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A new higher-order family of inclusion zero-finding methods

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Abstract

Starting from a suitable fixed point relation, a new one-parameter family of iterative methods for the simultaneous inclusion of complex zeros in circular complex arithmetic is constructed. It is proved that the order of convergence of this family is four. The convergence analysis is performed under computationally verifiable initial conditions. An approach for the construction of accelerated methods with negligible number of additional operations is discussed. To demonstrate convergence properties of the proposed family of methods, two numerical examples results are given.

MSC: 65H05; 65G20; 30C15

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1. Introduction

In this paper we derive a new fixed point relation which is the base for the construction of a new one-parameter family of iterative methods for the simultaneous determination of complex zeros of a polynomial. The basic method realized in circular complex arithmetic has the convergence order equal to four. Convergence analysis of the proposed method and numerical example are given. A discussion on the construction of modified methods with very fast convergence is also included.

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The presentation of the paper is organized as follows. Some basic definitions and operations of circular complex interval arithmetic, necessary for the convergence analysis and the construction of inclusion methods, are given at the end of Introduction. In Section 2 we derive a fixed point relation of square root type which is the basis for the construction of iterative methods for the simultaneous inclusion of simple complex zeros presented in Section 3. The convergence analysis of the basic fourth-order interval method is given in Section 4. In Section 5 we discuss some modified interval methods with the accelerated convergence of high computational efficiency. Numerical results obtained for various values of the involved parameter are given in Section 6.

In the construction of inclusion methods and the convergence analysis we will estimate some complex quantities using an approach by circular complex arithmetic which deals with disks in the complex plane. A disk Z with center $\operatorname{mid} Z = c$ and radius $\operatorname{rad} Z = r$, that is $Z := \{z : |z - c| \leqslant r\}$, will be denoted briefly by the parametric notation $Z = \{c; r\}$. For more details about properties of circular complex interval arithmetic see the books [1, Chapter 5; 10, Chapter 1].

Consider now the inversion of a disk $Z = \{c; r\}$ which does not contain the origin, that is, $|c| > r$ holds. Under the transformation $w(z) = 1/z$ this disk maps into the disk

$$Z^{-1} = \left\{ \frac{1}{z} : z \in \{c; r\} \right\} = \left\{ \frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2} \right\}. \tag{1.1}$$

Aside from the inverse disk Z^{-1}, another type of inversion (so-called centered form)

$$Z^{I_c} = \left\{ \frac{1}{c}; \frac{r}{|c|(|c| - r)} \right\} \tag{1.2}$$

is often used. It is easy to check that $\operatorname{rad} Z^{I_c} > \operatorname{rad} Z^{-1}$ and $|\operatorname{mid} Z^{I_c} - \operatorname{mid} Z^{-1}| = \operatorname{rad} Z^{I_c} - \operatorname{rad} Z^{-1}$. According to this and a geometric construction we infer that

$$Z^{-1} \subset Z^{I_c} \quad (0 \notin Z).$$

In circular complex arithmetic the following simple properties are valid [5]:

$$z \in \{c; r\} \;\Rightarrow\; |c| - r \leqslant |z| \leqslant |c| + r, \tag{1.3}$$

$$\{c_1; r_1\} \pm \{c_2; r_2\} = \{c_1 \pm c_2; r_1 + r_2\}, \tag{1.4}$$

$$\alpha\{c; r\} = \{\alpha c; |\alpha| r\} \quad (\alpha \in \mathbb{C}), \tag{1.5}$$

$$\{c_1; r_1\} \cap \{c_2; r_2\} = \emptyset \;\Leftrightarrow\; |c_1 - c_2| > r_1 + r_2, \tag{1.6}$$

$$\{c_1; r_1\} \cdot \{c_2; r_2\} = \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\}. \tag{1.7}$$

Following (1.7) and inversions (1.1) and (1.2), division is defined as

$$Z_1 : Z_2 = Z_1 \cdot Z_2^{-1} \quad \text{or} \quad Z_1 : Z_2 = Z_1 \cdot Z_2^{I_c} \quad (0 \notin Z_2).$$

The square root of a disk $\{c; r\}$ in the centered form, where $c = |c| e^{i\theta}$ and $|c| > r$, is defined as the union of two disjoint disks (see [3]):

$$\{c; r\}^{1/2} := \left\{ \sqrt{|c|} e^{i\theta/2}; \sqrt{|c|} - \sqrt{|c| - r} \right\} \cup \left\{ -\sqrt{|c|} e^{i\theta/2}; \sqrt{|c|} - \sqrt{|c| - r} \right\}. \tag{1.8}$$

In what follows, disks in the complex plane will be denoted by capital letters.

2. Fixed-point relation

Let α ($\neq -1$) be a complex parameter whose size will be discussed later. Particular case $\alpha = -1$ was considered in detail in [7,13] so that the corresponding methods will not be analyzed in this paper. Let P be a monic polynomial with simple zeros $\zeta_1, \ldots, \zeta_n$ and let $z_1, \ldots, z_n$ be their approximations. For the point $z = z_i$ let us introduce the notations:

$$\Sigma_{\lambda,i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{(z_i - \zeta_j)^\lambda}, \quad s_{\lambda,i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{(z_i - z_j)^\lambda} \quad (\lambda = 1, 2),$$

$$\delta_{1,i} = \frac{P'(z_i)}{P(z_i)}, \quad \delta_{2,i} = \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2}, \tag{2.1}$$

$$f_i^* = (\alpha+1)\Sigma_{2,i} - \alpha(\alpha+1)\Sigma_{1,i}^2, \quad f_i = (\alpha+1)s_{2,i} - \alpha(\alpha+1)s_{1,i}^2,$$

$$\varepsilon_i = z_i - \zeta_i, \quad \varepsilon = \max_{1\leqslant i\leqslant n} |\varepsilon_i|. \tag{2.2}$$

Lemma 2.1. *For $i \in I_n := \{1, \ldots, n\}$ the following identity is valid*:

$$(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i^* = \left(\frac{\alpha+1}{\varepsilon_i} - \alpha\delta_{1,i}\right)^2. \tag{2.3}$$

Proof. Starting from the identities

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z - \zeta_j} \tag{2.4}$$

and

$$\frac{P'(z)^2 - P(z)P''(z)}{P(z)^2} = -\left(\frac{P'(z)}{P(z)}\right)' = \sum_{j=1}^{n} \frac{1}{(z-\zeta_j)^2}, \tag{2.5}$$

we obtain

$$\begin{aligned}(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i^* &= (\alpha+1)\left(\frac{1}{\varepsilon_i^2} + \Sigma_{2,i}\right) - \alpha\left(\frac{1}{\varepsilon_i} + \Sigma_{1,i}\right)^2 - (\alpha+1)\Sigma_{2,i} + \alpha(\alpha+1)\Sigma_{1,i}^2 \\ &= \frac{1}{\varepsilon_i^2} + \alpha^2\Sigma_{1,i}^2 - \frac{2\alpha}{\varepsilon_i}\Sigma_{1,i} = \left(\frac{1}{\varepsilon_i} - \alpha\Sigma_{1,i}\right)^2 \\ &= \left(\frac{\alpha+1}{\varepsilon_i} - \alpha\delta_{1,i}\right)^2. \qquad \square\end{aligned}$$

From identity (2.3) we obtain the following fixed point relation:

$$\zeta_i = z_i - \frac{\alpha+1}{\alpha\delta_{1,i} + [(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i^*]^{1/2}} \quad (i \in I_n), \tag{2.6}$$

assuming that two values of the square root have to be taken in (2.6).

Let us assume that we have found mutually disjoint disks $Z_1, \ldots, Z_n$ with centers $z_i = \text{mid } Z_i$ and radii $r_i = \text{rad } Z_i$ such that $\zeta_i \in Z_i \ (i \in I_n)$. Let us substitute the zeros ζ_j by their inclusion disks Z_j in the expression for f_i^*. In this way we obtain a circular extension F_i of f_i^*,

$$F_i = (\alpha+1) \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{1}{z_i - Z_j}\right)^2 - \alpha(\alpha+1) \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{z_i - Z_j}\right)^2 \tag{2.7}$$

with $f_i^* \in F_i$ for each $i \in I_n$.

Using the inclusion isotonicity property, from the fixed point relation (2.6) we get

$$\zeta_i \in \hat{Z}_i := z_i - \frac{\alpha+1}{\alpha\delta_{1,i} + [(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - F_i]^{1/2}} \quad (i \in I_n). \tag{2.8}$$

If the denominator in (2.8) is a disk not containing 0, then $\hat{Z}_i$ is a new outer circular approximation to the zero ζ_i, that is, $\zeta_i \in \hat{Z}_i \ (i \in I_n)$.

3. Family of methods in circular complex arithmetic

Let us introduce some notations:

1. The circular inclusion approximations $Z_1^{(m)}, \ldots, Z_n^{(m)}$ of the zeros at the *m*th iterative step will be briefly denoted by $Z_1, \ldots, Z_n$, and the new approximations $Z_1^{(m+1)}, \ldots, Z_n^{(m+1)}$, obtained by some simultaneous inclusion iterative method, by $\hat{Z}_1, \ldots, \hat{Z}_n$, respectively;
2. $S_{k,i}(\boldsymbol{A}, \boldsymbol{B}) = \sum_{j=1}^{i-1} (\text{INV}(z_i - A_j))^k + \sum_{j=i+1}^{n} (\text{INV}(z_i - B_j))^k, \ z_i = \text{mid } Z_i,$
   $F_i(\boldsymbol{A}, \boldsymbol{B}) = (\alpha+1) S_{2,i}(\boldsymbol{A}, \boldsymbol{B}) - \alpha(\alpha+1) S_{1,i}^2(\boldsymbol{A}, \boldsymbol{B}),$

where $\boldsymbol{A} = (A_1, \ldots, A_n)$ and $\boldsymbol{B} = (B_1, \ldots, B_n)$ are some vectors of disks and $\text{INV} \in \{()^{-1}, ()^{I_c}\}$. If $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{Z} = (Z_1, \ldots, Z_n)$, then we will write $S_{k,i}(\boldsymbol{Z}, \boldsymbol{Z}) = S_{k,i}$ and $F_i(\boldsymbol{Z}, \boldsymbol{Z}) = F_i$.

3. $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$ (current disk approximations),
   $\widehat{\boldsymbol{Z}} = (\widehat{Z}_1, \ldots, \widehat{Z}_n)$ (new disk approximations).

Starting from (2.8), we obtain a new one-parameter family of iterative processes for the simultaneous inclusion of all simple zeros of a polynomial. In our consideration of a new family we will always suppose that $\alpha \neq -1$. However, the particular case $\alpha = -1$ reduces (by applying a limiting process) to the already known Halley-like interval method which was studied in [7,12–14]. For the total-step methods ("Jacobi" or parallel mode) and single-step methods (serial or "Gauss–Seidel" mode) the abbreviations TS and SS will be used.

First, following (2.8), we will construct the family of total-step methods:

Basic total-step method (TS):

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{\alpha+1}{\alpha\delta_{1,i}^{(m)} + [(\alpha+1)\delta_{2,i}^{(m)} - \alpha[\delta_{1,i}^{(m)}]^2 - F_i^{(m)}(\mathbf{Z},\mathbf{Z})]_*^{1/2}}$$
$$(i \in I_n;\ \ m = 0, 1, \ldots). \tag{TS}$$

The symbol $*$ indicates that one of the two disks (say, $U_{1,i} = \{u_{1,i}; d_i\}$ and $U_{2,i} = \{u_{2,i}; d_i\}$, where $u_{1,i} = -u_{2,i}$) has to be chosen according to some suitable criterion. That disk will be called a "*proper*" disk. From (2.3) and the inclusion $f_i^* \in F_i$ we conclude that the proper disk is one which contains the complex number $(\alpha+1)/\varepsilon_i - \alpha\delta_{1,i}$. The choice of the proper sign in front of the square root in (TS) was considered in detail in [3] (see, also, [6, Chapter 3]). The following criterion for the choice of a proper disk of a square root (between two disks) can be stated:

If the disks $Z_1, \ldots, Z_n$ are reasonably small, then we have to choose that disk (*between $U_{1,i}$ and $U_{2,i}$*) *whose center minimizes $|P'(z_i)/P(z_i) - u_{k,i}|$ $(k = 1, 2)$.*

Now we will present some special cases of this family total-step iterative methods (omitting the iteration indices):

$\alpha = 0$, *Ostrowski-like method*:

$$\widehat{Z}_i = z_i - \frac{1}{[\delta_{2,i} - S_{2,i}(\mathbf{Z},\mathbf{Z})]_*^{1/2}} \quad (i \in I_n). \tag{3.1}$$

$\alpha = 1/(n-1)$, *Laguerre-like method*:

$$\widehat{Z}_i = z_i - \frac{n}{\delta_{1,i} + \left[(n-1)\left(n\delta_{2,i} - \delta_{1,i}^2 - \left(nS_{2,i}(\mathbf{Z},\mathbf{Z}) - \dfrac{n}{n-1}S_{1,i}^2(\mathbf{Z},\mathbf{Z})\right)\right)\right]_*^{1/2}} \quad (i \in I_n). \tag{3.2}$$

$\alpha = 1$, *Euler-like method*:

$$\widehat{Z}_i = z_i - \frac{2}{\delta_{1,i} + [2\delta_{2,i} - \delta_{1,i}^2 - 2(S_{2,i}(\mathbf{Z},\mathbf{Z}) - S_{1,i}^2(\mathbf{Z},\mathbf{Z}))]_*^{1/2}} \quad (i \in I_n). \tag{3.3}$$

$\alpha = -1$, *Halley-like method*:

$$\widehat{Z}_i = z_i - \frac{2\delta_{1,i}}{\delta_{2,i} + \delta_{1,i}^2 - S_{2,i}(\mathbf{Z},\mathbf{Z}) - S_{1,i}^2(\mathbf{Z},\mathbf{Z})} \quad (i \in I_n). \tag{3.4}$$

The Halley-like method is obtained for $\alpha \to -1$ applying a limiting operation.

The names come from the similarity with the quoted classical methods. For instance, omitting the sum in (3.1) we obtain the well-known Ostrowski method $\hat{z}_i = z_i - 1/[\delta_{2,i}]_*^{1/2}$.

4. Convergence analysis

In this section we give the convergence analysis of the interval method (TS). In the sequel we will always assume that $n \geqslant 3$.

Let us introduce the abbreviation $a = |\alpha|$. Also, for disjoint disks $Z_1, \ldots, Z_n$ let us define

$$r = \max_{1 \leqslant i \leqslant n} r_i, \quad \rho = \min_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} \{|z_i - z_j| - r_j\}, \quad z_i = \text{mid } Z_i, \quad r_i = \text{rad } Z_i.$$

Lemma 4.1. *Let*

$$\eta = \frac{|\alpha + 1|(n-1)r}{\rho^3}\,(2 + 3a(n-1))$$

and let the inequality

$$\rho > 4(n-1)r \tag{4.1}$$

hold. Then

$$F_i \subset \{f_i; \eta\}. \tag{4.2}$$

Proof. We use the following inclusion derived in [6, Chapter 3]:

$$\left(\frac{1}{z_i - Z_j}\right)^k \subset \left\{\frac{1}{(z_i - z_j)^k}; \frac{kr}{\rho^{k+1}}\right\} \quad (k = 1, 2, \ldots). \tag{4.3}$$

Since

$$|z_i - \zeta_j| \geqslant |z_i - z_j| - |z_j - \zeta_j| \geqslant |z_i - z_j| - r_j \geqslant \rho,$$

we have

$$\frac{1}{|z_i - \zeta_j|} \leqslant \frac{1}{\rho} \quad \text{and} \quad \frac{1}{|z_i - z_j|} \leqslant \frac{1}{\rho}. \tag{4.4}$$

Using (4.1), (4.4) and the inclusion (4.3), from (2.7) we obtain

$$\begin{aligned}
F_i &\subset (\alpha+1)\sum_{j \neq i}\left\{\frac{1}{(z_i - z_j)^2}; \frac{2r}{\rho^3}\right\} - \alpha(\alpha+1)\left(\sum_{j\neq i}\left\{\frac{1}{z_i - z_j}; \frac{r}{\rho^2}\right\}\right)^2 \\
&\subseteq \left\{(\alpha+1)\sum_{j\neq i}\frac{1}{(z_i - z_j)^2}; \frac{2|\alpha+1|(n-1)r}{\rho^3}\right\} - \alpha(\alpha+1)\left\{\sum_{j\neq i}\frac{1}{z_i - z_j}; \frac{(n-1)r}{\rho^2}\right\}^2 \\
&\subset \left\{(\alpha+1)\sum_{j\neq i}\frac{1}{(z_i - z_j)^2}; \frac{2|\alpha+1|(n-1)r}{\rho^3}\right\} - \alpha(\alpha+1)\left\{\left(\sum_{j\neq i}\frac{1}{z_i - z_j}\right)^2; \frac{3(n-1)^2 r}{\rho^3}\right\} \\
&= \left\{f_i; \frac{|\alpha+1|(n-1)r}{\rho^3}\,(2 + 3a(n-1))\right\} = \{f_i; \eta\}. \qquad \square
\end{aligned}$$

Let

$$y_i = (\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i, \quad v_i = \frac{f_i^* - f_i}{\left(\dfrac{\alpha+1}{\varepsilon_i} - \alpha\delta_{1,i}\right)^2}, \quad d = 12|\alpha+1|(n-1)^2\,\frac{r^2}{\rho^3}.$$

Lemma 4.2. *Let inequality* (4.1) *hold and* $a = |\alpha| < 1.13$. *Then*

(i) $|y_i| > \frac{15-11a-a^2}{16r^2} > 0$;
(ii) $|y_i| - \eta > \frac{119-92a-11a^2}{128r^2} > 0$;
(iii) $\sqrt{\{y_i;\eta\}} \subset \left\{\sqrt{y_i}; d\right\}$;
(iv) $\sqrt{1+v_i} \in \{1; \frac{|\alpha+1|(2a+1)|\varepsilon_i|}{(a-4)^2\rho}\}$.

Proof. (i) Using the above notation and identities (2.4) and (2.5), we obtain

$$\begin{aligned} y_i &= (\alpha+1)\left(\frac{1}{\varepsilon_i^2} + \Sigma_{2,i}\right) - \alpha\left(\frac{1}{\varepsilon_i} + \Sigma_{1,i}\right)^2 - ((\alpha+1)s_{2,i} - \alpha(\alpha+1)s_{1,i}^2) \\ &= \frac{1-2\alpha\varepsilon_i\Sigma_{1,i}}{\varepsilon_i^2} + (\alpha+1)\Sigma_{2,i} - \alpha\Sigma_{1,i}^2 - (\alpha+1)s_{2,i} + \alpha(\alpha+1)s_{1,i}^2. \end{aligned}$$

Starting from this expression and using the estimates

$$|\Sigma_{k,i}| \leqslant \frac{n-1}{\rho^k}, \quad |s_{k,i}| \leqslant \frac{n-1}{\rho^k} \quad (k=1,2)$$

and (4.1), we find

$$\begin{aligned} |y_i| &\geqslant \left|\frac{1-2\alpha\varepsilon_i\Sigma_{1,i}}{\varepsilon_i^2}\right| - |(\alpha+1)\Sigma_{2,i}| - |\alpha\Sigma_{1,i}|^2 - |(\alpha+1)s_{2,i}| - |\alpha(\alpha+1)s_{1,i}^2| \\ &\geqslant \frac{1-2ar\cdot\dfrac{n-1}{\rho}}{r^2} - \frac{(a+1)(n-1)}{\rho^2} - \frac{a(n-1)^2}{\rho^2} - \frac{(a+1)(n-1)}{\rho^2} - \frac{a(a+1)(n-1)^2}{\rho^2} \\ &> \frac{1-\dfrac{a}{2}}{r^2} - \frac{a+1}{8(n-1)r^2} - \frac{a}{16r^2} - \frac{a(a+1)}{16r^2} \geqslant \frac{15-11a-a^2}{16r^2} > 0 \end{aligned}$$

for $a < 1.13$.

(ii) Using (i) we obtain

$$\begin{aligned} |y_i| - \eta &> \frac{15-11a-a^2}{16r^2} - \frac{(a+1)(2+3a(n-1))}{64(n-1)^2r^2} > \frac{15-11a-a^2}{16r^2} - \frac{3a^2+4a+1}{128r^2} \\ &= \frac{-11a^2-92a+119}{128r^2} > 0 \end{aligned}$$

for $a < 1.13$.

(iii) Using (1.8) we find

$$\sqrt{\{y_i;\eta\}} = \left\{ y_i^{1/2};\ \sqrt{|y_i|} - \sqrt{|y_i| - \eta} \right\} = \{y_i^{1/2};\ R_i\},$$

where

$$R_i = \frac{\eta}{\sqrt{|y_i|} + \sqrt{|y_i| - \eta}}.$$

By (i) and (ii) we estimate

$$R_i = \frac{\dfrac{|\alpha+1|(n-1)^2 r}{\rho^3}\left(\dfrac{2}{n-1} + 3a\right)}{\sqrt{|y_i|} + \sqrt{|y_i| - \eta}} < 12|\alpha+1|(n-1)^2\,\frac{r^2}{\rho^3}.$$

Therefore,

$$\sqrt{\{y_i;\eta\}} = \left\{\sqrt{y_i};\ R_i\right\} \subset \left\{\sqrt{y_i};\ 12|\alpha+1|(n-1)^2\,\frac{r^2}{\rho^3}\right\} = \left\{\sqrt{y_i};\ d\right\}.$$

(iv) Starting from the expressions for f_i and f_i^* we find

$$f_i^* - f_i = -(\alpha+1)\sum_{j\neq i} \frac{\varepsilon_j}{(z_i-\zeta_j)(z_i-z_j)}\left(\frac{1}{z_i-\zeta_j} + \frac{1}{z_i-z_j}\right) + \alpha(\alpha+1)\left(\sum_{j\neq i}\frac{\varepsilon_j}{(z_i-\zeta_j)(z_i-z_j)}\right)\left(\sum_{j\neq i}\frac{1}{z_i-\zeta_j} + \sum_{j\neq i}\frac{1}{z_i-z_j}\right).$$

Hence, by (4.4) and the inequalities $|\varepsilon_i| \leqslant r$, $|z_i - z_j| \geqslant \rho$ we get

$$|f_i^* - f_i| \leqslant \frac{|\alpha+1|(n-1)r}{\rho^2}\left(\frac{1}{\rho} + \frac{1}{\rho}\right) + \frac{a|\alpha+1|(n-1)r}{\rho^2}\left(\frac{n-1}{\rho} + \frac{n-1}{\rho}\right) = \frac{2|\alpha+1|(n-1)r}{\rho^3}(1 + a(n-1)). \tag{4.5}$$

Using (4.1) and (4.4) we estimate

$$|\alpha+1-\alpha\varepsilon_i\delta_{1,i}| = \left|\alpha+1-\alpha\varepsilon_i\left(\sum_{j\neq i}\frac{1}{z_i-\zeta_j} + \frac{1}{\varepsilon_i}\right)\right| \geqslant 1 - a|\varepsilon_i|\sum_{j\neq i}\frac{1}{|z_i-\zeta_j|} \geqslant 1 - a\,\frac{(n-1)r}{\rho} > \frac{4-a}{4}. \tag{4.6}$$

In the similar way we find

$$|\alpha+1-\alpha\varepsilon_i\delta_{1,i}| \leqslant 1 + a|\varepsilon_i|\sum_{j\neq i}\frac{1}{|z_i-\zeta_j|} < \frac{4+a}{4}. \tag{4.7}$$

By (4.5) and (4.6) we estimate

$$|v_i| \leqslant \frac{|f_i^* - f_i|}{\left|\dfrac{\alpha+1}{\varepsilon_i} - \alpha\delta_{1,i}\right|^2} < \frac{\dfrac{2|\alpha+1|(n-1)r}{\rho^3}(1+a(n-1))}{\dfrac{1}{|\varepsilon_i|^2}|\alpha+1-\alpha\varepsilon_i\delta_{1,i}|^2}$$
$$\leqslant \frac{32|\alpha+1|(n-1)(1+a(n-1))}{(4-a)^2}\frac{r^2}{\rho^2}\frac{|\varepsilon_i|}{\rho} =: \gamma_i(\alpha).$$

Let us observe that

$$\gamma_i(\alpha) < \frac{32|\alpha+1|(n-1)(1+a(n-1))}{(4-a)^2} \cdot \frac{1}{64(n-1)^3} < \frac{|\alpha+1|\left(\dfrac{1}{n-1}+a\right)}{2(4-a)^2(n-1)}$$
$$< \frac{|\alpha+1|(1+2a)}{8(a-4)^2} < 1$$

for $|\alpha| < 1.13$.

Let $V_i := \{0; \gamma_i(\alpha)\}$, then $v_i \in V_i$ and, by the inclusion isotonicity property and (1.8), we find

$$\sqrt{1+v_i} \in \sqrt{1+V_i} = \sqrt{\{1;\gamma_i(\alpha)\}} = \left\{1; 1-\sqrt{1-\gamma_i(\alpha)}\right\} = \left\{1; \frac{\gamma_i(\alpha)}{1+\sqrt{1-\gamma_i(\alpha)}}\right\} \subset \{1;\gamma_i(\alpha)\}.$$

Finally, we obtain

$$\sqrt{1+v_i} \in \left\{1; \frac{|\alpha+1|(2a+1)|\varepsilon_i|}{(a-4)^2\rho}\right\}. \qquad \square$$

Using Lemmas 4.1 and 4.2 we are now able to prove that the order of convergence of the inclusion method (TS) is four.

Theorem 4.1. *Let the interval sequences* $\{Z_i^{(m)}\}$ $(i \in I_n)$ *be defined by the iterative formula* (TS), *where* $|\alpha| < 1.13$. *Then, under the condition*

$$\rho^{(0)} > 4(n-1)r^{(0)}, \tag{4.8}$$

for each $i \in I_n$ *and* $m = 0, 1, \ldots$ *we have*

1. $\zeta_i \in Z_i^{(m)}$;
2. $r^{(m+1)} < \frac{14(n-1)^2(r^{(m)})^4}{(\rho^{(0)} - \frac{17}{11}r^{(0)})^3}$.

Proof. We will prove assertion 1 by induction. Suppose that $\zeta_i \in Z_i^{(m)}$ for $i \in I_n$ and $m \geqslant 1$. Then

$$f_i^* = (\alpha+1)\sum_{j\neq i}\frac{1}{(z_i^{(m)}-\zeta_j)^2} - \alpha(\alpha+1)\left(\sum_{j\neq i}\frac{1}{z_i^{(m)}-\zeta_j}\right)^2 \in F_i^{(m)}$$

and, according to (2.8), it follows

$$\zeta_i \in z_i^{(m)} - \frac{\alpha+1}{\alpha\delta_{1,i}^{(m)} + [(\alpha+1)\delta_{2,i}^{(m)} - \alpha(\delta_{1,i}^{(m)})^2 - F_i^{(m)}]_*^{1/2}} = Z_i^{(m+1)}.$$

The symbol $*$ points that the proper disk has to be chosen. Since $\zeta_i \in Z_i^{(0)}$, we obtain by induction that $\zeta_i \in Z_i^{(m)}$ for each $m = 0, 1, 2, \ldots$.

Let us prove now that the interval method (TS) has the order of convergence equal to four (assertion 2). We use induction and start with $m = 0$. For simplicity, all indices are omitted and all quantities in the first iteration are denoted by $\wedge$.

Using circular arithmetic operations, inclusion (4.2) and assertion (iii) of Lemma 4.2, from the iterative formula (TS) (omitting iteration indices) we obtain

$$\widehat{Z}_i \subset z_i - \frac{\alpha+1}{\alpha\delta_{1,i} + \left\{\sqrt{y_i}; d\right\}_*} = z_i - \frac{\alpha+1}{\{u_i; d\}}, \tag{4.9}$$

where we put $u_i = \alpha\delta_{1,i} + [y_i]_*^{1/2}$. By (4.7) and (iv) of Lemma 4.2, we obtain

$$\begin{aligned} u_i &= \alpha\delta_{1,i} + [y_i]_*^{1/2} \\ &= \alpha\delta_{1,i} + \left(\frac{\alpha+1}{\varepsilon_i} - \alpha\delta_{1,i}\right)\sqrt{1+v_i} \in \alpha\delta_{1,i} + \left(\frac{\alpha+1}{\varepsilon_i} - \alpha\delta_{1,i}\right)\left\{1; \frac{|\alpha+1|(2a+1)}{(a-4)^2} \cdot \frac{|\varepsilon_i|}{\rho}\right\} \\ &\subset \left\{\frac{\alpha+1}{\varepsilon_i}; \frac{(4+a)(2a+1)|\alpha+1|}{32(a-4)^2} \cdot \frac{1}{r}\right\} =: U_i. \end{aligned}$$

Hence, by virtue of (1.3),

$$\begin{aligned} |u_i| &> |\mathrm{mid}\, U_i| - \mathrm{rad}\, U_i = \frac{|\alpha+1|}{|\varepsilon_i|} - \frac{(4+a)(2a+1)|\alpha+1|}{32(a-4)^2 r} \\ &> \frac{|\alpha+1|}{r} \cdot \frac{30a^2 - 265a + 508}{32(a-4)^2} > 0 \end{aligned} \tag{4.10}$$

for $|\alpha| < 1.13$. Using (4.1) and (4.10) we estimate

$$|u_i| - d > \frac{|\alpha+1|}{r}\left(\frac{30a^2 - 265a + 508}{32(a-4)^2} - \frac{3}{32}\right) = \frac{|\alpha+1|}{r}\left(\frac{27a^2 - 241a + 460}{32(a-4)^2}\right) > 0$$

for $|\alpha| < 1.13$.

By (1.1) from (4.9) we find

$$\widehat{Z}_i \subset z_i - (\alpha+1)\left\{\frac{\overline{u_i}}{|u_i|^2 - d^2}; \frac{d}{|u_i|^2 - d^2}\right\},$$

whence

$$\widehat{r}_i = \mathrm{rad}\, \widehat{Z}_i < \frac{|\alpha+1|d}{|u_i|^2 - d^2}. \tag{4.11}$$

Using the lower bound for $|u_i|$ given by (4.10), from (4.11) we obtain

$$\widehat{r_i} < \frac{12|\alpha+1|^2(n-1)^2\,\dfrac{r^2}{\rho^3}}{\left(\dfrac{|\alpha+1|}{r}\left(\dfrac{30a^2-265a+508}{32(a-4)^2}\right)\right)^2-\left(12|\alpha+1|(n-1)^2\,\dfrac{r^2}{\rho^3}\right)^2}$$
$$< \frac{12(n-1)^2\cdot\dfrac{r^4}{\rho^3}}{\left(\dfrac{30a^2-265a+508}{32(a-4)^2}\right)^2-\left(\dfrac{3}{32}\right)^2},$$

wherefrom

$$\widehat{r} < \frac{14(n-1)^2 r^4}{\rho^3} \tag{4.12}$$

and

$$\hat{r} < 14(n-1)^2\,\frac{r^3}{\rho^3}\,r < \frac{14}{64(n-1)}\,r < 0.11r < \frac{3}{25}\,r. \tag{4.13}$$

According to a geometric construction and the fact that the disks $Z_i^{(m)}$ and $Z_i^{(m+1)}$ must have at least one point in common (the zero ζ_i), the following relation can be derived (see [4]):

$$\rho^{(m+1)} \geqslant \rho^{(m)} - r^{(m)} - 3r^{(m+1)}. \tag{4.14}$$

Using inequalities (4.13) and (4.14) (for $m=0$), we find

$$\rho^{(1)} \geqslant \rho^{(0)} - r^{(0)} - 3r^{(1)} > 4(n-1)r^{(0)} - r^{(0)} - \tfrac{9}{25}\,r^{(0)} > \tfrac{25}{3}\,r^{(1)}(4(n-1)-1-\tfrac{9}{25}),$$

wherefrom it follows

$$\rho^{(1)} > 4(n-1)r^{(1)}. \tag{4.15}$$

This is condition (4.1) for the index $m=1$, which means that all assertions of Lemmas 4.1 and 4.2 are valid for $m=1$.

Using the definition of ρ and (4.15), for arbitrary pair of indices $i, j \in I_n$ $(i \neq j)$ we have

$$|z_i^{(1)} - z_j^{(1)}| \geqslant \rho^{(1)} > 4(n-1)r^{(1)} > 2r^{(1)} \geqslant r_i^{(1)} + r_j^{(1)}. \tag{4.16}$$

Therefore, in regard to (1.6), the disks $Z_1^{(1)}, \ldots, Z_n^{(1)}$ produced by (TS) are disjoint.

Applying mathematical induction with the argumentation used for the derivation of (4.12)–(4.14) and (4.16) (which makes the part of the proof with respect to $m=1$), we prove that, for each $m=0,1,\ldots,$

the disks $Z_1^{(m)}, \ldots, Z_n^{(m)}$ are disjoint and the following relations are true:

$$r^{(m+1)} < \frac{14(n-1)^2(r^{(m)})^4}{(\rho^{(m)})^3}, \tag{4.17}$$

$$r^{(m+1)} < \tfrac{3}{25} r^{(m)}, \tag{4.18}$$

$$\rho^{(m)} > 4(n-1)r^{(m)}. \tag{4.19}$$

In addition we note that the last inequality (4.19) means that the assertions of Lemmas 4.1 and 4.2 hold for each $m = 0, 1, 2, \ldots$.

For simplicity, let $\omega = 3/25$. Then

$$1 + 4(\omega + \omega^2 + \cdots + \omega^m) - \omega^m < 1 + \frac{4\omega}{1-\omega} = \frac{17}{11}. \tag{4.20}$$

By the successive application of (4.14) and (4.18) we obtain

$$\begin{aligned}
\rho^{(m)} &> \rho^{(m-1)} - r^{(m-1)} - 3\omega r^{(m-1)} = \rho^{(m-1)} - r^{(m-1)}(1+3\omega) \\
&> \rho^{(m-2)} - r^{(m-2)} - 3\omega r^{(m-2)} - \omega r^{(m-2)}(1+3\omega) \\
&= \rho^{(m-2)} - r^{(m-2)}(1 + 4\omega + 4\omega^2 - \omega^2) \\
&\ \ \vdots \\
&> \rho^{(0)} - r^{(0)}(1 + 4\omega + 4\omega^2 + \cdots + 4\omega^m - \omega^m) \\
&> \rho^{(0)} - \tfrac{17}{11} r^{(0)},
\end{aligned}$$

where we used (4.20). According to the last inequality and (4.17) we find

$$r^{(m+1)} < \frac{14(n-1)^2(r^{(m)})^4}{(\rho^{(0)} - \frac{17}{11} r^{(0)})^3}.$$

Therefore, assertion 2 of Theorem 4.1 holds. The last relation shows that the order of convergence of the inclusion method (TS) is four. □

Remark 1. The condition $|\alpha| < 1.13$ is only sufficient. This bound is used to provide the validity of some (not so sharp) inequalities and estimates in the presented convergence analysis. However, the value of $|\alpha|$ can be taken to be considerably larger in practice, as many numerical examples have shown.

5. Improved methods

The convergence rate of the total-step method (TS) can be accelerated using new circular approximations as soon as they are calculated in the current iteration (Gauss–Seidel approach or single-step mode). In this way we construct

Basic single-step method (SS):

$$\widehat{Z}_i = z_i - \frac{\alpha + 1}{\alpha\delta_{1,i} + [(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - F_i(\widehat{\mathbf{Z}}, \mathbf{Z})]_*^{1/2}} \quad (i \in I_n). \tag{SS}$$

The *R*-order of convergence of the single-step method (SS) is at least $3 + x_n$, where $x_n > 1$ is the unique positive root of the equation $x^n - x - 3 = 0$ (see [7]).

Let us introduce the following notations:

$$N_i = N(z_i) = 1/\delta_{1,i} = \frac{P(z_i)}{P'(z_i)} \quad \text{(Newton's correction)}, \tag{5.1}$$

$$H_i = H(z_i) = \left[\frac{P'(z_i)}{P(z_i)} - \frac{P''(z_i)}{2P'(z_i)}\right]^{-1} = \frac{2\delta_{1,i}}{\delta_{1,i}^2 + \delta_{2,i}} \quad \text{(Halley's correction)}, \tag{5.2}$$

$$\boldsymbol{Z}_N = (Z_{N,1}, \ldots, Z_{N,n}) \quad Z_{N,i} = Z_i - N(z_i) \quad \text{(Newton's disk approximations)},$$

$$\boldsymbol{Z}_H = (Z_{H,1}, \ldots, Z_{H,n}) \quad Z_{H,i} = Z_i - H(z_i) \quad \text{(Halley's disk approximations)}.$$

We recall that the correction terms (5.1) and (5.2) appear in the iterative formulas

$$\widehat{Z} = z - N(z) \quad \text{(Newton's method)}, \quad \text{and} \quad \widehat{Z} = z - H(z) \quad \text{(Halley's method)},$$

which have quadratic and cubic convergence, respectively.

The approximation F_i of f_i^* is obtained by substituting the zeros $\zeta_1, \ldots, \zeta_n$ by their approximations $Z_1, \ldots, Z_n$. If we apply the substitution procedure taking better approximations (compared to Z_i) $Z_{N,i}$ or $Z_{H,i}$ in the sums $\Sigma_{1,i}$ and $\Sigma_{2,i}$, then we will obtain the better approximations $F_i(\boldsymbol{Z}_N, \boldsymbol{Z}_N)$ and $F_i(\boldsymbol{Z}_H, \boldsymbol{Z}_H)$ to f_i^*. In this way we obtain algorithms with the improved convergence. For example, using Newton's disks $Z_{N,j}$ we can construct

Total-step method with Newton's correction (TSN):

$$\widehat{Z}_i = z_i - \frac{\alpha + 1}{\alpha\delta_{1,i} + [(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - F_i(\boldsymbol{Z}_N, \boldsymbol{Z}_N)]_*^{1/2}} \quad (i \in I_n). \tag{TSN}$$

The *R*-order of convergence of the modified method (TSN) depends on the type of the inversion of a disk used in the calculation of $F_{N,i}$ (cf. [2,8,9]). It is equal to $2 + \sqrt{7} \cong 4.646$ if the exact inversion (1.1) is applied and 5 if we apply the centered inversion (1.2).

Further acceleration of the convergence rate can be achieved applying Gauss–Seidel approach to the inclusion methods with corrections. An extensive study of these improved methods, together with detailed convergence analysis, is given in the forthcoming paper.

It is worth noting that the increase of the convergence rate of the methods with corrections is obtained with negligible number of additional calculations (since $P(z_i)$, $P'(z_i)$, $P''(z_i)$ are already evaluated for all $i \in I_n$), which means that these methods possess very high computational efficiency.

We recall that the symbol $*$ in the above methods denotes the proper value of the square root.

6. Numerical results

To test the convergence properties of the presented inclusion methods from the family (TS), we applied these methods to polynomial equations of various degrees. In order to save all significant digits of the obtained approximations and to control the enclosure of the sought zeros, we implemented the corresponding algorithms on PC PENTIUM IV using the programming package *Mathematica* 5 with multiple precision arithmetic.

For comparison purpose, beside methods (3.1)–(3.4), we tested the method (TS) which is obtained for $\alpha = 1/2$:

$$\widehat{Z}_i = z_i - \frac{3}{\delta_{1,i} + \sqrt{2}[3\delta_{2,i} - \delta_{1,i}^2 - 3S_{2,i}(\mathbf{Z},\mathbf{Z}) + \frac{3}{2}S_{1,i}^2(\mathbf{Z},\mathbf{Z})]_*^{1/2}} \quad (i \in I_n), \tag{6.1}$$

and the following inclusion method of the order four:

$$\hat{Z}_i = z_i - \frac{1}{\dfrac{P'(z_i)}{P(z_i)} - \sum_{\substack{j=1\\ j\neq i}}^{n} (z_i - Z_j + N_j)^{I_c}} \quad (i \in I_n). \tag{6.2}$$

([2]) and

$$\hat{Z}_i = z_i - \frac{W(z_i)}{1 + \sum_{\substack{j=1\\ j\neq i}}^{n} W(z_j)(Z_i - W_i - z_j)^{I_c}} \quad (i \in I_n). \tag{6.3}$$

([9]).

Some authors consider that it is always better to apply more iterations of Weierstrass' method of the second order [6, Chapter 3]

$$\hat{Z}_i = z_i - P(z_i)\left(\prod_{\substack{j=1\\ j\neq i}}^{n} (z_i - Z_j)\right)^{-1} \quad (i \in I_n) \tag{6.4}$$

than any higher-order method. To check this opinion, we also employed this method.

The performed numerical experiments demonstrated very fast convergence of the inclusion method even in the case of relatively large initial disks. In all tested examples the choice of initial disks was carried out under a weaker condition than (4.1); moreover, the ratio $\rho^{(0)}/r^{(0)}$ was most frequently two, three or more times less than $4(n-1)$. We have selected two typical examples.

Example 1. Inclusion methods (3.1)–(3.4), (6.1)–(6.4) were applied for the simultaneous approximation to the zeros of the polynomial

$$P(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300.$$

The exact zeros of this polynomial are $-3, \pm 1, \pm 2\mathrm{i}, \pm 2\mathrm{i} \pm \mathrm{i}$. The initial disks were selected to be $Z_i^{(0)} = \{z_i^{(0)}; 0.3\}$ with the centers

$$z_1^{(0)} = -3.1 + 0.2\mathrm{i}, \quad z_2^{(0)} = -1.2 - 0.1\mathrm{i}, \quad z_3^{(0)} = 1.2 + 0.1\mathrm{i},$$
$$z_4^{(0)} = 0.2 - 2.1\mathrm{i}, \quad z_5^{(0)} = 0.2 + 1.9\mathrm{i}, \quad z_6^{(0)} = -1.8 + 1.1\mathrm{i},$$
$$z_7^{(0)} = -1.8 - 0.9\mathrm{i}, \quad z_8^{(0)} = 2.1 + 1.1\mathrm{i}, \quad z_9^{(0)} = 1.8 - 0.9\mathrm{i}.$$

The entries of the maximal radii of the disks produced in the first three iterations, for different values of α, are given in Table 1, where the denotation $A(-q)$ means $A \times 10^{-q}$.

Table 1
The maximal radii of inclusion disks

Methods	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(TS) $\alpha = 1$	1.96(−2)	5.32(−9)	7.95(−39)
(TS) $\alpha = 0.5$	1.45(−2)	7.13(−10)	4.64(−43)
(TS) $\alpha = \frac{1}{n-1}$	9.03(−3)	3.96(−10)	4.81(−42)
(TS) $\alpha = 0$	8.09(−3)	3.20(−10)	1.70(−40)
(TS) $\alpha = -1$	2.38(−2)	4.28(−8)	4.62(−34)
(6.2)	5.38(−2)	1.11(−5)	4.90(−23)
(6.3)	1.12(−2)	9.97(−9)	3.38(−34)
(6.4)	diverges	—	—

Example 2. The same interval methods from Example 1 were applied for the determination of the eigenvalues of Hessenberg's matrix H (see [11]). Gerschgorin's disks were taken as initial regions containing these eigenvalues. It is known that these disks are of the form $\{h_{ii}; R_i\}$ $(i = 1, \dots, n)$, where h_{ii} are the diagonal elements of a matrix $[h_{ij}]$ and $R_i = \sum_{j \neq i} |h_{ij}|$. If these disks are mutually disjoint, then each of them contains one and only one eigenvalue, which is very convenient for the application of inclusion methods.

The methods were tested in the example of the matrix

$$H = \begin{bmatrix} 2+3\mathrm{i} & 1 & 0 & 0 & 0 \\ 0 & 4+6\mathrm{i} & 1 & 0 & 0 \\ 0 & 0 & 6+9\mathrm{i} & 1 & 0 \\ 0 & 0 & 0 & 8+12\mathrm{i} & 1 \\ 1 & 0 & 0 & 0 & 10+15\mathrm{i} \end{bmatrix},$$

whose characteristic polynomial is

$$g(\lambda) = \lambda^5 - (30+45\mathrm{i})\lambda^4 + (-425+1020\mathrm{i})\lambda^3 + (10\,350 - 2025\mathrm{i})\lambda^2 \\ - (32\,606 + 32\,880\mathrm{i})\lambda - 14\,641 + 71\,640\mathrm{i}.$$

We selected Gerschgorin's disks

$$Z_1 = \{2+3\mathrm{i}; 1\}, \ Z_2 = \{4+6\mathrm{i}; 1\}, \ Z_3 = \{6+9\mathrm{i}; 1\}, \ Z_4 = \{8+12\mathrm{i}; 1\}, \ Z_5 = \{10+15\mathrm{i}; 1\}$$

to be initial disks containing the zeros of g, that is, the eigenvalues of H. The radii $r_i^{(m)} = \operatorname{rad} Z_i^{(m)}$ $(m=1,2)$ of the produced disks are displayed in Table 2.

As in Example 1, Weierstrass' method (6.4) diverged.

From Table 2 we observe that the applied inclusion methods converges very fast. The explanation for this extremely rapid convergence lies in the fact that the eigenvalues of Hessenberg's matrix are very close to the diagonal elements. For the closeness to the desired zeros, the centers of initial disks cause very fast convergence of the sequences of centers of inclusion disks, which provides fast convergence of radii.

Table 2
The maximal radii of inclusion disks containing the eigenvalues of Hessenberg's matrix

	(TS) $\alpha = 1$	(TS) $\alpha = 0.5$	(TS) $\alpha = 1/(n-1)$	(TS) $\alpha = 0$	(TS) $\alpha = -1$	(6.2)	(6.3)
$r^{(1)}$	2.73(−10)	2.39(−10)	2.21(−10)	2.04(−10)	2.73(−10)	5.64(−7)	3.27(−7)
$r^{(2)}$	4.92(−43)	3.65(−43)	3.02(−43)	2.38(−43)	2.73(−43)	1.71(−37)	1.60(−28)

7. Conclusions

We presented a new one-parameter family of iterative methods for the simultaneous inclusion of simple complex zeros of a polynomial. Computationally verifiable initial conditions, which provide the inclusion of zeros at each iteration as well as the convergence of the fourth-order right from the start, are stated. The characteristics and advantages of this family can be summarized as follows:

(1) the produced enclosed disks enable automatic determination of rigorous error bounds of the obtained approximations;

(2) the proposed family is of general type and includes previously derived methods of the square-root type;

(3) numerical examples demonstrate stable and fast convergence of the family (TS); furthermore, the methods of this family compete the existing inclusion methods of the fourth order (6.2) and (6.3), sometimes they produce tighter disks; moreover, numerical experiments show that a variation of the parameter α can often provide a better approaching to the wanted zeros compared to (6.2) and (6.3). See Examples 1 and 2.

(4) quadratically convergent Weierstrass' method (6.4) diverged not only in the displayed examples, but also in the case of numerous polynomial equations. This means that the application of Newton-like methods (like Weierstrass' method (6.4)) is not always better than higher-order methods.

(5) the order of convergence of the proposed family of methods is four; it can be significantly increased by suitable (already calculated) corrections with negligible number of operations attaining in this way a high computational efficiency.

(6) a slight modification of the fixed point relation, which served as the base for the construction of the considered algorithm, can provide the simultaneous inclusion of multiple zeros.

Algorithms with corrections (item (5)) and algorithms for the inclusion of multiple complex zeros (item (6)) will be considered in details in the forthcoming papers.

References

[1] G. Alefeld, J. Herzberger, Introduction to Interval Computations, Academic Press, New York, 1983.

[2] C. Carstensen, M.S. Petković, An improvement of Gargantini's simultaneous inclusion method for polynomial roots by Schroeder's correction, Appl. Numer. Math. 13 (1994) 453–468.

[3] I. Gargantini, Parallel Laguerre iterations: the complex case, Numer. Math. 26 (1976) 317–323.

[4] I. Gargantini, Further application of circular arithmetic: Schröder-like algorithms with error bound for finding zeros of polynomials, SIAM J. Numer. Anal. 15 (1978) 497–510.

[5] I. Gargantini, P. Henrici, Circular arithmetic and the determination of polynomial zeros, Numer. Math. 18 (1972) 305–320.

[6] M.S. Petković, Iterative Methods for Simultaneous Inclusion of Polynomial Zeros, Springer, Berlin, Heidelberg, New York, 1989.
[7] M.S. Petković, On Halley-like algorithms for the simultaneous approximation of polynomial complex zeros, SIAM J. Numer. Anal. 26 (1989) 740–763.
[8] M.S. Petković, Halley-like method with corrections for the inclusion of polynomial zeros, Computing 62 (1999) 69–88.
[9] M.S. Petković, C. Carstensen, On some improved inclusion methods for polynomial roots with Weierstrass' corrections, Comput. Math. Appl. 25 (1993) 59–67.
[10] M.S. Petković, Lj.D. Petković, Complex Interval Arithmetic and its Applications, Wiley-VCH, Berlin, Weinheim, New York, 1998.
[11] J. Stoer, R. Bulirsch, Einführung in die Numerische Mathematik II, Springer, Berlin, 1973.
[12] X. Wang, S. Zheng, Parallel Halley iteration method with circular arithmetic for finding all zeros of a polynomial, Numer. Math. J. Chinese Univ. 4 (1985) 308–314.
[13] X. Wang, S. Zheng, A family of parallel and interval iterations for finding simultaneously all roots of a polynomial with rapid convergence (I), J. Comput. Math. 4 (1984) 70–76.
[14] X. Wang, S. Zheng, G. Shen, Bell's disk polynomial and parallel disk iteration, Numer. Math. J. Chinese Univ. 4 (1987) 328–345.