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Cluster combinatorics of d-cluster categories ☆

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ABSTRACT

We study the cluster combinatorics of d-cluster tilting objects in d-cluster categories. Using mutations of maximal rigid objects in d-cluster categories, which are defined in a similar way to mutations for d-cluster tilting objects, we prove the equivalences between d-cluster tilting objects, maximal rigid objects and complete rigid objects. Using the chain of $d+1$ triangles of d-cluster tilting objects in [O. Iyama, Y. Yoshino, Mutations in triangulated categories and rigid Cohen–Macaulay modules, Invent. Math. 172 (1) (2008) 117–168], we prove that any almost complete d-cluster tilting object has exactly $d+1$ complements, compute the extension groups between these complements, and study the middle terms of these $d+1$ triangles. All results are the extensions of corresponding results on cluster tilting objects in cluster categories established for d-cluster categories in [A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204 (2006) 572–618]. They are applied to the Fomin–Reading generalized cluster complexes of finite root systems defined and studied in [S. Fomin, N. Reading, Generalized cluster complexes and Coxeter combinatorics, Int. Math. Res. Not. 44 (2005) 2709–2757; H. Thomas, Defining an m-cluster category, J. Algebra 318 (2007) 37–46; K. Baur, R. Marsh, A geometric description of m-cluster categories, Trans. Amer. Math. Soc. 360 (2008) 5789–5803; K. Baur, R. Marsh, A geometric description of the m-cluster categories of type D_n, preprint, arXiv:math.RT/0610512; see also Int. Math. Res. Not. 2007 (2007), doi:10.1093/imrn/rnm011], and to that of infinite root systems [B. Zhu, Generalized cluster complexes via quiver representations, J. Algebraic Combin. 27 (2008) 25–54].

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1. Introduction

Cluster categories are introduced by Buan, Marsh, Reineke, Reiten, Todorov [BMRRT] for a categorified understanding of cluster algebras introduced by Fomin and Zelevinsky in [FZ1,FZ2], see also [CCS] for type A_n. We refer [FZ3] for a survey on cluster algebras and their combinatorics, see also [FR1]. Cluster categories are the orbit categories $\mathcal{D}/\tau^{-1}[1]$ of derived categories of hereditary categories by the automorphism group $\langle\tau^{-1}[1]\rangle$ generated by the automorphism $\tau^{-1}[1]$. They are triangulated categories [Ke]. Cluster categories, on the one hand, provide a successful model for acyclic cluster algebras and their cluster combinatoric; see, for example, [BMRRT,BMR,CC,CK1,CK2,IR,Zh1,Zh2]; on the other hand, they replace module categories as a new generalization of the classical tilting theory, see, for example, [KR1,KR2,IY,KZ]. Cluster tilting theory and its combinatorics are the essential ingredients in the connection between quiver representations and cluster algebras, and have now become a new part of tilting theory in the representation theory of algebras; we refer to the surveys [BM,Rin,Re] and the references there for recent developments and background on cluster tilting theory.

Let H be a finite dimensional hereditary algebra over a field K with n non-isomorphic simple modules, and let $\mathcal{C}(H)$ be the corresponding cluster category. In a triangulated category, there are three possible kinds of rigid objects: cluster tilting (maximal 1-orthogonal in the sense of Iyama [I]), maximal rigid, and complete rigid. It is well known that they are not equivalent to each other in general [BIKR,KZ]. But in the cluster category $\mathcal{C}(H)$, they are equivalent [BMRRT]. Compared with classical tilting modules, cluster tilting objects in cluster categories have nice properties [BMRRT]. For example, any almost complete cluster tilting object in a cluster category can be completed to a cluster tilting object in exactly two ways, but in mod H, there are at most two ways to complete an almost complete basic tilting module. Moreover, the two complements M, M^* of an almost complete basic cluster tilting object $\bar{T}$ are connected by two triangles

$$M^* \longrightarrow B \longrightarrow M \longrightarrow M^*[1],$$

$$M \longrightarrow B' \longrightarrow M^* \longrightarrow M[1]$$

in $\mathcal{C}(H)$, where respectively, $B \longrightarrow M$ and $B' \longrightarrow M^*$ are minimal right $\operatorname{add}\bar{T}$-approximations of M and M^* in $\mathcal{C}(H)$. It follows that M and M^* satisfy the condition $\dim_{D_M} \operatorname{Ext}^1_{\mathcal{C}(H)}(M, M^*) = 1 = \dim_{D_{M^*}} \operatorname{Ext}^1_{\mathcal{C}(H)}(M^*, M)$, where D_M (or D_{M^*}) is the endomorphism division ring of M (resp. M^*). Conversely, if two indecomposable rigid objects M, M^* satisfy the condition above, one can find an almost complete cluster-tilting object $\bar{T}$ such that M and M^* are the two complements of $\bar{T}$. In this case, $\bar{T} \oplus M^*$ is called a mutation of $\bar{T} \oplus M$. Any two cluster-tilting objects are connected through mutations, provided that the ground field K is algebraically closed.

Keller [Ke] introduced d-cluster categories $\mathcal{D}/\tau^{-1}[d]$ as a generalization of cluster categories for $d \in \mathbf{N}$. They are studied recently in [Th,Zh3,BaM1,BaM2,KR1,KR2,IY,HoJ1,HoJ2,J,Pa,ABST,T,Wr]. d-cluster categories are triangulated categories with Calabi–Yau dimension $d+1$ [Ke]. When $d=1$, ordinary cluster categories are recovered.

The aim of this paper is to study the cluster tilting theory in d-cluster categories. It is motivated by two factors. First, since some properties of cluster tilting objects in cluster categories do not hold in general in this generalized setting (for example, the endomorphism algebras of d-cluster tilting objects are not again Goreistein algebras of dimension at most d in general [KR1]), one natural question is to see whether other properties of cluster tilting objects hold in d-cluster categories. Second, in [Zh3] we use d-cluster categories to define a generalized cluster complexes of the root systems of the corresponding Kac–Moddy Lie algebras (see also [BMRRT] and [Zh1] for a quiver approach of cluster complexes). When H is of finite representation type, these complexes are the same as those defined by Fomin and Reading [FR2] using the combinatorics of the root systems, see also [Th]. We need the combinatorial properties of d-cluster tilting objects for these generalized cluster complexes.

In [Zh3], the second author of this paper proved that any basic d-cluster tilting object in a d-cluster category $\mathcal{C}_d(H)$ contains exactly n indecomposable direct summands, where n is the number of non-isomorphic simple H-modules, and that the number of complements of an almost complete

d-cluster tilting object is at least $d+1$. The present article is a completion of the result from [Zh3] mentioned above. Furthermore, it can be viewed as a generalization to d-cluster categories of (almost) all the results for cluster categories in [BMRRT].

The paper is organized as follows: In Section 2, we recall and collect some notion and basic results needed in this paper. In Section 3, we prove that the d-cluster tilting objects in d-cluster categories are equivalent to the maximal rigid objects, and also to the complete rigid objects (i.e. rigid objects containing n non-isomorphic indecomposable direct summands, where n is the number of simple modules over the associated hereditary algebra). In the Dynkin case, this equivalence was proved in [Th] using the fact that every indecomposable object is rigid. In Section 4, we compare two chains of $d+1$ triangles, from [Zh3] and [IY] respectively, in order to prove that a basic almost complete d-cluster tilting object has exactly $d+1$ non-isomorphic complements, which are connected by these $d+1$ triangles. The extension groups between the complements of an almost complete d-cluster tilting object are computed explicitly, and a necessary and sufficient condition for $d+1$ indecomposable rigid objects to be the complements of an almost complete d-cluster tilting object is obtained in Section 5. In Section 6, for an almost complete d-cluster tilting object, the middle terms of the $d+1$ triangles which are connected by the $d+1$ complements are proved to contain no direct summands common to them all. In the final section, we give an application of the results proved in these previous sections to the generalized cluster complexes defined by Fomin and Reading [FR2], studied in [Th], and [Zh3], and show that all the main properties of these generalized cluster complexes of finite root system in [FR2,Th] hold also for the generalized cluster complexes of arbitrary root systems defined in [Zh3].

After completing and submitting this work, we saw Wralsen's paper [Wr] (arXiv:0712.2870). The fact that maximal d-rigid objects and d-cluster tilting objects coincide and that almost complete d-cluster tilting objects have $d+1$ complements, have also been proved independently in [Wr], with different proofs.

2. Basics on d-cluster categories

In this section, we collect some basic definitions and fix notation that we will use throughout the paper.

Let H be a finite dimensional hereditary algebra over a field K. We denote by $\mathcal{H}$ the category of finite dimensional modules over H. It is a hereditary abelian category [DR]. The subcategory of $\mathcal{H}$ consisting of isomorphism classes of indecomposable H-modules is denoted by $\operatorname{ind}\mathcal{H}$. The bounded derived category of $\mathcal{H}$ will be denoted by $D^b(H)$ or $\mathcal{D}$. We denote the non-isomorphic indecomposable projective representations in $\mathcal{H}$ by $P_1, \dots, P_n$, and the simple representations with dimension vectors $\alpha_1, \dots, \alpha_n$ by $E_1, \dots, E_n$. We use $D(-)$ to denote $\operatorname{Hom}_K(-, K)$ which is a duality operation in $\mathcal{H}$.

The derived category $\mathcal{D}$ has Auslander–Reiten triangles, and the Auslander–Reiten translate τ is an automorphism of $\mathcal{D}$. Fix a positive integer d, and denote by $F_d = \tau^{-1}[d]$, it is an automorphism of $\mathcal{D}$. The d-cluster category of H is defined in [Ke]; we denote by $\mathcal{D}/F_d$ the corresponding factor category. Its objects are by definition the F_d-orbits of objects in $\mathcal{D}$, and the morphisms are given by

$$\operatorname{Hom}_{\mathcal{D}/F_d}(\widetilde{X}, \widetilde{Y}) = \bigoplus_{i \in \mathbf{Z}} \operatorname{Hom}_{\mathcal{D}}\bigl(X, F_d^i Y\bigr).$$

Here X and Y are objects in $\mathcal{D}$, and $\widetilde{X}$ and $\widetilde{Y}$ are the corresponding objects in $\mathcal{D}/F_d$ (although we shall sometimes write such objects simply as X and Y).

Definition 2.1. (See [Ke,Th].) The orbit category $\mathcal{D}/F_d$ is called the d-cluster category of $\mathcal{H}$ (or of H), and is denoted by $\mathcal{C}_d(\mathcal{H})$, or sometimes by $\mathcal{C}_d(H)$.

By [Ke], the d-cluster category is a triangulated category with shift functor [1] induced by the shift functor in $\mathcal{D}$; the projection $\pi : \mathcal{D} \longrightarrow \mathcal{D}/F$ is a triangle functor. When $d = 1$, this orbit category is called the cluster category of $\mathcal{H}$, and denoted by $\mathcal{C}(\mathcal{H})$, or sometimes by $\mathcal{C}(H)$.

$\mathcal{H}$ is a full subcategory of $\mathcal{D}$ consisting of complexes concentrated in degree 0. Passing to $\mathcal{C}_d(\mathcal{H})$ by the projection π, $\mathcal{H}$ is a (possibly not full) subcategory of $\mathcal{C}_d(\mathcal{H})$, and $\mathcal{C}(\mathcal{H})$ is also a (possibly not full) subcategory of $\mathcal{C}_d(\mathcal{H})$. For any $i \in \mathbf{Z}$, we use $(\mathcal{H})[i]$ to denote the copy of $\mathcal{H}$ under the ith shift $[i]$, considered as a subcategory of $\mathcal{C}_d(\mathcal{H})$. Thus, $(\operatorname{ind}\mathcal{H})[i] = \{M[i] \mid M \in \operatorname{ind}\mathcal{H}\}$. For any object M in $\mathcal{C}_d(\mathcal{H})$, let $\operatorname{add} M$ denote the full subcategory of $\mathcal{C}_d(\mathcal{H})$ consisting of direct summands of direct sums of copies of M.

For $X, Y \in \mathcal{C}_d(\mathcal{H})$, we will use $\operatorname{Hom}(X, Y)$ to denote the Hom-space $\operatorname{Hom}_{\mathcal{C}_d(\mathcal{H})}(X, Y)$ in the d-cluster category $\mathcal{C}_d(\mathcal{H})$ throughout the paper. We define $\operatorname{Ext}^i(X, Y)$ to be $\operatorname{Hom}(X, Y[i])$.

We summarize some known facts about d-cluster categories [BMRRT,Ke], see also [Zh3].

Proposition 2.2.

1. *$\mathcal{C}_d(\mathcal{H})$ has Auslander–Reiten triangles and Serre functor $\Sigma = \tau[1]$, where τ is the AR-translate in $\mathcal{C}_d(\mathcal{H})$, induced from the AR-translate in $\mathcal{D}$.*
2. *$\mathcal{C}_d(\mathcal{H})$ is a Calabi–Yau category of CY-dimension $d+1$.*
3. *$\mathcal{C}_d(\mathcal{H})$ is a Krull–Remak–Schmidt category.*
4. $\operatorname{ind}\mathcal{C}_d(\mathcal{H}) = \bigcup_{i=0}^{i=d-1} (\operatorname{ind}\mathcal{H})[i] \cup \{P_j[d] \mid 1 \leqslant j \leqslant n\}$.

Proof. See [Zh3]. $\square$

Using Proposition 2.2, we can define the degree for every indecomposable object in $\mathcal{C}_d(\mathcal{H})$ as follows [Zh3]:

Definition 2.3. For any indecomposable object $X \in \mathcal{C}_d(\mathcal{H})$, we call the non-negative integer $\min\{k \in \mathbf{Z}_{\geqslant 0} \mid X \cong M[k]$ in $\mathcal{C}_d(\mathcal{H})$, for some $M \in \operatorname{ind}\mathcal{H}\}$ the degree of X, denoted by $\deg X$. If $\deg X = k$, $k = 0, \ldots, d-1$, we say that X is of color $k+1$; if $\deg X = d$, we say that X is of color 1.

By Proposition 2.2, any indecomposable object X of degree k is isomorphic to $M[k]$ in $\mathcal{C}_d(\mathcal{H})$, where M is an indecomposable representation in $\mathcal{H}$, $0 \leqslant \deg X \leqslant d$, X has degree d if and only if $X \cong P[d]$ in $\mathcal{C}_d(\mathcal{H})$ for some indecomposable projective object $P \in \mathcal{H}$, and X has degree 0 if and only if $X \cong M[0]$ in $\mathcal{C}_d(\mathcal{H})$ for some indecomposable object $M \in \mathcal{H}$. Here $M[0]$ denotes the object M of $\mathcal{H}$, considered as a complex concentrated in degree 0.

Now we recall the notion of d-cluster tilting objects from [KR1,Th,Zh3,IY]. This notion is equivalent to the "maximal d-orthogonal subcategories" of Iyama [I,IY].

Definition 2.4. Let $\mathcal{C}_d(\mathcal{H})$ be the d-cluster category.

1. An object X in $\mathcal{C}_d(\mathcal{H})$ is called rigid if $\operatorname{Ext}^i(X, X) = 0$, for all $1 \leqslant i \leqslant d$.
2. An object X in $\mathcal{C}_d(\mathcal{H})$ is called maximal rigid if it satisfies the property: $Y \in \operatorname{add} X$ if and only if $\operatorname{Ext}^i(X \oplus Y, X \oplus Y) = 0$ for all $1 \leqslant i \leqslant d$.
3. An object X in $\mathcal{C}_d(\mathcal{H})$ is called completely rigid if it contains exactly n non-isomorphic indecomposable direct summands.
4. An object X in $\mathcal{C}_d(\mathcal{H})$ is called d-cluster tilting if it satisfies the property that $Y \in \operatorname{add} X$ if and only if $\operatorname{Ext}^i(X, Y) = 0$ for all $1 \leqslant i \leqslant d$.
5. An object X in $\mathcal{C}_d(\mathcal{H})$ is called an almost complete d-cluster tilting if there is an indecomposable object Y with $Y \notin \operatorname{add} X$ such that $X \oplus Y$ is a d-cluster tilting object. Such Y is called a complement of the almost complete d-cluster tilting object.

For a basic d-cluster tilting object T in $\mathcal{C}_d(H)$, an indecomposable object $X_0 \in \operatorname{add} T$ and its complement X such that $X_0 \oplus X = T$, then there is a triangle in $\mathcal{C}_d(H)$:

$$X_1 \xrightarrow{g} B_0 \xrightarrow{f} X_0 \longrightarrow X_1[1],$$

where f is the minimal right add X-approximation of X_0 and g is the minimal left add X-approximation of X_1. It is easy to see that $T' := X_1 \oplus X$ is a basic d-cluster tilting object (compare [IY]). We call T' is a mutation of T in the direction of X_0. We call two d-cluster tilting objects T, T' mutation equivalent provided that there are finitely many d-cluster tilting objects T_1 $(= T), T_2, \dots,$ T_n $(= T')$ such that T_{i+1} is a mutation of T_i for any $1 \leqslant i \leqslant n-1$.

From the proof of Theorem 4.6 in [Zh3], we know that every d-cluster tilting object is mutation equivalent to a d-cluster tilting object in $\mathcal{H}[0]$.

The following results are proved in [Zh3].

Proposition 2.5.

1. *Any indecomposable rigid object X in $\mathcal{C}_d(\mathcal{H})$ is either of the form $M[i]$, where M is a rigid module (i.e. $\mathrm{Ext}^1_H(M, M) = 0$) in $\mathcal{H}$ and $0 \leqslant i \leqslant d-1$, or of the form $P_j[d]$ for some $1 \leqslant j \leqslant n$. In particular, if Γ is a Dynkin graph, then any indecomposable object in $\mathcal{C}_d(\mathcal{H})$ is rigid.*
2. *Suppose $d \geqslant 2$. Then $\mathrm{End}_{\mathcal{C}_d(H)}(X)$ is a division algebra for any indecomposable rigid object X.*
3. *Let $d \geqslant 2$ and $X = M[i]$, $Y = N[j]$ be indecomposable objects of degree i, j respectively in $\mathcal{C}_d(H)$. Suppose that $\mathrm{Hom}(X, Y) \neq 0$. Then one of the following holds:*
   (1) *We have $i = j$ or $j-1$ (provided $j \geqslant 1$);*
   (2) *We have $i = 0$, $i = d$ (and $M = P$) or $d-1$ (provided $j = 0$).*
4. *Let $d \geqslant 2$ and $M, N \in \mathcal{H}$. Then any non-split triangle between $M[0]$ and $N[0]$ in $\mathcal{C}_d(H)$ is induced from a non-split exact sequence between M and N in $\mathcal{H}$.*

3. Equivalence of *d*-cluster tilting objects and maximal rigid objects

The equivalence between cluster tilting objects and maximal rigid objects in cluster categories was proved in [BMRRT]. For d-cluster categories, in the simply laced Dynkin case, the equivalence of d-cluster tilting objects and maximal rigid objects is easily obtained because any indecomposable object is rigid (compare [Th]). We will now prove it for arbitrary d-cluster categories. From the proof of Theorem 4.6 in [Zh3], we know that every d-cluster tilting object is mutation equivalent to one in $\mathcal{H}[0]$. If there is a similar result for mutations of maximal rigid objects, then we can get the equivalence by the obvious equivalence between d-cluster tilting objects and maximal rigid objects in $\mathcal{H}[0]$ (both are tilting modules in mod H).

Lemma 3.1. *Let $d \geqslant 2$, $T = X \oplus X_0$ be a basic maximal rigid object in $\mathcal{C}_d(\mathcal{H})$ and X_0 an indecomposable object. Then there are $d+1$ triangles*

$$X_{i+1} \xrightarrow{g_i} T_i \xrightarrow{f_i} X_i \xrightarrow{\delta_i} X_{i+1}[1], \qquad (*)$$

where $T_i \in \mathrm{add}\, X$, f_i is the minimal right $\mathrm{add}\, X$-approximation of X_i, g_i is the minimal left $\mathrm{add}\, X$-approximation of X_{i+1}, all the $X \oplus X_i$ are maximal rigid objects, and all X_i are distinct up to isomorphisms for $i = 0, \dots, d$.

Proof. First we prove that there is a triangle

$$X_1 \xrightarrow{g_0} T_0 \xrightarrow{f_0} X_0 \xrightarrow{\delta_0} X_1[1],$$

where $T_0 \in \mathrm{add}\, X$, f_0 is the minimal right add X-approximation of X_0, g is the minimal left add X-approximation of X_1, and $X \oplus X_1$ is a maximal rigid object.

Let $T_0 \xrightarrow{f_0} X_0$ be the minimal right add X-approximation of X_0, and let

$$X_1 \xrightarrow{g_0} T_0 \xrightarrow{f_0} X_0 \xrightarrow{\delta_0} X_1[1] \qquad (1)$$

be the triangle into which f embeds. By the discussion in [BMRRT], one can easily check that g_0 is the minimal left $\operatorname{add} X$-approximation of X_1, X_1 is indecomposable and $X_1 \notin \operatorname{add} X$. By applying $\operatorname{Hom}(X, -)$ to the triangle, we have $\operatorname{Ext}^i(X, X_1) = 0$, for $1 \leqslant i \leqslant d$ (for $i = 1$, because f is the minimal right $\operatorname{add} X$-approximation of X_0). By applying $\operatorname{Hom}(X_0, -)$ to the triangle, we get $\operatorname{Ext}^i(X_0, X_0) \cong \operatorname{Ext}^{i+1}(X_0, X_1)$, for $1 \leqslant i \leqslant d-1$. By applying $\operatorname{Hom}(-, X_1)$ to the triangle, we have $\operatorname{Ext}^i(X_1, X_1) \cong \operatorname{Ext}^{i+1}(X_0, X_1)$, for $1 \leqslant i \leqslant d-1$. So $\operatorname{Ext}^i(X_1, X_1) \cong \operatorname{Ext}^i(X_0, X_0) = 0$ for $1 \leqslant i \leqslant d-1$. Since $\mathcal{C}_d(\mathcal{H})$ is a Calabi–Yau category of CY-dimension $d+1$, $\operatorname{Ext}^d(X_1, X_1) \cong D\operatorname{Ext}^1(X_1, X_1) = 0$. We claim that $X \oplus X_1$ is a maximal rigid object. If not, we have an indecomposable object $Y_1 \notin \operatorname{add}(X \oplus X_1)$, such that $X \oplus X_1 \oplus Y_1$ is a rigid object. Then we have a triangle

$$Y_1 \xrightarrow{\psi} T_1 \xrightarrow{\varphi} Y_0 \longrightarrow X_1[1], \tag{2}$$

where ψ is the minimal left $\operatorname{add} X$-approximation of Y_1. It is easy to prove that φ is the minimal right $\operatorname{add} X$-approximation of Y_0, $Y_0 \notin \operatorname{add} X$, and $\operatorname{Ext}^i(Y_0, X \oplus Y_0) = 0$ for $1 \leqslant i \leqslant d$. We will prove that $\operatorname{Ext}^i(Y_0, X_0) = 0$ for $1 \leqslant i \leqslant d$; then $Y_0 \cong X_0$ due to the fact that $X \oplus X_0$ is a maximal rigid object. By applying $\operatorname{Hom}(-, Y_1)$ to the first triangle, we have $0 = \operatorname{Ext}^i(X_1, Y_1) \cong \operatorname{Ext}^{i+1}(X_0, Y_1)$ for $1 \leqslant i \leqslant d-1$. By applying $\operatorname{Hom}(X_0, -)$ to the second triangle, we have $\operatorname{Ext}^i(X_0, Y_0) \cong \operatorname{Ext}^{i+1}(X_0, Y_1) = 0$ for $1 \leqslant i \leqslant d-1$. So we have $\operatorname{Ext}^i(X_0, Y_0) = 0$ for $1 \leqslant i \leqslant d-1$, and thus $\operatorname{Ext}^i(Y_0, X_0) = 0$ for $2 \leqslant i \leqslant d$. By applying $\operatorname{Hom}(-, X_1)$ to the second triangle, we have $0 = \operatorname{Ext}^1(Y_1, X_1) \cong \operatorname{Ext}^2(Y_0, X_1)$. By applying $\operatorname{Hom}(Y_0, -)$ to the first triangle, we have $\operatorname{Ext}^1(Y_0, X_0) \cong \operatorname{Ext}^2(Y_0, X_1) = 0$. So $\operatorname{Ext}^1(Y_0, X_0) = 0$. In all, $\operatorname{Ext}^i(Y_0, X_0) = 0$ for $1 \leqslant i \leqslant d$. Therefore $Y_0 \cong X_0$ which induces an isomorphism between the triangles (1) and (2). Then $Y_1 \cong X_1$, a contradiction. This proves that $X \oplus X_1$ is a maximal rigid object.

Second we repeat this process to get $d+1$ triangles

$$X_{i+1} \xrightarrow{g_i} T_i \xrightarrow{f_i} X_i \xrightarrow{\delta_i} X_{i+1}[1], \tag{$*$}$$

where $T_i \in \operatorname{add} X$, f_i is the minimal right $\operatorname{add} X$-approximation of X_i, g_i is the minimal left $\operatorname{add} X$-approximation of X_{i+1}, and all the $X \oplus X_i$ are maximal rigid objects.

Third it is easy to see that $\delta_d[d]\delta_{d-1}[d-1]\cdots\delta_1[1]\delta_0 \neq 0$ (similar as that in Corollary 4.5 in [Zh3]). In particular, $\operatorname{Hom}(X_i, X_j[j-i]) \neq 0$ and $X_i \ncong X_j$, $\forall 0 \leqslant i < j \leqslant d$. This finishes the proof. □

With the help of Lemma 3.1, one can define mutations of maximal rigid objects similar to those of d-cluster tilting objects: Let

$$X_{i+1} \xrightarrow{g_i} T_i \xrightarrow{f_i} X_i \xrightarrow{\delta_i} X_{i+1}[1]$$

be the ith triangle in Lemma 3.1. We say that each of the maximal rigid objects $X \oplus X_i$, for $i = 1, \ldots, d$, is a mutation of the maximal rigid object $X \oplus X_0$. A maximal rigid object T is mutation equivalent to a maximal rigid object T' provided that there are finitely many maximal rigid objects T_1 $(= T), T_2, \ldots, T_{n-1}, T_n$ $(= T')$ such that T_i is a mutation of T_{i-1} for any i.

Lemma 3.2. *Let $d \geqslant 2$, $T = X \oplus X_0$ be a maximal rigid object and X_0 be an indecomposable object. Then T is mutation equivalent to a maximal rigid object in $\mathcal{H}[0]$.*

Proof. In the proof of Theorem 4.6 in [Zh3], we proved that any d-cluster tilting object is mutation equivalent to a d-cluster tilting object in $\mathcal{H}[0]$. The same proof works here (with the help of Lemma 3.1), after replacing d-cluster tilting objects by maximal rigid objects. We omit the details and refer to the proof of Theorem 4.6 in [Zh3]. □

Now we prove the main result in this section.

Theorem 3.3. *Let X be a basic rigid object in the d-cluster category $\mathcal{C}_d(\mathcal{H})$. Then the following statements are equivalent:*

1. *X is a d-cluster tilting object.*
2. *X is a maximal rigid object.*
3. *X is a complete rigid object, i.e. it contains exactly n indecomposable summands.*

Proof. We suppose that $d > 1$; the same statement was proved for $d = 1$ in [BMRRT]. We prove that the first two conditions are equivalent. A d-cluster tilting object must be a maximal rigid object by definition. Now we assume X is a maximal rigid object. Then X is mutation equivalent to a maximal rigid object $T'[0]$ in $\mathcal{H}[0]$ by Lemma 3.2. We have that $\mathrm{Ext}^k(T'[0], T'[0]) \cong \mathrm{Ext}^k_{\mathcal{D}}(T'[0], T'[0]) \cong \mathrm{Ext}^k_{\mathcal{H}}(T', T')$, $k = 1, \ldots, d-1$, and $\mathrm{Ext}^d(T'[0], T'[0]) \cong D\,\mathrm{Ext}(T'[0], T'[0]) \cong D\,\mathrm{Ext}_{\mathcal{H}}(T', T')$. So T' is a maximal rigid module in $\mathcal{H}$. Hence T' is a tilting module, and thus $T'[0]$ is a d-cluster tilting object. Therefore T is a d-cluster tilting object, since it is mutation equivalent to the d-cluster tilting object $T'[0]$.

Now we prove that the last two conditions are equivalent. In [Zh3], we know that every basic d-cluster tilting object has exactly n indecomposable summands. Conversely, any basic rigid object with n indecomposable summands will be a basic maximal rigid object, since otherwise it can be extended to a basic maximal rigid object that contains at least $n+1$ indecomposable summands. This is a contradiction. □

This theorem immediately yields the following important conclusion.

Corollary 3.4. *Let X be a rigid object in $\mathcal{C}_d(\mathcal{H})$. Then there exists an object Y such that $X \oplus Y$ is a d-cluster tilting object.*

4. Complements of almost complete basic *d*-cluster tilting objects

The number of complements of an almost complete cluster tilting object in a cluster category $\mathcal{C}(\mathcal{H})$ is exactly two [BMRRT]. From Corollary 4.5 in [Zh3], we know that the number of complements of an almost complete d-cluster tilting object is at least $d+1$. In this section, we will prove it is exactly $d+1$.

Let $T = X \oplus X_0$ be a basic d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$, and X an almost complete d-cluster tilting object. By Theorem 4.4 in [Zh3] and Theorem 3.10 in [IY], we have the following two chains of $d+1$ triangles:

$$X_{i+1} \xrightarrow{g_i} B_i \xrightarrow{f_i} X_i \xrightarrow{\delta_i} X_{i+1}[1], \tag{$*$}$$

where for $i = 0, 1, \ldots, d$, $B_i \in \operatorname{add} X$, the map f_i is the minimal right $\operatorname{add} X$-approximation of X_i and g_i is the minimal left $\operatorname{add} X$-approximation of X_{i+1}.

$$X'_{i+1} \xrightarrow{b_i} C_i \xrightarrow{a_i} X'_i \xrightarrow{c_i} X'_{i+1}[1], \tag{$**$}$$

where for $i = 0, 1, \ldots, d$, $C_i \in \operatorname{add} T$, the map a_i is the minimal right $\operatorname{add} T$-approximation of X'_i (except a_0, which is the sink map of X'_0 in $\operatorname{add} T$) and b_i is the minimal left $\operatorname{add} T$-approximation of X'_{i+1} (except b_d, which is the source map of X'_d in $\operatorname{add} T$), and $X'_0 = X'_{d+1} = X_0$.

In [IY], the authors show that $X_0 \notin \operatorname{add}(\bigoplus_{0 \leqslant i \leqslant d} C_i)$ is a sufficient condition for an almost complete d-cluster tilting object to have exactly $d+1$ complements. The main aim of this section is to prove that $B_i = C_i$ for all $0 \leqslant i \leqslant d$, which implies this sufficient condition. We will first study the properties of the degree of an indecomposable object in $\mathcal{C}_d(\mathcal{H})$ which is a useful tool for studying rigid objects in d-cluster categories.

Lemma 4.1.
Let X_i, $0 \leqslant i \leqslant d$, be the objects appearing in the triangles in $(*)$. *If* $\deg X_0 = 0$, *then*

(1) $\deg X_1 = 0$, *d or $d-1$, and*
(2) $\deg X_i \geqslant d - i$, *for any $2 \leqslant i \leqslant d$.*

Proof. (1) We have the fact that $\mathrm{Hom}(X_0, X_1[1]) = \mathrm{Ext}(X_0, X_1) \neq 0$. If $0 < \deg X_1 < d-1$ (which implies $d \geqslant 3$), then $2 \leqslant \deg X_1[1] \leqslant d-1$ and $\mathrm{Hom}(X_0, X_1[1]) = 0$ by Proposition 2.5(3). This is a contradiction.

(2) If $\deg X_1 = 0$, then $\deg X_2 = d$ or $d-1$ or $d-2$ (because X_0, X_1, X_2 cannot have the same degree by the proof of Theorem 4.6 in [Zh3]). Now we prove the assertion that $\deg X_{i+1} \geqslant d-(i+1)$ provided that $\deg X_i \geqslant d-i$ for some i ($1 \leqslant i \leqslant d-1$). If $\deg X_{i+1} < d-(i+1)$, then $1 \leqslant \deg X_{i+1}[1] < d-i$, which implies $d \geqslant 2$, and then $\mathrm{Hom}(X_i, X_{i+1}[1]) = 0$ by Proposition 2.5. This contradicts the fact $\mathrm{Ext}(X_i, X_{i+1}) \neq 0$. So by induction on i, we get the statement (2). □

Lemma 4.2. *Let $d \geqslant 2$ and $X = M[i]$, $Y = N[j]$ be indecomposable objects of degree i, j respectively in $\mathcal{C}_d(\mathcal{H})$. Suppose that $0 \leqslant j + k - i \leqslant d - 1$. Then*

(1) $\mathrm{Hom}(X, Y[k]) \cong \mathrm{Hom}_{\mathcal{D}}(X, Y[k])$, *and*
(2) $\mathrm{Hom}(X, \tau^{-1}Y[k]) \cong \mathrm{Hom}_{\mathcal{D}}(X, \tau^{-1}Y[k])$.

Proof. (1) $\mathrm{Hom}(X, Y[k]) = \bigoplus_{l \in \mathbf{Z}} \mathrm{Hom}_D(X, \tau^{-l}Y[k+ld])$.

When $l \geqslant 1$, $\mathrm{Hom}_{\mathcal{D}}(X, \tau^{-l}Y[k+ld]) \cong \mathrm{Hom}_{\mathcal{D}}(\tau^l M, N[k+ld-i+j]) = 0$, since $k+ld-i+j \geqslant ld \geqslant 2$.

When $l \leqslant -1$, $\mathrm{Hom}_{\mathcal{D}}(X, \tau^{-l}Y[k+ld]) \cong D\,\mathrm{Hom}_{\mathcal{D}}(\tau^{-l-1}N, M[-k-ld+i-j+1]) = 0$, since $-l-1 \geqslant 0$ and $-k-ld+i-j+1 \geqslant 2-(l+1)d \geqslant 2$.

It follows that $\mathrm{Hom}(X, Y[k]) \cong \mathrm{Hom}_{\mathcal{D}}(X, Y[k])$.

(2) $\mathrm{Hom}(X, \tau^{-1}Y[k]) = \bigoplus_{l \in \mathbf{Z}} \mathrm{Hom}_D(X, \tau^{-l-1}Y[k+ld])$.

When $l \geqslant 1$, $\mathrm{Hom}_{\mathcal{D}}(X, \tau^{-l-1}Y[k+ld]) \cong \mathrm{Hom}_{\mathcal{D}}(\tau^{l+1}M, N[k+ld-i+j]) = 0$, since $l+1 \geqslant 2$ and $k+ld-i+j \geqslant ld \geqslant 2$.

When $l = -1$, $\mathrm{Hom}_D(X, \tau^{-l-1}Y[k+ld]) = \mathrm{Hom}_D(M, N[k-d-i+j]) = 0$, since $k-d-i+j \leqslant -1$.

When $l \leqslant -2$, $\mathrm{Hom}_{\mathcal{D}}(X, \tau^{-l-1}Y[k+ld]) \cong D\,\mathrm{Hom}_{\mathcal{D}}(\tau^{-l-2}N, M[-k-ld+i-j+1]) = 0$, since $-l-2 \geqslant 0$ and $-k-ld+i-j+1 \geqslant 2-(l+1)d \geqslant 2$.

It follows that $\mathrm{Hom}(X, \tau^{-1}Y[k]) \cong \mathrm{Hom}_{\mathcal{D}}(X, \tau^{-1}Y[k])$. □

For convenience, we add a triangle below to the triangle chains $(*)$:

$$X_0 \xrightarrow{g_{-1}} B_{-1} \xrightarrow{f_{-1}} X_{-1} \xrightarrow{\delta_{-1}} X_0[1],$$

where f_{-1} is the right $\mathrm{add}\,X$-approximation and g_{-1} is the left $\mathrm{add}\,X$-approximation. Now we prove the main theorem in this section.

Theorem 4.3. *Let $d \geqslant 2$, $T = X \oplus X_0$ be a basic d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$, and X an almost complete d-cluster tilting object. Then there are exactly $d+1$ complements $\{X_i\}_{0 \leqslant i \leqslant d}$ of X, which are connected by the $d+1$ triangles* $(*)$.

Proof. The main step in the proof is to show that $X_0 \notin \mathrm{add}\,C_i$ for $0 \leqslant i \leqslant d$.

For $i = 0$ or $i = d$, since f_0 is the minimal right $\mathrm{add}\,X$-approximation of X_0 and $\mathrm{End}\,X_0$ is a division ring, for any map $h \in \mathrm{Hom}(T', X_0)$ that is not a retraction, where T' is some object in $\mathrm{add}\,T$, there exists $h' \in \mathrm{Hom}(T', B_0)$ such that $h = f_0 h'$. Therefore, f_0 is a sink map in $\mathrm{add}\,T$. By the uniqueness of the sink map, we get $C_0 \cong B_0$, $X_1 \cong X'_1$ and, dually $C_d \cong B_{-1}$, $X_{-1} \cong X'_d$. So $X_0 \notin \mathrm{add}\,C_0$ and $X_d \notin \mathrm{add}\,C_d$.

For $1 \leqslant i \leqslant d-2$ (this implies $d \geqslant 3$), if $i=1$, by applying $\mathrm{Hom}(X_0,-)$ to the triangle $X_2 \longrightarrow B_1 \longrightarrow X_1 \longrightarrow X_2[1]$, we have the exact sequence

$$\mathrm{Hom}(X_0, B_1) \longrightarrow \mathrm{Hom}(X_0, X_1) \longrightarrow \mathrm{Ext}(X_0, X_2) \longrightarrow 0.$$

We need to prove $\mathrm{Ext}(X_0, X_2) = 0$. If not, i.e. $\mathrm{Ext}(X_0, X_2) \neq 0$, then $\mathrm{Hom}(X_0, X_1) \neq 0$. Similarly, by applying $\mathrm{Hom}(-, X_2)$ to the triangle $X_1 \longrightarrow B_0 \longrightarrow X_0 \longrightarrow X_1[1]$, we have the exact sequence

$$\mathrm{Hom}(X_1, X_2) \longrightarrow \mathrm{Ext}(X_0, X_2) \longrightarrow 0,$$

so $\mathrm{Ext}(X_0, X_2) \neq 0$ implies $\mathrm{Hom}(X_1, X_2) \neq 0$. We know that $\mathrm{Ext}(X_0, X_1) \neq 0$ and $\mathrm{Ext}(X_1, X_2) \neq 0$. We may assume that the degree of X_0 is 0; then $\deg X_1 = 0$, d or $d-1$ by Lemma 4.1. But $\mathrm{Hom}(X_0, X_1) \neq 0$ implies that the degree of X_1 is not d or $d-1$, so it is 0. For the same reason, $\deg X_2 = 0$, which contradicts the fact that X_0, X_1, and X_2 do not all have the same degree (refer to the proof of Theorem 4.6 in [Zh3]).

If $2 \leqslant i \leqslant d-2$, then by applying $\mathrm{Hom}(X_0,-)$ to the triangle $X_{i+1} \longrightarrow B_i \longrightarrow X_i \longrightarrow X_{i+1}[1]$, we get the exact sequence

$$\mathrm{Hom}(X_0, B_i) \longrightarrow \mathrm{Hom}(X_0, X_i) \longrightarrow \mathrm{Ext}(X_0, X_{i+1}) \longrightarrow 0.$$

We want to prove that $\mathrm{Hom}(X_0, X_i) = 0$, which implies $\mathrm{Ext}(X_0, X_{i+1}) = 0$. We also assume that the degree of X_0 is 0. Since $\deg X_i \geqslant d-i \geqslant 2$ by Lemma 4.1, it follows that $\mathrm{Hom}(X_0, X_i) = 0$. So $\mathrm{Ext}(X_0, X_{i+1}) = 0$, and it follows that f_i is the minimal right $\mathrm{add}\, T$-approximation of X_i. By the uniqueness of the minimal approximation map, since $X_1 \cong X_1'$, we get $C_i \cong B_i$ and $X_{i+1} \cong X_{i+1}'$ for $1 \leqslant i \leqslant d-2$, so $X_0 \notin \mathrm{add}(\bigoplus_{1 \leqslant i \leqslant d-2} C_i)$.

For $i = d-1 \geqslant 1$ (which implies $d \geqslant 2$), we claim that in the triangle $X_d \xrightarrow{g_{d-1}} B_{d-1} \xrightarrow{f_{d-1}} X_{d-1} \longrightarrow X_d[1]$, the morphism f_{d-1} is the minimal right $\mathrm{add}(X \oplus X_0)$-approximation of X_{d-1}, which is equivalent to the fact that $\mathrm{Ext}(X_0, X_d) = 0$. Suppose that $\deg X_0 = 0$ and $\deg X_1 \neq 0$ (if $\deg X_0 = \deg X_1 = 0$, then $\deg X_2 \neq 0$, and we can replace X_0 by X_1). From Lemma 4.1(2), $\deg X_{d-1} \geqslant 1$. If $\deg X_{d-1} = 1$, then $\deg X_d = 1$ or 0 since $\mathrm{Hom}(X_{d-1}, X_d[1]) \neq 0$. So we divide the calculation of $\mathrm{Ext}(X_0, X_d)$ into three cases:

1. The case $\deg X_{d-1} \geqslant 2$. Then by Proposition 2.1(3) $\mathrm{Hom}(X_0, X_{d-1}) = 0$, which implies $\mathrm{Ext}(X_0, X_d) = 0$.
2. The case $\deg X_{d-1} = 1$ and $\deg X_d = 1$. By applying $\mathrm{Hom}(X_0,-)$ to the triangle $X_d \longrightarrow B_{d-1} \longrightarrow X_{d-1} \xrightarrow{\delta_{d-1}} X_d[1]$ we get the exact sequence
$$\mathrm{Hom}(X_0, X_{d-1}) \xrightarrow{\delta_{d-1}^*} \mathrm{Hom}\big(X_0, X_d[1]\big) \longrightarrow 0,$$
where $\delta_{d-1} \in \mathrm{Hom}(X_{d-1}, X_d[1]) \cong \mathrm{Hom}_{\mathcal{D}}(X_{d-1}, X_d[1])$ by Lemma 3.2. For any $\varphi \in \mathrm{Hom}(X_0, X_{d-1}) \cong \mathrm{Hom}_{\mathcal{D}}(X_0, X_{d-1})$, by Lemma 4.2, we have $\delta_{d-1}^*(\varphi) = \delta_{d-1}\varphi \in \mathrm{Hom}_{\mathcal{D}}(X_0, X_d[1]) = 0$. So $\delta_{d-1}^* = 0$. Thus $\mathrm{Ext}(X_0, X_d) = 0$.
3. The case $\deg X_{d-1} = 1$ and $\deg X_d = 0$. Consider the triangle $X_d' \longrightarrow C_{d-1} \longrightarrow X_{d-1}' \longrightarrow X_d'[1]$. Since $X_{-1} \cong X_d'$ and $X_{d-1} \cong X_{d-1}'$, the triangle is $X_{-1} \longrightarrow C_{d-1} \longrightarrow X_{d-1} \longrightarrow X_{-1}[1]$, where $C_{d-1} \in \mathrm{add}(X \oplus X_0)$. Analogously, we get a triangle
$$X_0 \longrightarrow Y \longrightarrow X_d \longrightarrow X_0[1],$$
where $Y \in \mathrm{add}(X \oplus X_1)$. Since $\deg X_0 = \deg X_d = 0$, then the degree of the indecomposable summands of Y is zero. But $\deg X_1 \neq 0$, so $X_1 \notin Y$, that is, $Y \in \mathrm{add}\, X$. By applying $\mathrm{Hom}(X_0,-)$ to the triangle above, we get the exact sequence

$$\mathrm{Ext}(X_0, Y) \longrightarrow \mathrm{Ext}(X_0, X_d) \longrightarrow \mathrm{Ext}^2(X_0, X_0) \longrightarrow X_0[1],$$

so $\mathrm{Ext}(X_0, X_d) = 0$ since $X_0 \oplus X$ is a d-cluster tilting object.

Then $C_{d-1} \cong B_{d-1}$ so $X_0 \notin \operatorname{add} C_{d-1}$.

In all, $X_0 \notin \operatorname{add}(\bigoplus_{0 \leqslant i \leqslant d} C_i)$, which satisfies the condition of Corollary 5.9 in [IY]. Therefore, X has exactly $d+1$ complements in $\mathcal{C}_d(\mathcal{H})$. $\square$

As a consequence of the proof of the theorem above, we have

Corollary 4.4. *The corresponding triangles in the chains* $(*)$ *and* $(**)$ *are isomorphic.*

Let $d \geqslant 2$. For a (basic) d-cluster tilting object $T = X \oplus X_0$ in $\mathcal{C}_d(\mathcal{H})$ with an almost complete d-cluster tilting object X, and for any i between 0 and d, the triangle

$$X_{i+1} \xrightarrow{g_i} B_i \xrightarrow{f_i} X_i \xrightarrow{\delta_i} X_{i+1}[1]$$

in $(*)$ is called the ith connecting triangle of the complements of X with respect to X_0. These $d+1$ triangles form a $d+1$-Auslander–Reiten triangle starting at X_0 (see [IY]).

Similar to the cluster categories in [BMRRT], one can associate to $\mathcal{C}_d(\mathcal{H})$ a mutation graph of d-cluster tilting objects: the vertices are the basic d-cluster tilting objects, and there is an edge between two vertices if the corresponding two basic d-cluster tilting objects in $\mathcal{C}_d(\mathcal{H})$ have all but one indecomposable summand in common. Exactly as in [BMRRT], we obtain the conclusion below, which means that over an algebraically closed field, any two d-cluster tilting objects in $\mathcal{C}_d(\mathcal{H})$ can be connected by a series of mutations.

Proposition 4.5. *Let K be an algebraically closed field. Given an indecomposable hereditary k-algebra H, the associated mutation graph of d-cluster tilting objects in $\mathcal{C}_d(\mathcal{H})$ is connected.*

5. Relations of complements

Let $T = X \oplus X_0$ be a basic d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$. The almost complete d-cluster object X has exactly $d+1$ complements X_i, $0 \leqslant i \leqslant d$, as shown in Theorem 4.3. When $d = 1$, the extension groups of between X_0 and X_1 were computed in [BMRRT]. In this section we will compute $\mathrm{Ext}^k(X_i, X_j)$. Throughout this section, we assume $d \geqslant 2$, and X is a basic almost complete d-cluster tilting object, the $d+1$ complements $X_0, \dots, X_d$ of X are connected by the $d+1$ triangles in $(*)$ in Section 4:

$$X_{i+1} \xrightarrow{g_i} B_i \xrightarrow{f_i} X_i \xrightarrow{\delta_i} X_{i+1}[1], \qquad (*)$$

where for $i = 0, 1, \dots, d$, $B_i \in \operatorname{add} X$, f_i is the minimal right $\operatorname{add} X$-approximation of X_i and g_i is the minimal left $\operatorname{add} X$-approximation of X_{i+1}.

Lemma 5.1. $\mathrm{Ext}^i(X_0, X_i) \cong \mathrm{Ext}(X_0, X_1) \cong \mathrm{End}_{\mathcal{H}}(X_0)$, *and* $\mathrm{Ext}^k(X_0, X_i) = 0$ *for* $1 \leqslant i \leqslant d$, *and* $k \in \{1, \dots, d\} \setminus \{i\}$.

Proof. By applying $\mathrm{Hom}(X_0, -)$ to the triangles $(*)$ we get the long exact sequences

$$\mathrm{Ext}^k(X_0, B_i) \longrightarrow \mathrm{Ext}^k(X_0, X_i) \longrightarrow \mathrm{Ext}^{k+1}(X_0, X_{i+1}) \longrightarrow \mathrm{Ext}^{k+1}(X_0, B_i),$$

where $i = 0, 1, \dots, d$, and $k = 1, 2, \dots, d-1$. Since $\mathrm{Ext}^k(X_0, B_i) = 0$ for $0 \leqslant i \leqslant d$ and $1 \leqslant k \leqslant d$, we have $\mathrm{Ext}^k(X_0, X_i) \cong \mathrm{Ext}^{k+1}(X_0, X_{i+1})$ for $0 \leqslant i \leqslant d$ and $1 \leqslant k \leqslant d-1$. So $\mathrm{Ext}^{i+1}(X_0, X_{i+1}) \cong$

$\mathrm{Ext}^i(X_0, X_i)$, for $1 \leqslant i \leqslant d-1$. Hence we get the left equation by induction on i. Applying $\mathrm{Hom}(X_0, -)$ to the triangle $X_1 \longrightarrow B_0 \longrightarrow X_0 \xrightarrow{\delta_0} X_1[1]$ induces the exact sequence

$$\mathrm{Hom}(X_0, X_0) \xrightarrow{\delta_0^*} \mathrm{Ext}(X_0, X_1) \longrightarrow 0.$$

Since $\mathrm{Hom}(X_0, X_0)$ is a division algebra for $d \geqslant 2$, it follows that $\delta_0^*(\varphi) = \delta_0\varphi$ is non-zero for any non-zero map φ in $\mathrm{End}\, X_0$, which must therefore be an isomorphism of X_0. Then δ_0^* is a monomorphism and hence an isomorphism. This gives the first part of the lemma.

For the second part, if $i < k$, we have $\mathrm{Ext}^k(X_0, X_i) \cong \mathrm{Ext}^{k-1}(X_0, X_{i-1}) \cong \cdots \cong \mathrm{Ext}^{k-i}(X_0, X_0) = 0$, since $0\langle k-i < d+1$, and if $i\rangle k$, we have $\mathrm{Ext}^k(X_0, X_i) \cong \mathrm{Ext}^{k+1}(X_0, X_{i+1}) \cong \cdots \cong \mathrm{Ext}^{k+d+1-i}(X_0, X_{d+1}) = \mathrm{Ext}^{k+d+1-i}(X_0, X_0) = 0$, since $0 < k+d+1-i < d+1$. □

Lemma 5.2. $\mathrm{End}\, X_i \cong \mathrm{End}\, X_0$ *as algebras, for* $0 \leqslant i \leqslant d$.

Proof. We only need to prove the ring isomorphism $\mathrm{End}\, X_1 \cong \mathrm{End}\, X_0$, since the others are done by induction. It is exactly the same as the proof of the case $d = 1$ in [BMRRT]. □

Lemma 5.3.

$$\dim_{\mathrm{End}\, X_i} \mathrm{Ext}^k(X_i, X_j) = \begin{cases} 1 & \text{if } i+k-j = 0 \bmod (d+1), \\ 0 & \text{otherwise}, \end{cases}$$

for $0 \leqslant k \leqslant d$. *If we fix an* $\mathrm{End}\, X_i$*-basis* $\{\delta_i\}$ *of* $\mathrm{Ext}^1(X_i, X_{i+1})$, *then for any* $0 \leqslant i \leqslant d$ *and* $0 \leqslant k \leqslant d$, $\mathrm{Ext}^k(X_i, X_{i+k})$ *has an* $\mathrm{End}(X_i)$*-basis* $\{\delta_{i+k}[k]\cdots\delta_{i+1}[1]\delta_i\}$, *where* $X_{i+k} = X_{i+k-(d+1)}$ *and* $\delta_{i+k} = \delta_{i+k-(d+1)}$, *for* $i+k > d$.

Proof. The case of $i = 0$ of the first part follows easily from the two lemmas above, and the case for arbitrary i follows from the same proof after replacing 0 by i. For the second part, it is easy to see that any morphisms $\delta_{i+k}[k]\cdots\delta_{i+1}[1]\delta_i$ are non-zero in $\mathrm{Ext}^k(X_i, X_{i+k})$, hence form a basis over $\mathrm{End}\, X_i$ of $\mathrm{Ext}^k(X_i, X_{i+k})$. □

Definition 5.4. A set of $d+1$ indecomposable objects $X_0, X_1, \ldots, X_d$ in $\mathcal{C}_d(\mathcal{H})$ is called an exchange team if they satisfy Lemma 5.3, i.e.

$$\dim_{\mathrm{End}\, X_i} \mathrm{Ext}^k(X_i, X_j) = \begin{cases} 1 & \text{if } i+k-j = 0 \bmod (d+1), \\ 0 & \text{otherwise}, \end{cases}$$

for $0 \leqslant k \leqslant d$. If we fix an $\mathrm{End}\, X_i$-basis $\{\delta_i\}$ of $\mathrm{Ext}^1(X_i, X_{i+1})$, then for any $0 \leqslant i \leqslant d$ and $0 \leqslant k \leqslant d$, $\mathrm{Ext}^k(X_i, X_{i+k})$ has an $\mathrm{End}\, X_i$-basis $\{\delta_{i+k}[k]\cdots\delta_{i+1}[1]\delta_i\}$, where $X_{i+k} = X_{i+k-(d+1)}$ and $\delta_{i+k} = \delta_{i+k-(d+1)}$, for $i+k > d$.

This is a generalization of the notation of exchange pairs in cluster categories, defined in [BMRRT]. Given an exchange team $\{X_i\}_{i=0}^d$, by definition we can find $d+1$ non-split triangles

$$X_{i+1} \xrightarrow{g_i} B_i \xrightarrow{f_i} X_i \longrightarrow X_{i+1}[1] \qquad (***)$$

in $\mathcal{C}_d(\mathcal{H})$, where we use the same notation as before. We will now start to prove that $B = \bigoplus_{0 \leqslant i \leqslant d} B_i$ is a rigid object.

Lemma 5.5. *With the notation above, we have*

$$\operatorname{Ext}^k(B \oplus X_i, B \oplus X_i) = 0,$$

for all $1 \leqslant k \leqslant d$ *and* $0 \leqslant i \leqslant d$.

Proof. Apply $\operatorname{Hom}(X_0, -)$ to the triangle $X_1 \longrightarrow B_0 \longrightarrow X_0 \xrightarrow{\delta_0} X_1[1]$ to get the exact sequence

$$\operatorname{Hom}(X_0, X_0) \xrightarrow{\alpha} \operatorname{Ext}(X_0, X_1) \longrightarrow \operatorname{Ext}(X_0, B_0) \longrightarrow \operatorname{Ext}(X_0, X_0).$$

Since $\alpha \neq 0$ ($\alpha(1_{X_0}) = \delta_0 \neq 0$) and $\dim_{\operatorname{End}(X_0)} \operatorname{Ext}(X_0, X_1) = 1$, while $\operatorname{Ext}(X_0, X_0) = 0$ by assumption, it follows that $\operatorname{Ext}(X_0, B_0) = 0$. By assumption, $\operatorname{Ext}^k(X_0, X_1) = 0$ and $\operatorname{Ext}^k(X_0, X_0) = 0$ for any $2 \leqslant k \leqslant d$, so it follows that $\operatorname{Ext}^k(X_0, B_0) = 0$ for any $2 \leqslant k \leqslant d$. Hence $\operatorname{Ext}^k(X_0, B_0) = 0$, for $1 \leqslant k \leqslant d$.

Apply $\operatorname{Hom}(X_0, -)$ to the triangle $X_{i+1} \xrightarrow{g_i} B_i \xrightarrow{f_i} X_i \longrightarrow X_{i+1}[1]$ to get the exact sequence

$$\begin{array}{l}
\longrightarrow \operatorname{Ext}(X_0, X_{i+1}) \longrightarrow \operatorname{Ext}(X_0, B_i) \longrightarrow \operatorname{Ext}(X_0, X_i) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\longrightarrow \operatorname{Ext}^i(X_0, X_{i+1}) \longrightarrow \operatorname{Ext}^i(X_0, B_i) \longrightarrow \operatorname{Ext}^i(X_0, X_i) \\
\longrightarrow \operatorname{Ext}^{i+1}(X_0, X_{i+1}) \longrightarrow \operatorname{Ext}^{i+1}(X_0, B_i) \longrightarrow \operatorname{Ext}^{i+1}(X_0, X_i) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\longrightarrow \operatorname{Ext}^d(X_0, X_{i+1}) \longrightarrow \operatorname{Ext}^d(X_0, B_i) \longrightarrow \operatorname{Ext}^d(X_0, X_i).
\end{array}$$

$\operatorname{Ext}^i(X_0, X_i) \longrightarrow \operatorname{Ext}^{i+1}(X_0, X_{i+1})$ is an isomorphism (because $f \in \operatorname{Ext}^{i+1}(X_0, X_{i+1})$ can be decomposed), and $\operatorname{Ext}^k(X_0, X_{i+1}) = 0 = \operatorname{Ext}^l(X_0, X_i)$ for $k \neq i+1$ and $l \neq i$, so $\operatorname{Ext}^k(X_0, B_i) = 0$ for any $1 \leqslant k \leqslant d$. Analogously, we get $\operatorname{Ext}^k(X_i, B_j) = 0$ for all $1 \leqslant k \leqslant d$ and $0 \leqslant i, j \leqslant d$.

Apply $\operatorname{Hom}(B, -)$ to the triangles $X_{i+1} \longrightarrow B_i \longrightarrow X_i \longrightarrow X_{i+1}[1]$ to get the exact sequences

$$\operatorname{Ext}^k(B, X_{i+1}) \longrightarrow \operatorname{Ext}^k(B, B_i) \longrightarrow \operatorname{Ext}^k(B, X_i).$$

Then $\operatorname{Ext}^k(B, B_i) = 0$ for all $0 \leqslant i \leqslant d$ and $1 \leqslant k \leqslant d$, so $\operatorname{Ext}^k(B, B) = 0$ for all $1 \leqslant k \leqslant d$. □

Note that this implies that the X_i cannot be direct summands of B (if $X_i \in \operatorname{add} B$ for some i, then $\operatorname{Ext}(X_i, X_{i+1})$ is a direct summand of $\operatorname{Ext}(B \oplus X_{i+1}, B \oplus X_{i+1}) = 0$, a contradiction) and B is a rigid object in $\mathcal{C}_d(\mathcal{H})$. Hence B can be extended to a d-tilting object by Corollary 3.4. Let $T = B \oplus T'$ be a d-cluster tilting object in $\mathcal{C}_d(\mathcal{H})$.

Lemma 5.6. *Under the same assumptions and notation as before, if* N *is an indecomposable summand of* T *and there exists some* j *such that* N *is not isomorphic to* X_i *for all* $i \neq j$, *then* $\operatorname{Ext}^k(N, X_j) = 0$ *for any* $1 \leqslant k \leqslant d$.

Proof. Assume by contradiction that $\operatorname{Ext}^k(N, X_j) \neq 0$ for some $1 \leqslant k \leqslant d$, and there is some indecomposable summand N of T with $N \ncong X_i$ for all $i \neq j$. Applying $\operatorname{Hom}(N, -)$ to the $d+1$ triangles $(***)$, we get $\operatorname{Ext}^1(N, X_{j-k+1}) \cong \operatorname{Ext}^k(N, X_j) \neq 0$. Without loss of generality, we may assume that $j - k = 0$. So we have $\operatorname{Hom}(N, X_1[1]) = \operatorname{Ext}^1(N, X_1) \neq 0$ and an exact sequence

$$\operatorname{Hom}(N, X_0) \longrightarrow \operatorname{Hom}\big(N, X_1[1]\big) \longrightarrow 0,$$

which implies that there exists a non-zero morphism $t \in \operatorname{Hom}(N, X_0) \neq 0$ such that $\delta_0 t \neq 0$. Applying $\operatorname{Hom}(N, -)$ to the $d+1$ triangles $(***)$, we get $\operatorname{Ext}^d(N, X_d) \cong \operatorname{Ext}^{d-1}(N, X_{d-1}) \cong \cdots \cong \operatorname{Ext}^1(N, X_1) \neq 0$, and then $\delta_d[d] \cdots \delta_1[1]\delta_0 t \neq 0$. Denote by

$$X_0[d] \longrightarrow A \xrightarrow{r} X_0 \longrightarrow X_0[d+1]$$

the AR-triangle ending at X_0 in $\mathcal{C}_d(\mathcal{H})$. Consider the commutative diagram

$$\begin{array}{ccccccc} X_0[d] & \longrightarrow & A & \xrightarrow{r} & X_0 & \longrightarrow & X_0[d+1] \\ \| & & \downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle b_2} & & \| \\ X_0[d] & \xrightarrow{g_d[d]} & B_d[d] & \xrightarrow{f_d[d]} & X_d[d] & \xrightarrow{\delta_d[d]} & X_0[d+1], \end{array}$$

where the map b_1 exists since $\delta_d[d] \neq 0$ (thus $g_d[d]$ is not a section), and hence there exists a map b_2 such that the diagram commutes. From Definition 5.4, we know that $\operatorname{Hom}(X_0, X_d[d])$ has an $\operatorname{End} X_0$-basis $\{\delta_d[d]\cdots\delta_1[1]\delta_0\}$. Since $b_2 \in \operatorname{Hom}(X_0, X_d[d])$ is not zero, there exists an isomorphism $\phi \in \operatorname{End}(X_0)$ such that $b_2 = \delta_d[d]\cdots\delta_1[1]\delta_0\phi$. Let $s = \phi^{-1}t \in \operatorname{Hom}(N, X_0)$, then $b_2 s \neq 0$. Since $N \ncong X_0$, there is some map $s' : N \longrightarrow A$, such that $s = rs'$. Note that $b_2 s = b_2 rs' = f_d[d]b_1 s'$ is a non-zero map, and consequently $b_1 s' \neq 0$. But this contradicts $\operatorname{Hom}(N, B_d[d]) = 0$. This completes the proof of the lemma. □

Lemma 5.7. *If* $\operatorname{add}(\bigoplus_{1\leqslant i\leqslant d, i\neq j} X_i) \cap \operatorname{add} T = \{0\}$ *for some* $1 \leqslant j \leqslant d$, *then* X_j *is a direct summand of* T. *Writing* T *as* $X_j^k \oplus \overline{T}$, *where the* X_j *are not direct summands of* $\overline{T}$, *then* $X_i \oplus \overline{T}$ *is also a d-cluster tilting object for any* $0 \leqslant i \leqslant d$.

Proof. The first assertion follows directly from Lemma 5.6. The second follows from Theorem 4.3 and Lemma 5.6. □

In summary, we have the following main result:

Theorem 5.8. *The* $d+1$ *rigid indecomposable objects* $\{X_i\}_{0\leqslant i\leqslant d}$ *form the set of complements of an almost complete d-cluster tilting object in* $\mathcal{C}_d(\mathcal{H})$ *if and only if they form an exchange team.*

Since the chain of $d+1$-triangles of the complements of an almost complete d-cluster tilting object form a cycle, their distribution is uniform. In particular there are two cases: either every complement has a different degree, or that the degree of any complement is smaller than $d-1$ and only two complements have the same degree. We can summarize the cases as follows.

Proposition 5.9. *Suppose* $\deg X_0 = 0$ *and* $\deg X_1 \neq 0$. *Then there exists some k, with* $0 \leqslant k \leqslant d$, *such that*

$$\deg X_i = \begin{cases} d-i & \text{if } 1 \leqslant i \leqslant k, \\ d+1-i & \text{if } k+1 \leqslant i \leqslant d. \end{cases}$$

Proof. By Lemma 3.1, we know that $\deg X_i \geqslant d-i$ for $1 \leqslant i \leqslant d$. Since $d+1$-triangle chains form a cycle, analyzing the degree in the opposite direction from X_0, we get $\deg X_i \leqslant d-i+1$ for $1 \leqslant i \leqslant d$. If $\deg X_1 = d$, then $\deg X_2 = d-1$, since $\operatorname{Hom}(X_1, X_2[1]) \neq 0$ forces $\deg X_2 \geqslant d-1$. By induction, $\deg X_i = d-i+1$ for $1 \leqslant i \leqslant d$. This situation is equivalent to $k=0$. If $\deg X_1 = d-1$, then there exists some k such that $\deg X_k = \deg X_{k+1}$. By the way of the case $\deg X_1 = d$, we obtain the conclusion. □

6. Middle terms of the $d+1$ triangles

Throughout this section, we assume that $d \geqslant 2$. We assume that X is a basic almost complete d-cluster tilting object, and that the $d+1$ complements $X_0, \dots, X_d$ of X are connected by the $d+1$ triangles in $(*)$ in Section 4:

$$X_{i+1} \xrightarrow{g_i} B_i \xrightarrow{f_i} X_i \xrightarrow{\delta_i} X_{i+1}[1], \tag{$*$}$$

where for $i = 0, 1, \dots, d$, $B_i \in \operatorname{add} X$, the map f_i is the minimal right $\operatorname{add} X$-approximation of X_i and g_i is the minimal left $\operatorname{add} X$-approximation of X_{i+1}.

In [BMRRT], there was a conjecture that the sets of indecomposables of B_i appeared in the triangles $(*)$ are disjoint in cluster categories. That has been solved in [BMR]. We will prove the same statement for d-cluster categories. Prior to this, we need some preparatory work. For a tilting module T in $\mathcal{H}$, any two non-isomorphic summands T_1, T_2 of T have the following property: $\operatorname{Hom}(T_1, T_2) = 0$ or $\operatorname{Hom}(T_2, T_1) = 0$ (see [Ker]). The same property holds for d-cluster tilting objects in d-cluster categories when $d \geqslant 3$.

Lemma 6.1. *Suppose $d \geqslant 3$. Let T_1, T_2 be two non-isomorphic summands of a d-cluster tilting object T in $\mathcal{C}_d(\mathcal{H})$. Then* $\operatorname{Hom}(T_1, T_2) = 0$ *or* $\operatorname{Hom}(T_2, T_1) = 0$.

Proof. If not, then $\operatorname{Hom}(T_1, T_2) \neq 0$ and $\operatorname{Hom}(T_2, T_1) \neq 0$. Then $\deg T_1 = \deg T_2$ by the fact that $d \geqslant 3$ and Lemma 4.7 in [Zh3]. Let k denote this common value. Then T_1, T_2 are of the forms $T_1'[k]$, $T_2'[k]$ respectively, where T_1' and T_2' are partial tilting modules in $\mathcal{H}$. Hence $\operatorname{Hom}(T_1, T_2) \cong \operatorname{Hom}_{\mathcal{D}}(T_1', T_2') \neq 0$ and $\operatorname{Hom}(T_2, T_1) \cong \operatorname{Hom}_{\mathcal{D}}(T_2', T_1') \neq 0$ [Ker]. That is a contradiction. □

As a consequence, we get the following simple result.

Lemma 6.2. *Let $d \geqslant 3$. Then* $\operatorname{Hom}(X_i, X_{i+1}) = 0$.

Proof. Apply $\operatorname{Hom}(X_i, -)$ to the triangle $X_{i+1} \longrightarrow B_i \longrightarrow X_i \longrightarrow X_{i+1}[1]$ to get the exact sequence

$$\operatorname{Hom}\bigl(X_i, X_i[-1]\bigr) \longrightarrow \operatorname{Hom}(X_i, X_{i+1}) \longrightarrow \operatorname{Hom}(X_i, B_i).$$

In this exact sequence, $\operatorname{Hom}(X_i, X_i[-1]) = 0$ since $d \geqslant 3$. Since $B_i \longrightarrow X_i$ is the minimal right $\operatorname{add} X$-approximation, $\operatorname{Hom}(Y, X_i) \neq 0$ for any indecomposable direct summand Y of B_i. It follows from Lemma 6.1 that $\operatorname{Hom}(X_i, B_i) = 0$. Thus $\operatorname{Hom}(X_i, X_{i+1}) = 0$. □

Now we are able to prove the main conclusion in this section.

Theorem 6.3. *Let $\{B_i\}_{0 \leqslant i \leqslant d}$ be as above. Then the sets of indecomposable summands of B_i, for $i = 0, \dots, d$, are disjoint.*

Proof. We divide the proof into two cases:

(1) The case when $d = 2$. Suppose $\deg X_0 = 0$. Assume by contradiction that two of B_0, B_1, B_2 have non-trivial intersection. Without loss of generality, we suppose that there exists an indecomposable object $T_1 \in \operatorname{add} B_0 \cap \operatorname{add} B_1$. Then $\operatorname{Hom}(X_1, T_1) \neq 0 \neq \operatorname{Hom}(T_1, X_1)$, which implies that $\deg X_1 \neq \deg T_1$ (see [Ker]). We claim that $\deg X_1 = 1$, $\deg X_2 = 0$, and $\deg T_1 = 0$. If $\deg X_1 = 0$, then $\deg T_1 = 0$ by Lemma 4.9 in [Zh3], a contradiction. If $\deg X_1 = 2$ and $\deg T_1 = 0$, then $\operatorname{Hom}(T_1, X_1) = 0$ by Lemma 4.7 in [Zh3], a contradiction. If $\deg X_1 = 2$ and $\deg T_1 = 1$, then $\operatorname{Hom}(X_1, T_1) = 0$ by Lemma 4.7 in [Zh3], a contradiction. So $\deg X_1 = 1$, and then $\deg T_1 = 0$ (otherwise, $\deg T_1 = 2$ which implies $\operatorname{Hom}(T_1, X_1) = 0$, a contradiction). From Proposition 5.9, we have $\deg X_2 = 0$. Hence the degree of any indecomposable summands of B_2 is zero. Then $\operatorname{Hom}(X_2, B_2) = 0 = \operatorname{Hom}(B_2, X_0)$ (see the

discussion in the proof of Lemma 6.2). Apply $\mathrm{Hom}(X_2, -)$ to the triangle $X_0 \longrightarrow B_2 \longrightarrow X_2 \longrightarrow X_0[1]$ to get the exact sequence

$$\mathrm{Hom}\big(X_2, X_2[-1]\big) \longrightarrow \mathrm{Hom}(X_2, X_0) \longrightarrow \mathrm{Hom}(X_2, B_2),$$

where $\mathrm{Hom}(X_2, B_2) = 0$, so $\mathrm{Hom}(X_2, X_0) = 0$ (for any map $r \in \mathrm{Hom}(X_2, X_0)$, there exists $s \in \mathrm{Hom}(X_2, X_2[-1]) \cong \mathrm{Hom}(X_2, \tau^{-1}X_2[1]) \cong \mathrm{Hom}_{\mathcal{D}}(X_2, \tau^{-1}X_2[1]) \cong \mathrm{Hom}_{\mathcal{D}}(\tau X_2[-2], X_2[-1])$ and $t \in \mathrm{Hom}(X_2[-1], X_0) \cong \mathrm{Hom}(X_2, X_0[1]) \cong \mathrm{Hom}_{\mathcal{D}}(X_2, X_0[1]) \cong \mathrm{Hom}_{\mathcal{D}}(X_2[-1], X_0)$ (both of the second isomorphisms come from Lemma 4.2), such that $r = ts \in \mathrm{Hom}_{\mathcal{D}}(\tau X_2[-2], X_0) = 0$). Write the second triangle in $(*)$ as

$$X_2 \xrightarrow{\binom{h}{f}} B_1' \oplus T_1 \xrightarrow{(\alpha,\beta)} X_1 \longrightarrow X_2[1],$$

where $\beta \in \mathrm{Hom}(T_1, X_1) \cong \mathrm{Hom}_{\mathcal{D}}(T_1, X_1)$. Let g be a non-zero map in $\mathrm{Hom}(T_1, X_0)$ (such a map exists because T_1 is a direct summand of B_0). Then we get $(0, g)\binom{h}{f} = gf \in \mathrm{Hom}(X_2, X_0) = 0$, so there exists a map $\varphi \in \mathrm{Hom}(X_1, X_0) \cong \mathrm{Hom}(X_1, \tau^{-1}X_0[2]) \cong \mathrm{Hom}_{\mathcal{D}}(X_1, \tau^{-1}X_0[2])$ (the second isomorphism come from Lemma 4.2) such that $\varphi(\alpha, \beta) = (0, g)$. Then $g = \varphi\beta \in \mathrm{Hom}_{\mathcal{D}}(T_1, \tau^{-1}X_0[2]) = 0$. This is a contradiction.

(2) The case when $d \geqslant 3$. Suppose T_1 is an indecomposable summand of both B_i and B_j, $i < j$. Define $d(B_i, B_j) = \min\{j - i, i - j + d + 1\}$.

If $d(B_i, B_j) = 1$, then without loss of generality we may suppose that $i = 0$ and $j = 1$; then $\mathrm{Hom}(X_1, T_1) \neq 0$ and $\mathrm{Hom}(T_1, X_1) \neq 0$. But X_1 and T_1 are two non-isomorphic indecomposable summands of a d-cluster tilting object $X_1 \oplus X$, which is impossible by Lemma 6.1.

If $d(B_i, B_j) = 2$, then without loss of generality we may suppose that $i = 1$ and $j = 3$; then $\deg X_2 = \deg X_3 = \deg T_1$. Let k denote this common value. Then $\deg X_4 = k - 1$ when $k \geqslant 1$, and $\deg X_4 = d - 1$ when $k = 0$. Apply $\mathrm{Hom}(X_2, -)$ to the triangle $X_4 \xrightarrow{g_3} B_3 \xrightarrow{f_3} X_3 \xrightarrow{\delta_3} X_4[1]$ to get an exact sequence

$$\mathrm{Hom}(X_2, X_4) \longrightarrow \mathrm{Hom}(X_2, B_3) \longrightarrow \mathrm{Hom}(X_2, X_3).$$

Then $\mathrm{Hom}(X_2, X_4) \longrightarrow \mathrm{Hom}(X_2, B_3)$ is an epimorphism since $\mathrm{Hom}(X_2, X_3) = 0$. Since $T_1 \in \mathrm{add}\, B_1$, there exists a non-zero morphism $s \in \mathrm{Hom}(X_2, T_1)$, so the morphism $\binom{s}{0} : X_2 \longrightarrow T_1 \oplus B_3'$ is not zero, where $B_3 = B_3' \oplus T_1$. Hence there exists $r \in \mathrm{Hom}(X_2, X_4)$ such that $s = g_3 r$. Let $g_3 = \binom{h}{h'} : X_4 \longrightarrow T_1 \oplus B_3'$, where $h \in \mathrm{Hom}(X_4, T_1)$, then $s = hr$. Since $\mathrm{Hom}(X_2, X_4) \cong \mathrm{Hom}_{\mathcal{D}}(X_2, \tau^{-1}X_4[d])$ and $\mathrm{Hom}(X_4, T_1) \cong \mathrm{Hom}_{\mathcal{D}}(\tau^{-1}X_4[d], \tau^{-1}T_1[d])$, it follows that $hr \in \mathrm{Hom}_{\mathcal{D}}(X_2, \tau^{-1}T_1[d]) = 0$, a contradiction.

If $d(B_i, B_j) \geqslant 3$, then the degrees of the summands of B_i and B_j are distinct. Hence the sets of indecomposable summands of B_i are disjoint, for $i = 0, \ldots, d$. □

7. Cluster combinatorics of *d*-cluster categories

Denote by $\mathcal{E}(\mathcal{H})$ the set of isomorphism classes of indecomposable rigid modules in $\mathcal{H}$. The set $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ of isoclasses of indecomposable rigid objects in $\mathcal{C}_d(\mathcal{H})$ is the (disjoint) union of the subsets $\mathcal{E}(\mathcal{H})[i]$, $i = 0, 1, \ldots, d-1$, with $\{P_j[d] \mid 1 \leqslant j \leqslant n\}$ (see Section 4 in [Zh3]). A subset $\mathcal{M}$ of $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ is called rigid if for any $X, Y \in \mathcal{M}$, $\mathrm{Ext}^i(X, Y) = 0$ for all $i = 1, \ldots, d$. Denote by $\mathcal{E}_+(\mathcal{C}_d(\mathcal{H}))$ the subset of $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ consisting of all indecomposable exceptional objects other than $P_1[d], \ldots, P_n[d]$.

Now we recall the definition of simplicial complexes associated to the d-cluster category $\mathcal{C}_d(\mathcal{H})$ and the root system Φ from [Zh3].

Definition 7.1. The cluster complex $\Delta^d(\mathcal{H})$ of $\mathcal{C}_d(\mathcal{H})$ is a simplicial complex with $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ as its set of vertices, and the rigid subsets of $\mathcal{C}_d(\mathcal{H})$ as its simplices. The positive part $\Delta^d_+(\mathcal{H})$ is the subcomplex of $\Delta^d(\mathcal{H})$ on the subset $\mathcal{E}_+(\mathcal{C}_d(\mathcal{H}))$.

From the definition, the facets (maximal simplices) are exactly the d-cluster tilting subsets (i.e. the sets of indecomposable objects of $\mathcal{C}_d(\mathcal{H})$ (up to isomorphism) whose direct sum is a d-cluster tilting object).

As consequences of results in Sections 3–5, we have that:

Proposition 7.2.

1. *A face of the cluster complex $\Delta^d(\mathcal{H})$ is a facet if and only if it contains exactly n vertices. In particular, all facets in $\Delta^d(\mathcal{H})$ are of size n.*
2. *Every codimension 1 face of $\Delta^d(\mathcal{H})$ is contained in exactly $d+1$ facets.*
3. *Any codimension 1 face in $\Delta^d(\mathcal{H})$ has complements of each color.*

Throughout the rest of this section, we assume that $\mathcal{H}$ is the category of finite dimensional representations of a valued quiver $(\Gamma, \Omega, \mathcal{M})$. For basic material about valued quivers and their representations, we refer to [DR].

Let Φ be the root system of the Kac–Moody Lie algebra corresponding to the graph Γ. We assume that $P_1, \ldots, P_n$ are the non-isomorphic indecomposable projective representations in $\mathcal{H}$, and $E_1, \ldots, E_n$ are the simple representations with dimension vectors $\alpha_1, \ldots, \alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are the simple roots in Φ. We use $\Phi_{\geqslant -1}$ to denote the set of almost positive roots, i.e. the set of positive roots together with the $-\alpha_i$.

Fix a positive integer d, for any $\alpha \in \Phi^+$, following [FR2], we call $\alpha^1, \ldots, \alpha^d$ the d "colored" copies of α.

Definition 7.3. (See [FR2].) The set of colored almost positive roots is

$$\Phi^d_{\geqslant -1} = \{\alpha^i:\ \alpha \in \Phi_{>0},\ i \in \{1, \ldots, d\}\} \cup \{(-\alpha_i)^1:\ 1 \leqslant i \leqslant n\}.$$

We now define a map $\gamma^d_{\mathcal{H}}$ from $\operatorname{ind}\mathcal{C}_d(\mathcal{H})$ to $\Phi^d_{\geqslant -1}$. Note that any indecomposable object X of degree i in $\mathcal{C}_d(\mathcal{H})$ has the form $M[i]$, for some $M \in \operatorname{ind}\mathcal{H}$, and if $i = d$ then $M = P_j$, an indecomposable projective representation.

Definition 7.4. Let $\gamma^d_{\mathcal{H}}$ be defined as follows. Let $M[i] \in \operatorname{ind}\mathcal{C}_d(\mathcal{H})$, where $M \in \operatorname{ind} H$ and $i \in \{1, \ldots, d\}$ (note that if $i = d$ then $M = P_j$ for some j). We set

$$\gamma^d_{\mathcal{H}}\big(M[i]\big) = \begin{cases} (\underline{\dim}\, M)^{i+1} & \text{if } 0 \leqslant i \leqslant d-1; \\ (-\alpha_j)^1 & \text{if } i = d. \end{cases}$$

Note that if Γ is a Dynkin diagram, then $\gamma^d_{\mathcal{H}}$ is a bijection.

We denote by $\Phi^{sr}_{>0}$ the set of real Schur roots of (Γ, Ω), i.e.

$$\Phi^{sr}_{>0} = \{\underline{\dim}\, M:\ M \in \operatorname{ind}\mathcal{E}(\mathcal{H})\}.$$

Then the map $M \mapsto \underline{\dim}\, M$ gives a 1–1 correspondence between $\mathcal{E}(\mathcal{H})$ and $\Phi^{sr}_{>0}$ [Rin].

If we denote the set of colored almost positive real Schur roots by $\Phi^{sr,d}_{\geqslant -1}$ (which consists by definition of d copies of the set $\Phi^{sr}_{>0}$ together with one copy of the negative simple roots), then the map

$\gamma^d_{\mathcal{H}}$ gives a bijection from $\mathcal{E}(\mathcal{C}_d(\mathcal{H}))$ to $\Phi^{sr,d}_{\geqslant -1}$. $\Phi^{sr,d}_{\geqslant -1}$ contains a subset $\Phi^{sr,d}_{>0}$ consisting of all colored positive real Schur roots. The restriction of $\gamma^d_{\mathcal{H}}$ gives a bijection from $\mathcal{E}_+(\mathcal{C}_d(\mathcal{H}))$ to $\Phi^{sr,d}_{>0}$.

Using this bijection, in [Zh3] we defined, for any root system Φ and $\mathcal{H}$, an associated simplicial complex $\Delta^{d,\mathcal{H}}(\Phi)$ on the set $\Phi^{sr,d}_{>0}$, which is called the generalized cluster complex of Φ and is a generalization of the generalized cluster complexes defined by Fomin and Reading [FR2], see also [Th] for finite root systems Φ. It was proved that $\gamma^d_{\mathcal{H}}$ defines an isomorphism from the simplicial complex $\Delta^d(\mathcal{H})$ to the generalized cluster complex $\Delta^{d,\mathcal{H}}(\Phi)$, which sends vertices to vertices, and k-faces to k-faces [Zh3].

Corollary 7.5.

1. *A face of the generalized cluster complex $\Delta^{d,\mathcal{H}}(\Phi)$ is a facet if and only it contains exactly n vertices. In particular, $\Delta^{d,\mathcal{H}}(\Phi)$ is of pure dimension $n-1$.*
2. *Any codimension 1 face of $\Delta^{d,\mathcal{H}}(\Phi)$ is contained in exactly $d+1$ facets.*
3. *For any codimension 1 face of $\Delta^{d,\mathcal{H}}(\Phi)$, there are complements of each color.*

Proof. Combining Proposition 7.2 with the fact that $\gamma^d_{\mathcal{H}}$ is an isomorphism from $\Delta^d(\mathcal{H})$ to $\Delta^{d,\mathcal{H}}(\Phi)$ [Zh3], we have all the conclusions in the corollary. □

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