INFORMATION AND CONTROL 27, 289–328 (1975)

Minimal Representations of Some Classes of Dynamic Programming

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It is known that various discrete optimization problems can be represented by finite state models called sequential decision processes (sdp's). A subclass of sdp's, the class of monotone sdp's (msdp's), is particularly important since the method of dynamic programming is applicable to obtain optimal policies. Several subclasses of msdp's have also been introduced from the viewpoint of computational complexity for obtaining optimal policies. For each of these classes of sdp's, optimal policies are usually obtained (if possible at all) in fewer steps if a given optimization problem is represented by a model with fewer states.

Thus we are naturally led to the problem of finding a minimal (with the fewest states) representation of a given optimization problem by an sdp of a specified class. This paper investigates the existence or nonexistence of such minimization algorithms (in the sense of the theory of computation) for various classes of sdp's. It is shown that there exist minimization algorithms for some classes of sdp's, but there exist no algorithms for others.

The nonuniqueness of a minimal representation is also proved for each class of sdp's.

1. INTRODUCTION

In order to discuss the dynamic programming on a rigorous mathematical basis, it is commonly taken to consider the essence of dynamic programming as a certain monotonicity property possessed by some classes of sequential decision processes (sdp's) (Bellman, 1957; Mitten, 1964; Nemhauser, 1966; Denardo, 1967; Karp and Held, 1967; Elmaghraby, 1970; Ibaraki, 1972, 1973a, 1973b, 1974.) An sdp is a system consisting of finite states, a transition rule from one state to another corresponding to each decision, and a cost function associated with each transition.

Karp and Held (1967) showed that the functional equations of dynamic programming hold if an sdp satisfies a certain monotonicity condition. Thus representing a given problem by an sdp of this class may be identified with a formulation of the problem by dynamic programming. However, it was shown in Ibaraki (1974) that there exists no algorithm (in the sense of the theory of computation) for solving the resulting functional equations of dynamic programming for an arbitrarily given such sdp, under somewhat different mathematical formulation. To avoid this difficulty, Ibaraki (1973a) introduced three subclasses of sdp's for which the functional equations are always solvable to obtain an optimal policy. For these classes it was also demonstrated that such functional equations can be solved in fewer steps if the sdp under consideration has fewer states.

Therefore we are naturally led to the problem of finding a representation of a given problem by an sdp (of the class under consideration) with the fewest states. In this paper, we will consider algorithms for solving such minimization problems for various classes of sdp's. For some classes, it will be shown that such algorithms exist, whereas no such algorithms exist for others. These results may also be interpreted as indicating an aspect of the complexity hierarchy of various classes of sdp's.

In the following discussion, we are primarily concerned with the existence of an algorithm for each minimization problem. Thus, attention is not paid to the efficiency of the resulting algorithm obtained for a solvable problem. Attempts for improving them, however, are currently under way and some results will be reported elsewhere.

2. Definitions

This section introduces various definitions and notations which will be used in the subsequent discussions.

We assume that a problem is originally given in the form of a *discrete decision process* (ddp) (Karp and Held, 1967; Ibaraki, 1972) or a *recursive* ddp (r-ddp) (Ibaraki, 1974), which is considered as a general description of a discrete deterministic optimization problem. A ddp Υ is the system (Σ, S, f), where

Σ: a finite nonempty set of *primitive decisions* (alphabet); Σ^* denotes the set of all *policies* (strings) obtained by concatenating decisions $\in \Sigma$; ϵ stands for the null policy (string), i.e., $(\forall x \in \Sigma^*)(x\epsilon = \epsilon x = x)$.

$S \subset \Sigma^*$: a set of feasible policies of Υ;

$f: S \to E$, where E is the set of real numbers; f is called the *cost function* of Υ.

A policy $x \in \Sigma^*$ is *feasible* if $x \in S$, and *optimal* if $x \in S \wedge (\forall y \in S)(f(x) \leqslant f(y))$. The set of optimal policies of Y is denoted by $O(Y)$.

A *recursive* ddp (r-ddp) Y is a ddp with the additional restrictions that (1) S is regular (see the definition of a finite automaton below), (2) $f(x)$ takes on only integral values for $x \in S$, and (3) f: $\Sigma^* \to Z$ is a partial recursive function on Σ^* with $\mathrm{Dom}(f)(\equiv\{x \mid f(x) \text{ is defined}\}) = S$, where Z denotes the set of all integers. Here f: $\Sigma^* \to Z$ is a *partial recursive function* if f': $Z_+^2 \to Z_+^2$ (Z_+ denotes the set of nonnegative integers and $Z_+^2 = Z_+ \times Z_+$) defined by $f'(\varphi(gn(x))) = \varphi(f(x))$ is a partial recursive function in the ordinary sense (e.g., Davis, 1958) with $\mathrm{Dom}(f') = \varphi(gn(\mathrm{Dom}(f)))$. gn: $\Sigma^* \to Z$ is Gödel numbering (e.g., Davis, 1958) that is a one-to-one mapping from Σ^* to the set of positive integers, and φ, a one-to-one mapping from Z to Z_+^2, is for example given by

$$\varphi(\xi) = \begin{cases} (2, \xi) & \text{if } \xi \geqslant 0 \\ (1, |\xi|) & \text{if } \xi < 0. \end{cases}$$

A partial recursive function f with $\mathrm{Dom}(f) = S$ may be intuitively considered as a function whose value $f(x)$ can be computed in a finite number of steps if $x \in S$. A partial recursive function f with $\mathrm{Dom}(f) = \Sigma^*$ (i.e., total) is said *recursive*.

A *finite automaton* (fa) M is the system $(Q, \Sigma, q_0, \lambda, Q_F)$, where

- Q: a finite nonempty set of states,
- Σ: a finite nonempty alphabet and may be identified with Σ of a ddp Y,
- $q_0 \in Q$: an initial state of M,
- λ: $Q \times \Sigma \to Q$ is a state transition function,
- $Q_F \subset Q$: a set of final states.

λ can be extended to $Q \times \Sigma^* \to Q$ inductively by

$$(\forall q \in Q)(\forall x \in \Sigma^*)(\forall a \in \Sigma)(\lambda(q, \epsilon) = q \wedge \lambda(q, xa) = \lambda(\lambda(q, x), a)).$$

$\bar{\lambda}(x) \equiv \lambda(q_0, x)$ is used for convenience. $F(M) = \{x \mid \bar{\lambda}(x) \in Q_F\}$ denotes the set of strings *accepted* by M. $B \subset \Sigma^*$ is said *regular* if there exists an fa M such that $B = F(M)$. Some other properties of fa's and regular sets will be discussed in Section 5.

A (finite-state) *sequential decision process* (sdp)Π is the system (M, h, ξ_0), where

M: an fa $(Q, \Sigma, q_0, \lambda, Q_F)$,

h: $E \times Q \times \Sigma \to E$, and h is called the *cost function* of Π,

$\xi_0 \in E$: an initial cost value of the initial state q_0.

h can be extended to $h: E \times Q \times \Sigma^* \to E$ by

$$(\forall \xi \in E)(\forall q \in Q)(\forall x \in \Sigma^*)(\forall a \in \Sigma)(h(\xi, q, \epsilon) = \xi \wedge h(\xi, q, xa) = h(h(\xi, q, x), \lambda(q, x), a)).$$

$\bar{h}(x) \equiv h(\xi_0, q_0, x)$ is used for convenience. The set of *feasible policies* of an sdp Π is given by $F(\Pi) = F(M)$, and the set of *optimal policies* of Π by $O(\Pi) = \{x \in F(\Pi) \mid (\forall y \in F(\Pi))(\bar{h}(x) \leqslant \bar{h}(y))\}$. An sdp can be considered as a general model of discrete deterministic decision processes with finite states.

A recursive sdp (r-sdp) $\Pi = (M, h, \xi_0)$ is an sdp with additional restrictions that (1) h takes on only integral values, i.e., $h: Z \times Q \times \Sigma \to Z$, and (2) h is a partial recursive function with $\mathrm{Dom}(h) \supset L_\Pi$, where

$$L_\Pi = \{(\bar{h}(x), \bar{\lambda}(x), a) \mid x \in \Sigma^*, a \in \Sigma\}.$$

The partial recursiveness of h is defined similarly to f of Y (Ibaraki, 1974). Obviously, $\bar{h}$ of any r-sdp Π is a recursive function on Σ^* (i.e., $\bar{h}(x)$ can be computed in a finite number of steps for any $x \in \Sigma^*$).

If h of an sdp $\Pi = (M, h, \xi_0)$ satisfies

$$(\forall \xi_1, \xi_2 \in E)(\forall q \in Q)(\forall a \in \Sigma)(\xi_1 \leqslant \xi_2 \Rightarrow h(\xi_1, q, a) \leqslant h(\xi_2, q, a)),$$

h is said *monotone* and Π is called a *monotone* sdp (msdp). Of course, h: monotone $\Leftrightarrow (\forall \xi_1, \xi_2 \in E)(\forall q \in Q)(\forall x \in \Sigma^*)(\xi_1 \leqslant \xi_2 \Rightarrow h(\xi_1, q, x) \leqslant h(\xi_2, q, x))$. In particular, $(\forall x, y \in \Sigma^*)(\bar{\lambda}(x) = \bar{\lambda}(y) \wedge \bar{h}(x) \leqslant \bar{h}(y) \Rightarrow (\forall z \in \Sigma^*)(\bar{h}(xz) \leqslant \bar{h}(yz)))$.

If h of an r-dsp $\Pi = (M, h, \xi_0)$ satisfies

$$(\forall (\xi_1, q, a), (\xi_2, q, a) \in L_\Pi)(\xi_1 \leqslant \xi_2 \Rightarrow h(\xi_1, q, a) \leqslant h(\xi_2, q, a)),$$

h is said *monotone* and Π is called a monotone r-sdp (r-msdp). For an r-msdp $\Pi = (M, h, \xi_0)$,

$$(\forall x, y \in \Sigma^*)(\bar{\lambda}(x) = \bar{\lambda}(y) \wedge \bar{h}(x) \leqslant \bar{h}(y) \Rightarrow (\forall z \in \Sigma^*)(\bar{h}(xz) \leqslant \bar{h}(yz)))$$

holds.

In Karp and Held (1967), many interesting problems in operations research are formulated as sdp's and msdp's.

For msdp and r-msdp, it is known that the functional equations of dynamic programming hold (Karp and Held, 1967; Ibaraki, 1974), and hence they may be considered as general models of sequential decision processes to which the method of dynamic programming is applicable.

Three subclasses of the class of r-msdp's are now introduced (Ibaraki, 1973a). Let

$$\Pi = (M, h, \xi_0)$$

be an r-mdsp. Π is a *loop-free* r-msdp (r-lmsdp) if $F(\Pi)$ is finite. An r-lmsdp may be considered as a generalization of multistage decision processes. Π is a *strictly monotone* r-sdp (r-smsdp) if h of Π satisfies

$$(\forall(\xi_1, q, a), (\xi_2, q, a) \in L_\Pi)(\xi_1 < \xi_2 \Rightarrow h(\xi_1, q, a) < h(\xi_2, q, a)).$$

In this case,

$$(\forall x, y \in \Sigma^*)(\bar{\lambda}(x) = \bar{\lambda}(y) \wedge h(x) < h(y) \Rightarrow (\forall z \in \Sigma^*)(\bar{h}(xz) < \bar{h}(yz)))$$

holds. Π is a *positively monotone* r-sdp (r-pmsdp) if h of Π satisfies

$$(\forall(\xi, q, a) \in L_\Pi)(h(\xi, q, a) \geqslant \xi).$$

Obviously, $(\forall x, y \in \Sigma^*)(\bar{h}(x) \leqslant \bar{h}(xy))$ holds for an r-pmsdp. An r-pmsdp may be considered as a generalization of shortest path problems with non-negative arc lengths.

Let $Y = (\Sigma, S, f)$ be an r-ddp and $\Pi = (M, h, \xi_0)$ be an r-sdp. Then Π *weakly represents* (w-represents) Y if $O(\Pi) = O(Y)$ holds. Π *strongly represents* (s-represents) Y if $F(\Pi) = S \wedge (\forall x \in S)(\bar{h}(x) = f(x))$. Two r-sdp's $\Pi_1 = (M_1, h_1, \xi_{01})$ and $\Pi_2 = (M_2, h_2, \xi_{02})$ are *weakly equivalent* (w-equivalent) if $O(\Pi_1) = O(\Pi_2)$. Π_1 and Π_2 are *strongly equivalent* (s-equivalent) if $F(\Pi_1) = F(\Pi_2) \wedge (\forall x \in F(\Pi_1))(\bar{h}_1(x) = \bar{h}_2(x))$. An r-sdp Π is a *minimal w-representation* of an r-ddp Y by an r-sdp if Π w-represents Y and there exists no r-sdp Π' which is w-equivalent to Π and has fewer states than Π. Similarly an r-sdp Π is a *minimal s-representation* of an r-ddp Y by an r-sdp if Π s-represents Y and there exists no r-sdp Π' which is s-equivalent to Π and has fewer states than Π.

The above concepts can be similarly defined for other classes of sdp's such as r-msdp, r-lmsdp, r-smsdp, and r-pmsdp.

Since we will be concerned with the solvability (the existence of an algorithm) or the unsolvability of each minimization problem, ddp, sdp, and msdp will not be explicitly considered in the subsequent discussion, since the computability (recursiveness) of cost functions is not assumed in these models.

3. Problem Statement

It is now possible to present the precise meaning of minimization problems which will be discussed in this paper. Our final object is of course to find a minimal w (or s)-representation of a given r-ddp by an sdp of the specified class such as r-sdp, r-msdp, and so forth. For that, we want to know whether an algorithm, which obtains a minimal w (or s)-representation of any r-ddp by an r-sdp of the specified class, exists or not.

We see that two types of minimization problems (and algorithms to solve them) are conceivable for each class of problems. The first type is the one which first decides whether a given r-ddp can be w (or s)-represented by an r-sdp of the specified class, and then gives a minimal w (or s)-representation by an r-sdp of the specified class in case it is w (or s)-representable. On the other hand, the second type assumes that an r-sdp of the specified class Π w (or s)-representing a given r-ddp is given, and then finds a minimal r-sdp of the specified class w (or s)-equivalent to Π. The problem of the first type is of course more difficult than the second in the sense that the algorithm of the first type can be used in place of an algorithm of the second type.

A number of minimization problems are now defined depending on whether the representation is w or s, whether the specified class is r-sdp, r-msdp, r-lmsdp, r-smsdp, or r-pmsdp, and whether the type of algorithm is the first one or the second one.

As summarized in Table 1 in Section 10, some of these minimization problems are solvable, while the rest are all unsolvable.

Finally it should be noted here that a minimal representation by each class of r-sdp's is not usually unique, as shown in Section 11. Thus the algorithms presented in this paper provide only one of the minimal representations, even if the minimization problem under consideration is solvable.

Before proceeding to the minimization problems, we need two preparatory sections, one outlines some of the results obtained in the earlier papers, and the other summarizes fundamental properties of regular sets.

4. Further Definitions and Review of the Earlier Results

Let R be an equivalence relation on Σ^*. R is *right invariant* if $(\forall x, y \in \Sigma^*)(xRy \Rightarrow (\forall z \in \Sigma^*)(xzRyz))$. For equivalence relations R and T, T *refines* R if $(\forall x, y \in \Sigma^*)(xTy \Rightarrow xRy)$, and this is denoted by $T \leqslant R$. Let $B \subset \Sigma^*$. If $(\forall x, y \in \Sigma^*)(xRy \Rightarrow (x \in B \Leftrightarrow y \in B))$, then R *refines* B. For $B \subset \Sigma^*$ and an equivalence relation R, B/R stands for the set of equivalence classes of B under R. $| B/R |$ is the number of equivalence classes in B/R. $\Lambda(B)$ denotes the set of right invariant equivalence relations which refine B. In particular, $\Lambda(\Sigma^*)$ denotes the set of right invariant equivalence relations.

For equivalence relations R_1 and R_2, $R = R_1 \wedge R_2$ is defined by $(\forall x, y \in \Sigma^*)(xRy \Leftrightarrow xR_1y \wedge xR_2y)$. Obviously (e.g., Ibaraki, 1972), $R \in \Lambda(\Sigma^*)$ if $R_1, R_2 \in \Lambda(\Sigma^*)$. R refines B if either R_1 or R_2 refines B.

For $B \subset \Sigma^*$, define the equivalence relation R_B by $(\forall x, y \in \Sigma^*)(xR_By \Leftrightarrow (\forall z \in \Sigma^*)(xz \in B \Leftrightarrow yz \in B)$ (or equivalently $\{x\}\backslash B = \{y\}\backslash B$, where $A\backslash C = \{y \mid (\exists x \in A)(xy \in C)\}$)). R_B satisfies $R_B \in \Lambda(B)$ and $R \leqslant R_B$ for any $R \in \Lambda(B)$.

A subset of $\Lambda(B)$, $\Lambda_F(B)$, is particularly important in automata theory, where $\Lambda_F(B) = \{R \in \Lambda(B) \mid | \Sigma^*/R | < \infty\}$. It is known (Rabin and Scott, 1960) that (1) $\Lambda_F(B)$ is nonempty if and only if B is regular, and (2) each $C_j \in \Sigma^*/T$ is regular for $T \in \Lambda_F(B)$.

The next lemma proved in Ibaraki (1974) plays a fundamental role in deriving minimal representations.

Lemma 4.1. *Let $h': \Sigma^* \to Z$ be a given recursive function. Then there exists an* r-msdp $\Pi = (M, h, \xi_0)$ *satisfying* $(\forall x \in \Sigma^*)(\bar{h}(x) = h'(x))$ *and having n states if and only if there exists $T \in \Lambda_F(\Sigma^*)$ with $| \Sigma^*/T | = n$ such that* $(\forall x, y \in \Sigma^*)(xTy \wedge h'(x) \leqslant h'(y) \Rightarrow (\forall z \in \Sigma^*)(h'(xz) \leqslant h'(yz)))$. *(In fact, the standard construction (see Section 5 for its definition) of T may be used as fa M.)*

Let $\Psi = \{U_i \subset \Sigma^* \mid i = 1, 2,..., m\}$, where U_i are mutually disjoint. Then $T \in \Lambda(\Sigma^*)$ *jointly separates* (J-separates) Ψ if $(\forall x, y \in \Sigma^*)(x \in U_i \wedge y \in U_j \wedge xTy \Rightarrow i = j)$, i.e., each $C_k \in \Sigma^*/T$ intersects at most one $U_i \in \Psi$. In particular, T must satisfy $| \Sigma^*/T | \geqslant m$ to J-separate Ψ.

For $U \subset \Sigma^*$, define a binary relation $\preccurlyeq_U$ on Σ^* by $(\forall x, y \in \Sigma^*)(x \preccurlyeq_U y \Leftrightarrow (\forall z \in \Sigma^*)(yz \in U \Rightarrow xz \in U)$(or equivalently $x \preccurlyeq_U y \Leftrightarrow \{x\}\backslash U \supset \{y\}\backslash U$)). $\preccurlyeq_U$ can also be defined on B/R, where $B \subset \Sigma^*$ and $R \in \Lambda(U)$, by $(\forall A_i, A_j \in B/R)(A_i \preccurlyeq_U A_j \Leftrightarrow (\exists x \in A_i)(\exists y \in A_j)(x \preccurlyeq_U y))$(or equivalently $A_i \preccurlyeq_U A_j \Leftrightarrow (\forall x \in A_i)(\forall y \in A_j)(x \preccurlyeq_U y)$). $\preccurlyeq_U$ on Σ^* or on B/R is a pseudo ordering,[1] while $\leqslant_U$ on B/R_U is a partial ordering.[1] A set $B \subset \Sigma^*$ is *monotone* with respect to U if $(\forall x, y \in B)(x \leqslant_U y \vee y \leqslant_U x)$. Obviously this is equivalent to saying that $\leqslant_U$ defined on B/R_U is a total ordering.[1]

The next w-representation theorem was obtained in Ibaraki (1972).

THEOREM 4.2. *Let Y be a ddp with $U \equiv O(Y)$. Then there exists an sdp w-representing Y if and only if there exists $T \in \Lambda_F(\Sigma^*)$ such that (i) T J-separates U/R_U. Furthermore there exists an msdp w-representing Y if and only if there exists $T \in \Lambda_F(\Sigma^*)$ which satisfies (i) and for which (ii) each equivalence class $C_i \in \Sigma^*/T$ is monotone with respect to U.*

To obtain an s-representation counterpart of Theorem 4.2, define an equivalence relation R_Y for a ddp $Y = (\Sigma, S, f)$ by $(\forall x, y \in \Sigma^*)(xR_Y y \Leftrightarrow xR_S y \wedge (\forall xz, yz \in S)(f(xz) = f(yz)))$. (Note that if $xR_Y y$, then $(\forall z \in \Sigma^*)(xz \in S \Leftrightarrow yz \in S)$ holds since $xR_S y$.) As proved in Ibaraki (1972), $R_Y \in \Lambda(S)$ holds. Let

$$\Psi_p = \{A_j \in S/R_Y \mid (\forall x \in A_j)(f(x) = p)\}.$$

Define an ordering relation $\leqslant_Y$ on Σ^* by

$$(\forall x, y \in \Sigma^*)(x \leqslant_Y y \Leftrightarrow xR_S y \wedge (\forall xz, yz \in S)(f(xz) \leqslant f(yz))),$$

for a ddp $Y = (\Sigma, S, f)$. $\leqslant_Y$ can also be defined on B/R, where $B \subset \Sigma^*$ and $R \in \Lambda(S) \wedge R \leqslant R_Y$, by

$$(\forall A_i, A_j \in B/R)(A_i \leqslant_Y A_j \Leftrightarrow (\exists x \in A_i)(\exists y \in A_j)(x \leqslant_Y y)).$$

Although $\leqslant_Y$ on B/R is a pseudo ordering, $\leqslant_Y$ on B/R_Y is a partial ordering. A set $B \subset \Sigma^*$ is *monotone* with respect to Y if $(\forall x, y \in B)(x \leqslant_Y y \vee y \leqslant_Y x)$ (i.e., $\leqslant_Y$ on B/R_Y is a total ordering).

THEOREM 4.3 (Karp and Held, 1967; Ibaraki, 1972). *Let $Y = (\Sigma, S, f)$ be a ddp. Then there exists an sdp s-representing Y if and only if there exists $T \in \Lambda_F(S)$ such that (i) T J-separates Ψ_p for every $p \in E$. Furthermore, there exists an msdp s-representing Y if and only if there exists $T \in \Lambda_F(S)$ which satisfies (i) and for which (ii) each equivalence class $C_i \in \Sigma^*/T$ is monotone with respect to Y.*

[1] A binary relation $\leqslant$ on set A is a *pseudo ordering* if (i) $(\forall x \in A)(x \leqslant x)$ and (ii) $(\forall x, y, z \in A)(x \leqslant y \wedge y \leqslant z \Rightarrow x \leqslant z)$ hold. It is a *partial ordering* if (iii) $(\forall x, y \in A)(x \leqslant y \wedge y \leqslant x \Rightarrow x = y)$ holds in addition to (i) and (ii). It is a *total ordering* if (iv) $(\forall x, y \in A)(x \leqslant y \vee y \leqslant x)$ holds in addition to (i), (ii), and (iii).

As noticed in Ibaraki (1974), it is difficult to directly extend the above representation theorems to the case of r-msdp. However, it is of course possible to use the conditions in the above theorems as necessary conditions for an r-ddp to be w (or s)-represented by an r-msdp.

Representation theorems for three subclasses of the class of r-msdp's are now given (Ibaraki, 1973a).

THEOREM 4.4. *Let $Y = (\Sigma, S, f)$ be an r-ddp. Then (1) Y is w-representable by an r-lmsdp if and only if $O(Y)$ is finite, and (2) Y is s-representable by an r-lmsdp if and only if S is finite.*

THEOREM 4.5. *Let $Y = (\Sigma, S, f)$ be an r-ddp. Then Y is w-representable by an r-smsdp and by an r-pmsdp, respectively, if and only if $O(Y)$ is regular.*

The next theorem is also important (Ibaraki, 1973a).

THEOREM 4.6. *There exist algorithms to obtain $O(\Pi)$ for any r-lmsdp, r-smsdp, and r-pmsdp, respectively.*

Before concluding this section, we add a further property of $\preccurlyeq_U$ and $\preccurlyeq_Y$. Let $\preccurlyeq$ be a binary relation on Σ^*/R, where $R \in \Lambda(\Sigma^*)$. $\preccurlyeq$ is *right invariant* on Σ^*/R if $(\forall[x], [y] \in \Sigma^*/R)([x] \preccurlyeq [y] \Rightarrow (\forall z \in \Sigma^*)([xz] \preccurlyeq [yz]))$, where $[w]$ denotes the equivalence class of Σ^*/R containing $w \in \Sigma^*$.

PROPOSITION 4.7 (Ibaraki, 1972). *Let $U \subset \Sigma^*$. Then $\preccurlyeq_U$ on Σ^*/R, $R \in \Lambda(U)$, is right invariant.*

PROPOSITION 4.8 (Ibaraki, 1972). *Let Y be a ddp. Then $\preccurlyeq_Y$ on Σ^*/R, $R \in \Lambda(\Sigma^*) \wedge R \leqslant R_Y$, is right invariant.*

5. PROPERTIES OF REGULAR SETS

A regular set was defined in Section 2 in terms of an fa. This section provides some properties of regular sets. Since most of them are known in automata theory, we omit all proofs except for those given for new results. Omitted proofs may be found in Rabin and Scott (1960) or in textbooks such as Booth (1967), Hopcroft and Ullman (1970), and Harrison (1965). These properties will be used in the subsequent discussion.

As exhibited in Section 4, a regular set $B \subset \Sigma^*$ and $\Lambda_F(B)$ are closely related. In fact it is possible to construct an fa M satisfying $F(M) = B$ from any $T \in \Lambda_F(B)$. Let $T \in \Lambda_F(B)$ and define fa $M = (Q, \Sigma, q_0, \lambda, Q_F)$ such that $Q = \{[C_i] \mid C_i \in \Sigma^*/T\}$, $q_0 = [\epsilon]$, and $\lambda: Q \times \Sigma \to Q$ satisfies $(\forall x \in \Sigma^*)(\forall a \in \Sigma)(\lambda([x], a) = [xa])$, where $[x]$ denotes the state $[C_i] \in Q$ corresponding to $C_i \in \Sigma^*/T$ containing x. (Q_F of M is not explicitly specified.) This M is called the *standard construction* of T.

PROPOSITION 5.1. *Let $B \subset \Sigma^*$ be a regular set and let $M = (Q, \Sigma, q_0, \lambda, Q_F)$ be the standard construction of $T \in \Lambda_F(B)$. Then* (1) $(\forall x, y \in \Sigma^*)(\bar{\lambda}(x) = \bar{\lambda}(y) \Leftrightarrow xTy)$, (2) *if we let $Q_F = \{[C_i] \mid C_i \in B/T\}$, then $F(M) = B$, and* (3) *if $T = R_B$ holds and Q_F is defined as in* (2), *then M is the minimal fa* (*i.e., with the fewest states*) *which accepts B. Furthermore, the minimal fa accepting B is unique up to isomorphism* (*i.e., a renaming of the states*).

PROPOSITION 5.2. *Let $M = (Q, \Sigma, q_0, \lambda, Q_F)$ be an fa. Let T be defined by $(\forall x, y \in \Sigma^*)(xTy \Leftrightarrow \bar{\lambda}(x) = \bar{\lambda}(y))$, then $T \in \Lambda_F(F(M))$ and M is the standard construction of T.*

Some closure properties of the class of regular sets relevant to our discussion are summarized next.

PROPOSITION 5.3. *Let A, $B \subset \Sigma^*$ be regular. Then $A \cap B$, $A \cup B$, $\bar{A}(=\Sigma^* - A)$, and $A \backslash B(=\{y \mid (\exists x \in A)(xy \in B)\}$ are also regular.*

It is known in the literature that many decision problems concerning regular sets and fa's are solvable. The next proposition lists some of solvable decision problems.

PROPOSITION 5.4. *There exist algorithms for solving the following problems.* (1) *Decide if a regular set B is empty, finite, or infinite.* (2) *Decide if $A = B$ holds for two regular sets A and B.* (3) *Decide if $A \subset B$ holds for two regular set A and B.* (4) *Obtain fa's accepting $A \cap B$, $A \cup B$, $\bar{A}$, and $A \backslash B$ for regular sets A and B.* (5) *Obtain all equivalence classes of B/R_U for regular sets B and U.* (*Note that all equivalence classes in B/R_U are regular.*) (6) *Obtain the standard construction of $T \in \Lambda_F(B)$.* (7) *Obtain $T \in \Lambda_F(F(M))$* (*i.e., all equivalence classes of Σ^*/T*) *satisfying $(\forall x, y \in \Sigma^*)(xTy \Leftrightarrow \bar{\lambda}(x) = \bar{\lambda}(y))$ for any fa $M = (Q, \Sigma, q_0, \lambda, Q_F)$.*

From these results it is not difficult to see the followings.

PROPOSITION 5.5. *There exist algorithms for solving the following problems.* (1) *Decide if a given $T \in \Lambda_F(\Sigma^*)$ J-separates B/R_U, where B and U are regular.*

(2) *Decide if* $A \leqslant_U B$ *holds for regular sets* A, B, *and* U. (3) *Decide if all* $C_i \in \Sigma^*/T$, $T \in \Lambda_F(\Sigma^*)$ *is monotone with respect to a regular set* U.

Proof. We give an outline of the algorithm for solving (1). Other cases can be similarly treated. Let $\Sigma^*/T = \{C_1, C_2, ..., C_n\}, B/R_U = \{B_1, B_2, ..., B_m\}$. Decide if

$$(\exists C_j \in \Sigma^*/T)(\forall B_i, B_k \in B/R_U)(C_j \cap B_i \neq \varnothing \wedge C_j \cap B_k \neq \varnothing \wedge i \neq k).$$

If such C_j exists, T does not J-separate B/R_U; otherwise T J-separates B/R_U. Of course this can be decided in a finite number of steps by Proposition 5.4 (4) and (1) since $|\Sigma^*/T|$ and $|B/R_U|$ are both finite. Q.E.D.

Now let $M = (Q, \Sigma, q_0, \lambda, Q_F)$ be an fa and let $T \in \Lambda_F(\Sigma^*)$ be given by $(\forall x, y \in \Sigma^*)(xTy \Leftrightarrow \bar{\lambda}(x) = \bar{\lambda}(y))$. Assume that a binary relation $\leqslant$ is defined on Σ^*/T. Let $M' = (Q', \Sigma, q_0', \lambda', Q_F')$ be also an fa and let $T' \in \Lambda_F(\Sigma^*)$ be given by $(\forall x, y \in \Sigma^*)(xT'y \Leftrightarrow \bar{\lambda}'(x) = \bar{\lambda}'(y))$. Then *M′ covers M* with respect to $\leqslant$ if $(\forall A_i, A_k \in \Sigma^*/T)(\forall C_j \in \Sigma^*/T')(C_j \cap A_i \neq \varnothing \wedge C_j \cap A_k \neq \varnothing \Rightarrow A_i \leqslant A_k \vee A_k \leqslant A_i)$ holds. An fa M' is a *minimal cover* of M with respect to $\leqslant$ (satisfying condition A) if M' covers M with respect to $\leqslant$ (and satisfies condition A), and there exists no fa M'' with fewer states than M' which covers M with respect to $\leqslant$ (and satisfies condition A). Condition A in this statement can be used to represent any other conditions which each fa is required to satisfy. For example, in Section 6, the condition that fa M' satisfies $|F(M')| < \infty$ is used as condition A.

PROPOSITION 5.6. *There exists an algorithm to obtain a minimal cover of* M *with respect to* $\leqslant$, *satisfying condition* A, *for any fa* $M = (Q, \Sigma, q_0, \lambda, Q_F)$ *and binary relation* $\leqslant$ *on* Σ^*/T, *where* T *is given by* $(\forall x, y \in \Sigma^*)(xTy \Leftrightarrow \bar{\lambda}(x) = \bar{\lambda}(y))$, *under the assumption that* (*i*) *there is an algorithm to decide whether an fa* M' *satisfies condition* A, *and that* (*ii*) *there is at least one fa which covers* M *with respect to* $\leqslant$ *and satisfies condition* A.

Proof. First note that there exists an algorithm to decide if an fa M' covers M with respect to $\leqslant$ and satisfies condition A. This is because the property $(\forall A_i, A_j \in \Sigma^*/T)(\forall C_j \in \Sigma^*/T')(C_j \cap A_i \neq \varnothing \wedge C_j \cap A_k \neq \varnothing \Rightarrow A_i \leqslant A_k \vee A_k \leqslant A_i)$ can be checked in a finite number of steps since $|\Sigma^*/T|$ and $|\Sigma^*/T'|$ are both finite and A_i, A_k, C_j are all regular. (Note also that there exists an algorithm to decide whether M' satisfies condition A, by assumption.) Next let $\{M_1, M_2, ...\}$ be an effective enumeration of all fa's with alphabet Σ, in the nondecreasing order of the number of states. Although we will omit the details, such an enumeration obviously exists since there are only finite number of fa's with a fixed number of states. (Thus consider an enumeration which first enumerates all fa's with one state, then all fa's with two states and so on.) Then the next algorithm finds a minimal cover of M with respect to $\preccurlyeq$, satisfying condition A.

Step 1. Let $k = 1$. Go to Step 2.

Step 2. If M_k covers M with respect to $\preccurlyeq$ and satisfies condition A, terminate. M_k is a minimal cover of M with respect to $\preccurlyeq$, satisfying condition A. Otherwise go to Step 3.

Step 3. Increase k by one and return to Step 2.

This computation terminates in a finite number of steps (hence it is an algorithm) since each step is obviously a finite computation and the existence of an fa M' which covers M with respect to $\preccurlyeq$ and satisfies condition A is assumed. Q.E.D.

6. Minimal w-Representation of an r-ddp Y by an r-lmsdp

As was proved in Ibaraki (1974), there exists no algorithm for deciding whether an arbitrarily given r-ddp Y is w-representable by an r-lmsdp. Thus there exists no algorithm of the first type described in Section 3 for the w-representation by an r-lmsdp. This section shows, however, that there exists an algorithm for finding a minimal r-lmsdp which is w-equivalent to an arbitrarily given r-lmsdp (an algorithm of the second type described in Section 3).

Let $\Pi = (M, h, \xi_0)$ be an r-lmsdp with $U \equiv O(\Pi)$. Note that there exists an algorithm to obtain U by Theorem 4.6. As shown in Theorem 4.4, U is a finite set for any r-lmsdp. Since U is a regular set, $R_U \in \Lambda_F(U)$ and hence Σ^*/R_U consists of finite equivalence classes.

In Section 4, it was mentioned that $\preccurlyeq_U$ on Σ^*/R_U is a partial ordering. Let $Y \equiv \Sigma^*/R_U$ and define $P_U \subset Y \times Y$ by

$$(\forall A_i, A_j \in Y)((A_i, A_j) \in P_U \Leftrightarrow A_i \preccurlyeq_U A_j).$$

Thus P_U characterizes $\preccurlyeq_U$ on Y. In describing P_U, we often omit pairs $(A_i, A_i) \in P_U$ for simplicity, since $(\forall A_i \in Y)((A_i, A_i) \in P_U)$ always holds. P_U is alternatively illustrated by graph Γ_U in which each node corresponds to $A_i \in Y$, and $A_j \in Y$, $j \neq i$, is placed above $A_i \in Y$ with arc (A_i, A_j) if $(A_i, A_j) \in P_U$ and there exists no $A_k \in Y$ such that

$$(A_i, A_k) \in P_U \wedge (A_k, A_j) \in P_U \wedge k \neq i \wedge k \neq j.$$

P_U' is defined from P_U by

$$P_U' = P_U - \{(A_i, A_j) \in P_U \mid A_j \subset U \wedge A_j \subset U \wedge i \neq j\}.$$

P_U' is also a partial ordering. Graph Γ_U' is defined for P_U' in a manner similar to Γ_U. For P_U', define N_U' by

$$N_U' = \{K \subset Y \mid (\forall A_i, A_j \in Y)(A_i, A_j \in K \Rightarrow (A_i, A_j) \notin P_U' \wedge (A_j, A_i) \notin P_U'\}$$

EXAMPLE 6.1. Let $\Pi = (M, h, \xi_0)$ be an r-lmsdp given by $M = (Q, \Sigma, q_0, \lambda, Q_F)$; $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$, $\Sigma = \{a, b\}$, $Q_F = \{q_1, q_3, q_4\}$, and λ is given in Fig. 1; $\xi_0 = 0$; $h(\xi, q_0, \{a, b\}) = \xi$; $h(\xi, q_1, \{a, b\}) = \xi$;

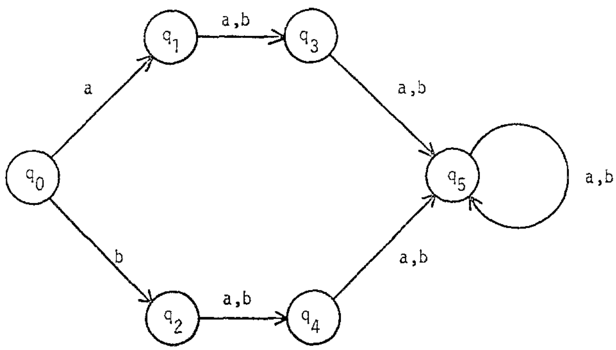

FIG. 1. State transition diagram of fa M of Example 6.1.

$h(\xi, q_2, a) = \xi + 1$; $h(\xi, q_2, b) = \xi$; $h(\xi, \{q_3, q_4, q_5\}, \{a, b\}) = \xi$. Obviously, $F(\Pi) = \{a, aa, ab, ba, bb\}$ and $U(\equiv O(\Pi)) = \{a, ab, aa, bb\}$. Σ^*/R_U consists of the following equivalence classes: $A_0 = \{\epsilon\}$ where $A_0 \backslash U = U$; $A_1 = \{a\}$ where $A_1 \backslash U = \{\epsilon, a, b\}$; $A_2 = \{b\}$ where $A_2 \backslash U = \{b\}$; $A_3 = \{aa, ab, bb\}$ where $A_3 \backslash U = \{\epsilon\}$; $A_4 = \Sigma^* - A_0 \cup A_1 \cup A_2 \cup A_3$ where $A_4 \backslash U = \varnothing$. P_U is given by $\{(A_1, A_2), (A_1, A_3), (A_0, A_4), (A_1, A_4), (A_2, A_4), (A_3, A_4)\}$ and Γ_U is shown in Fig. 2(a). By definition, $P_U' = P_U - \{(A_1, A_3)\}$ and Γ_U' is shown in Fig. 2(b). $K \in N_U'$ are $\{A_1, A_3, A_0\}$, $\{A_3, A_2, A_0\}$, $\{A_0, A_3\}$ and so forth.

The next lemma was proved in Ibaraki (1972). It is useful to find $T \in \Lambda_F(\Sigma^*)$ satisfying two conditions in Theorem 4.2.

LEMMA 6.1. *Let Y be a ddp with $U \equiv O(Y)$. $T \in \Lambda_F(\Sigma^*)$ J-separates U/R_U and each $C_j \in \Sigma^*/T$ is monotone with respect to U if and only if T*

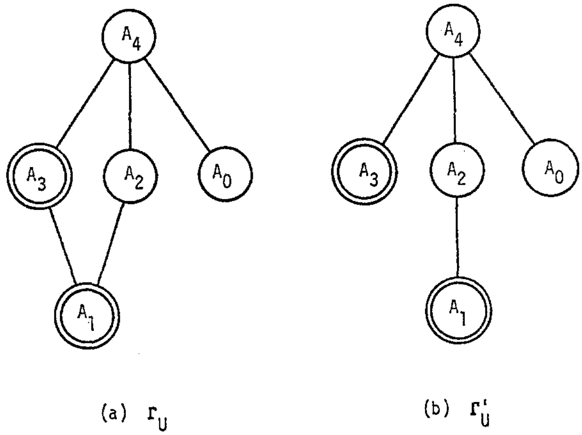

FIG. 2. Γ_U and Γ_U' of the r-lmsdp Π of Example 6.1. (Double circles denote $A_i \in Y$ such that $A_i \subset U$.)

J-separates every $K \in N_U'$ *(i.e.,* $(\forall A_i, A_k \in Y)(\forall C_j \in \Sigma^*/T)(A_i \cap C_j \neq \varnothing \wedge A_k \cap C_j \neq \varnothing \Rightarrow (A_i, A_k) \in P_U' \vee (A_k, A_i) \in P_U'))$.

Thus a minimal w-representation of Y by an msdp can be obtained by finding $T \in \Lambda_F(\Sigma^*)$ satisfying Lemma 6.1 with the fewest $|\Sigma^*/T|$. Although there exists no algorithm to find such T for an arbitrary given ddp Y, there does exist such an algorithm if Y is restricted to be the one w-representable by an r-lmsdp (i.e., $O(Y)$ is finite) and an r-lmsdp w-representing Y is given (or $O(Y)$ is given). As shown below, this is possible in both cases in which the resulting minimal msdp is restricted to be an r-lmsdp and in which the resulting msdp can be any r-msdp.

From now on, assume that $U(\equiv O(Y) = O(\Pi))$ be given (see Theorem 4.6) and Π is an r-lmsdp w-representing Y.

LEMMA 6.2. *Let* $T \in \Lambda_F(\Sigma^*)$ *J-separate all* $K \in N_U'$, *where* U *is finite. Then there exists an r-msdp* $\Pi = (M, h, \xi_0)$ *which satisfies* $O(\Pi) = U$ *and in which* M *is the standard construction of* T. *Furthermore, there exists an algorithm to obtain such r-msdp* Π *for any* T *and any finite set* U. *In this case, if* T *satisfies the condition* $|\cup\{C_j \mid C_j \in \Sigma^*/T \wedge C_j \cap U \neq \varnothing\}| < \infty$, *the resulting r-msdp* Π *is an r-lmsdp.*

Proof. Although this is almost a restatement of Theorem 4.2, we give a construction method of Π to show that the restriction to an r-msdp (instead of an msdp) is not really a restriction in this case. Let $T \in \Lambda_F(\Sigma^*)$ J-separate all $K \in N_U'$. Let $W = \Sigma^*/R_U \wedge T$ where $R_U \wedge T \in \Lambda_F(U)$ by definition.

Define a recursive function $h'\colon W \to Z$ so that (i) $(\forall D_i, D_j \in W)(D_i\ T\ D_j \wedge h'(D_i) \leqslant h'(D_j) \Leftrightarrow D_i\ T\ D_j \wedge D_i \preccurlyeq_U D_j)$ and (ii) $(\forall D_i \in W)(D_i \subset U \Rightarrow h'(D_i) = 0 \wedge D_i \not\subset U \Rightarrow h'(D_i) > 0)$, where $D_i\ T\ D_j$ stands for $(\forall x \in D_i)(\forall y \in D_j)(xTy)$. Such h' exists because (a) W is finite, (b) $R_U \wedge T \leqslant T$, (c) $\preccurlyeq_U$ on each $C_j \in \Sigma^*/T$ is monotone with respect to U (see Lemma 6.1 and the discussion prior to Theorem 4.2), and (d) $(\forall D_i, D_j \in W)(D_i \subset U \wedge D_j \not\subset U \wedge D_i\ T\ D_j \Rightarrow D_i \preccurlyeq_U D_j)$ by definition of U (note that $\epsilon \in D_i \backslash U$ but $\epsilon \notin D_j \backslash U$). h' is a recursive function since W is finite and each $D_i \in W$ is regular. $h'\colon W \to Z$ can then be extended to $\Sigma^* \to Z$ by $(\forall x \in \Sigma^*)(x \in D_i \Rightarrow h'(x) = h'(D_i))$. For any h' satisfying (i) and (ii), it holds that $(\forall D_i, D_j \in W)(\forall x \in D_i)(\forall y \in D_j)(xTy \wedge h'(x) \leqslant h'(y) \Leftrightarrow D_i\ T\ D_j \wedge h'(D_i) \leqslant h'(D_j) \Leftrightarrow D_i\ T\ D_j \wedge D_i \preccurlyeq_U D_j \Rightarrow (\forall z \in \Sigma^*)([D_i z]\ T\ [D_j z] \wedge [D_i z] \preccurlyeq_U [D_j z])$ (since $\preccurlyeq_U$ is right invariant on W by Proposition 4.7, where $[D_i z]$ denotes $D_k \in W$ such that $D_i z$ $(=\{xz \mid x \in D_i\}) \subset D_k) \Rightarrow (\forall z \in \Sigma^*)(xzTyz \wedge h'([D_i z]) \leqslant h'([D_j z])) \Rightarrow (\forall z \in \Sigma^*)(xzTyz \wedge h'(xz) \leqslant h'(yz)))$. Therefore, there exists an r-msdp $\Pi = (M(Q, \Sigma, q_0, \lambda, Q_F), h, \xi_0)$ satisfying $(\forall x \in \Sigma^*)(\bar{h}(x) = h'(x))$ by Lemma 4.1, where M is the standard construction of T. Π satisfies $O(\Pi) = U$ by condition (ii) if we let

$$Q_F = \{[C_j] \mid C_j \in \Sigma^*/T \wedge C_j \cap U \neq \varnothing\}.$$

It is obvious that an algorithm to obtain the above Π exists since each process described above can be done in a finite number of steps. The last lemma statement is also obvious since $F(M) = \cup\{C_j \mid C_j \in \Sigma^*/T \wedge C_j \cap U \neq \varnothing\}$, which is assumed to be finite. Q.E.D.

Now we move to the next lemma.

LEMMA 6.3. *Let U be a given finite set and M_U be the standard construction of R_U. Let $T \in \Lambda_F(\Sigma^*)$ be given and M be its standard construction. Then T J-separates all $K \in N_U'$ if and only M covers M_U with respect to*[2] P_U'.

Proof. T J-separates all $K \in N_U' \Leftrightarrow (\forall A_i, A_k \in \Sigma^*/R_U)(\forall C_j \in \Sigma^*/T)(C_j \cap A_i \neq \varnothing \wedge C_j \cap A_k \neq \varnothing \Rightarrow (A_i, A_k) \in P_U' \vee (A_k, A_i) \in P_U') \Leftrightarrow M$ covers M_U with respect to P_U'. Q.E.D.

By Lemma 6.3, we will have the following algorithm.

Algorithm for obtaining a minimal r-msdp (r-lmsdp) Π satisfying $O(\Pi) = U$, for a finite set U.

[2] This is equivalent to saying that M covers M_U with respect to $\preccurlyeq$, where $\preccurlyeq$ is defined by $(\forall A_i, A_j \in \Sigma^*/R_U)(A_i \preccurlyeq A_j \Leftrightarrow (A_i, A_j) \in P_U')$.

Step 1. Obtain M_U.

Step 2. Obtain a minimal cover M of M_U with respect to P_U' (satisfying $|F(M)| < \infty$). Then construct r-msdp (r-lmsdp) Π satisfying $O(\Pi) = U$ from M.

Note that condition $|F(M)| < \infty$ of Step 2 is active only when a minimal r-lmsdp is required. Step 1 and Step 2 can be done in a finite number of steps by Propositions 5.4(5)(6), 5.5, and 5.6, and Lemma 6.2. (Note that at least one fa M which covers M_U with respect to P_U' (and satisfies $|F(M)| < \infty$) exists since M_U itself is such an fa.)

THEOREM 6.4. *For any r-lmsdp Π, there exists an algorithm to obtain a minimal r-lmsdp (or r-msdp) w-equivalent to Π.*

Proof. Obtain $U(\equiv O(\Pi))$ (see Theorem 4.6) and then apply the algorithm given above. Q.E.D.

It should be noted that, in many cases, a minimal r-lmsdp (or r-msdp) can be obtained by ad hoc method without enumerating fa's. Such an example will be given next.

EXAMPLE 6.2. Let $U = \{a, ab, aa, bb\}$ (see Example 6.1) and find a minimal r-msdp and a minimal r-lmsdp $\tilde{\Pi}$ satisfying $O(\tilde{\Pi}) = U$. P_U' was obtained in Example 6.1 (see Fig. 2(b) for Γ_U'). Since $K = \{A_0, A_1, A_3\} \in N_U'$, any $T \in \Lambda_F(\Sigma^*)$ J-separating all $K \in N_U'$ satisfies $|\Sigma^*/T| \geqslant 3$. Let us first obtain a minimal r-msdp. Consider $T \in \Lambda_F(\Sigma^*)$ given by $\Sigma^*/T = \{C_1, C_2, C_3\}$, where

$$C_1 = \{\epsilon\} = A_0,$$
$$C_2 = \{a, b\} = A_1 \cup A_2,$$

and

$$C_3 = \{x \in \Sigma^* \mid |x| \geqslant 2\}^3 = A_3 \cup A_4.$$

This T J-separates all $K \in N_U'$ as obvious from Fig. 3(a). The standard construction $\tilde{M}$ of T is shown in Fig. 4 together with $\tilde{q}_0$ and $\tilde{Q}_F$. Then r-msdp $\tilde{\Pi}$ with $O(\tilde{\Pi}) = U$ is constructed by Lemma 6.2. The resulting r-msdp $\Pi = (\tilde{M}, \tilde{h}, \tilde{\xi}_0)$ is given by $\tilde{\xi}_0 = 1$; $\tilde{h}(\xi, [C_1], a) = 0$; $\tilde{h}(\xi, [C_1], b) = 1$; $\tilde{h}(\xi, [C_2], a) = 0$ if $\xi = 0$, 1 if $\xi = 1$; $\tilde{h}(\xi, [C_2], b) = 0$; $\tilde{h}(\xi, [C_3], \{a, b\}) = 1$. (The values of h' introduced in the proof of Lemma 6.2 are also indicated in Fig. 3(a).) This $\tilde{\Pi}$ is minimal since it has three states. Next let us obtain

[3] $|x|$ denotes the length of string $x \in \Sigma^*$.

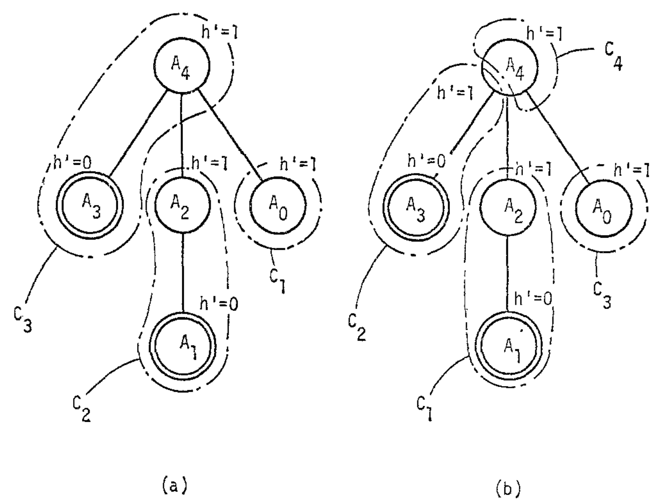

FIG. 3. $T \in \Lambda_F(\Sigma^*)$ giving (a) a minimal r-msdp and (b) a minimal r-lmsdp of Example 6.2. (C_j denotes an equivalence class of T.)

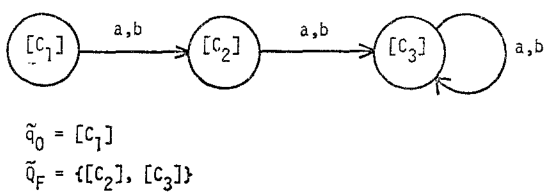

FIG. 4. The standard construction $\tilde{M}$ of $T \in \Lambda_F(\Sigma^*)$ obtained for the minimal r-msdp of Example 6.2.

a minimal r-lmsdp $\tilde{\Pi}$ satisfying $O(\tilde{\Pi}) = U$. By condition $|F(\tilde{\Pi})| < \infty$, we must find $T \in \Lambda_F(\Sigma^*)$ satisfying

$$| \cup\{C_j \mid C_j \in \Sigma^*/T \wedge C_j \cap U \neq \varnothing\}| < \infty. \tag{1}$$

It is shown as follows that $| \Sigma^*/T | \geqslant 4$ must hold to satisfy condition (1). Let $T \in \Lambda_F(\Sigma^*)$ satisfy (1) and J-separate all $K \in N_U'$. Note that A_0, A_1, A_2, $A_3 \in \Sigma^*/R_U$ are all finite and only $A_4 \in \Sigma^*/R_U$ is infinite. Let C_1, $C_2 \in \Sigma^*/T$ satisfy $C_1 \cap A_1 \neq \varnothing$ and $C_2 \cap A_3 \neq \varnothing$, where A_1, $A_3 \subset U$. Then $| C_1 \cap A_4 | < \infty$ and $| C_1 \cap A_4 | < \infty$ by condition (1). Since T J-separates all $K \in N_U'$, another equivalence class $C_3 \in \Sigma^*/T$ satisfying $C_3 \supset A_0 = \{\epsilon\}$ is required. This C_3 satisfies $C_3 \cap A_4 = \varnothing$ by condition (1) because $\epsilon \in C_3 \wedge x \in A_4 \cap C_3 \Rightarrow (\forall i \geqslant 0)(x^i \in C_3)$ (since T is right invariant) $\Rightarrow$

$(\forall i \geqslant 0)(x^i a \in C_1) \Rightarrow |C_1| = \infty$, a contradiction. Consequently we have $A_4 - C_1 \cup C_2 \cup C_3 \neq \varnothing$ implying that at least one more equivalence class $C_4 \in \Sigma^*/T$ is necessary to include the rest of A_4. $T \in \Lambda_F(\Sigma^*)$ with $|\Sigma^*/T| = 4$ J-separating all $K \in N_U'$ and satisfying condition (1) above is for example given by $\Sigma^*/T = \{C_1, C_2, C_3, C_4\}$, where

$$C_1 = \{a, b\} = A_1 \cup A_2, \qquad C_2 = \{aa, ba, ab, bb\} = A_3 \cup \{ba\},$$
$$C_3 = \{\epsilon\}, \qquad C_4 = \Sigma^* - C_1 \cup C_2 \cup C_3$$

(see Fig. 3(b)). The standard construction $\tilde{M}$ of T is shown in Fig. 5 together with $\tilde{q}_0$ and $\tilde{Q}_F$. $\tilde{h}$ and $\tilde{\xi}_0$ of r-lmsdp $\tilde{\Pi} = (\tilde{M}, \tilde{h}, \tilde{\xi}_0)$ are given by: $\tilde{\xi}_0 = 1$; $\tilde{h}(\xi, [C_3], a) = 0$; $\tilde{h}(\xi, [C_3], b) = 1$; $\tilde{h}(\xi, [C_1], a) = \xi$; $\tilde{h}(\xi, [C_1], b) = 0$; $\tilde{h}(\xi, [C_2], \{a, b\}) = \tilde{h}(\xi, [C_4], \{a, b\}) = 1$.

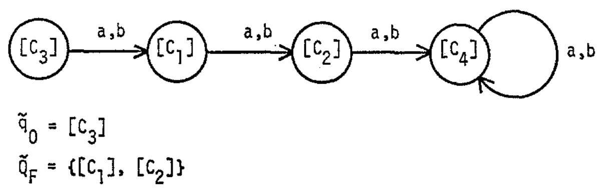

FIG. 5. Standard construction $\tilde{M}$ of T giving a minimal r-lmsdp of Example 6.2.

In concluding this section, a remark is given here. Although we assumed that U is a finite set throughout this section, most of the above argument is also valid even if U is a regular set rather than a finite set. In particular, the above algorithm for obtaining a minimal r-msdp can be applied to any regular set U (note that there exists no r-lmsdp $\tilde{\Pi}$ satisfying $F(\tilde{\Pi}) = U$ if U is an infinite set). This extension may be important from the practical point of view since many sequential decision processes Π have regular $O(\Pi)$ as exhibited by Theorem 4.5.

THEOREM 6.5. *Let $U \subset \Sigma^*$ be regular. Then there exists an algorithm to obtain a minimal r-msdp $\tilde{\Pi}$ satisfying $O(\tilde{\Pi}) = U$.*

7. MINIMAL s-REPRESENTATION OF AN r-ddp BY AN r-lmsdp

This section shows that there exists an algorithm which first decides whether an arbitrarily given r-ddp Y is s-representable by an r-lmsdp and then obtains a minimal s-representation of Y by an r-lmsdp in case it is s-representable (i.e., an algorithm of the first type described in Section 3).

For an r-ddp Y, relation $\leqslant_Y$ was defined in Section 4 prior to Theorem 4.3. The partial ordering $\leqslant_Y$ on Σ^*/R_Y is particularly important in the following discussion. Let $Y \equiv \Sigma^*/R_Y$. $\leqslant_Y$ on Y is characterized by $P_Y \subset Y \times Y$ defined by

$$(\forall A_i, A_j \in Y)((A_i, A_j) \in P_Y \Leftrightarrow A_i \leqslant_Y A_j).$$

In describing P_Y, pairs $(A_i, A_i) \in P_Y$ are usually omitted for simplicity, since $(\forall A_i \in Y)((A_i, A_i) \in P_Y)$. P_Y is alternatively illustrated by graph Γ_Y defined in a manner similar to Γ_U by replacing $A_i \in \Sigma^*/R_U$ by $A_i \in \Sigma^*/R_Y$ and $\leqslant_U$ by $\leqslant_Y$. P_Y' is defined from P_Y by

$$P_Y' = P_Y - \{(A_i, A_j) \in P_Y \mid (\exists p \in Z)(A_i, A_j \in \Psi_p \wedge A_i \neq A_j)\}.$$

(Ψ_p was defined in Section 4.) Graph Γ_Y' is defined for P_Y' similarly to Γ_Y. N_Y' is given by

$$N_Y' = \{K \subset Y \mid (\forall A_i, A_j \in Y)(A_i, A_j \in K \Rightarrow (A_i, A_j) \notin P_Y' \wedge (A_j, A_i) \notin P_Y'\}.$$

EXAMPLE 7.1. Let $Y = (\Sigma, S, f)$ be an r-ddp with $\Sigma = \{a, b\}$, $S = \{a, b, aa, ab, ba, bb\}$ and

$$f(a) = f(b) = f(ab) = f(ba) = f(bb) = 1$$
$$f(aa) = 0.$$

S is regular since it is finite. R_S is given by $\Sigma^*/R_S = \{C_0, C_1, C_2, C_3\}$, where
$C_0 = \{\epsilon\}$, $C_1 = \{a, b\}$, $C_2 = \{aa, ab, ba, bb\}$, and $C_3 = \Sigma^* - C_0 - C_1 - C_2$. Σ^*/R_Y consists of the following equivalence classes: $A_0 = \{\epsilon\}$, $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{aa\}$, $A_4 = \{ab, ba, bb\}$, and $A_5 = C_3$. Obviously, $A_1 \leqslant_Y A_2$ and $A_3 \leqslant_Y A_4$ since $f(aa) < f(ba)$. Thus $P_Y = \{(A_1, A_2), (A_3, A_4)\}$. Γ_Y is shown in Fig. 6. Furthermore,

$$\Psi_0 = \{A_3\}$$
$$\Psi_1 = \{A_1, A_2, A_4\}.$$

Thus we have $P_Y' = \{(A_3, A_4)\}$. Γ_Y' is shown in Fig. 7. For example, $\{A_0\}$, $\{A_0, A_1, A_2\}$, $\{A_0, A_1, A_2, A_3, A_5\}$ belong to N_Y'.

The following lemma was proved in Ibaraki (1972). They are useful in finding $T \in \Lambda_F(S)$ which satisfies two conditions in Theorem 4.3.

LEMMA 7.1. *For an r-ddp $\Upsilon = (\Sigma, S, f)$, $T \in \Lambda_F(S)$ J-separates Ψ_p for every $p \in E$ and each $C_j \in \Sigma^*/T$ is monotone with respect to Υ if and only if T J-separates all $K \in N_\Upsilon'$, i.e., $(\forall A_i, A_k \in \Sigma^*/R_\Upsilon)(\forall C_j \in \Sigma^*/T)(A_i \cap C_j \neq \varnothing \wedge A_k \cap C_j \neq \varnothing \Rightarrow (A_i, A_k) \in P_\Upsilon' \vee (A_k, A_i) \in P_\Upsilon')$.*

As a result, a minimal s-representation of Υ by an msdp is in principle obtained by finding $T \in \Lambda_F(S)$ which J-separates all $K \in N_\Upsilon'$ and has the minimum $|\Sigma^*/T|$. It was proved in Ibaraki (1974), however, that there is no algorithm to find such T for an arbitrarily given ddp (or r-ddp) Υ. If Υ is s-representable by an r-lmsdp, however, we have the next theorem.

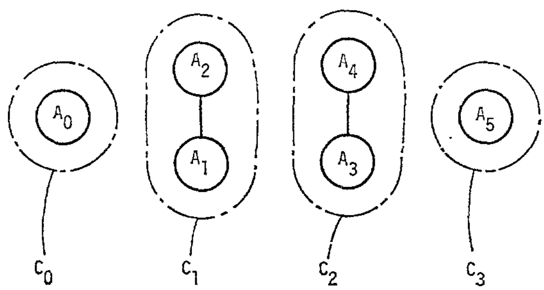

FIG. 6. Γ_Υ of r-ddp Υ of Example 7.1.

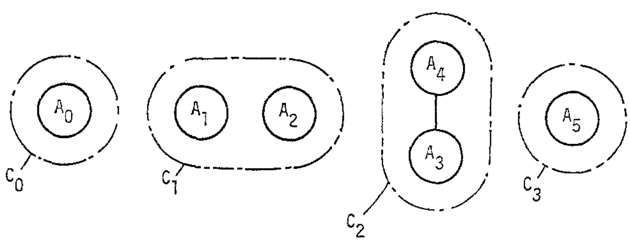

FIG. 7. Γ_Υ' of r-ddp Υ of Example 7.1.

THEOREM 7.2. (1) *There exists an algorithm for deciding whether an arbitrarily given r-ddp Υ is s-representable by an r-lmsdp.* (2) *If an r-ddp Υ is s-representable by an r-lmsdp, there exists an algorithm to obtain a minimal s-representation of Υ by an r-lmsdp.*

Proof. (1) An r-ddp $\Upsilon = (\Sigma, S, f)$ is s-representable by an r-lmsdp if and only if S is finite (see Theorem 4.4). By Proposition 5.4 (1), there exists an algortihm to decide whether a regular set S is finite or not. (2) The existence of such an algorithm will be shown in the following discussion (Theorem 7.6). Q.E.D.

Now let $Y = (\Sigma, S, f)$ be an r-ddp with S finite. Let $Y \equiv \Sigma^*/R_Y$. Y has an equivalence class $A_d \in Y$ such that $A_d = \{x \in \Sigma^* \mid (\forall y \in \Sigma^*)(xy \notin S)\}$. Since $\Sigma^* - A_d = \{x \in \Sigma^* \mid (\exists y \in \Sigma^*)(xy \in S)\}$ is also finite by the finiteness of S, Y consists of finite equivalence classes A_d and A_i, $i = 1, 2,\ldots, m$, where $\bigcup_{i=1}^{m} A_i = \Sigma^* - A_d$. Thus we have $R_Y \in \Lambda_F(S)$ (note $R_Y \in \Lambda(S)$ as mentioned in Section 4). Let $M_Y = (Q, \Sigma, q_0, \lambda, Q_F)$ be the standard construction of R_Y where $Q_F = \{[A_i] \mid A_i \in S/R_Y\}$. M_Y satisfies $F(M_Y) = S$ by Proposition 5.1.

LEMMA 7.3. *There exists an algorithm to obtain M_Y from an arbitrarily given r-ddp $Y = (\Sigma, S, f)$ with S finite.*

Proof. Since S is finite, there exist algorithms to obtain A_d and to decide whether $x R_Y y$ holds for any $x, y \in \Sigma^* - A_d$ (a finite set). (Note that $A_d \backslash S = \varnothing$ and hence $(\forall x, y \in A_d)(x R_Y y)$.) Thus there exists an algorithm to obtain $R_Y \in \Lambda_F(S)$. M_Y can be obtained from R_Y in a finite number of steps by Proposition 5.4 (6). Q.E.D.

LEMMA 7.4. *Let $Y = (\Sigma, S, f)$ be an r-ddp with S finite. Then for any $T \in \Lambda_F(\Sigma^*)$ J-separating all $K \in N_Y'$, there exists an algorithm to obtain an r-lmsdp $\Pi = (M, h, \xi_0)$ s-representing Y, where M is the standard construction of T.*

Proof. Since T J-separates all $K \in N_Y'$, $T \in \Lambda_F(S)$ follows by definition of N_Y' (note that $(\forall A_i \subset S)(\forall A_j \not\subset S)((A_i, A_j) \notin P_Y' \wedge (A_j, A_i) \notin P_Y')$ holds). $T \in \Lambda_F(S)$ and $R_Y \in \Lambda_F(S)$ (as proved above) implies $R = R_Y \wedge T \in \Lambda_F(S)$. Now define $h'\colon \Sigma^* \to Z$ satisfying

(i) $h'(x) = f(x)$ for $x \in S$,

(ii) $(\forall x, y \in \Sigma^*)([x]\ T[y] \wedge [x] \leqslant_Y [y] \Leftrightarrow xTy \wedge h'(x) \leqslant h'(y))$, where $[w]$ is the equivalence class of Σ^*/R containing $w \in \Sigma^*$.

Such h' exists since $\leqslant_Y$ on C_j/R $(=C_j/R_Y)$, $C_j \in \Sigma^*/T$, is a total ordering (note that C_j is monotone with respect to Y since T J-separates all $K \in N_Y'$), and $(\forall x, y \in S)(xTy \wedge f(x) \leqslant f(y) \Leftrightarrow [x]\ T[y] \wedge [x] \leqslant_Y [y])$ by the fact that T J-separates Ψ_p for every $p \in Z$. Then $(\forall x, y \in \Sigma^*)(xTy \wedge h'(x) \leqslant h'(y) \Rightarrow [x]\ T[y] \wedge [x] \leqslant_Y [y] \Rightarrow (\forall z \in \Sigma^*)([xz]\ T[yz] \wedge [xz] \leqslant_Y [yz])$ (since $\leqslant_Y$ is right invariant by Proposition 4.8) $\Rightarrow (\forall z \in \Sigma^*)(xzTyz \wedge h'(xz) \leqslant h'(yz)))$. h' is obviously a recursive function since $|\Sigma^*/R| < \infty$.

Thus by Lemma 4.1, there exists an r-msdp $\Pi = (M, h, \xi_0)$ satisfying $(\forall x \in \Sigma^*)(\bar{h}(x) = h'(x))$. If we define Q_F of $M = (Q, \Sigma, q_0, \lambda, Q_F)$ by

$$Q_F = \{[C_j] \mid C_j \in S/T\},$$

then $F(\Pi) = S$ holds (hence Π is an r-lmsdp) and Π s-represents Y by condition (i) given above. Note that the above r-lmsdp Π can be constructed from T in a finite number of steps since (a) R_Y can be obtained in a finite number of steps as mentioned in the proof of Lemma 7.3, (b) h' can be defined in a finite number of steps since Σ^*/R is finite, and (c) r-lmsdp Π can be constructed from T and h' in a finite number of steps. Q.E.D.

Note here that there always exists at least one $T \in \Lambda_F(\Sigma^*)$ J-separating all $K \in N_Y'$ for any r-ddp $Y = (\Sigma, S, f)$ with S finite. For example $T = R_Y$ J-separates all $K \in N_Y'$.

The next lemma is a basis for the algorithm to obtain a minimal r-lmsdp.

LEMMA 7.5. *Let $Y = (\Sigma, S, f)$ be an r-ddp with S finite. Let $T \in \Lambda_F(\Sigma^*)$ be given and M be the standard construction of T. Then T J-separates all $K \in N_Y'$ if and only if M covers M_Y with respect to P_Y'.*

Proof. T J-separates all $K \in N_Y' \Leftrightarrow (\forall C_j \in \Sigma^*/T)(\forall A_i, A_k \in \Sigma^*/R_Y)$ $(C_j \cap A_i \neq \varnothing \wedge C_j \cap A_k \neq \varnothing \Rightarrow (A_i, A_k) \in P_Y' \vee (A_k, A_i) \in P_Y') \Leftrightarrow M$ covers M_Y with respect to P_Y'. Q.E.D.

Based on these lemmas, we have the following algorithm.

Algorithm for obtaining a minimal r-lmsdp Π s-representing a given r-ddp $Y = (\Sigma, S, f)$ with S finite.

Step 1. Obtain M_Y.

Step 2. Obtain a minimal cover $\tilde{M}$ of M_Y with respect to P_Y'.

Step 3. Obtain r-lmsdp $\tilde{\Pi} = (\tilde{M}, \tilde{h}, \xi_0)$ s-representing Y by following the proof of Lemma 7.4. Terminate.

Steps 1, 2, 3 are finite computations, respectively, by Lemma 7.3, Proposition 5.6, and Lemma 7.4.

THEOREM 7.6. *There exists an algorithm to obtain a minimal s-representation of an r-ddp $Y = (\Sigma, S, f)$ with S finite by an r-lmsdp.*

Proof. The above algorithm eventually reaches Step 3 and terminates, since for example M_Y covers M_Y with respect to P_Y'. $\tilde{\Pi}$ is minimal by Lemmas 7.4 and 7.5. Q.E.D.

THEOREM 7.7. *There exists an algorithm to obtain a minimal r-lmsdp s-equivalent to an arbitrarily given r-lmsdp Π (i.e., the algorithm of the second type described in Section* 3).

Proof. Obtain r-ddp $\Upsilon = (\Sigma, S, f)$ s-represented by Π (this is of course a finite computation). Then obtain a minimal r-lmsdp $\tilde{\Pi}$ s-representing Υ by means of the above algorithm. $\tilde{\Pi}$ is a desired r-lmsdp. Q.E.D.

EXAMPLE 7.2. Let us obtain a minimal s-representation of r-ddp Υ given in Example 7.1. R_S, R_Υ, P_Υ, P_Υ' were obtained in Example 7.1. First obtain the standard construction $M_\Upsilon = (Q, \Sigma, q_0, \lambda, Q_F)$ of $R_\Upsilon \in \mathcal{A}_F(S)$. Since $\Sigma^*/R_\Upsilon = \{A_0, A_1, ..., A_5\}$, M_Υ has six states $Q = \{[A_0], [A_1], ..., [A_5]\}$, and $q_0 = [A_0]$ since $\epsilon \in A_0$. $\lambda: Q \times \Sigma \to Q$ and Q_F are given in Fig. 8. Note that $P_\Upsilon' = \{(A_3, A_4)\}$. A minimal $\tilde{M}$ which covers M_Υ with respect to P_Υ' is given by $\tilde{M} = (\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{\lambda}, \tilde{Q}_F)$, where $\tilde{Q} = \{[A_0], [A_1], [A_2], [A_3 \cup A_4], [A_5]\}$, $\tilde{q}_0 = [A_0]$, and $\tilde{\lambda}$, $\tilde{Q}_F$ are given in Fig. 9. The minimality

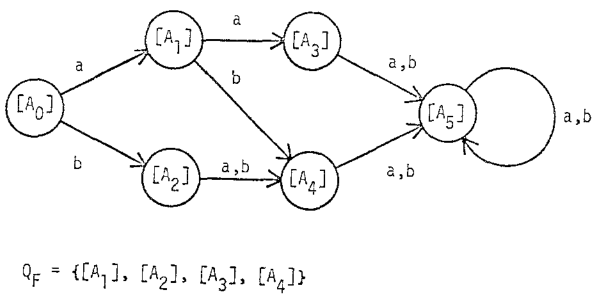

FIG. 8. State transition diagram of M_Υ in Example 7.2.

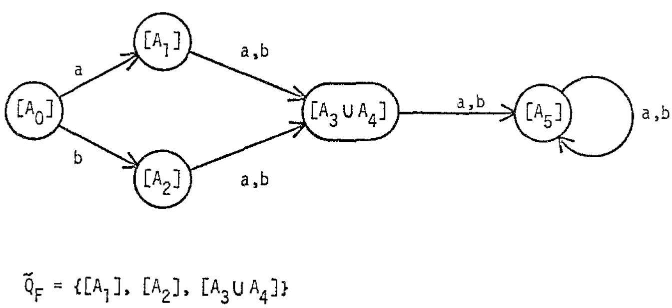

FIG. 9. State transition diagram of a minimal cover $\tilde{M}$ of M_Υ with respect to P_Υ' in Example 7.2.

of $\tilde{M}$ follows from the fact that $\{A_0, A_1, A_2, A_3, A_5\} \in N_Y'$ and hence any $T \in \Lambda_F(S)$ J-separating all $K \in N_Y'$ satisfies $| \Sigma^*/T | \geqslant 5$. Since $R_Y \leqslant T$, $\Sigma^*/R(\equiv \Sigma^*/R_Y \wedge T) = \Sigma^*/R_Y$ follows in this case. h': $\Sigma^*/R \to Z$ introduced in the proof of Lemma 7.4 is determined as follows:

(a) $h'(A_1) = h'(A_2) = 1$, $h'(A_3) = 0$, $h'(A_4) = 1$ (by condition (i) of the proof of Lemma 7.4)

(b) $h'(A_0) = h'(A_5) = 0$ (by condition (ii) of the proof of Lemma 7.4).

From $\tilde{M}$ and h', we obtain an r-lmsdp $\tilde{\Pi} = (\tilde{M}, \tilde{h}, \tilde{\xi}_0)$ satisfying $(\forall x \in \Sigma^*)$ $(\tilde{\tilde{h}}(x) = h'(x))$ by following Lemma 7.4. $\tilde{h}$ and $\tilde{\xi}_0$ are given by $\tilde{\xi}_0(=h'(A_0)) = 0$; $\tilde{h}(\xi, [A_0], \{a, b\}) = \xi + 1$; $\tilde{h}(\xi, [A_1], a) = \xi - 1$; $\tilde{h}(\xi, [A_1], b) = \xi$; $h(\xi, [A_2], \{a, b\}) = \xi$; $\tilde{h}(\xi, [A_3 \cup A_4], \{a, b\}) = 0$; $\tilde{h}(\xi, [A_5], \{a, b\}) = \xi$. This $\tilde{\Pi}$ is a minimal s-representation of Y by an r-lmsdp.

8. Minimal w-Representation of an r-ddp by an r-smsdp

As was shown in Ibaraki (1973a), there exists no algorithm to decide whether an arbitrarily given r-ddp Y is w-representable or s-representable by an r-smsdp respectively. Furthermore, it will be shown in Section 10 that there exists no algorithm to obtain a minimal r-smsdp s-equivalent to an arbitrarily given r-smsdp. However, there does exist an algorithm to obtain a minimal r-smsdp w-equivalent to an arbitrarily given r-smsdp, as will be shown in this section.

Let Π be an r-smsdp with $U \equiv O(\Pi)$. Since U is regular by Theorem 4.5, $R_U \in \Lambda_F(U)$. Let $M_U = (Q, \Sigma, q_0, \lambda, Q_F)$ be the standard construction of R_U with $Q_F = \{[C_j] \mid C_j \in U/R_U\}$. A state $q_d \in Q$ is said to be a *dead state* if $(\forall x \in \Sigma^*)(\lambda(q_d, x) \notin Q_F)$. M_U has at most one dead state. Define $M_U^* = (Q^*, \Sigma, q_0, \lambda^*, Q_F)$ from M_U as follows: if M_U has no dead state or $U = \varnothing$, then $M_U^* = M_U$, while if M_U has a dead state and $U \neq \varnothing$, M_U^* is given by $Q^* = Q - \{q_d\}$ and

$$\lambda^*(q, a) = \begin{cases} \lambda(q, a) & \text{if} \quad q \in Q^* \wedge a \in \Sigma \wedge \lambda(q, a) \in Q^* \\ \lambda(q, a) = q & \text{if} \quad q \in Q^* \wedge a \in \Sigma \wedge \lambda(q, a) = q_d . \end{cases}$$

Let $\Pi_U^* = (M_U^*, h, \xi_0)$ be an r-smsdp given by $\xi_0 = 0$ and

$$h(\xi, q, a) = \begin{cases} \xi + 1 & \text{if} \quad q \in Q^* \wedge a \in \Sigma \wedge \lambda(q, a) \notin Q^* \text{ (i.e., } \lambda(q, a) = q_d) \\ \xi & \text{otherwise.} \end{cases}$$

By definition, it holds that $(\forall x \in \Sigma^*)(\bar{h}(x) \geqslant 1 \Leftrightarrow \bar{\lambda}(x) = q_d$ in $M_U)$, and

$$O(\Pi_U{}^*) = \{x \in \Sigma^* \mid \bar{\lambda}^*(x) \in Q_F \wedge \bar{h}(x) = 0\}$$
$$= \{x \in \Sigma^* \mid \bar{\lambda}(x) \in Q_F\} = U.$$

This proves that $\Pi_U{}^*$ is w-equivalent to Π.

Based on the next lemma proved in Ibaraki (1973a), we will show that $\Pi_U{}^*$ is actually minimal.

LEMMA 8.1. *For any r-smsdp Π with n states, there exists an fa M satisfying (i) $F(M) = O(\Pi)$, and (ii) M has at most $n + 1$ states if M has a dead and at most n states if M has no dead state.*

THEOREM 8.2. *The $\Pi_U{}^*$ defined above for an arbitrarily given r-smsdp Π with $U(\equiv O(\Pi))$ is a minimal r-smsdp w-equivalent to Π.*

Proof. Let $\tilde{\Pi} = (\tilde{M}, \tilde{h}, \tilde{\xi}_0)$ be a minimal r-smsdp w-equivalent to Π. Assume that $\tilde{\Pi}$ has n states. We prove that $\Pi_U{}^*$ also has n states. By Lemma 8.1, there exists an fa M with $F(M) = U$ having at most $n + 1$ states if M has a dead state, and at most n states if M has no dead state. Now note that M_U given above for U is the minimal fa accepting U (see Proposition 5.1 (3)). Since $F(M_U) = F(M)$, M_U has at most $n + 1$ states if M_U has a dead state, and at most n if M_U has no dead state. (Since $F(M_U) = F(M)$, M_U has a dead state if and only if M has a dead state.) Finally $\Pi_U{}^*$ has at most n states in either case, as obvious from the construction given above. This proves the minimality of $\Pi_U{}^*$. Q.E.D.

As a result, the next algorithm is obtained.

Algorithm for obtaining a minimal r-smsdp w-equivalent to an r-smsdp Π.

Step 1. Obtain $U(\equiv O(\Pi))$.

Step 2. Obtain $\Pi_U{}^*$.

Step 1 is a finite computation by Theorem 4.6. Step 2 can also be done in a finite number of steps since M_U can be obtained from U in a finite number of steps by Proposition 5.4 (6), and $\Pi_U{}^*$ is obviously obtained from M_U in a finite number of steps.

THEOREM 8.3. *There exists an algorithm to obtain a minimal r-smsdp w-equivalent to an arbitrarily given r-smsdp Π.*

9. Minimal w-Representation of an r-ddp by an r-pmsdp

As was shown in Ibaraki (1973a), there exists no algorithm to decide whether an arbitrarily given r-ddp is w-representable or s-representable by an r-pmsdp, respectively. It will be also shown in Section 10 that there exists no algorithm to obtain a minimal r-pmsdp s-equivalent to an arbitrarily given r-pmsdp. However, there does exist an algorithm to obtain a minimal r-pmsdp w-equivalent to an arbitrarily given r-pmsdp, as will be shown in this section.

Let Π be an r-pmsdp with $U \equiv O(\Pi)$. Then U is regular by Theorem 4.5 and hence $R_U \in \Lambda_F(U)$. Let $\Sigma^*/R_U = \{A_1, A_2, ..., A_n\}$. For any $T \in \Lambda_F(\Sigma^*)$ such that T J-separates U/R_U and each $C_j \in \Sigma^*/T$ is monotone with respect to U, graph $\Gamma_{U;T}$ is defined as follows: Let $\Sigma^*/R_U \wedge T = \{D_1, D_2, ..., D_m\}$ (note that $R_U \wedge T \in \Lambda_F(U)$). $\Gamma_{U;T}$ has nodes $D_1, D_2, ..., D_m$, and has three types of arcs.

(1) (D_i, D_j) is an arc of type 1 if $D_i T D_j \wedge D_i \neq D_j \wedge D_i \leqslant_U D_j$. ($D_i T D_j$ stands for $(\forall x \in D_i)(\forall y \in D_j)(xTy)$ and $D_i \leqslant_U D_j$ for $(\forall x \in D_i)(\forall y \in D_j)(x \leqslant_U y) \Leftrightarrow (\forall A_k, A_l \in \Sigma^*/R_U)(D_i \subset A_k \wedge D_j \subset A_l \Rightarrow A_k \leqslant_U A_l)$.)

(2) (D_i, D_j) is an arc of type 2 if $(\exists a \in \Sigma)(D_i a \subset D_j)$ (i.e, there is a transition from state $[D_i]$ to $[D_j]$ in the standard construction of $R_U \wedge T$).

(3) (D_i, D_j) is an arc of type 3 if $D_i \subset U \wedge D_j \subset U \wedge D_i \neq D_j$.

A *path* in $\Gamma_{U;T}$ is a sequence of arcs $\beta = (D_{i_1}, D_{i_2})(D_{i_2}, D_{i_3}) \cdots (D_{i_{k-1}}, D_{i_k})$. β is a *circuit* if $D_{i_1} = D_{i_k}$ holds. A circuit β is an *I-circuit* (an *inconsistent circuit*) if β contains an arc of type 1.

Theorem 9.1. *Let $U \subset \Sigma^*$ be a regular set. Assume that an r-pmsdp $\Pi = (M(Q, \Sigma, q_0, \lambda, Q_F), h, \xi_0)$ satisfies $O(\Pi) = U$ and let $T \in \Lambda_F(\Sigma^*)$ be given by $(\forall x, y \in \Sigma^*)(xTy \Leftrightarrow \bar{\lambda}(x) = \bar{\lambda}(y))$. Then (i) each $C_j \in \Sigma^*/T$ is monotone with respect to U, and (ii) $\Gamma_{U;T}$ has no I-circuit.*[4] *Conversely, for any $T \in \Lambda_F(\Sigma^*)$ satisfying conditions (i) and (ii), there exists an r-pmsdp $\tilde{\Pi} = (\tilde{M}, \tilde{h}, \tilde{\xi}_0)$ such that $O(\tilde{\Pi}) = U$ and $\tilde{M}$ is the standard construction of T. In this case, there exists an algorithm to obtain such r-pmsdp $\tilde{\Pi}$ from T.*

Proof. Necessity. Condition (i) follows from Theorem 4.2 since r-pmsdp is an msdp. Let (D_i, D_j) be an arc of type 1 in $\Gamma_{U;T}$. Then $(\forall x \in D_i)(\forall y \in D_j)(\bar{h}(x) < \bar{h}(y))$ holds, since otherwise $xTy \wedge \bar{h}(x) \geqslant \bar{h}(y) \Rightarrow (\forall z \in \Sigma^*)(xzTyz \wedge \bar{h}(xz) \geqslant \bar{h}(yz)) \Rightarrow (xz \in U \Rightarrow yz \in U) \Rightarrow D_j \leqslant_U D_i$, a contradiction.

[4] It follows from (ii) that T J-separates U/R_U.

Assume that there exists an I-circuit

$$\beta = (D_{i_1}, D_{i_2})(D_{i_2}, D_{i_3}) \cdots (D_{i_{k-1}}, D_{i_k})$$

in $\Gamma_{U;T}$, where $D_{i_1} = D_{i_k}$ and (D_{i_1}, D_{i_2}) is assumed to be of type 1 without loss of generality. Then we can find $x_{i_j} \in D_{i_j}$ for $j = 1, 2,..., k-1$ and $x_{j_k} = x_{j_1}$ such that $x_{i_j} \lessdot_U x_{i_{j+1}}$ if $(D_{i_j}, D_{i_{j+1}})$ is of type 1, and $x_{i_{j+1}} = x_{i_j}a$ if $(D_{i_j}, D_{i_{j+1}})$ is of type 2 and satisfies $D_{i_j}a \subset D_{i_{j+1}}$ for $a \in \Sigma$, and $x_{i_j} \in U \wedge x_{i_{j+1}} \in U$ if $(D_{i_j}, D_{i_{j+1}})$ is of type 3. We have $\bar{h}(x_{i_j}) < \bar{h}(x_{i_{j+1}})$ for $(D_{i_j}, D_{i_{j+1}})$ of type 1 as proved above, $\bar{h}(x_{i_j}) \leqslant \bar{h}(x_{i_{j+1}})$ for $(D_{i_j}, D_{i_{j+1}})$ of type 2 since Π is an r-pmsdp, and $\bar{h}(x_{i_j}) = \bar{h}(x_{i_{j+1}}) = h^*$ (the value of an optimal policy) for $(D_{i_j}, D_{i_{j+1}})$ of type 3. Consequently, $\bar{h}(x_{j_1}) < \bar{h}(x_{j_2}) \leqslant \bar{h}(x_{j_3}) \leqslant \cdots \leqslant \bar{h}(x_{j_k}) = \bar{h}(x_{j_1})$ follows and this is a contradiction.

Sufficiency. Denote $\Sigma^*/R_U \wedge T$ by Y, where $Y = \{D_1, D_2, ..., D_m\}$. Define a numbering h': $Y \to Z$ which satisfies the next conditions.

(a) $(\forall D_i, D_j \in Y)((D_i, D_j)$ is an arc in $\Gamma_{U;T} \Rightarrow h'(D_i) \leqslant h'(D_j))$,

(b) $(\forall D_i, D_j \in Y)((D_i, D_j)$ is an arc of type 1 in $\Gamma_{U;T} \Rightarrow h'(D_i) < h'(D_j))$.

This numbering is possible since $\Gamma_{U;T}$ is a finite graph and has no I-circuit. If there is a circuit β in $\Gamma_{U;T}$ (not an I-circuit), any D_i, D_j in β satisfies $h'(D_i) = h'(D_j)$ by condition (a). In particular, any $D_i, D_j \subset U$ satisfies $h'(D_i) = h'(D_j)$ since there always exists a circuit which consists of arcs of type 3 and includes D_i and D_j. Next extend h' to h': $\Sigma^* \to Z$ by $(\forall x \in D_i \in Y)$ $(h'(x) = h'(D_i))$. h' is obviously a recursive function since Y is finite and $D_i \in Y$ is a regular set by assumption. Let $[x]$ denote the equivalence class in Y containing x. We have $(\forall x, y \in \Sigma^*)(xTy \wedge h'(x) \leqslant h'(y) \Rightarrow [x]\ T[y] \wedge [x] \lessdot_U [y]$ (by the fact that each $C_j \in \Sigma^*/T$ is monotone with respect to U and by condition (a) above) $\Rightarrow (\forall z \in \Sigma^*)([xz]\ T[yz] \wedge [xz] \lessdot_U [yz])$ (by Proposition 4.7) $\Rightarrow (\forall z \in \Sigma^*)(xzTyz \wedge h'(xz) \leqslant h'(yz))$ (by conditions (a) (b). Note that $[xz] = [yz]$ possibly holds.)). Thus by Lemma 4.1, we have an r-msdp $\tilde{\Pi} = (\tilde{M}(\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{\lambda}, \tilde{Q}_F), \tilde{h}, \tilde{\xi}_0)$ satisfying $(\forall x \in \Sigma^*)(\bar{\tilde{h}}(x) = h'(x))$, where $\tilde{M}$ is the standard construction of T. Note h' defined above satisfies $(\forall x, z \in \Sigma^*)(\bar{\tilde{h}}(x) \leqslant \bar{\tilde{h}}(xz))$ since there is a path consisting of arcs of type 2 from $[x]$ to $[xz]$. Thus $\tilde{\Pi}$ is an r-pmsdp. Furthermore, if we let

$$\tilde{Q}_F = \{[C_j] \mid C_j \in \Sigma^*/T \wedge C_j \cap U \neq \varnothing\},$$

we have

$$\begin{aligned} O(\tilde{\Pi}) &= \{x \mid \tilde{\lambda}(x) \in \tilde{Q}_F \wedge (\forall y \in F(\tilde{\Pi}))(\bar{\tilde{h}}(x) \leqslant \bar{\tilde{h}}(y))\} \\ &= \{x \mid (\exists y \in U)([x]\ T[y]) \wedge (\forall [z] \in Y)([x]\ T[z] \Rightarrow [x] \lessdot_U [z])\} \\ &\quad \text{(see the property of } h' \text{ given above)} \\ &= \{x \mid [x] \subset U\} \text{ (by definition of } \lessdot_U) = U. \end{aligned}$$

Consequently $\tilde{\Pi}$ w-represents Π. Finally, it is also obvious that there exists an algorithm to obtain such $\tilde{\Pi}$ by following the above construction, since $\Gamma_{U;T}$ is a finite graph and hence the above h': $Y \to Z$ can be constructed in a finite number of steps. Q.E.D.

Now assume that a regular set U and an equivalence relation $T \in \Lambda_F(\Sigma^*)$ are given. Then by Proposition 5.5 (3) and by the fact that $\Gamma_{U;T}$ is a finite graph, there exists an algorithm to decide whether two conditions in Theorem 9.1 are satisfied. Next let $\{T_1, T_2, ...\}$ be an effective enumeration of all $T \in \Lambda_F(\Sigma^*)$ in the nondecreasing order of $|\Sigma^*/T|$. Such an enumeration may be obtained from the effective enumeration of all fa's $\{M_1, M_2, ...\}$ discussed in the proof of Proposition 5.6, by $(\forall x, y \in \Sigma^*)(xT_iy \Leftrightarrow \lambda_i(x) = \lambda_i(y))$, where $M_i = (Q_i, \Sigma, q_{0i}, \lambda_i, Q_{Fi})$, $i = 1, 2, ...$.

With these preparations, we have the following algorithm.

Algorithm for obtaining a minimal r-pmsdp $\tilde{\Pi}$ w-equivalent to a given r-pmsdp Π.

Step 1. Obtain $O(\Pi)$ and let $U = O(\Pi)$. Let $k = 1$ and go to Step 2.

Step 2. Check if T_k satisfies conditions: (i) each $C_j \in \Sigma^*/T_k$ is monotone with respect to U, and (ii) $\Gamma_{U;T_k}$ has no I-circuit. If yes, go to Step 4; otherwise go to Step 3.

Step 3. Increase k by one and return to Step 2.

Step 4. Obtain r-pmsdp $\tilde{\Pi}$ from T_k, by Theorem 9.1. Terminate.

Step 1 and Step 2 are, respectively, finite computations by Theorem 4.6 and by the remark mentioned above. Step 4 is also a finite computation by Theorem 9.1. This leads to the next theorem.

THEOREM 9.2. *There exists an algorithm to obtain a minimal r-pmsdp w-equivalent to an arbitrarily given r-pmsdp.*

Proof. In the above algorithm, Step 4 is eventually reached because there always exists an r-pmsdp w-equivalent to Π (for example consider Π itself). $\tilde{\Pi}$ obtained in Step 4 is then obviously minimal. Q.E.D.

The above algorithm, however, is extremely inefficient, though it always terminates in a finite number of steps. In many practical problems, a minimal r-pmsdp can be discovered by ad hoc method. The next proposition is sometimes useful to prove the minimality of a given r-pmsdp.

PROPOSITION 9.3. *Let* $U \subset \Sigma^*$ *be a regular set and let*

$$\Sigma^*/R_U = \{A_1, A_2, ..., A_n\}.$$

If $A_i \subset U \wedge A_k \not\subset U \wedge A_i \leqslant_U A_k \wedge A_k \backslash U \neq \varnothing$, *then any* $T \in A_F(\Sigma^*)$, *having* $C_j \in \Sigma^*/T$ *which satisfies* $C_j \cap A_i \neq \varnothing \wedge C_j \cap A_k \neq \varnothing$, *has an I-circuit in* $\Gamma_{U;T}$.

Proof. Let $D_i = A_i \cap C_j$ and $D_k = A_k \cap C_j$. Then there is arc (D_i, D_k) of type 1 in $\Gamma_{U;T}$ since $D_i \leqslant_U D_k$ (note $A_i \leqslant_U A_k$). Furthermore, since $A_k \backslash U \neq \varnothing$, there is a path consisting of arcs of type 2 from D_k to D_l, where $D_l \subset U$. If $D_l = D_i$, the statement is proved. If $D_l \neq D_i$, there is arc (D_l, D_i) of type 3 by definition. Therefore, in both cases, we have an I-circuit. Q.E.D.

EXAMPLE 9.1. Let

$$\Sigma = \{a, b, c\} \quad \text{and} \quad U = \{ab(a \cup b \cup c)^* \cup (b \cup c)(a \cup b)(a \cup b \cup c)^*\}.$^{[5]}$$

Let us obtain a minimal r-pmsdp Π satisfying $O(\Pi) = U$. Σ^*/R_U consists of the following equivalence classes : $A_0 = \{\epsilon\}$ where $A_0 \backslash U = U$; $A_1 = \{a\}$ where $A_1 \backslash U = \{b(a \cup b \cup c)^*\}$; $A_2 = \{b, c\}$ where $A_2 \backslash U = \{(a \cup b)(a \cup b \cup c)^*\}$; $A_3 = U$ where $A_3 \backslash U = \{(a \cup b \cup c)^*\}$; $A_4 = \{aa(a \cup b \cup c)^* \cup (a \cup b \cup c)\, c(a \cup b \cup c)^*\}$ where $A_4 \backslash U = \varnothing$. Thus Γ_U as shown in Fig. 10

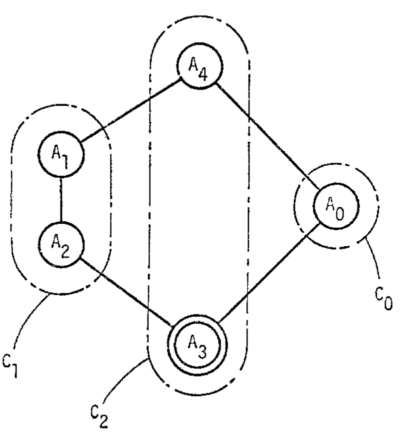

FIG. 10. Γ_U of U defined in Example 9.1.

[5] Notations $\cup$ and * in this expression, respectively, stand for the union and star operation of the regular expression (Kleene, 1956). $\{P\}$ stands for the set represented by regular expression P.

is obtained. Since $\{A_0, A_1\} \in N_U'$ for example, at least two states are required (see Theorem 4.2 and Lemma 6.1). Now we will give an r-pmsdp Π with three states, satisfying $O(\Pi) = U$, and after we will prove the minimality of Π. Define $T \in \Lambda_F(\Sigma^*)$ by $\Sigma^*/T = \{C_0, C_1, C_2\}$ where $C_0 = \{\epsilon\} = A_0$, $C_1 = \{a, b, c\} = A_1 \cup A_2$, $C_2 = \{(a \cup b \cup c)(a \cup b \cup c)(a \cup b \cup c)^*\} = A_3 \cup A_4$. This T is also illustrated in Fig. 10. Obviously, each $C_j \in \Sigma^*/T$ is monotone with respect to U and T J-separates $U/R_U = \{A_3\}$. $\Sigma^*/R_U \wedge T(=\Sigma^*/R_U)$ also consists of the equivalence classes A_0, A_1, A_2, A_3, and A_4. Then $\Gamma_{U;T}$ is obtained and shown in Fig. 11, in which

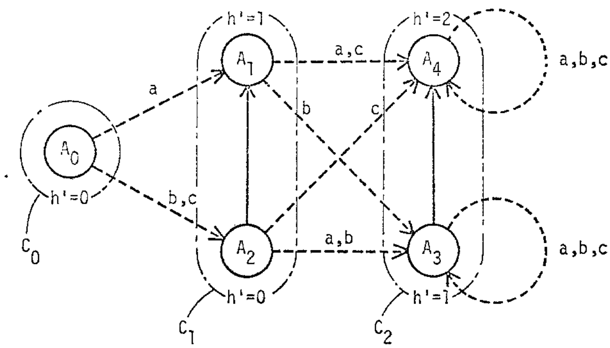

FIG. 11. $\Gamma_{U\ T}$ of U and T given in Example 9.1.

solid arcs indicate those of type 1 and broken arcs indicate those of type 2. There is no arc of type 3, in this case. From Fig. 11, it immediately follows that there is no I-circuit in $\Gamma_{U;T}$. Thus by Theorem 9.1, we see that an r-pmsdp with $|\Sigma^*/T| = 3$ exists. A numbering h' on $\Sigma^*/R_U \wedge T$ satisfying conditions in the proof of Theorem 9.1 is also illustrated in Fig. 11. An r-pmsdp $\tilde{\Pi} = (\tilde{M}(\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{\lambda}, \tilde{Q}_F), \tilde{h}, \tilde{\xi}_0)$ is then obtained from h' by following the proof of Theorem 9.1. The state transition diagram of $\tilde{M}$ and $\tilde{Q}_F$ are shown in Fig. 12. $\tilde{h}$ and $\tilde{\xi}_0$ are given by $\tilde{\xi}_0 = 0$; $\tilde{h}(\xi, [C_0], a) = \xi + 1$; $\tilde{h}(\xi, [C_0], \{b, c\}) = \xi$; $\tilde{h}(\xi, [C_1], a) = \xi + 1$; $\tilde{h}(\xi, [C_1], b) = \xi$ if $\xi \geqslant 1$, $\xi + 1$ if $\xi \leqslant 0$; $\tilde{h}(\xi, [C_1], c) = \xi + 1$ if $\xi \geqslant 1$, $\xi + 2$ if $\xi \leqslant 0$; $\tilde{h}(\xi, [C_2], \{a, b, c\}) = \xi$. Obviously, $O(\Pi) = \{x \in F(\Pi) \mid (\forall y \in F(\Pi))(\tilde{h}(x) \leqslant \tilde{h}(y))\} = \{x \in C_2 \mid (\forall y \in C_2)(\tilde{h}(x) \leqslant \tilde{h}(y))\} = \{x \in C_2 \mid \tilde{h}(x) = 1\} = A_3 = U$. Finally, we will show that $\tilde{\Pi}$ is a minimal r-pmsdp with $O(\tilde{\Pi}) = U$. Since at least two states are required as mentioned above, assume that there exists an r-pmsdp $\Pi' = (M', h', \xi_0')$ which has two states and satisfies $O(\Pi') = U$. Let M' be the standard construction of $T' \in \Lambda_F(\Sigma^*)$, where $|\Sigma^*/T'| = 2$.

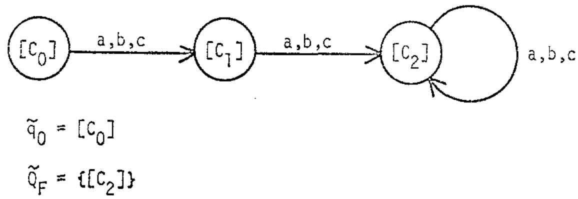

FIG. 12. State transition diagram of fa $\tilde{M}$ of r-pmsdp constructed in Example 9.1.

Then for each $C_i \in \Sigma^*/T'$ to be monotone with respect to U, there exists $C_i \in \Sigma^*/T'$ such that $C_i \cap A_3 \neq \varnothing \wedge (\exists j \in \{0, 1, 2\})(C_i \cap A_j \neq \varnothing)$ (see Fig. 10). Since $A_0 \backslash U$, $A_1 \backslash U$ and $A_2 \backslash U$ are not $\varnothing$, respectively, there is an I-circuit in $\Gamma_{U;T}$, by Proposition 9.3. This is a contradiction.

As was shown in Ibaraki (1973a), an r-pmsdp Π with $O(\Pi) = U$ is easily constructed from the standard construction of R_U, if U is regular: $\Pi = (M, h, \xi_0)$ where $M = (Q, \Sigma, q_0, \lambda, Q_F)$ is the standard construction of R_U with $Q_F = \{[A_i] \mid A_i \in U/R_U\}$, $(\forall q \in Q)(\forall a \in \Sigma)(h(\xi, q, a) = \xi)$ and $\xi_0 = 0$. (Obviously $O(\Pi) = \{x \in F(\Pi) \mid (\forall y \in F(\Pi))(\bar{h}(x) \leqslant \bar{h}(y))\} = \{x \in F(\Pi) \mid \bar{h}(x) = 0\} = F(\Pi) = F(M) = U$.) Although this r-pmsdp is not always minimal as exhibited by Example 9.1, there are many cases in which it is actually minimal. The next theorem shows one situation in which its minimality is guaranteed.

THEOREM 9.4. *Let U be a regular set. If Σ^*/R_U has no equivalence class A_i such that $A_i \backslash U = \varnothing$, the above r-pmsdp Π constructed from the standard construction of R_U is a minimal r-pmsdp satisfying $O(\Pi) = U$. Furthermore, M of the minimal r-pmsdp $\Pi = (M, h, \xi_0)$ with $O(\Pi) = U$ is unique except for a renaming of the states.*

Proof. Assume that a minimal r-pmsdp $\tilde{\Pi}$ with $O(\tilde{\Pi}) = U$ is constructed from $T \in \Lambda_F(\Sigma^*)$ by following Theorem 9.1. We will show that there exists no $C_j \in \Sigma^*/T$ satisfying $(\exists A_i, A_k \in \Sigma^*/R_U)(A_i \cap C_j \neq \varnothing \wedge A_k \cap C_j \neq \varnothing \wedge A_i \subset U \wedge A_k \not\subset U)$. First if such C_j exists and $A_i \preccurlyeq_U A_k$, $\Gamma_{U;T}$ has an I-circuit by Proposition 9.3. On the other hand, if such C_j exists and neither $A_i \preccurlyeq_U A_k$ nor $A_k \preccurlyeq_U A_i$ holds, C_j is not monotone with respect to U. Finally, $A_k \preccurlyeq_U A_i$ cannot hold since $A_i \subset U$ and $A_k \not\subset U$ (note that $\epsilon \in A_i \backslash U$ but $\epsilon \notin A_k \backslash U$). Thus we have $(\forall C_j \in \Sigma^*/T)(C_j \subset U \vee C_j \subset \Sigma^* - U)$ and this implies that $T \in \Lambda_F(U)$. Since R_U is the unique equivalence relation in $\Lambda_F(U)$ with the fewest equivalence classes (note that $R \leqslant R_U$ holds for any $R \in \Lambda(U)$), $T = R_U$ follows. Thus r-pmsdp Π constructed from M_U

is minimal. This also proves the uniqueness of M of a minimal r-pmsdp $\Pi = (M, h, \xi_0)$ satisfying $O(\Pi) = U$ (except for a renaming of the states). (See Proposition 5.1(3).) Q.E.D.

On the other hand, if R_U has an equivalence class $A_i \in \Sigma^*/R_U$ such that $A_i \backslash U = \varnothing$, the number of states can always be reduced at least by one.

THEOREM 9.5. *Let U be a regular set with an equivalence class $A_i \in \Sigma^*/R_U$ such that $A_i \backslash U = \varnothing$. Then it is possible to construct an r-pmsdp Π which satisfies $O(\Pi) = U$ and has $(|\Sigma^*/R_U| - 1)$ states.*

Proof. Let $\Sigma^*/R_U = \{A_1, A_2, ..., A_n\}$ and $A_n \backslash U = \varnothing$, without loss of generality. Let $M = (Q, \Sigma, q_0, \lambda, Q_F)$ be the standard construction of R_U with $Q_F = \{[A_i] \mid A_i \in U/R_U\}$. From M, define an fa $M' = (Q', \Sigma, q_0', \lambda', Q_F')$ by

$$Q' = Q - \{[A_n]\}$$

$$q_0' = q_0$$

$$\lambda'([A_i], a) = \begin{cases} \lambda([A_i], a) & \text{if } \lambda([A_i], a) \neq [A_n] \\ [A_i] & \text{if } \lambda([A_i], a) = [A_n] \end{cases}$$

$$\text{for } i = 1, 2, ..., n-1 \text{ and } a \in \Sigma.$$

$$Q_F' = Q_F.$$

Then an r-pmsdp $\Pi' = (M', h', \xi_0)$ is defined by

$$\xi_0' = 0$$

$$h'(\xi, [A_i], a) = \begin{cases} 0 & \text{if } \xi = 0 \wedge \lambda([A_i], a) \neq [A_n] \\ 1 & \text{if } \xi = 0 \wedge \lambda([A_i], a) = [A_n] \\ \xi & \text{if } \xi \neq 0. \end{cases}$$

It is not difficult to see that $\bar{h}'(x) = 1$ if and only if there exists a prefix[6] y of x such that $\bar{\lambda}(y) = [A_n]$. Therefore, $(\forall x \in \Sigma^*)(x \in F(M) \Leftrightarrow \bar{\lambda}(x) \in Q_F \Leftrightarrow \bar{\lambda}(x) \in Q_F \wedge (\forall y = \text{prefix of } x)(\bar{\lambda}(y) \neq [A_n]) \Leftrightarrow \bar{\lambda}'(x) \in Q_F' \wedge \bar{h}'(x) = 0$ (by definitions of λ' and h' above) $\Leftrightarrow x \in O(\Pi'))$. This proves that $O(\Pi') = F(M) = U$. Π' has $(|\Sigma^*/R_U| - 1)$ states. Q.E.D.

Sometimes the number of states can be further reduced. Example 9.1 is such an example

[6] y is a *prefix* of x if $x = yz$ for some $z \in \Sigma^*$.

10. Undecidability of Certain Minimization Problems

The previous sections have discussed five solvable cases of minimization problems associated with r-lmsdp, r-smsdp, and r-pmsdp. It will be shown in this section that the rest of minimization problems introduced in Section 3 are all unsolvable (i.e., there exist no algorithms to solve them, respectively). These undecidabilities are all proved by reducing them to the well-known undecidable problem: the halting problem of a Turing machine (e.g., Davis, 1958).

Theorem 10.1. (1) *There exist no algorithms which first decide whether an arbitrarily given r-ddp Y is w-representable by an r-sdp, an r-msdp, an r-smsdp, an r-pmsdp, and an r-lmsdp, respectively, and then obtain a minimal w-representation of Y by an sdp of the corresponding class in case it is w-representable.* (2) *There exist no algorithms which first decide whether an arbitrarily given r-ddp Y is s-representable by an r-sdp, an r-msdp, an smsdp, and an r-pmsdp, respectively, and then obtain a minimal s-representation of Y by an sdp of the corresponding class in case it is s-representable.*

Proof. Both statements are immediate consequences of the facts proved in Ibaraki (1973a, 1974) that there exists no algorithm which decides whether an arbitrarily given r-ddp Y is w(or s)-representable by each of sdp's listed above. Q.E.D.

Theorem 10.2. (1) *There exist no algorithms to obtain minimal r-sdp and r-msdp which are, respectively, w-equivalent to arbitrarily given r-sdp and r-msdp.* (2) *There exist no algorithms to obtain minimal r-sdp, r-msdp, r-smsdp, and r-pmsdp which are, respectively, s-equivalent to arbitrarily given r-sdp, r-msdp, r-smsdp, and r-pmsdp.*

Proof. Let Ξ be the set of all Turing machines. For $\alpha \in \Xi$, let S_α denote the number of steps required until Turing machine α halts. S_α takes on ∞ if α never halts. It is known (e.g., Davis, 1958) that there exists no algorithm to decide whether $S_\alpha < \infty$ or $S_\alpha = \infty$ for an aribtrarily given $\alpha \in \Xi$ (the halting problem of a Turing machine).

(a) We first prove (1) and (2) for an r-sdp and an r-msdp. Define an r-msdp (hence an r-sdp) $\Pi_\alpha = (M, h_\alpha, \xi_0)$ given by $M = (Q, \Sigma, q_0, \lambda, Q_F)$ where $Q = \{q_0, q_1, ..., q_k\}$ (k is a given constant greater than 1), $\Sigma = \{a, b\}$, λ and Q_F are given in Fig. 13; $\xi_0 = 0$; $h_\alpha(\xi, q_0, a) = \xi - 1$; $h_\alpha(\xi, q_0, b) = 0$ if $S_\alpha < |\xi|$, 1 if $S_\alpha \geqslant |\xi|$; $h_\alpha(\xi, \{q_1, q_2, ..., q_{k-1}\}, b) = \xi$; $h_\alpha(\xi, q_k, b) = 1$;

$h_\alpha(\xi, q, c) = 1$ for all other combinations of $q \in Q$ and $c \in \Sigma$. $\bar{h}_\alpha : \Sigma^* \to Z$ obviously satisfies

$$\bar{h}_\alpha(x) = \begin{cases} -i & \text{if } x = a^i \\ 0 & \text{if } x = a^i b^j \wedge 1 \leqslant j \leqslant k \wedge S_\alpha < i \\ 1 & \text{for all other } x \in \Sigma^*, \end{cases}$$

and $F(\Pi_\alpha) = \Sigma^* - \{a^i \mid i \geqslant 0\}$, $U_\alpha(\equiv O(\Pi_\alpha)) = \{a^i b^j \mid 1 \leqslant j \leqslant k \wedge S_\alpha < i\}$ if $S_\alpha < \infty$, $\{a^* b(a \cup b)^*\}$ if $S_\alpha = \infty$. $\bar{h}_\alpha$ is a recursive function since $\bar{h}_\alpha(x)$ is computed for any $x \in \Sigma^*$ in a finite number of steps as follows: (i) If

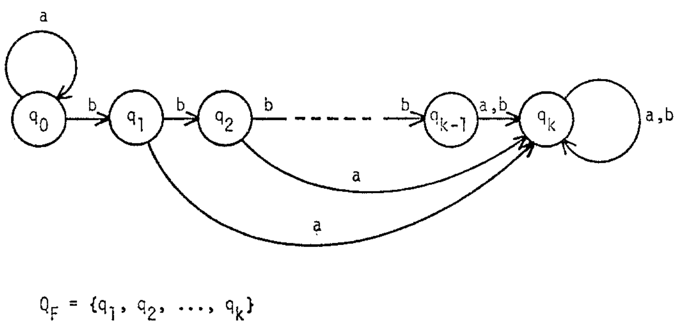

FIG. 13. State transition diagram of fa M used in the proof of Theorem 10.2(a).

$x = a^i$ for some i, $\bar{h}_\alpha(x) = -i$; otherwise go to (ii). (ii) If $x \neq a^i b^j$ for any $i \geqslant 0$, $j \geqslant 0$, $\bar{h}(x) = 1$; otherwise go to (iii). (iii) For $x = a^i b^j$, let Turing machine α operate i steps. If α halts in less than i steps, $\bar{h}_\alpha(x) = 0$; otherwise $\bar{h}_\alpha(x) = 1$. Now assume that $S_\alpha < \infty$. Then the above Π_α is a minimal r-sdp (hence a minimal r-msdp) s-equivalent (or w-equivalent) to Π_α, as proved next. U_α/R_{U_α} consists of the following k equivalence classes: $A_j = \{a^i b^j \mid i > S_\alpha\}, j = 1, 2, \ldots, k$. Thus any $T \in \Lambda_F(\Sigma^*)$ J-separating U_α/R_{U_α} must have at least k equivalence classes of Σ^*/T. Furthermore T requires at least $k + 1$ equivalence classes since $a^l \notin U_\alpha$, $l > S_\alpha$, cannot satisfy $a^l T a^l b^j$, for $j = 1, 2, \ldots, k$, by the fact that $a^l b \in A_1$ and $a^l b^j b \in A_s$, $s > 1$. (Note that T is right invariant.) Thus any r-sdp w-representing (and hence s-representing) Y_α has at least $k + 1$ states and it proves the minimality of Π_α under the assumption $S_\alpha < \infty$. Next consider the case in which $S_\alpha = \infty$. Then $\Pi = (M, h, \xi_0)$ is s-equivalent (hence w-equivalent) to Π_α, where Π is defined by $M = (Q, \Sigma, q_0, \lambda, Q_F)$; $Q = \{q_0, q_1\}$; λ is given by $\lambda(q_0, a) = q_0$, $\lambda(q_0, b) = \lambda(q_1, \{a, b\}) = q_1$; $Q_F = \{q_1\}$; $\xi_0 = 0$; and $h(\xi, q, c) = 1$ for all $\xi \in Z$, $q \in Q$, and $c \in \Sigma$. Consequently, for $k > 1$, a minimal r-sdp (or r-msdp) s (or w)-equivalent to Π_α has no more than two states if and only if $S_\alpha = \infty$. Therefore, if there were an algorithm as stated in (1) and (2) for an r-sdp or an r-msdp, the halting problem of a Turing machine would be solved as follows: (i) obtain a minimal r-sdp (or r-msdp) s (or w)-equivalent to Π_α. (ii) $S_\alpha = \infty$ if and only if the resulting r-sdp (or r-msdp) has at most two states. This is of course a contradiction.

(b) We prove (2) for an r-smsdp and an r-pmsdp. Let $\Pi_\alpha = (M, h_\alpha, \xi_0)$ be an r-smsdp (and an r-pmsdp) defined by $M = (Q, \Sigma, q_0, \lambda, Q_F)$; $Q = \{q_0, q_1\}$, $\Sigma = \{a, b\}$, λ and Q_F be given by Fig. 14; $\xi_0 = 0$; $h_\alpha(\xi, q_0, a) = h_\alpha(\xi, \{q_0, q_1\}, b) = \xi + 1$; $h_\alpha(\xi, q_1, a) = \xi + 1$ if $\xi < S_\alpha$, $\xi + 2$ if $\xi \geqslant S_\alpha$. Then $F(\Pi_\alpha) = \Sigma^*$ and $\bar{h}_\alpha$ satisfies

$$\bar{h}_\alpha(x) = \begin{cases} |x| & \text{if } |x| \leqslant S_\alpha \\ |x| + N_{aa}(z) & \text{if } |x| > S_\alpha, \end{cases}$$

$$\text{where} \quad x = yz \wedge |y| = \max[S_\alpha - 1, 0],$$

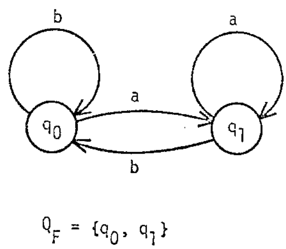

FIG. 14. State transition diagram of fa M used in the proof of Theorem 10.2(b).

where $|w|$ for $w \in \Sigma^*$ denotes the length of w, and $N_{aa}(z)$ denotes the number of subsequence aa's in z (for example aaa is counted as two subsequences of aa). The recursiveness of $\bar{h}_\alpha$ can be proved in a manner similar to $\bar{h}_\alpha$ of (a). Now, assume $S_\alpha < \infty$. Then Π_α is a minimal r-smsdp (and r-pmsdp) s-equivalent to Π_α, as proved below. Let Y_α be the r-ddp s-represented by Π_α. Then we have $\Psi_m = \{A, B\}$, where $m = \max[S_\alpha, 1]$, $A = \{xa \mid x \in \Sigma^* \wedge |x| = m - 1\}$ and $B = \{xb \mid x \in \Sigma^* \wedge |x| = m - 1\}$ (Ψ_p was defined in Section 4 prior to Theorem 4.3). Thus any s-representation of Y_α by an r-sdp (hence by an r-smsdp or by an r-pmsdp) requires at least two states since any $T \in \Lambda_F(\Sigma^*)$ J-separating Ψ_m satisfies $|\Sigma^*/T| \geqslant 2$ (see Theorem 4.3 and Lemma 7.1). On the other hand, if $S_\alpha = \infty$, there exists a minimal r-smsdp (and r-pmsdp) Π with one state s-equivalent to Π_α. Such Π is given by (M, h, ξ_0) where $M = (\{q_0\}, \Sigma, q_0, \lambda, \{q_0\})$; $\lambda(q_0, \{a, b\}) = q_0$; $h(\xi, q_0, \{a, b\}) = \xi + 1$; $\xi_0 = 0$. (Π is s-equivalent to Π_α since

$(\forall x \in \Sigma^*)(\bar{h}(x) = |x| = \bar{h}_\alpha(x))$.) Consequently there exists a minimal r-smsdp (and r-pmsdp) with one state which is s-equivalent to Π_α if and only if $S_\alpha = \infty$. Therefore, if there were an algorithm as stated in (2) for an r-smsdp or for an r-pmsdp, the halting problem of a Turing machine would be again solvable. This is a contradiction. Q.E.D.

TABLE 1

Solvability of Each Minimization Problem[a]

Represen-tation	Type of Algorithm	r-sdp	r-msdp	r-imsdp	r-smsdp	r-pmsdp	r-lmsdp
w	1[b]	*U*	*U*	*U*	*U*	*U*	*U*
	2[c]	*U*	*U*	*U*	*S*	*S*	*S*
s	1[b]	*U*	*U*	*U*	*U*	*U*	*S*
	2[c]	*U*	*U*	*U*	*U*	*U*	*S*

[a] *U*: Unsolvable, *S*: Solvable.

[b] Algorithm 1 first decides whether a given r-ddp Υ is *-representable[d] by an sdp of the corresponding type (column), and then finds a minimal *-representation[d] of Υ by an sdp of the corresponding type (column) in case it is *-representable.[d]

[c] Algorithm 2 obtains a minimal sdp of the corresponding type (column) *-equivalent[d] to a given sdp of the corresponding type (column).

[d] * should read w or s depending on the row of the entry (see the first column) under consideration.

Summing up the results obtained so far, we have Table 1 which shows the solvability or the unsolvability of each of the minimization problems. Although the definition of r-imsdp is not given in this paper (see Ibaraki, 1974), its results are also included because proofs of Theorems 10.1 and 10.2 can be directly modified for an r-imsdp.

11. Nonuniqueness of Minimal Representations

This section shows the nonuniqueness of various minimal representations.

THEOREM 11.1. *Minimal s-representations of an r-ddp Υ by an r-sdp, r-msdp, r-smsdp, r-pmsdp, and r-lmsdp are not generally unique, respectively.*

Proof. Consider an r-ddp $Y = (\Sigma, \Sigma^*, f)$ where $\Sigma = \{a\}$ and $f(\epsilon) = f(a) = 0$, $f(a^k) = k - 1$ for $k = 2, 3, \ldots$. Then Σ^*/R_Y consists of the equivalence classes $A_i = \{a^i\}$, $i = 0, 1, 2, \ldots$. Since $\Psi_0 = \{A_0, A_1\}$, any r-sdp s-representing Y has at least two states since $T \in A_F(S)$ J-separating Ψ_0 has at least two equivalence classes (see Theorem 4.3). Thus the following two s-representations are both minimal.

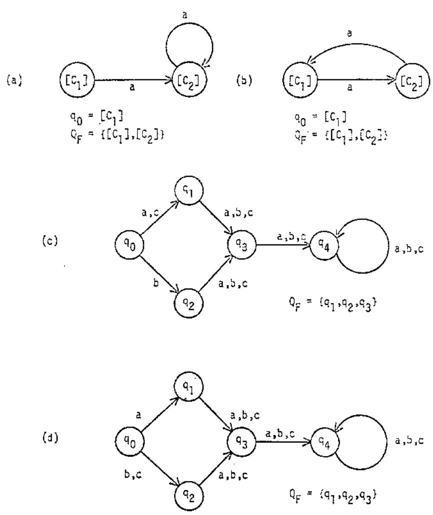

FIG. 15. State transition diagrams of fa's used in the proof of Theorem 11.1.

(a) $\Pi = (M(Q, \Sigma, q_0, \lambda, Q_F), h, \xi_0)$ where λ is given by Fig. 15(a) for two states corresponding to $C_1 = A_0$ and $C_2 = \bigcup_{i=1}^{\infty} A_i$. ξ_0 and h are given by $\xi_0 = 0$, $h(\xi, [C_1], a) = \xi$ and $h(\xi, [C_2], a) = \xi + 1$.

(b) λ is given by Fig. 15(b) for two states corresponding to $C_1 = \bigcup \{A_i \mid i = 0, 2, 4, \ldots\}$ and $C_2 = \bigcup \{A_i \mid i = 1, 3, 5, \ldots\}$. ξ_0 and h are given by $\xi_0 = 0$, $h(\xi, [C_1], a) = \xi$ if $\xi \leqslant 0$, $\xi + 1$ if $\xi > 0$, and $h(\xi, [C_2], a) = \xi + 1$.

Since (a) and (b) are both r-msdp's, r-smsdp's, r-pmsdp's as well as r-sdp's, the theorem is proved for them. To prove the theorem for an r-lmsdp, let $Y = (\Sigma, S, f)$ be an r-ddp with $\Sigma = \{a, b, c\}$, $S = \{x \in \Sigma^* \mid |x| = 1 \text{ or } 2\}$ and $f(a) = f(b) = 0$, $f(c) = 1$, $f(aa) = 1$, $f(x) = 2$ for other $x \in \Sigma^*$ with $|x| = 2$. Σ^*/R_S consists of the equivalence classes C_1, C_2, C_3, C_4 where $C_1 = \{\epsilon\}$, $C_2 = \{a, b, c\}$, $C_3 = \{x \mid |x| = 2\}$, $C_4 = \Sigma^* - C_1 - C_2 - C_3$. Σ^*/R_Y consists of the following equivalence classes: $A_0 = \{\epsilon\}$, $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{c\}$, $A_4 = \{aa\}$, $A_5 = \{x \in \Sigma^* \mid |x| = 2 \wedge x \neq aa\}$, $A_6 = \Sigma^* - \bigcup_{i=0}^{5} A_i$. $C_1 = A_0$, $C_2 = A_1 \cup A_2 \cup A_3$, $C_3 = A_4 \cup A_5$, and

$C_4 = A_6$ hold. Furthermore, $A_1 \preccurlyeq_Y A_2$, $A_2 \preccurlyeq_Y A_3$, $A_1 \preccurlyeq_Y A_3$, $A_4 \preccurlyeq_Y A_5$, and $\Psi_0 = \{A_1, A_2\}$. Since $|\Sigma^*/R_S| = 4$ and $A_1, A_2 \in C_1$, at least five states are required to s-represent Y by an r-msdp ($T \in \Lambda_F(S)$ satisfying Theorem 4.3 has at least five equivalence classes: four from the fact $T \leqslant R_S$ and one more to J-separate Ψ_0). Then we have the following minimal r-lmsdp's s-representing Y.

(c) $\Pi = (M, h, \xi_0)$ where M is given in Fig. 15(c). $\xi_0 = 0$ and $h(\xi, q_0, \{a, b\}) = 0$, $h(\xi, q_0, c) = 1$, $h(\xi, q_1, a) = 1$ if $\xi \leqslant 0$, 2, if $\xi \geqslant 1$, $h(\xi, q, d) = 2$ for all other $q \in Q$, $d \in \Sigma$.

(d) $\Pi = (M, h, \xi_0)$ where M is given in Fig. 15(d). $\xi_0 = 0$ and $h(\xi, q_0, \{a, b\}) = 0$, $h(\xi, q_0, c) = 1$, $h(\xi, q_1, a) = 1$, $h(\xi, q, d) = 2$ for all other $q \in Q$ and $d \in \Sigma$.

This proves the theorem for an r-lmsdp. Q.E.D.

THEOREM 11.2. *Minimal w-representations of an r-ddp Y by an r-sdp, r-msdp, r-smsdp, r-pmsdp, and r-lmsdp are not generally unique, respectively.*

Proof. Consider the first r-ddp Y used in the proof of Theorem 11.1. Since $U(=O(Y)) = \{\epsilon, a\}$ and $U/R_U = \{A_0, A_1\}$ where $A_0 = \{\epsilon\}$ and $A_1 = \{a\}$, any w-representation by an r-sdp requires at least two states (see Theorem 4.2). Thus r-sdp's (a) and (b) used in the proof of Theorem 11.1 are also minimal w-representations by an r-sdp, r-msdp, r-smsdp, or r-pmsdp. This proves the theorem for an r-sdp, r-msdp, r-smsdp, and r-pmsdp. Finally to prove the theorem for an r-lmsdp, let $Y = (\Sigma, S, f)$ be an r-ddp with $\Sigma = \{a, b, c\}$, and $U(=O(Y)) = \{a, b, aa, ab, ba, ca\}$. Σ^*/R_U consists of the equivalence classes $A_0 = \{\epsilon\}$, $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{c\}$, $A_4 = \{aa, ab, ba, ca\}$, $A_5 = \Sigma^* - \bigcup_{i=0}^{4} A_i$, and $P_U' = \{(A_i, A_3), (A_j, A_5) \mid i = 0, 1, 2, j = 0, 1, ..., 4\}$. Then $T \in \Lambda_F(\Sigma^*)$ that gives rise to an r-lmsdp w-representing Y has at least five equivalence classes: four to J-separate $\{A_0, A_1, A_2, A_4\} \in N_U'$ and one more to include A_5, since $|F(\Pi)| < \infty$. Then the following two r-lmsdp's w-representing Y are minimal.

(a) $\Pi = (M, h, \xi_0)$, where M is given by Fig. 15(c); $\xi_0 = 0$; $h(\xi, q_0, \{a, b\}) = \xi$, $h(\xi, q_0, c) = \xi + 1$, $h(\xi, \{q_1, q_2\}, a) = 0$, $h(\xi, q_1, b) = \xi$, $h(\xi, q_1, c) = h(\xi, q_2, \{b, c\}) = \xi + 1$, $h(\xi, q, d) = \xi$ for other $q \in Q$, $d \in \Sigma$.

(b) $\Pi = (M, h, \xi_0)$, where M is given by Fig. 15(d); $\xi_0 = 0$; h is the same as h of (a).

This proves the theorem for an r-lmsdp. Q.E.D.

The nonuniqueness of w-representation may appear somewhat trivial since we can easily obtain an infinite number of different r-sdp's (or other classes of r-sdp's) by simply adjusting the cost function h of $\Pi = (M, h, \xi_0)$ without altering $O(\Pi)$. However, the above theorem says that the w-representation is nonunique in the stronger sense that there are w-representations having essentially different fa's M of $\Pi = (M, h, \xi_0)$.

Conclusion

Various minimization problems associated with each class of sequential decision processes were considered in this paper. As summarized in Table 1, there are five solvable cases and all the other problems are unsolvable. It should also be emphasized that, except for the problem of finding a minimal r-smsdp w-equivalent to a given r-smsdp, all algorithms presented in the paper are extremely inefficient (though finite) because they are based on the effective enumeration of fa's or T's in $\Lambda_F(\Sigma^*)$. Thus the improvement of their efficiency would be one of the main subjects in the future research. Since it is noticed that our problems have certain similarity with the well-known minimization problem of incompletely specified sequential machines, an attempt is being made to increase the efficiency through the use of techniques developed for incompletely specified sequential machines.

Although the minimization of a single sequential decision process was investigated throughout this paper, it may be more important in practice to consider the minimization of a family of sequential decision processes, which are obtained by changing some parameters involved in the processes. (A model of this type was discussed in Karp and Held (1967).) It is expected that the results obtained in this paper also work as a basis for such problems.

Acknowledgment

The author wishes to appreciate Prof. H. Mine of Kyoto University for his persistent encouragement and helpful comments on the subject.

Received: November 21, 1972

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