Available online at www.sciencedirect.com
SCIENCE DIRECT
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 49 (2005) 1669–1678
www elsevier com/locate/camwa

# The Number of Limit Cycles for a Family of Polynomial Systems

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(Received May 2003, revised and accepted February 2005)

**Abstract**—In this paper, the number of limit cycles in a family of polynomial systems was studied by the bifurcation methods With the help of a computer algebra system (e.g., MAPLE 7 0), we obtain that the least upper bound for the number of limit cycles appearing in a global bifurcation of systems (2.1) and (2.2) is $5n + 5 + (1 - (-1)^n)/2$ for $c \neq 0$ and $n$ for $c \equiv 0$. © 2005 Elsevier Ltd All rights reserved.

**Keywords**—Hilbert's 16th problem, Global bifurcation, Abelian integrals, Limit cycles

## 1. PRELIMINARY LEMMAS

In the qualitative theory of real planar differential systems, a typical problem is to determine limit cycles (see [1,2] for more details). A classical approach to generate limit cycles is perturbing a system, which has a center, so that limit cycles bifurcate in the perturbed system from some periodic orbits of the unperturbed system (see [3–7] for example).

Consider a planar system of the form,

$$\begin{aligned} \dot{x}(t) &= H_y + \varepsilon f(x, y, \varepsilon, a), \\ \dot{y}(t) &= -H_x + \varepsilon g(x, y, \varepsilon, a), \end{aligned} \tag{1.1}$$

$H$, $f$, $g$ are $C^\infty$ functions in a region $G \subset R^2$, $\varepsilon \in R$ is a small parameter, and $a \in D \subset R^n$ with $D$ compact. For $\varepsilon = 0$, (1.1) becomes Hamiltonian with the Hamiltonian function $H(x, y)$. Suppose there exists a constant $H_0 > 0$, such that for $0 < h < H_0$, the equation $H(x, y) = h$ defines a smooth closed curve $L_h \subset G$ surrounding the origin and shrinking to the origin as $h \to 0$. Hence, $H(0, 0) = 0$ and for $\varepsilon = 0$ (1.1) has a center at the origin.

Let

$$\Phi(h, a) = \oint_{L_h} (g\,dx - f\,dy)_{\varepsilon=0} = \oint_{L_h} (H_y g + H_x f)_{\varepsilon=0}\,dt, \tag{1.2}$$

which is called the first-order Melnikov function or Abelian integral of (1.1). This function plays an important role in the study of limit cycles bifurcation (see [8–14] for example). In the case

We thank the referees for many useful suggestions that helped us to improve the work.
Supported by National Natural Sciences Foundation of China (10371072)

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doi 10.1016/j camwa.2005 02.007

that (1.1) is a polynomial system, a well-known problem is to determine the least upper bound of the number of zeros of $\Phi$. This is the weakened Hilbert's $16^{\text{th}}$ problem (see [1]).

In this paper, we first state some preliminary lemmas that can be used to find the maximal number of limit cycles by using zeros of $\Phi$. These lemmas are known results or based on known results. Then, we study the global bifurcations of limit cycles for some polynomial systems, and obtain the least upper bound for the number of limit cycles.

For Hopf bifurcation, we have the following lemma.

LEMMA 1.1. *(See [5].) Let $H(x,y) = K(x^2+y^2) + O(|x,y|^3)$ with $K > 0$ for $(x,y)$ near the origin. Then, the function $\Phi$ is of class $C^\infty$ in $h$ at $h = 0$. If $\Phi(h,a_0) = K_1(a_0)h^{k+1} + O(h^{k+2})$, $K_1(a_0) \neq 0$ for some $a_0 \in D$, then (1.1) has at most $k$ limit cycles near the origin for $|\varepsilon| + |a - a_0|$ sufficiently small*

The following lemma is well-known result (see [2] for example).

LEMMA 1.2. *If $\Phi(h,a_0) = K_2(a_0)(h-h_0)^k + O(|h-h_0|^{k+1})$, $K_2(a_0) \neq 0$ for some $a_0 \in D$ and $h_0 \in (0, H_0)$, then (1.1) has at most $k$ limit cycles near $L_{h_0}$ for $|\varepsilon| + |a - a_0|$ sufficiently small.*

Let $L_0$ denote the origin and set

$$S = \bigcup_{0 \le h < H_0} L_h. \tag{1 3}$$

It is obvious that $S$ is a simply connected open subset of the plane. We suppose that the function $\Phi$ has the following form,

$$\Phi(h,a) = I(h)N(h,a), \tag{1.4}$$

where $I \in C^\infty$ for $h \in [0,\, H_0)$ and satisfies

$$I(0) = 0, \quad I'(0) \neq 0, \quad \text{and} \quad I(h) \neq 0, \qquad \text{for } h \in (0, H_0). \tag{1.5}$$

Using the above two lemmas, Xiang and Han [11] proved the next lemma.

LEMMA 1.3. *Let (1.4) and (1.5) hold. If there exists a positive integer $k$ such that for every $a \in D$ the function $N(h,a)$ has at most $k$ zeros in $h \in [0, H_0)$ (multiplicities taken into account), then for any given compact set $V \subset S$, there exists $\varepsilon_0 = \varepsilon_0(V) > 0$, such that for all $0 < |\varepsilon| < \varepsilon_0$, $a \in D$ the system (1.1) has at most $k$ limit cycles in $V$.*

REMARK 1.1. As we know, if there exists $a_0 \in D$, such that the function $N(h,a_0)$ has exactly $k$ simple zeros $0 < h_1 < \cdots < h_k < H_0$ with $N(0,a_0) \neq 0$, then for any compact set $V$ satisfying $L_{h_k} \subset \operatorname{int} V$ and $V \subset S$, there exists $\varepsilon_0 > 0$, such that for all $0 < |\varepsilon| < \varepsilon_0, |a - a_0| < \varepsilon_0$, (1.1) has precisely $k$ limit cycles in $V$.

REMARK 1.2. The conclusions of Lemma 1.1 and Lemma 1.2 are local with respect to both the parameter $a$ and the set $S$, while the conclusion of Lemma 1.3 is global because it holds in any compact set of $S$ and uniformly in $a \in D$.

## 2. THE NUMBER OF LIMIT CYCLES IN A FAMILY OF POLYNOMIAL SYSTEMS

In this section, we consider a family of real planar polynomial systems of the form,

$$\begin{aligned} \dot{x} &= y\left(1 + ax + by + cx\left(x^2+y^2\right)\right) + \varepsilon \sum_{0 \le i+j \le n} a_{ij} x^i y^j, \\ \dot{y} &= -x\left(1 + ax + by + cx\left(x^2+y^2\right)\right) + \varepsilon \sum_{0 \le i+j \le n} b_{ij} x^i y^j, \end{aligned} \tag{2.1}$$

where $a$, $b$, $c$ are real with $a^2 + b^2 \neq 0$, and $a_{ij}, b_{ij}$ satisfy $|a_{ij}| \le K$, $|b_{ij}| \le K$ with $K$, a positive constant and $n$, a positive integer. Let $B_K = \{(a_{ij}, b_{ij}) \mid \; |a_{ij}| \le K, |b_{ij}| \le K\}$.

On the region $\Omega = \{(x,y) \mid 1+ax+by+cx(x^2+y^2) \neq 0\}$, (2.1) is equivalent to

$$\dot{x} = y + \frac{\varepsilon}{(1+ax+by+cx(x^2+y^2))} \sum_{0\le i+j\le n} a_{ij}x^i y^j, \\ \dot{y} = -x + \frac{\varepsilon}{(1+ax+by+cx(x^2+y^2))} \sum_{0\le i+j\le n} b_{ij}x^i y^j. \tag{2.2}$$

Let $\Phi(h)$ denote the first-order Melnikov function of (2.2) for $0 \le h < H_0$ with $H_0$ satisfying $1-((a+cH_0)^2+b^2)H_0 = 0$. Then, we have the following main results.

THEOREM 2.1. *Suppose $c \neq 0$. For any $K>0$ and compact set $V$ in $\Omega$, if $\Phi(h)$ is not identically zero for $(a_{ij}, b_{ij})$ varying in a compact set $D$ in $B_K$, then there exists an $\varepsilon_0 > 0$, such that for $0<|\varepsilon|<\varepsilon_0$, $(a_{ij}, b_{ij}) \in D$, (2.1) or (2.2) has at most $5n+5+(1-(-1)^n)/2$ limit cycles in $V$.*

THEOREM 2.2. *Suppose $c=0$.*

(1) *For any $K>0$ and compact set $V$ in $\Omega$, if $\Phi(h)$ is not identically zero for $(a_{ij}, b_{ij})$ varying in a compact set $D$ in $B_K$, then there exists an $\varepsilon_0>0$, such that for $0<|\varepsilon|<\varepsilon_0$, $(a_{ij}, b_{ij}) \in D$, (2.1) or (2.2) has at most $n$ limit cycles in $V$.*
(2) *For any $K>0$ and compact set $V$ in $\Omega$, there exists an $\varepsilon_0>0$ and $(a^0_{ij}, b^0_{ij}) \in B_K$, such that for all $0<\varepsilon|<\varepsilon_0$, $|a_{ij}-a^0_{ij}|<\varepsilon_0$, $|b_{ij}-b^0_{ij}|<\varepsilon_0$, (2.1) or (2.2) has precisely $n$ limit cycles in $V$.*

Before proving the above theorems, we first give some lemmas.
Let

$$I_{i,j}(h) = \oint_{L_h} \frac{x^i y^j \, dt}{(1+ax+by+cx(x^2+y^2))}, \qquad i \ge 0, \quad j \ge 0, \tag{2.3}$$

$$\Phi_{ij}(h) = a_{ij} I_{i+1,j}(h) + b_{ij} I_{i,j+1}(h), \qquad i \ge 0, \quad j \ge 0, \tag{2.4}$$

where

$$L_h : x = \sqrt{h}\sin t, \qquad y = \sqrt{h}\cos t.$$

Let

$$K_0 = 2\pi, \qquad K_{2j} = \frac{(2j-1)!!}{(2j)!!} 2\pi, \qquad \mathbb{C}^j_k = \frac{k!}{j!\,(k-j)!},$$

and

$$(a+ch+bi)\,i = \rho(h)\, e^{i\theta(h)}, \qquad (a+ch)^2+b^2 = \rho^2(h), \qquad z_{1,2} = \frac{1 \mp \sqrt{1-\rho^2(h)\,h}}{\sqrt{h}\rho(h)\, e^{i\theta(h)}}, \tag{2.5}$$

where $i = \sqrt{-1}$.

LEMMA 2.1. *For $k \ge 0$, we have*

$$I_{2k,0}(h) = \frac{(-1)^k 2\pi \left(\sqrt{h}\right)^{2k}}{2^{2k}\sqrt{1-\rho^2(h)\,h}} \left\{ C^0_{2k} \left( \frac{1-\sqrt{1-\rho^2(h)\,h}}{\sqrt{h}\rho(h)} \right)^{2k} \cos 2k\theta(h) \right.$$
$$- C^1_{2k} \left( \frac{1-\sqrt{1-\rho^2(h)\,h}}{\sqrt{h}\rho(h)} \right)^{2k-2} \cos(2k-2)\,\theta(h) + \cdots \tag{2.6}$$
$$\left. + (-1)^{k-1} C^{k-1}_{2k} \left( \frac{1-\sqrt{1-\rho^2(h)\,h}}{\sqrt{h}\rho(h)} \right)^2 \cos 2\theta(h) + (-1)^k C^k_{2k} \frac{1}{2} \right\}$$

*and*

$$I_{2k+1,0}(h) = \frac{(-1)^k 2\pi \left(\sqrt{h}\right)^{2k+1}}{2^{2k+1}\sqrt{1-\rho^2(h)h}} \left\{ C_{2k+1}^0 \left(\frac{1-\sqrt{1-\rho^2(h)h}}{\sqrt{h}\rho(h)}\right)^{2k+1} \sin(2k+1)\theta(h) \right.$$
$$-C_{2k+1}^1 \left(\frac{1-\sqrt{1-\rho^2(h)h}}{\sqrt{h}\rho(h)}\right)^{2k-1} \sin(2k-1)\theta(h) + \cdots \tag{2.7}$$
$$\left. + (-1)^k C_{2k+1}^k \left(\frac{1-\sqrt{1-\rho^2(h)h}}{\sqrt{h}\rho(h)}\right) \sin\theta(h) \right\},$$

*where*

$$\cos 2k\theta(h) = \frac{1}{\rho^{2k}(h)} \sum_{j=0}^{k} (-1)^j C_{2k}^{2j} b^{2k-2j} (a+ch)^{2j} \tag{2.8}$$

*and*

$$\sin(2k+1)\theta(h) = \frac{1}{\rho^{2k+1}(h)} \sum_{j=0}^{k} (-1)^j C_{2k+1}^{2j+1} b^{2k-2j} (a+ch)^{2j+1}, \qquad k = 0,1,2,\dots. \tag{2.9}$$

PROOF. We use the residue theorem to compute the integral $I_{k,0}(h)$.

By the definitions of $L_h$ and $I_{k,0}(h)$, we have

$$I_{k,0}(h) = \oint_{L_h} \frac{x^k\,dt}{1+ax+by+cx(x^2+y^2)}$$
$$= \int_0^{2\pi} \frac{\left(\sqrt{h}\sin t\right)^k dt}{1+a\left(\sqrt{h}\sin t\right)+b\left(\sqrt{h}\cos t\right)+ch\left(\sqrt{h}\sin t\right)}.$$

Let $e^{it} = z$. Then, $dt = 1/iz\,dz$, $\sin t = (z^2-1)/2iz$, where $i = \sqrt{-1}$. Then, the above formula becomes

$$I_{k,0}(h) = \oint_{|z|=1} \frac{\left(\sqrt{h}\left((z^2-1)/2iz\right)\right)^k dz/iz}{1+(a+ch)\left(\sqrt{h}\left((z^2-1)/2iz\right)\right)+b\left(\sqrt{h}\left((z^2+1)/2z\right)\right)}$$
$$= \frac{2\left(\sqrt{h}/2i\right)^k}{\sqrt{h}(a+ch+bi)} \oint_{|z|=1} f(z)\,dz \tag{2.10}$$
$$= \frac{-4\pi\left(\sqrt{h}/2i\right)^k}{\sqrt{h}\rho(h)e^{i\theta}} \left\{\operatorname{Res}[f(z),z_1] + \operatorname{Res}[f(z),0]\right\},$$

where

$$f(z) = \frac{\left((z^2-1)/z\right)^k}{(z-z_1)(z-z_2)}, \qquad z_{1,2} = \frac{1 \mp \sqrt{1-\rho^2(h)h}}{\sqrt{h}\rho(h)e^{i\theta(h)}}, \qquad (a+ch+bi)i = \rho(h)e^{i\theta(h)}.$$

By the definition of residue, we have

$$\operatorname{Res}[f(z),0] = C_{-1},$$

where $C_{-1}$ is the coefficient of power of $-1$ in Laurent series of $f(z)$ being in the neighborhood of the point $z=0$. In fact,

$$\begin{aligned} C_{-1} = \frac{(-1)^k}{z_1 - z_2} \Bigg\{ & \left(-\frac{1}{z_1^k} + \frac{1}{z_2^k}\right) - C_k^1 \left(-\frac{1}{z_1^{k-2}} + \frac{1}{z_2^{k-2}}\right) + \cdots \\ & + (-1)^{[(k-1)/2]} C_k^{[(k-1)/2]} \left(-\frac{1}{z_1^{k-2[(k-1)/2]}} + \frac{1}{z_2^{k-2[(k-1)/2]}}\right) \Bigg\}, \end{aligned} \tag{2.11}$$

where $[r]$ denotes the integer part of $r$.

Similarly, we have

$$\operatorname{Res}[f(z), z_1] = \frac{(z_1^2 - 1)^k}{z_1^k (z_1 - z_2)} = \frac{(-1)^k}{z_1 - z_2} \sum_{j=0}^{k} (-1)^j C_k^j \frac{1}{z_1^{k-2j}}. \tag{2.12}$$

From (2.11) and (2.12), we have

$$\begin{aligned} \operatorname{Res}[f(z), z_1] + \operatorname{Res}[f(z), 0] = \frac{(-1)^k}{z_1 - z_2} \Bigg\{ & \left(\frac{1}{z_2^k} + (-z_1)^k\right) - C_k^1 \left(\frac{1}{z_2^{k-2}} + (-z_1)^{k-2}\right) \\ & + \cdots + (-1)^{[(k-1)/2]} C_k^{[(k-1)/2]} \left(\frac{1}{z_2^{k-2[(k-1)/2]}} + \left(-z_1^{k-2[(k-1)/2]}\right)\right) \Bigg\}. \end{aligned} \tag{2.13}$$

By the definition of $z_{1,2}$, we have

$$\begin{aligned} \frac{1}{z_2^k} + z_1^k &= \left(\frac{\sqrt{h}\rho(h)e^{i\theta(h)}}{1 + \sqrt{1 - \rho^2(h)h}}\right)^k + \left(\frac{1 - \sqrt{1 - \rho^2(h)h}}{\sqrt{h}\rho(h)e^{i\theta(h)}}\right)^k \\ &= \left(\frac{1 - \sqrt{1 - \rho^2(h)h}}{\sqrt{h}\rho(h)}\right)^k \left(e^{ik\theta(h)} + e^{-ik\theta(h)}\right) \\ &= 2\left(\frac{1 - \sqrt{1 - \rho^2(h)h}}{\sqrt{h}\rho(h)}\right)^k \cos k\theta(h) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \frac{1}{z_2^k} - z_1^k &= \left(\frac{\sqrt{h}\rho(h)e^{i\theta(h)}}{1 + \sqrt{1 - \rho^2(h)h}}\right)^k + \left(\frac{1 - \sqrt{1 - \rho^2(h)h}}{\sqrt{h}\rho(h)e^{i\theta(h)}}\right)^k \\ &= \left(\frac{1 - \sqrt{1 - \rho^2(h)h}}{\sqrt{h}\rho(h)}\right)^k \left(e^{ik\theta(h)} - e^{-ik\theta(h)}\right) \\ &= 2i\left(\frac{1 - \sqrt{1 - \rho^2(h)h}}{\sqrt{h}\rho(h)}\right)^k \sin k\theta(h). \end{aligned} \tag{2.15}$$

where $i = \sqrt{-1}$, $k = 1, 2, 3, \ldots$. Hence, for $k$ even, from (2.10), (2.13), and (2.14), we obtain that

$$\begin{aligned} I_{k,0}(h) &= \frac{-4\pi\left(\sqrt{h}/2i\right)^k}{\sqrt{h}\rho(h)e^{i\theta(h)}} \{\operatorname{Res}[f(z), z_1] + \operatorname{Res}[f(z), 0]\} \\ &= \frac{(-1)^{k/2} 2\pi \left(\sqrt{h}\right)^k}{2^k \sqrt{1 - \rho^2(h)h}} \Bigg\{ \left(\frac{1 - \sqrt{1 - \rho^2(h)h}}{\sqrt{h}\rho(h)}\right)^k \cos k\theta(h) \\ &\quad - C_k^1 \left(\frac{1 - \sqrt{1 - \rho^2(h)h}}{\sqrt{h}\rho(h)}\right)^{k-2} \cos(k-2)\theta(h) \\ &\quad + \cdots + (-1)^{k/2-1} C_k^{k/2-1} \left(\frac{1 - \sqrt{1 - \rho^2(h)h}}{\sqrt{h}\rho(h)}\right)^2 \cos 2\theta(h) + (-1)^{k/2} C_k^{k/2} \frac{1}{2} \Bigg\}, \end{aligned}$$

and for $k$ odd, by using (2.15), (2.10) becomes

$$I_{k,0}(h) = \frac{(-1)^{k-1} 2\pi \left(\sqrt{h}\right)^k}{2^k \sqrt{1-\rho^2(h)h}} \left\{ \left( \frac{1-\sqrt{1-\rho^2(h)h}}{\sqrt{h}\rho(h)} \right)^k \sin k\theta(h) \right.$$
$$-C_k^1 \left( \frac{1-\sqrt{1-\rho^2(h)h}}{\sqrt{h}\rho(h)} \right)^{k-2} \sin(k-2)\theta(h)$$
$$\left. +\cdots+(-1)^{(k-1)/2} C_k^{(k-1)/2} \left( \frac{1-\sqrt{1-\rho^2(h)h}}{\sqrt{h}\rho(h)} \right) \sin\theta(h) \right\}.$$

Then, (2.6),(2.7) follow from the formula,

$$\rho^k(h)\{\cos k\theta(h) + \imath \sin k\theta(h)\} = \{(a+ch+bi)\,i\}^k. \tag{2.16}$$

The proof is completed.

Suppose $c=0$ and let $\sqrt{1-\rho^2 h} = r$. Similarly, we can prove

LEMMA 2.2. *Suppose $c=0$. For $k \ge 0$, we have*

$$I_{2k,0}(h) = \frac{(-1)^k 4\pi (1-r)^k}{r(2\rho)^{2k}} \left\{ c_k^{(2k,0)} r^k + \cdots + c_1^{(2k,0)} r^1 + c_0^{(2k,0)} \right\} \tag{2.17}$$

*and*

$$I_{2k+1,0}(h) = \frac{(-1)^{k+1} 4\pi (1-r)^k}{r(2\rho)^{2k+1}} \left\{ c_{k+1}^{(2k+1,0)} r^{k+1} + \cdots + c_1^{(2k+1,0)} r^1 + c_0^{(2k+1,0)} \right\}, \tag{2.18}$$

*where*

$$c_\jmath^{(2k,0)} = (-1)^\jmath \left\{ C_k^\jmath \cos 2k\theta - C_{2k}^1 \left( C_{k-1}^\jmath - C_{k-1}^{\jmath-1} \right) \cos(2k-2)\theta \right.$$
$$\left. +\cdots+(-1)^{k-\jmath-1} C_{2k}^{k-1} \left( C_{k-1}^\jmath - C_{k-1}^{\jmath-1} \right) \cos 2\theta + (-1)^{k-\jmath} \frac{1}{2} C_{2k}^k C_k^\jmath \right\}$$

*and*

$$c_\jmath^{(2k+1,0)} = (-1)^\jmath \left\{ C_k^\jmath \sin(2k+1)\theta - C_{2k+1}^1 \left( C_{k-1}^\jmath - C_{k-1}^{\jmath-1} \right) \sin(2k-1)\theta \right.$$
$$\left. +\cdots+(-1)^{k-\jmath-1} C_{2k+1}^{k-1} \left( C_{k-1}^\jmath - C_{k-1}^{\jmath-1} \right) \sin 3\theta + (-1)^{k-\jmath} C_{2k+1}^k C_k^\jmath \sin\theta \right\},$$

*where $j = 0, 1, \dots,$ and $\sqrt{1-\rho^2 h} = r$, $\rho e^{\imath\theta} = (a+bi)i$.*

LEMMA 2.3. *If $b \neq 0$, then for $k \ge 0$, we have*

$$I_{2k,1}(h) = \frac{1}{b}\left( K_{2k} h^k - I_{2k,0}(h) - (a+ch) I_{2k+1,0}(h) \right), \tag{2.19}$$

*and*

$$I_{2k+1,1}(h) = \frac{1}{b}\left( -I_{2k+1,0}(h) - (a+ch) I_{2k+2,0}(h) \right). \tag{2.20}$$

PROOF. By the definition of $I_{\imath,\jmath}(h)$, we have

$$\oint_{L_h} x^k\,dt = \oint_{L_h} \frac{x^k\,dt}{1+ax+by+cx(x^2+y^2)} + \oint_{L_h} \frac{(a+ch)x^{k+1}\,dt}{1+ax+by+cx(x^2+y^2)}$$
$$+ \oint_{L_h} \frac{bx^k y\,dt}{1+ax+by+cx(x^2+y^2)}$$
$$= I_{k,0}(h) + (a+ch) I_{k+1,0}(h) + b I_{k,1}(h).$$

Hence, formulae (2.19) and (2.20) follow from the above formula. The proof is completed.

Similarly, by the definition of $I_{\imath,\jmath}(h)$, we have the following.

LEMMA 2.4. *It holds that*

$$I_{k,2i}(h) = \sum_{j=0}^{i} (-1)^j C_i^j h^{i-j} I_{k+2j,0}(h) \tag{2.21}$$

*and*

$$I_{k,2i+1}(h) = \sum_{j=0}^{i} (-1)^j C_i^j h^{i-j} I_{k+2j,1}(h). \tag{2.22}$$

LEMMA 2.5. *For $k \geq 0$, we have*

$$\sum_{i+j=2k-1} \Phi_{ij}(h) = \sum_{i=0}^{k} \tilde{a}_i^{(2k-1)} I_{2(k-i),i}(h) h^i + \sum_{i=0}^{k-1} \tilde{b}_i^{(2k-1)} I_{2(k-i)-1,i}(h) h^i \tag{2.23}$$

*and*

$$\sum_{i+j=2k} \Phi_{ij}(h) = \sum_{i=0}^{k} \tilde{a}_i^{(2k)} I_{2(k-i)+1,i}(h) h^i + \sum_{i=0}^{k} \tilde{b}_i^{(2k)} I_{2(k-i),i}(h) h^i, \tag{2.24}$$

*where*

$$\tilde{a}_i^{(2k)} = \sum_{j=i}^{k} (-1)^{j-i} C_j^i a_j^{(2k)}, \qquad 0 \leq i \leq k,$$

$$\tilde{b}_i^{(2k)} = \sum_{j=i}^{k} (-1)^{j-i} C_j^i b_j^{(2k)}, \qquad 0 \leq i \leq k,$$

$$\tilde{a}_i^{(2k-1)} = \sum_{j=i}^{k} (-1)^{j-i} C_j^i a_j^{(2k-1)}, \qquad 0 \leq i \leq k,$$

$$\tilde{b}_i^{(2k-1)} = \sum_{j=i}^{k} (-1)^{j-i} C_j^i b_j^{(2k-1)}, \qquad 0 \leq i \leq k-1,$$

*and*

$$a_j^{(2k)} = \begin{cases} a_{2k-2j,2j} + b_{2k-2j+1,2j-1}, & j > 0, \\ a_{2k,0}, & j = 0, \end{cases}$$

$$b_j^{(2k)} = \begin{cases} a_{2k-2j-1,2j+1} + b_{2k-2j,2j}, & j \neq k, \\ b_{0,2k}, & j = k, \end{cases}$$

$$a_j^{(2k-1)} = \begin{cases} a_{2k-2j-1,2j} + b_{2k-2j,2j-1}, & j > 0, \\ a_{2k-1,0}, & j = 0, \end{cases}$$

$$b_j^{(2k-1)} = \begin{cases} a_{2k-2j-2,2j+1} + b_{2k-2j-1,2j}, & j \neq k, \\ b_{2k-1,0}, & j = k. \end{cases}$$

PROOF. By the definition (2.5) of $\Phi_{ij}(h)$, we have

$$\begin{aligned} \sum_{i+j=2k} \Phi_{ij}(h) &= \sum_{i=1}^{k} \left(\Phi_{2k-2i,2i}(h) + \Phi_{2k-2i+1,2i-1}(h)\right) + \Phi_{2k,0}(h) \\ &= \sum_{i=0}^{k} \left\{ a_i^{(2k)} I_{2k-2i+1,2i}(h) + b_i^{(2k)} I_{2k-2i,2i+1}(h) \right\}, \end{aligned}$$

so the formula (2.24) follows from Lemma 2.4 and the above formula. Similarly, (2.23) can be proved too The proof is completed.

Furthermore, if $c \neq 0$ from the above lemmas, we have

$$\sum_{i+j=2k-1} \Phi_{ij}(h) = \frac{1}{\sqrt{1-\rho^2(h)h}\,(h^2+b^2)^{2k}} \left\{ P_{5k+1}^{(2k-1)}(h) + \sqrt{1-\rho^2(h)h}\,P_{5k-2}^{(2k-1)}(h) \right\} \tag{2.25}$$

and

$$\sum_{i+j=2k} \Phi_{ij}(h) = \frac{1}{\sqrt{1-\rho^2(h)h}\,(h^2+b^2)^{2k+1}} \left\{ P_{5k+2}^{(2k)}(h) + \sqrt{1-\rho^2(h)h}\,P_{5k+2}^{(2k)}(h) \right\}, \tag{2.26}$$

where $P_m^{(k)}(h)$ denotes a polynomial of $h$ of degree $m$ whose coefficients are linear combinations of $a_{ij}$, $b_{ij}$ with $i+j=k$, and $\rho^2(h) = (a+ch)^2 + b^2$.

Suppose $c = 0$ and let $\sqrt{1-\rho^2 h} = r$. By Lemma 2.5, we have the following.

LEMMA 2.6. *Suppose $c = 0$. For $k \geq 0$, we have*

$$\sum_{i+j=2k-1} \Phi_{ij}(h) = \frac{(1-r)^k}{r} \left\{ c_k^{(2k-1)} r^k + \cdots + c_1^{(2k-1)} r + c_0^{(2k-1)} \right\} \tag{2.27}$$

*and*

$$\sum_{i+j=2k} \Phi_{ij}(h) = \frac{(1-r)^k}{r} \left\{ c_{k+1}^{(2k)} r^{k+1} + \cdots + c_1^{(2k)} r + c_0^{(2k)} \right\}, \tag{2.28}$$

*where $c_j^{(2k-1)}$ and $c_j^{(2k)}$ are linear combinations of $(a_{ij}, b_{ij})$ with $i+j = 2k-1$ and $i+j = 2k$ respectively, and $c_k^{(2k-1)} \cdot c_{k+1}^{(2k)} \neq 0$.*

Now, we are in position to prove the main results.

PROOF OF THEOREM 2.1. In this following, we first suppose $n = 2s$. In this case, by (1.2), the Melinikov function $\Phi(h)$ of system (2.2) has the following form,

$$\begin{aligned} \Phi(h) &= \oint_{L_h} \frac{1}{(1+ax+by+cx(x^2+y^2))} \sum_{0 \leq i+j \leq 2s} \left( a_{ij} x^{i+1} y^j + b_{ij} x^i y^{j+1} \right) \\ &= \sum_{0 \leq i+j \leq 2s} \Phi_{ij}(h) = \sum_{k=1}^{s} \left( \sum_{i+j=2k-1} \Phi_{ij}(h) + \sum_{i+j=2k} \Phi_{ij}(h) \right) + \Phi_{0,0}(h). \end{aligned} \tag{2.29}$$

Then, by the above lemmas and formulae (2.25),(2.26), the function $\Phi(h)$ has the form,

$$\begin{aligned} \Phi(h) &= \sum_{k=0}^{s} \left\{ \frac{1}{\sqrt{1-\rho^2(h)h}(h^2+b^2)^{2k}} \left( P_{5k+1}^{(2k-1)}(h) + \sqrt{1-\rho^2(h)h}\,P_{5k-2}^{(2k-1)}(h) \right) \right. \\ &\quad \left. + \frac{1}{\sqrt{1-\rho^2(h)h}(h^2+b^2)^{2k+1}} \left( P_{5k+2}^{(2k)}(h) + \sqrt{1-\rho^2(h)h}\,P_{5k+2}^{(2k)}(h) \right) \right\} \\ &= \frac{1}{\sqrt{1-\rho^2(h)h}\,(h^2+b^2)^{2s+1}} \left( P_{5s+3}(h) + \sqrt{1-\rho^2(h)h}\,P_{5s}(h) \right). \end{aligned} \tag{2.30}$$

Here, $P_{5s+3}(h)$ and $P_{5s}(h)$ denote the polynomials of $h$ of degree $5s+3$ and $5s$, respectively, whose coefficients are linear combinations of $a_{ij}, b_{ij}$ with $0 \leq i+j \leq 2s$, and $\rho^2(h) = (a+ch)^2 + b^2$.

Obviously, all the zeros of (2.30) satisfy

$$(P_{5s+3}(h))^2 = (1-(a+ch)^2 h - b^2 h)(P_{5s}(h))^2.$$

Hence, the number of zeros of $\Phi(h)$ are not larger than $10s+6$.

For the case of $n = 2s - 1$, similarly, we can prove that the number of zeros of $\Phi(h)$ are not larger than $10s + 2$.

Notice that $\Phi(h) = 0$ at $h = 0$ in (2.30). Hence, from Lemma 1.3 we know that there exists an $\varepsilon_0 > 0$, such that when $0 < |\varepsilon| < \varepsilon_0$ and $(a_{ij}, b_{ij})$ satisfies $|a_{ij}| \le K, |b_{ij}| \le K$, system (2.2) has at most $5n + 5 + (1 - (-1)^n)/2$ limit cycles. The proof is completed.

PROOF OF THEOREM 2.2. Suppose $c = 0$. First, let $n = 2s$.

(1) By (1.2), (2.27), (2.28), and (2.29), the Melinikov function $\Phi(h)$ of system (2.2) has the following form,

$$\begin{aligned}\Phi(h) &= \oint_{L_h} \frac{1}{(1+ax+by)} \sum_{0\le i+j\le 2s} \left(a_{ij}x^{i+1}y^j + b_{ij}x^i y^{j+1}\right) \\ &= \sum_{k=1}^{s} \frac{(1-r)^k}{r} \left\{ \left(c_k^{(2k-1)} r^k + \cdots + c_1^{(2k-1)} r + c_0^{(2k-1)}\right) \right. \\ &\quad \left. + \left(c_{k+1}^{(2k)} r^{k+1} + \cdots + c_1^{(2k)} r + c_0^{(2k)}\right)\right\} + a_{00} I_{1,0} + b_{00} I_{0,1} \\ &= \frac{1-r}{r}\left\{c_{2s} r^{2s} + \cdots + c_1 r + c_0\right\} \\ &= \frac{1-r}{r} P_{2s}(r),\end{aligned} \tag{2.31}$$

where $P_{2s}(r)$ is a polynomial of $r$ of degree $2s$.

Hence, from Lemma 1.3, we know that there exists an $\varepsilon_0 > 0$, such that when $0 < |\varepsilon| < \varepsilon_0$ and $(a_{ij}, b_{ij})$ satisfies $|a_{ij}| \le K, |b_{ij}| \le K$, system (2.2) has at most $2s$ limit cycles.

(2) From the proof of Theorem 2.1 we know that the coefficients $c_{2s}, \ldots, c_1, c_0$ of (2.31) satisfy

$$\begin{aligned} c_{2s} &= L\left(c_{s+1}^{(2s)}\right), \\ c_{2s-1} &= L\left(c_{s+1}^{(2s)}, c_s^{(2s-1)}\right), \\ &\vdots \\ c_1 &= L\left(c_{s+1}^{(2s)}, c_s^{(2s-1)}, \ldots, c_1^{(1)}\right), \\ c_0 &= L\left(c_{s+1}^{(2s)}, c_s^{(2s-1)}, \ldots, c_1^{(1)}, c_0^{(0)}\right), \end{aligned} \tag{2.32}$$

where $L(\cdots)$ denotes the linear function and $c^{(k)}$ is linear combinations of $a_{ij}, b_{ij}$ with $i + j = k$, for $k = 0, 1, \ldots, 2s$

From (2.32), we have that the linear map,

$$K : (a_{ij}, b_{ij}) \to (c_{2s}, c_{2s-1}, \cdot \cdot, c_1, c_0),$$

is surjective. That is, the coefficients $c_{2s}, c_{2s-1}, \ldots, c_1, c_0$ of $P_{2s}(r)$ are independently varied, so there exists $(a_{ij}^0, b_{ij}^0)$, such that $P_{2s}(r)$ has exact $2s$ simple zeros for $\varepsilon_0 > 0$ and $|a_{ij} - a_{ij}^0|, |b_{ij} - b_{ij}^0|$ small. Then, from Lemma 1.3 and Remark 1.1, we have proved that there exists an $\varepsilon_0 > 0$, such that for $0 <| \varepsilon |< \varepsilon_0$, $|a_{ij} - a_{ij}^0| < \varepsilon_0$, $|a_{ij} - a_{ij}^0| < \varepsilon_0$, system (2.2) has precisely $2s$ limit cycles.

For the case of $n = 2s - 1$, a similar proof can be given.

This finishes the proof.

REMARK 2.1. In fact, for the system,

$$\begin{aligned}\dot{x} &= y\left(1 + ax + by + c\left(x^2 + y^2\right)\right), \\ \dot{y} &= -x\left(1 + ax + by + c\left(x^2 + y^2\right)\right),\end{aligned}$$

where $a, b, c$ are real with $a^2 + b^2 \neq 0$, $c \neq 0$, we can use a similar argument to prove that the number of limit cycles of the above system under the perturbations of the polynomials of order $n$, is at most $N$, where

$$N = \begin{cases} 2n+1, & a^2 + b^2 \neq 4c, \\ n, & a^2 + b^2 = 4c. \end{cases}$$

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