $A \in \mathcal{B}(H(D_M)^m)$, converges weakly, as $T \to \infty$, to

$$\mathbf{Q}^{\underline{L}}(A) := m_H \left\{ \omega \in \Omega : \underline{L}(s, \omega) \in A \right\}$$

for $A \in \mathcal{B}(H(D_M)^m)$.

Hence, in order to prove Theorem 1.2 it remains to determine the support of the measure $\mathbf{Q}_T^{\underline{L}}$, which is implied by Hurwitz's classical result on zeros of uniformly convergent sequence of functions. Let us recall that the support of the probabilistic space $(S, \mathcal{B}(S), \mathbf{P})$ is the minimal closed set with measure 1. It means that the support consists of all elements $x \in S$ satisfying $\mathbf{P}(V) > 0$ for every neighborhood V of x. By using (7), [21, Lemma 12.7] and the definition of support, and modifying the proof of [21, Lemma 12.6], we have the following lemma.

Lemma 3.3 *The support of the measure $\mathbf{Q}_T^{\underline{L}}$ is the set*

$$\mathbf{S}_M := \{\underline{\varphi} := (\varphi_1, \ldots, \varphi_m) \in H(D_M)^m : \underline{\varphi}(s) \neq 0 \text{ for } s \in D_M, \text{ or } \underline{\varphi} \equiv 0\}.$$

Now, we are ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2 By Mergelyan's approximation theorem it suffices (see the proof of [21, Theorem 12.5]) to assume that $g_1, \ldots, g_m$ have non-vanishing analytic continuation to D_M, where $M > 0$ is such that $K_1, \ldots, K_m \subset D_M$. Then, by the last lemma, $(g_1, \ldots, g_m)$ is an element of the support $\mathbf{S}_M$. Therefore, using the fact that $\mathbf{Q}_T^{\underline{L}}$ converges weakly to $\mathbf{Q}^{\underline{L}}$ and the fact that the set Φ of functions $\underline{\varphi} \in H(D_M)^m$ satisfying

$$\max_{1 \le j \le m} \max_{s \in K_j} |\varphi_j(s) - g_j(s)| < \varepsilon$$

is open, yields

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \max_{1 \le j \le m} \max_{s \in K_j} |L_j(s + i\tau) - g_j(s)| < \varepsilon \right\}$$
$$= \liminf_{T \to \infty} \mathbf{Q}_T^L(\Phi) \ge \mathbf{Q}^L(\Phi) > 0,$$

which completes the proof. $\qquad\square$

4 Examples

In this section we give examples of L-functions from analytic number theory satisfying Selberg's conjecture, and, particularly, the assumptions of Theorem 1.2.

Let us start with a general discussion about joint universality of the Riemann zeta function $\zeta(s)$ and L-function $L(s)$ from the Selberg class. In this case, it suffices to assume that $L(s)$ satisfies (4) and

$$\sum_{p \le x} a_L(p) \ll \frac{x}{(\log x)^A}, \qquad \text{for arbitrary } A > 0.$$

It is well known, that there is a strong relation between the error term in the above estimation and zero-free region of $L(s)$. For example, [7, Theorem 5.13] states that the prime number theorem for general L-function holds under the assumption of existence of the zero-free region. More precisely, one can deduce that for any function $1 \ne L \in \mathcal{S}$ with polynomial Euler product we have

$$\sum_{p \le x} a_L(p) \ll m_L \operatorname{li} x + O\left(x e^{-c'\sqrt{\log x}}\right) \qquad \text{for some } c' > 0,$$

provided there exists $c > 0$ such that

$$L(\sigma + it) \ne 0 \quad \text{for } \sigma > 1 - \frac{c}{\log(|t| + 2)}, \ t \in \mathbb{R} \tag{12}$$

except a real zero $\beta < 1$. Therefore, we can easily deduce joint universality of the Riemann zeta function $\zeta(s)$ and any entire L-function from the Selberg class with zero-free region of the form (12). It means that, for example, we can show $\zeta(s)$ and any Hecke L-function $L_{\mathbf{K}}(s; \chi)$ associated to a finite extension $\mathbf{K}$ of $\mathbb{Q}$ and a non-principle primitive grössencharacker χ are jointly universal in the strip $\sigma_{\mathrm{m}}(L_{\mathbf{K}}(s; \chi)) < \sigma < 1$. Similarly, we can show that the Riemann zeta function and Artin L-function associated to a finite Galois extension are jointly universal. The last example of this kind can be delivered by the theory of classical automorphic L-functions. For instance, the normalized L-function $L(s, f)$ associated to holomorphic primitive cusp form. Here, we refer to Iwaniec and Kowalski [7, Chapter 5] for the proofs of needed prime number theorems for Hecke, Artin and automorphic L-functions and more examples of L-functions jointly universal with the Riemann zeta function.

Next, consider the joint universality property for $\zeta(s)$ and L-function $L(s)$ with a pole at $s = 1$ of order m_L satisfying $0 < m_L < d_L$. Then it turns out that instead of (5) it suffices to assume the truth of Selberg's conjecture (3) with $R(x) \ll 1$ and the existence of a zero-free region for $L(s)$. Indeed, it is well known (see [4] or [5, Theorem 2.4.1]) that every function in $\mathcal{S}$ can be factored into primitive elements. Let us recall that $F \in \mathcal{S}$ is primitive if $F = F_1 F_2$ for $F_1, F_2 \in \mathcal{S}$ implies $F_1 = 1$ or $F_2 = 1$. Furthermore, Selberg's conjecture (3) with $R(x) \ll 1$ implies that the Riemann zeta function is the only primitive element of $\mathcal{S}$

with a pole (see [4] or [5, Theorem 2.5.2]). More precisely, under Selberg's Conjecture 1.1, every given function $L \in \mathcal{S}$ with a pole at $s = 1$ of order m_L can be factored into m_L-th power of $\zeta(s)$ and an entire function from $\mathcal{S}$. Therefore, assuming (12) for a given $L \in \mathcal{S}$ with $0 < m_L < d_L$ and recalling again [7, Theorem 5.13] gives that we can factor $L(s)$ into $\zeta(s)^{m_L}$ and an entire function $1 \neq L^*(s) \in \mathcal{S}$, which, obviously, has no zeros at least in the same region as $L(s)$ and satisfies (5). Moreover, L^* satisfies (4) as L does, since one can easily observe that Selberg's conjecture (3) with $R(x) \ll 1$ gives

$$
\begin{aligned}
m_l \log\log x + O(1) &= \sum_{p \leq x} \frac{|a_L(p)|^2}{p} = \sum_{p \leq x} \frac{|a_{\zeta^{m_L}}(p) + a_{L^*}(p)|^2}{p} \\
&= \sum_{p \leq x} \frac{|a_{\zeta^{m_L}}(p)|^2}{p} + \sum_{p \leq x} \frac{|a_{L^*}(p)|^2}{p} + O(1) \\
&= m_L \log\log x + \sum_{p \leq x} \frac{|a_{L^*}(p)|^2}{p} + O(1).
\end{aligned}
$$

Since, additionally, L^* is entire, we can show, by the previous reasoning, that $\zeta(s)$ and $L^*(s)$ are jointly universal in the strip $\sigma_m(L^*) < \sigma < 1$. Thus, it is easy to see that $\zeta(s)$ and $L(s)$ are jointly universal in the same strip, provided $L(s)$ satisfies (4), (12) and Selberg's conjecture (3) holds for every L-function with $R(x) \ll 1$.

As an example of application of this observation, we can consider Dedekind zeta function $\zeta_{\mathbf{K}}(s)$ associated to any algebraic number field $\mathbf{K}$. Then it is known that (12), (4) and (5) hold for any algebraic number field (cf. [7, Section 5.10]). Hence, $\zeta_{\mathbf{K}}$ can be written as $\zeta(s)L^*(s)$, which implies the joint universality theorem for $\zeta(s)$ and $\zeta_{\mathbf{K}}(s)$ in the strip $\sigma_m(L^*) < \sigma < 1$ under the assumption of Selberg's orthonormality conjecture.

Let us note that usually the abscissa of the mean-square is smaller for L-functions of smaller degree d_L, namely [21, Corollary 6.11] says that $\sigma_m(L) < \max(\frac{1}{2}, 1 - \frac{1}{d_L})$. Therefore, the above approach by factorization of L-function usually gives universality for a wider strip than the direct proof of joint universality for given L-functions. For example, following [14, Section 2] let us consider a normalized holomorphic Hecke eigen cusp form f, the automorphic L-function $L(s, f)$ and the symmetric square L-function $L(s, \text{sym}^2 f)$ (for the definition see [14, Eq. (2.4) and (2.6)]). It is known that the Rankin-Selberg L-function $L(s, f \otimes g)$ is a function of degree 4 and it is universal (see [13] and [16]) in the strip $3/4 < \sigma < 1$. However, one can easily show that

$$
L(s, f \otimes f) = \zeta(s)L(s, \text{sym}^2 f)
$$

and it is known that the abscissa of the mean-square of $L(s, \text{sym}^2 f)$ is at most 2/3. Therefore, using [14, Eq. (3.8)], we obtain joint universality for $\zeta(s)$ and $L(s, \text{sym}^2 f)$ in the strip. It implies joint universality for the Riemann zeta function and the automorphic L-function in the wider strip $2/3 < \sigma < 1$.

It turns out that the theory of the Rankin-Selberg convolution delivers more examples for application of our Theorem 1.2. It is known that the Rankin-Selberg convolution and the Rankin-Selberg square are powerful tools to investigate the existence of prime number theorem for automorphic L-functions. For example, Iwaniec and Kowalski [7, Section 5] showed that the existence of the Rankin-Selberg L-function $L(s, f \otimes g)$ implies the existence of its zero-free region, provided some additional conditions related to automorphic forms f, g hold. Moreover, they proved that zero-free region for automorphic L-function gives prime number theorem (see [7, Theorem 5.13]). Note that the coefficients $\lambda_{f \otimes g}(p)$ of the Rankin-Selberg convolution $L(s, f \otimes g)$ satisfy $\lambda_{f \otimes g}(p) = \lambda_f(p)\lambda_g(p)$, where $\lambda_f(p)$ and $\lambda_g(p)$

are coefficients of automorphic L-functions $L(s, f)$ and $L(s, g)$ associated to automorphic forms f and g, respectively. In particular, the coefficients of the Rankin-Selberg square $L(s, f \otimes \overline{f})$ satisfy $\lambda_{f \otimes \overline{f}}(p) = |\lambda_f(p)|^2$. Therefore, we obtain that the existence of the Rankin-Selberg convolution and the Rankin-Selberg square implies the strong version of Selberg's conjecture, namely

$$\sum_{p \leq x} |\lambda_f(p)|^2 = \kappa_f \operatorname{li} x + O\left(x e^{-c\sqrt{\log x}}\right) \qquad (\kappa_f > 0), \tag{13}$$

$$\sum_{p \leq x} \lambda_f(p)\overline{\lambda_g(p)} = O\left(x e^{-c\sqrt{\log x}}\right) \qquad (f \neq g). \tag{14}$$

The existence of the Rankin-Selberg convolution and square as well as zero-free region are well investigated for many automorphic L-functions. For example, it is known (see [7, Theorem 5.41]) that $L(s, f \otimes g)$ has no zero in the region (12) except possibly a one simple zero $\beta < 1$, provided f and g are classical primitive modular forms. Hence, we get that (13) and (14) hold and we get joint universality for any collection of automorphic L-function $L(s, f_1), \ldots, L(s, f_m)$ with distinct classical primitive modular forms, provided they belong to $\mathcal{S}$.

Similarly, the result of Liu and Ye [12, Theorem 2.3] implies joint universality for a quite general automorphic L-functions $L(s, \pi_j)$, $j = 1, 2, \ldots, m$, associated to irreducible unitary cuspidal representation π_j of $GL_m(\mathbb{Q}_A)$ satisfying $\pi_i \not\cong \pi_j \otimes |\det|^{i\tau}$ for any $\tau \in \mathbb{R}$, provided they are elements of the Selberg class.

It should be noted that, most likely, the Selberg class consists only of automorphic L-functions in which case it is widely believed and known for many examples that instead of Selberg's Conjecture 1.1 we can expect (13) and (14). It means that probably there is no example of L-functions from Selberg class satisfying Selberg's Conjecture 1.1, which do not fulfill (4) and (5). Thus, we conjecture that we do not loss of generality by assuming the stronger version of Selberg's conjecture.

References

1. Achieser, N.I.: Letures on the Theory of Approximation. Frederick Ungar Publishing Company, New York (1956)
2. Bauer, H.: The value distribution of Artin L-series and zeros of zeta-functions. J. Number Theory **98**, 254–279 (2003)
3. Bombieri, E., Hejhal, D.A.: On the distribution of zeros of linear combinations of Euler products. Duke Math. J. **80**, 821–862 (1995)
4. Conrey, J.B., Ghosh, A.: On the Selberg class of Dirichlet series: small degrees. Duke Math. J. **72**, 673–693 (1993)
5. Friedlander, J.B., Heath-Brown, D.R., Iwaniec, H., Kaczorowski, J., Analytic number theory. In: Perelli, A., Viola, C. (eds.) Lecture Notes in Mathematics 1891, Lectures from the C.I.M.E. Summer School held in Cetraro, July 11–18, 2002. Springer, Berlin (2006)
6. Gaier, D.: Vorlesungen über Approximation im Komplexen. Birkhäuser, Basel (1980)
7. Iwaniec, H., Kowalski, E.: Analytic Number Theory. American Mathematical Society, Colloquium Publications 53. AMS Providence, RI (2004)

8. Kaczorowski, J., Kulas, M.: On the non-trivial zeros off the critical line for L-functions from extended Selberg class. Mon. Math. **150**, 217–232 (2007)
9. Karatsuba, A.A., Voronin, S.M.: The Riemann Zeta Function. de Gruyter, Berlin (1992)
10. Laurinčikas, A., Matsumoto, K.: The universality of zeta-functions attached to certain cusp forms. Acta Arith. **98**, 345–359 (2001)
11. Lee, Y., Nakamura, T., Pańkowski, Ł.: Joint universality for Lerch zeta functions. J. Math. Soc. Jpn. 2015 arXiv:1503.06001
12. Liu, J., Ye, Y.: Perron's formula and the prime number theorem for automorphic L-functions. Pure. Appl. Math. Q. **3**(2), 481–497 (2007)
13. Matsumoto, K.: The mean values and the universality of Rankin–Selberg L-function. In: The Proceedings of the Turku Symposium in Number Theory in Memory of Kustaa Inkeri, Walter de Gruyter, pp. 201–221 (2001)
14. Mishou, H.: Joint universality theorems for pairs of automorphic zeta functions. Math. Z. **277**, 1113–1154 (2014)
15. Nakamura, T.: Joint value approximation and joint universality for several types of zeta functions. Acta Arith. **134**(1), 67–82 (2008)
16. Nagoshi, H.: Value-distribution of Rankin–Selberg L-functions. In: New Directions in Value-Distribution Theory of Zeta and L-functions: Proceedings of Würzburg Conference, October 6–10, 2008, pp. 275–287. Shaker Verlag (2009)
17. Nagoshi, H., Steuding, J.: Universality for L-functions in the Selberg class. Lith. Math. J. **50**(3), 293–311 (2010)
18. Potter, H.S.A.: The mean values of certain Dirichlet series I. Proc. Lond. Math. Soc. **46**, 467–468 (1940)
19. Queffélec, H., Queffélec, M.: Diophantine Approximation and Dirichlet Series. HRI Lecture Notes Series 2, AMS (2013)
20. Reich, A.: Werteverteilung von Zetafunktionen. Arch. Math. **34**, 440–451 (1980)
21. Steuding, J.: Value-Distribution of L-Functions. Springer, Berlin (2007)
22. Voronin, S.M.: Theorem on the universality of the Riemann zeta function, Izv. Akad. Nauk SSSR Ser Mat **39**, 475–486 (in Russian). Math. USSR Izv. **9**, 443–453 (1975)
23. Voronin, S.M.: Analytic Properties of Dirichlet Generating Functions of Arithmetic Objects. Ph.D. Thesis, Moscow, Steklov Mathematical Institute (1977) (Russian)