An anisotropic visco-hyperelastic model for ligaments at finite strains. Formulation and computational aspects

E. Peña, B. Calvo, M.A. Martínez, M. Doblare *

Group of Structural Mechanics and Materials Modelling, Aragón Institute of Engineering Research (I3A), University of Zaragoza, María de Luna, 3, E-50018 Zaragoza, Spain

Received 29 July 2005; received in revised form 10 April 2006
Available online 16 May 2006

Abstract

In this paper we present a fully three-dimensional finite strain anisotropic visco-hyperelastic model for ligaments and tendons. The structural model is formulated within the framework of non-linear continuum mechanics and is well-suited for its finite element implementation. This model is based on a local additive decomposition of the stress tensor into initial and non-equilibrium parts as resulted from the assumed structure of the free-energy density function that generalizes Kelvin–Voigt linear viscous models. Also, we use a local multiplicative decomposition of the deformation gradient into volume-preserving and dilatational parts that permits to model the incompressible properties of soft biological tissues. To simulate the viscoelastic properties of this kind of tissues, we consider different viscoelastic behaviours for the matrix and the different families of fibers. A second-order accurate numerical integration procedure is used, established entirely in the reference configuration. Expressions for the stress and elasticity tensors in the spatial description are also presented.

Of all soft tissues, we have focused in ligaments due to the importance of their viscoelastic properties in the clinical practise. In order to show clearly the performance of the constitutive model, we present 3D simulations of the behaviour of the anterior cruciate ligament and patellar tendon graft. The model was also tested for various multi-axial loading situations. The relaxation and creep responses and the strain rate dependent behaviour of anterior cruciate ligament and patellar tendon graft were accurately predicted.

© 2006 Elsevier Ltd. All rights reserved.

Keywords: Visco-hyperelasticity; Internal variables; Finite strains; Ligaments; Finite element method

1. Introduction

Biological soft tissues sustain large deformations, rotations and displacements, have a highly non-linear behaviour, posses anisotropic mechanical properties and show a clear time and strain rate dependency. Their typical anisotropic behaviour is caused by several collagen fiber families (usually one or two fibers coincide at each point) that are arranged in a matrix of soft material named ground substance (Holzapfel and Gasser,
Typical examples of fibered soft biological tissues are blood vessels, tendons, ligaments, cornea and cartilage. The time-rate dependent material behaviour of this kind of tissues has been well-documented and quantified in the literature. This includes works on ligaments (Puso and Weiss, 1998), tendons (Johnson et al., 1996), blood vessels (Humphrey, 1995), cornea (Pinsky and Datye, 1991) and articular cartilage (Hayes and Mockros, 1971). This behaviour can arise from the fluid flow inside the tissue, from the inherent viscoelasticity of the solid phase, or from viscous interactions between the tissue phases (Mak, 1986).

Many constitutive models have been proposed for biological soft tissues. A theory of quasilinear viscoelasticity was early proposed by Fung and is still widely used in the field of biomechanics to describe soft tissue viscoelastic behaviour (Fung, 1993). This model has however an important drawback, such as the information must be saved at every previous time step in order to compute the stress response at the current one. Puso and Weiss (1998) formulated a time discretization algorithm of the convolution integral in which the relaxation function and the elastic constitutive behaviour were split by means of a multiplicative decomposition, thus reducing the non-linear response of the tissue to the latter, maintaining the viscous behaviour within the framework of linear viscoelasticity. They used an anisotropic strain-energy function to model the fibered behaviour of the ligaments but did not consider the different viscoelastic behaviour of collagen fibers and ground substance. A fully three-dimensional finite strain viscoelastic model not restricted to isotropy was developed by Simo (1987). The model described therein was based on the concept of internal variables and allows a very general description of materials involving irreversible effects. This constitutive model was applied to model the mechanical behaviour of isotropic elastomers. More recent papers by Pioletti et al. (1998) and Limbert and Middleton (2004) modelled the isotropic and transversely isotropic visco-hyperelastic behaviour of ligaments. These phenomenological approaches used elastic and viscous potentials which involved 10 and 15 invariants, respectively. Other approaches for viscoelasticity are due to LeTallec et al. (1993), Provenzano et al. (2002), Johnson et al. (1996), Kaliske (2000), Quaglini et al. (2004) and many others.

All these models have in common that they do not specifically deal with the deformation of ligament as a fiber-reinforced composite material. Sasaki and Odajima (1996) and Puxkandl et al. (2002) demonstrated the different viscoelastic behaviour of collagen tissue and ground substance in ligaments. Only, and following Simo’s constitutive framework, Holzapfel and Gasser (2001) proposed a viscoelastic model of fiber-reinforced composites at finite strains that considered different viscoelastic behaviour for the matrix and the fibers. However, Holzapfel’s model was of Maxwell-type, while in Sasaki and Odajima (1996), Puxkandl et al. (2002) or Vita and Slaughter (2005) ligaments were proved to exhibit a Kelvin–Voigt-type viscoelastic constitutive behaviour.

In this paper, we present a fully three-dimensional finite strain anisotropic visco-hyperelastic Kelvin–Voigt model for ligaments. This model has been applied to simulate some clinical applications. The structural model is formulated within Simo’s constitutive framework. It is based on a local additive decomposition of the stress tensor into initial and non-equilibrium parts and consider different viscoelastic behaviours for the matrix and the fibers. We have considered a material reinforced by one family of fibers continuously distributed in a compliant solid isotropic matrix (Spencer, 1954). Since the mechanical response of biological tissues is almost isochoric (Simo and Taylor, 1991), we assume uncoupled volumetric and deviatoric responses over any range of deformation. This is achieved by a local multiplicative decomposition of the deformation gradient into volume-preserving and dilatational parts. Although the description of the constitutive model is established in the reference configuration we also express the stress and elasticity tensors in the spatial description for a more simple FE implementation. Incremental objectivity is trivially satisfied by establishing the numerical integration procedure entirely in the reference configuration (Holzapfel, 1996). To our knowledge, the anisotropic visco-hyperelastic model based on the local additive decomposition of the stress tensor into initial and non-equilibrium parts and the explicit expression for the stress and fourth-order elasticity tensor in the spatial description for time-dependent finite strains in fibered materials have not been recorded previously in the literature.

The paper is organized as follows. In Section 2 the constitutive equations of finite elasticity with uncoupled volume response are briefly revisited. We represent the free-energy density function in terms of five independent invariants. In Section 3 the anisotropic visco-hyperelastic model is presented. Section 4 shows the integration algorithm for the constitutive equations. The application of this model to some examples is presented in Section 5. Comparison with experimental data and numerical simulations are included to illustrate the effectiveness of the proposed formulation. Finally, Section 6 includes some concluding remarks.
2. Finite elasticity with uncoupled volume response

As a first step towards the development of a non-linear anisotropic visco-hyperelastic model, we consider the formulation of finite strain hyperelasticity in terms of invariants with uncoupled volumetric/deviatoric responses, first suggested in Flory (1961), generalized in Simo and Taylor (1991) and employed for anisotropic soft biological tissues in Weiss et al. (1996), Holzapfel et al. (2000) and Peña et al. (2005).

Let $x = x(X, t) : \Omega_0 \times \mathbb{R} \to \mathbb{R}^3$ denote the motion mapping and let $F$ be the associated deformation gradient. Here $X$ and $x$ define the respective positions of a particle in the reference $\Omega_0$ and current $\Omega$ configurations such as $F = \frac{\partial x}{\partial X}$. Further, let $J \equiv \det F$ be the Jacobian of the motion. To properly define volumetric and deviatoric responses in the non-linear range, we introduce the following kinematic decomposition (Flory, 1961):

\[
\begin{align*}
F &= J^T F, \quad \overline{F} = J^{-1} F \\
C &= F^T F, \quad \overline{C} = J^{-\frac{3}{2}} C = F^T F
\end{align*}
\]

The term $J^T F$ is associated with volume-changing deformations, while $\overline{F}$ is associated with volume-preserving deformations. We shall call $\overline{F}$ and $\overline{C}$ the modified deformation gradient and the modified right Cauchy–Green tensors, respectively.

The direction of a fiber at a point $X \in \Omega_0$ is defined by a unit vector field $m_0(X)$, $|m_0| = 1$. It is usually assumed that, under deformation, the fiber moves with the material points of the continuum body. Therefore, the stretch $\lambda$ of the fiber defined as the ratio between its lengths at the deformed and reference configurations can be expressed as

\[
\lambda m(x, t) = \overline{F}(X, t)m_0(X), \quad \lambda^2 = m_0 \cdot F^T F \cdot m_0 = m_0 \cdot C \cdot m_0
\]

where $m$ is the unit vector of the fiber in the deformed configuration.

To characterize isothermal processes, we postulate the existence of a unique decoupled representation of the strain-energy density function $\Psi$ (Simo and Taylor, 1985). Based on the kinematic assumption (1) and following Spencer (1954) it can be shown that five modified invariants are necessary to form the integrity bases of the tensors $\overline{C}, m_0 \otimes m_0$. Then, the free energy can be written in a decoupled form as

\[
\Psi = \Psi_{\text{vol}}(J) + \Psi_{\text{iso}}(\overline{C}, m_0 \otimes m_0) = \Psi_{\text{vol}}(J) + \Psi_{\text{iso}}(\overline{I}_1, \overline{I}_2, \overline{I}_4, \overline{I}_5)
\]

where

\[
\begin{align*}
\overline{I}_1 &= \text{tr} \, \overline{C}, \quad \overline{I}_2 = \frac{1}{2} (\text{tr} \, \overline{C})^2 - \text{tr} \overline{C}^2 \\\n\overline{I}_4 &= m_0 \cdot \overline{C} \cdot m_0, \quad \overline{I}_5 = m_0 \cdot \overline{C} \cdot m_0
\end{align*}
\]

with $\overline{I}_1$ and $\overline{I}_2$ the first two strain invariants of the symmetric modified Cauchy–Green tensor and the pseudo-invariants. $\overline{I}_4$, $\overline{I}_5$ characterize the anisotropy constitutive response of the fibers: $\overline{I}_4$ has a clear physical meaning since it is the square of the stretch along the fibers. In order to reduce the number of material parameters and to work with physically motivated invariants, we shall omit the dependency of the free energy $\Psi$ on $\overline{I}_5$. This hypothesis is usually used in biomechanical modelling (Holzapfel et al., 2002).

The stress response is then obtained from the derivatives of the stored-energy function, getting

\[
S = 2 \frac{\partial \Psi}{\partial C} = S_{\text{vol}} + S_{\text{iso}} = J p C^{-1} + J^{-\frac{3}{2}} (1 - 1/3 C^{-1} \otimes C) \cdot \overline{\Sigma}
\]

where $p$ is the hydrostatic pressure and $\overline{\Sigma}$ the modified second Piola–Kirchhoff stress tensor

\[
p = \frac{d \Psi_{\text{vol}}(J)}{dJ}, \quad \overline{\Sigma} = 2 \frac{\partial \Psi_{\text{iso}}(\overline{C}, m_0)}{\partial C}
\]

The Cauchy stress tensor $\sigma$ is $1/J$ times the push-forward of $S$ $(\sigma = J^{-1} \chi(S))$, see Holzapfel (2000). The expression of the Cauchy stress tensor $\sigma$ is included in the Appendix.

We conclude our development of uncoupled volumetric/deviatoric finite deformation elasticity with one family of fibers by recording the explicit expressions for the elastic tangent moduli.
second Piola–Kirchhoff stress tensor \( \mathbf{S} \) at a certain point. Its variation with respect to the right Cauchy–Green tensor \( \mathbf{C} \) is the elasticity tensor in the material description or the referential tensor of elasticities and may be written as (Simo and Hughes, 1998)

\[
\mathbf{C} = 2 \frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} = \mathbf{C}_{\text{vol}} + \mathbf{C}_{\text{iso}} = 2 \frac{\partial \mathbf{S}_{\text{vol}}}{\partial \mathbf{C}} + 2 \frac{\partial \mathbf{S}_{\text{iso}}}{\partial \mathbf{C}}
\]  

(8)

The elasticity tensor in the spatial description or the spatial tensor of elasticities, denoted by \( \mathbf{c} \), is defined as the push-forward of \( \mathbf{C} \) times a factor \( J^{-1} \), so that

\[
\mathbf{c} = J^{-1} \mathbf{X}(\mathbf{C}) = \mathbf{c}_{\text{vol}} + \mathbf{c}_{\text{iso}}
\]  

(9)

The expression of the spatial tensor \( \mathbf{c} \) is included in the Appendix.

3. Anisotropic visco-hyperelastic model

In this section, we develop the basic structure of the proposed finite strain anisotropic visco-hyperelastic model. In order to describe viscoelastic effects we apply the concept of internal variables (Simo, 1987; Holzapfel, 1996). For other approaches to describe the viscoelastic behaviour see Christensen (1982). Following Simo (1987), we postulate the existence of an uncoupled free-energy function of the form

\[
\psi(\mathbf{C}, \mathbf{m}_0, \mathbf{Q}_i) = \psi^{\text{vol}}(J) + \psi^{\text{iso}}_{\text{int}}(\mathbf{C}, \mathbf{m}_0 \otimes \mathbf{m}_0) - \sum_{i=1}^{n} \left[ \frac{1}{2} \mathbf{C} : \mathbf{Q}_i + \mathcal{E} \left( \sum_{i=1}^{n} \mathbf{Q}_i \right) \right]
\]  

(10)

where \( \psi^{\text{vol}} \) and \( \psi^{\text{iso}} \) are the volumetric and deviatoric parts of the initial elastic stored-energy function \( \psi^{\text{vol}}, \mathbf{Q}_i \) play the role of internal variables (not accessible to direct observation) corresponding to the reference configuration and \( \mathcal{E} \) is a certain function of the internal variables.

Restricting our attention to the isothermal case and exploiting the Clausius–Duhem inequality \( \mathcal{D}_{\text{int}} = -\dot{\psi} + \frac{1}{2} \mathbf{S} : \dot{\mathbf{C}} \geq 0 \) (Marsden and Hughes, 1994), we get

\[
\mathbf{S} = 2 \frac{\partial \psi(\mathbf{C}, \mathbf{m}_0, \mathbf{Q}_i)}{\partial \mathbf{C}} - J^{-1} \mathbf{F} \mathbf{c} \mathbf{F}^T = 2 \frac{\partial \psi^{\text{vol}}}{\partial \mathbf{C}} - \sum_{i=1}^{n} \mathbf{Q}_i
\]  

(11)

\[
\mathcal{D}_{\text{int}} = -\sum_{i=1}^{n} \frac{\partial \psi(\mathbf{C}, \mathbf{m}_0, \mathbf{Q}_i)}{\partial \mathbf{Q}_i} : \dot{\mathbf{Q}_i} = \sum_{i=1}^{n} \left[ \frac{1}{2} \mathbf{C} - \frac{\partial \mathcal{E}(\mathbf{Q}_i)}{\partial \mathbf{Q}_i} \right] : \dot{\mathbf{Q}_i} \geq 0
\]  

(12)

\( \mathbf{Q}_i \) may be interpreted as non-equilibrium stresses, in the sense of non-equilibrium thermodynamics, and remain unaltered under superposed spatial rigid body motions (Simo and Hughes, 1998). This fundamental requirement is the same invariance property classically placed on the second Piola–Kirchhoff tensor \( \mathbf{S} \) and automatically ensures frame indifference of the constitutive relationship (11).

Since \( \mathbf{\sigma} = \frac{1}{2} \mathbf{F} \mathbf{S} \mathbf{F}^T \), in the spatial description the expression (11) may be recast in the equivalent form

\[
\sigma = p \mathbf{I} + \frac{1}{2} \mathbf{F} \left( 2 \frac{\partial \psi^{\text{vol}}}{\partial \mathbf{C}} - \sum_{i=1}^{n} \mathbf{Q}_i \right) \mathbf{F}^T
\]  

(13)

Note that the visco-hyperelastic model proposed herein is based on a local additive decomposition of the stress tensor into initial and non-equilibrium parts. Based on previous studies (Sasaki and Odajima, 1996; Puxkandl et al., 2002; Vita and Slaughter, 2005), the ligament is assumed to have a Kelvin–Voigt-type viscoelastic constitutive behaviour. On the contrary, the model in Holzapfel and Gasser (2001) supposes the additive split of the stress tensor into equilibrium and non-equilibrium parts assuming Maxwell-type viscoelastic behaviour for blood vessels.

Motivated by Holzapfel and Gasser (2001) and in order to consider different contributions of the matrix material and families of fibers on the non-equilibrium part, we divide the internal variables in

\[
\mathbf{Q}_i = \sum_{j=1, j\neq 3}^{5} \mathbf{Q}_{ij}
\]  

(14)
where $\mathbf{Q}_{11}$ and $\mathbf{Q}_{12}$ are the isotropic contribution due to the matrix material associated to $I_1$ and $I_2$ invariants and $\mathbf{Q}_{44}, \mathbf{Q}_{55}$ are the anisotropic contribution due to the two families of fibers associated to $\mathbf{T}_4, \mathbf{T}_5$ invariants.

We now formulate the evolution equations separately for each contribution. We consider the following set of rate equations governing the evolution of internal variables $\mathbf{Q}_{ij}$ (Simo, 1987)

$$
\dot{\mathbf{Q}}_{ij} + \frac{1}{\tau_{ij}} \mathbf{Q}_{ij} = \frac{\gamma_{ij}}{\tau_{ij}} \text{DEV} \left[ 2\delta \Psi_0^{(ij)}(\mathbf{C}, \mathbf{m}_0) \right], \quad \lim_{t \to -\infty} \dot{\mathbf{Q}}_{ij} = 0
$$

(15)

with $\gamma_{ij} \in [0, 1]$ free-energy factors associated with relaxation times $\tau_{ij} > 0$ and $\delta \Psi_0^{(ij)} = \frac{\partial \Psi_0^{(ij)}}{\partial \mathbf{Q}_{ij}}$.

The evolution equations (15) are linear and, therefore, explicitly lead to the following convolution representation:

$$
\mathbf{Q}_{ij}(t) = \frac{\gamma_{ij}}{\tau_{ij}} \int_{-\infty}^{t} \exp \left[ -\frac{(t-s)}{\tau_{ij}} \right] \text{DEV} \left[ 2\delta \Psi_0^{(ij)} \right] ds
$$

(16)

We can determine $\Xi$ from the condition of thermodynamic equilibrium (Simo, 1987). It is clear that, given the rate equations (15), equilibrium is achieved for

$$
\dot{\mathbf{Q}}_{ij} = 0 \Rightarrow \mathbf{Q}_{ij} = \gamma_{ij} \text{DEV} \left[ 2\delta \Psi_0^{(ij)}(\mathbf{C}, \mathbf{m}_0) \right],
$$

$$
\frac{\partial \delta \Psi_0^{(ij)}}{\partial \mathbf{Q}_{ij}} = 0 \Rightarrow \sum_{i=1}^{n} \sum_{j=1, j \neq 3}^{5} \left[ -\frac{1}{2} \mathbf{C} + \frac{\partial \Xi}{\partial \mathbf{Q}_{ij}} \right] = 0
$$

(17)

defines $\Xi$ as the Legendre transformation of the function $\Psi_0^{(ij)}$ in the sense that

$$
\Xi(\mathbf{Q}_{ij}) = \sum_{i=1}^{n} \left[ \sum_{j=1, j \neq 3}^{5} \left[ -2\gamma_{ij} \delta \Psi_0^{(ij)} \right] + \frac{1}{2} \mathbf{C} : \sum_{j=1, j \neq 3}^{5} \mathbf{Q}_{ij} \right]
$$

(18)

Substitution of (16) into (11) and integrating by parts then yields the following equivalent expression:

$$
\mathbf{S} = J \rho \mathbf{C}^{-1} + J^{-\frac{3}{2}} \sum_{j=1, j \neq 3}^{5} \left[ \left( 1 - \sum_{i=1}^{n} \gamma_{ij} \right) \text{DEV} \left\{ 2\delta \Psi_0^{(ij)}(\mathbf{C}, \mathbf{m}_0) \right\} \right] \\
+ \sum_{i=1}^{n} \sum_{j=1, j \neq 3}^{5} \left[ J^{-\frac{3}{2}} \gamma_{ij} \int_{-\infty}^{t} \exp \left[ -\frac{(t-s)}{\tau_{ij}} \right] \frac{d}{ds} \left\{ \text{DEV} \left[ 2\delta \Psi_0^{(ij)}(\mathbf{C}, \mathbf{m}_0) \right] \right\} ds \right]
$$

(19)

Note that $\mathbf{S}$ attains its equilibrium value (17) as $\tau_{ij} \to \infty$. The corresponding value of the equilibrium stress is a fraction of the initial stress; that is

$$
\lim_{\tau_{ij} \to -\infty} \mathbf{S} = J \rho \mathbf{C}^{-1} + J^{-\frac{3}{2}} \sum_{j=1, j \neq 3}^{5} \left[ \left( 1 - \sum_{i=1}^{n} \gamma_{ij} \right) \text{DEV} \left\{ 2\delta \Psi_0^{(ij)}(\mathbf{C}, \mathbf{m}_0) \right\} \right]
$$

(20)

The convolution representation (19) in terms of the Cauchy stress tensor takes the form

$$
\mathbf{\sigma} = p \mathbf{1} + \frac{1}{J} \left[ \sum_{j=1, j \neq 3}^{5} \left[ \left( 1 - \sum_{i=1}^{n} \gamma_{ij} \right) \text{dev} \left\{ \mathbf{F} \left[ 2\delta \Psi_0^{(ij)}(\mathbf{C}, \mathbf{m}_0) \right] \mathbf{F}^{\top} \right\} \right] \\
+ \sum_{i=1}^{n} \sum_{j=1, j \neq 3}^{5} \left[ \gamma_{ij} \int_{-\infty}^{t} \exp \left[ -\frac{(t-s)}{\tau_{ij}} \right] \frac{d}{ds} \left\{ \text{dev} \left\{ \mathbf{F} \left[ 2\delta \Psi_0^{(ij)}(\mathbf{C}, \mathbf{m}_0) \right] \mathbf{F}^{\top} \right\} \right\} ds \right]
$$

(21)

4. Integration algorithm for the constitutive equations

The basic idea in the numerical integration of the constitutive equations is to evaluate the convolution integral in (19) through a recursive relation. A related procedure was first suggested by Herrmann and Peterson (1968) and Taylor et al. (1970) and modified by Simo (1987). The key idea is to transform the convolution
representation discussed in the preceding section into a two-step recursive formula involving internal variables stored at the quadrature points of a finite-element mesh (Simo and Hughes, 1998).

First at all, we introduce the following internal algorithmic history variables:

\[
\mathbf{H}^{(ij)} = \int_{t}^{t + \infty} \exp \left[ \frac{-\tau_{ij}}{\tau_{ij}} \right] \frac{d}{ds} \left\{ \text{DEV} \left[ 2\theta \mathbf{y}^{(ij)}_{\text{iso}} (\mathbf{C}, \mathbf{m}_{0}) (s) \right] \right\} ds
\]

(22)

Let \([t_0, T] \subset \mathbb{R}\), with \(t_0 < T\), be the time interval of interest. Without loss of generality, we take \(t_0 = -\infty\). Further, let \([t_0, T] = \bigcup_{k \in \mathbb{N}} [t_k, t_{k+1}]\), be a partition of the interval \([t_0, T]\) with \(\mathbb{N}\) an appropriate subset of the natural numbers and \(\Delta t_k = t_{k+1} - t_k\) the associated time increment. From an algorithmic standpoint, the problem is defined in the usual strain-driven format and we assume that at certain times \(t_n\) and \(t_{n+1}\) all relevant kinematic quantities are known.

Using the semigroup property of the exponential function, the property of additivity of the integral over the interval of integration and the midpoint rule to approximate the integral over \([t_n, t_{n+1}]\) we can arrive to the update formula (Simo and Hughes, 1998)

\[
\mathbf{H}_{n+1}^{(ij)} = \exp \left[ -\frac{\Delta t_n}{\tau_{ij}} \right] \mathbf{H}_n^{(ij)} + \exp \left[ -\frac{\Delta t_n}{2\tau_{ij}} \right] \left( \mathbf{S}_{n+1}^{(ij)} - \mathbf{S}_n^{(ij)} \right)
\]

(23)

where \(\mathbf{S}_{n+1}^{(ij)} = \text{DEV} \left[ 2\theta \mathbf{y}^{(ij)}_{\text{iso}} (\mathbf{C}, \mathbf{m}_{0}) (s) \right]\) is the term of the initial stress response corresponding to \(I_n\), i.e., \(\mathbf{S}_{n+1}^{(1)}\) and \(\mathbf{S}_{n+1}^{(2)}\) are due to the matrix material and \(\mathbf{S}_{n+1}^{(3)}, \mathbf{S}_{n+1}^{(4)}\) are due to the fibers, see (A.2).

Following the convolution representation (19), the algorithmic approximation for the second Piola–Kirchhoff stress takes the form

\[
\mathbf{S}_{n+1} = J_{n}^{-1} \mathbf{p}_{n+1} \mathbf{C}_{n+1}^{-1} + J_{n+1}^{-\frac{3}{2}} \sum_{j=1, j\neq 3}^{5} \left( 1 - \sum_{i=1}^{n} \gamma_{ij} \right) \mathbf{S}_{n+1}^{(ij)} + J_{n+1}^{-\frac{3}{2}} \sum_{j=1, j\neq 3}^{5} \left( 1 + \sum_{i=1}^{n} \gamma_{ij} \right) \left\{ \text{DEV} \left[ \mathbf{H}_{n+1}^{(ij)} \right] \right\}
\]

(24)

Also, we can calculate the Cauchy stress tensor as

\[
\mathbf{\sigma}_{n+1} = \mathbf{p}_{n+1} \mathbf{C}_{n+1}^{-1} + \left[ 1 - \sum_{j=1}^{n} \gamma_{ij} \right] \left\{ \text{DEV} \left[ \mathbf{F}_{n+1}^{(ij)} \right] \right\}
\]

(25)

The tangent modulus plays a crucial role in the numerical solution of the boundary value problem by Newton-type iterative methods (Zienkiewicz and Taylor, 1994). The use of consistently linearized moduli is essential to preserve the quadratic rate of the asymptotic convergence that characterizes full Newton’s method (Hughes, 2000).

In order to obtain an easier recursive update procedure, we rewrite the update formula (23) as (Simo and Hughes, 1998)

\[
\tilde{\mathbf{H}}_{n+1}^{(ij)} = \mathbf{H}_{n+1}^{(ij)} - \exp \left[ -\frac{\Delta t_n}{\tau_{ij}} \right] \mathbf{S}_{n+1}^{(ij)}
\]

(26)

\[
\mathbf{H}_{n+1}^{(ij)} = \tilde{\mathbf{H}}_{n}^{(ij)} + \exp \left[ -\frac{\Delta t_n}{\tau_{ij}} \right] \mathbf{S}_{n+1}^{(ij)}
\]

(27)

With this notation

\[
\mathbf{S}_{n+1} = J_{n+1} \mathbf{p}_{n+1} \mathbf{C}_{n+1}^{-1} + J_{n+1}^{-\frac{3}{2}} \sum_{j=1, j\neq 3}^{5} \left( 1 - \gamma_{ij} + \sum_{i=1}^{n} \gamma_{ij} \right) \mathbf{S}_{n+1}^{(ij)} + \sum_{i=1}^{n} \gamma_{ij} \left\{ \text{DEV} \left[ \mathbf{H}_{n+1}^{(ij)} \right] \right\}
\]

(28)

\[
\mathbf{\sigma}_{n+1} = \mathbf{p}_{n+1} \mathbf{C}_{n+1}^{-1} + \left[ 1 - \gamma_{ij} + \sum_{i=1}^{n} \gamma_{ij} \right] \mathbf{S}_{n+1}^{(ij)} + \sum_{i=1}^{n} \gamma_{ij} \left\{ \mathbf{\sigma}_{n+1}^{(ij)} \right\}
\]

(29)
where $y_j = \sum_{i=1}^{n} y_{ij}$ and $v_j = \sum_{i=1}^{n} v_{ij} \exp \left[ -\frac{\Delta t}{\tau_j} \right]$. Note that $\tilde{H}_n^{(i)}$ is a constant at time $t_{n+1}$ in the linearization process.

Using (8) and (28) we obtain

$$
\mathbf{c}_{n+1} = \mathbf{c}_{vol_{n+1}}^0 + \sum_{j=1, j \neq 3}^{5} \left[ (1 - \gamma_j + v_j) \mathbf{c}_{iso_{n+1}}^{(j)} - \frac{2}{3} \gamma_j \sum_{i=1}^{n} \gamma_{ij} \left\{ \text{DEV} \left[ \tilde{H}_n^{(ij)} \right] \otimes \mathbf{c}_{n+1}^{-1} + \mathbf{c}_{n+1}^{-1} \otimes \text{DEV} \left[ \tilde{H}_n^{(ij)} \right] \right\} ight] - \left( \tilde{h}_n^{(ij)} : \tilde{c} \right) \left( \mathbb{I}_{c_{n+1}} - \frac{1}{3} \mathbf{c}_{n+1}^{-1} \otimes \mathbf{c}_{n+1}^{-1} \right) \right] 
$$

and the spatial tangent modulus defined in (9) takes the form

$$
\mathbf{c}_{n+1} = \mathbf{c}_{vol_{n+1}}^0 + \sum_{j=1, j \neq 3}^{5} \left[ (1 - \gamma_j + v_j) \mathbf{c}_{iso_{n+1}}^{(j)} - \frac{2}{3} \gamma_j \sum_{i=1}^{n} \gamma_{ij} \left\{ \text{dev} \left[ \tilde{h}_n^{(ij)} \right] \otimes \mathbf{I}_{n+1} + \mathbf{I}_{n+1} \otimes \text{dev} \left[ \tilde{h}_n^{(ij)} \right] ight\} - \text{tr} \left[ \tilde{h}_n^{(ij)} \right] \left( \mathbb{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right) \right] 
$$

where $\tilde{h}_n^{(ij)} = \tilde{F}_n^{(ij)} \tilde{h}_n^{(ij)} \tilde{F}_n^{(ij)^T}$.

Note that in the recursive update procedure presented herein it is necessary to know at time $t_{n+1}$ the variables $\mathbf{H}_n$ and $\mathbf{S}_n^{(i)}$, so, the integration algorithm for the constitutive equations does not oblige to a constant time increment during the simulation. Frequently, biomechanical problems include highly large deformations and consequently it is convenient to use a variable time increment approach. In the case of $\Delta t_n$ constant, it is not necessary to store $\mathbf{S}_n^{(i)}$ being only needed to compute and store $\mathbf{H}_n$ and the computational cost of the recursive update procedure is lower.

For the reader’s convenience, we have summarized the overall implementation of the developed algorithm in Table 1.

### 5. Numerical examples

In order to illustrate the performance and the physical mechanisms involved in the constitutive model presented herein, we analyzed five numerical examples. The aim of the first two is to show the basic behaviour of the anisotropic visco-hyperelastic model in simple cases. The last three correspond to different applications in ligaments. We have focused on ligaments due to the importance of their viscoelastic properties in the clinical practise. In particular, in the third example we show the performance of the model to reproduce the strain rate influence in the response of ligaments. In the fourth, the evolution of the initial prestress in bone-patellar tendon-bone graft is studied. In the last example, we show the effect of the non-equilibrium part of the constitutive model in patellar tendon graft under cyclic loading. In all the examples, we omit the dependency of the free-energy function $\Psi$ on $I_5$ in order to reduce the number of material parameters and to work with physically motivated invariants.

#### 5.1. Influence of viscoelastic parameters in the strain–stress response

In order to study the influence of the viscoelastic parameters in the stress–strain response, we considered a transversely isotropic and hyperelastic material with its constitutive behaviour defined by the initial elastic stored-energy function (Weiss et al., 1996)

$$
\Psi^0 (\mathbf{C}, \mathbf{m}_0 \otimes \mathbf{m}_0) = \Psi_{vol}^0 (J) + \Psi_{iso}^0 (\mathbf{C}) + \Psi_{iso}^{(ij)} (\mathbf{C}, \mathbf{m}_0 \otimes \mathbf{m}_0) = \frac{1}{D} (\ln(J))^2 + C_1 (J_1 - 3) + C_2 (J_1 - 3)^2 + \frac{C_3}{C_4} \left[ e^{C_3 (\lambda - 1)} - 1 \right] \right]
$$

where $C_3 \geq 0$ is a stress-like material parameter and $C_4 \geq 0$ is a dimensionless parameter. Three sets of elastic material constants were chosen (Table 2) and only one internal variable ($i = 1$) was considered.

Uniaxial relaxation test was simulated for a strain rate of 3.6% s$^{-1}$ up to a stretch ratio of $\lambda = 1.36$. The viscoelastic parameters are summarized also in Table 2. The first ideal case, with the viscoelastic parameters
Table 1

Algorithmic procedure

(1) Database at each Gaussian point

\( \{ \mathbf{S}_n^{(i)}, \mathbf{H}_n^{(i)} \} \) \( i = 1 \ldots \) internal variables and \( j = 1 \ldots \) number of invariants

(2) Compute the initial elastic stress Cauchy tensor

\[
\text{dev}_{n+1} \left[ \sigma_{n+1}^{(0)} \right] = \frac{1}{J_{n+1}} \text{dev} \left\{ \mathbf{F}_{n+1} \left[ 2 \frac{\partial \psi_{\text{vol}}^{(0)}(\mathbf{C}_{n+1}, \mathbf{m}_0)}{\partial \mathbf{C}} \right] \mathbf{F}_{n+1}^T \right\}
\]

(3) Update algorithmic internal variables

\[
\mathbf{S}_{n+1}^{(0)} = \mathbf{F}_{n+1} \left( J_{n+1} \text{dev} \left[ \sigma_{n+1}^{(0)} \right] \right) \mathbf{F}_{n+1}^T
\]

\[
\tilde{\mathbf{H}}_n^{(j)} = \exp \left[ -\frac{\Delta t}{\tau_i} \right] \mathbf{H}_n^{(j)} - \exp \left[ -\frac{-\Delta t}{2\tau_i} \right] \mathbf{S}_n^{(j)}
\]

\[
\mathbf{H}_n^{(j)} = \tilde{\mathbf{H}}_n^{(j)} + \exp \left[ -\frac{\Delta t}{2\tau_i} \right] \mathbf{S}_n^{(j)}
\]

(4) Compute the Cauchy stress tensor

\[
P_{n+1} = \left. \frac{d \psi_{\text{vol}}(J_{n+1})}{d J} \right|_{n+1}
\]

\[
\mathbf{h}_n^{(j)} = \sum_{i=1}^n \gamma_{ij} \text{dev} \left[ \mathbf{F}_{n+1} \tilde{\mathbf{H}}_n^{(j)} \mathbf{F}_{n+1}^T \right]
\]

\[
\mathbf{\tilde{h}}_n^{(j)} = \sum_{i=1}^n \gamma_{ij} \text{tr} \left[ \mathbf{F}_{n+1} \tilde{\mathbf{H}}_n^{(j)} \mathbf{F}_{n+1}^T \right]
\]

\[
\sigma_{n+1} = n_{n+1} + \sum_{j=1, j \neq 3}^5 \left[ (1 - \gamma_j + v_j) \text{dev} \left[ \sigma_{n+1}^{(0)} \right] + \frac{1}{J_{n+1}} \mathbf{\tilde{h}}_n^{(j)} \right]
\]

(5) Compute initial elastic modulus

\( c_{\text{vol},n+1}^{(0)} \) and \( c_{\text{iso},n+1}^{(0)} \)

(6) Introduce viscoelastic effects

\[
c_{\text{iso},n+1} = \sum_{j=1, j \neq 3}^5 \left[ (1 - \gamma_j + v_j) c_{\text{iso},n+1}^{(0)} - \frac{2}{3 J_{n+1}} \left[ \mathbf{h}_n^{(j)} \otimes \mathbf{I}_{n+1} + \mathbf{I}_{n+1} \otimes \mathbf{\tilde{h}}_n^{(j)} - \mathbf{\tilde{h}}_n^{(j)} \left( -\frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) \right] \right]
\]

(7) Compute elastic modulus

\[
c_{n+1} = c_{\text{vol},n+1}^{(0)} + c_{\text{iso},n+1}
\]

equal for matrix and fibers, corresponded to an isotropic viscoelastic material. In cases II and III, the viscoelastic parameters of the matrix were assumed to be very small with respect to those of the fibers. In cases IV and V the viscoelastic parameters of the fibers were assumed to be very small with respect to those of the matrix. Finally, in cases VI and VII \( \gamma_{ij} \) were increased for the matrix and fibers respectively.

Fig. 1a illustrates the evolution of the stress response with time for the set I of constants. As can be observed, changing the viscoelastic parameters of the matrix did not produce changes in the stress evolution nor in the thermodynamic equilibrium stress (cases I, II, III and VI). On the contrary, when we decreased the free-energy factor \( \gamma_{14} \) (case IV) there was an increase in the initial (6\%) and equilibrium stresses (29\%). In addition, when the relaxation time decreased until \( \tau_{14} = 0.1 \) s (case V) equilibrium was achieved very fast. This is due to the small contribution of the matrix to the stress response with respect to the fibers. On the contrary,
Table 2
Viscoelastic material parameters

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set I</td>
<td>10</td>
<td>10</td>
<td>100</td>
<td>1</td>
<td>0.0036844</td>
</tr>
<tr>
<td>Set II</td>
<td>10</td>
<td>10</td>
<td>0.1</td>
<td>1</td>
<td>0.0036844</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case I</th>
<th>$\gamma_{11}$</th>
<th>$\gamma_{11}(s)$</th>
<th>$\gamma_{12}$</th>
<th>$\gamma_{12}(s)$</th>
<th>$\gamma_{14}$</th>
<th>$\gamma_{14}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>0.3</td>
<td>10.0</td>
<td>0.3</td>
<td>10.0</td>
<td>0.3</td>
<td>10.0</td>
</tr>
<tr>
<td>Case II</td>
<td>0.05</td>
<td>10.0</td>
<td>0.05</td>
<td>10.0</td>
<td>0.3</td>
<td>10.0</td>
</tr>
<tr>
<td>Case III</td>
<td>0.05</td>
<td>0.1</td>
<td>0.05</td>
<td>0.1</td>
<td>0.3</td>
<td>10.0</td>
</tr>
<tr>
<td>Case IV</td>
<td>0.3</td>
<td>10.0</td>
<td>0.3</td>
<td>10.0</td>
<td>0.05</td>
<td>10.0</td>
</tr>
<tr>
<td>Case V</td>
<td>0.3</td>
<td>10.0</td>
<td>0.3</td>
<td>10.0</td>
<td>0.05</td>
<td>0.1</td>
</tr>
<tr>
<td>Case VI</td>
<td>0.6</td>
<td>10.0</td>
<td>0.6</td>
<td>10.0</td>
<td>0.3</td>
<td>10.0</td>
</tr>
<tr>
<td>Case VII</td>
<td>0.3</td>
<td>10.0</td>
<td>0.3</td>
<td>10.0</td>
<td>0.6</td>
<td>10.0</td>
</tr>
</tbody>
</table>

Fig. 1. Results of the influence of viscoelastic parameters in the stress–strain response.
when the set II of constants was used an almost isotropic material response was obtained in agreement with a
Mooney–Rivlin material that provokes that changes in the viscoelastic parameters of the fibers do not affect to
the stress response (Fig. 1b).

5.2. Relaxation test of a fiber-reinforced rubber cube at finite strains

In this example we deal with the numerical simulation of a composite rubber block bonded with two perfect-
ly rigid plates as shown in Fig. 2. Due to the symmetry, only 1/8 of the global geometry was modelled. The
fiber distribution was aligned with the axial direction. The purpose of this simulation is to demonstrate the
effectiveness of the numerical algorithm and finite element implementation discussed in Sections 3 and 4.
For a strain rate of 3.6% s⁻¹, displacements were applied on the top to deform the mesh to a nominal vertical
strain of 75% and then fixed at this strain, while the bottom surface was fixed. After that, the relaxation pro-
cess of the cube was computed until thermodynamic equilibrium was obtained with sufficiently small time
steps in order to avoid errors in the numerical integration (0.01 s). We considered the strain-energy function
defined in (32). The viscoelastic response of the cube was modelled with only one internal variable, \( i = 1 \), and
the relaxation process and the associated parameters were chosen in order get an almost isotropic Neo-Hook-
ean material response at \( t \rightarrow \infty \) (Table 3).

The distribution of the Cauchy stress in the axial direction at different times is shown in Fig. 3. At time
\( t = 0.0^+ \) corresponding to the beginning of the relaxation process, there appeared a stress concentration at
the bottom of the cube and high stresses at the top. This stress concentration relaxed until \( t \rightarrow \infty \), where
the stress reduced up to a 35%. Due to the chosen parameters, the relaxation process was mainly associated
to the fibers. For this reason, the anisotropic contribution of the fibers decreased for increasing time until get-
ing an almost isotropic material, Fig. 3f.

The only numerical results we have found on a similar topic are those presented by Holzapfel and Gasser
(2001). Those authors were interested in arteries where a Maxwell-type viscoelastic model is more appropriate.
In addition, two families of fibers were considered in their examples. Therefore, the comparison with such
examples is only qualitative. During the relaxation process, similar stress behaviour was observed. The

![Fig. 2. Cube dimensions and symmetry planes.](image)

Table 3
Viscoelastic material parameters of the cube example

<table>
<thead>
<tr>
<th></th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>0.0036844</td>
</tr>
<tr>
<td>( \gamma_{11} )</td>
<td>( \gamma_{11}(s) )</td>
<td>( \gamma_{12} )</td>
<td>( \gamma_{12}(s) )</td>
<td>( \gamma_{14} )</td>
<td>( \gamma_{14}(s) )</td>
</tr>
<tr>
<td>0.001</td>
<td>10</td>
<td>0.0</td>
<td>0</td>
<td>0.9999</td>
<td>100</td>
</tr>
</tbody>
</table>
response of the rubber bar after initial elongation was anisotropic and tended towards the isotropic behaviour when the equilibrium was achieved.

5.3. Human ligament under different strain rates

Of all the knee ligaments, the anterior cruciate is the most frequent totally disrupted. Sports (skiing, basketball, soccer) and traffic accidents are the most important causes of ligament injury. The strain rate during injury is very important regarding the magnitude of the lesion. Therefore, the stress–strain behaviour of the ligament is an essential factor. The strain rate during non-physiological loads is determinant in the risk of damage in ligaments. Non-physiological movements under low strain rates do not usually provoke ligament damage that under high strain rates occurs (Fu et al., 1994).

Ligaments are usually considered as transversely isotropic and hyperelastic due to the presence of one family of fibers and their constitutive behaviour defined by the initial elastic stored-energy function

$$ W_{\text{iso}}(C, m) = W_{\text{iso}}^m(J) + W_{\text{iso}}^f(C, m) $$

where $W_{\text{iso}}^m$ represents the deviatoric mechanical contribution of the tissue matrix, $W_{\text{iso}}^f$ that of the fibers and $W_{\text{vol}}$ is a penalty function to enforce quasi-incompressibility. In this example, we used a strain-energy density function earlier proposed by Weiss et al. (1996) where $W_{\text{vol}}$ took the form $W_{\text{vol}} = \frac{1}{2}(\ln(J))^2$ (Gardiner and Weiss, 2003), being $D$ the inverse of the bulk modulus and $J$ the Jacobian. A Neo-Hookean model was considered for the matrix part of the strain-energy function defined by $W_{\text{iso}}^m = C_1(I_1 - 3)$. $C_1$ is the constant of the Neo-Hookean model and $I_1$ the first modified strain invariant of the symmetric modified Cauchy–Green tensor $C$. Following Limbert and Middleton (2004), the specific form of $W_{\text{iso}}^f$ was

$$ W_{\text{iso}}^f = \frac{C_3}{2C_4} \left\{ e^{[C_4[I_4^{-1} - 1]} - 1 \right\} $$

Fig. 3. Distribution of the Cauchy stress in the axial direction at different times (MPa).
where \( T_A = m \cdot \hat{m} = m_0 \cdot \hat{C} \cdot \hat{m}_0 \) is the square of the stretch of the fiber; \( m_0 \) the fiber direction in the reference configuration; \( C_3 \) scales the exponential stress and \( C_4 \) is related to the rate of collagen uncrimping.

The strain rate effect is visible in the stress–strain curves obtained by Pioletti et al. (1998) for the human ACL (Fig. 5). In order to fit the material parameters and compare the strain rate curves obtained, we reproduced the tensile test in Pioletti et al. (1998) Using the proposed anisotropic visco-hyperelastic model, the stress–strain curves obtained at different strain rates (0.012% s\(^{-1}\), 25% s\(^{-1}\), 38% s\(^{-1}\) and 50% s\(^{-1}\) were well fit, see Fig. 5a. On the contrary, when an isotropic visco-hyperelastic model was used, the stress–strain curve under low strain rate was not well fit. This is due to the significance of the viscoelastic properties of the matrix under low strain rates being not possible to simulate the behaviour of the ligament using the same properties of the matrix and fibers. The elastic and viscoelastic parameters obtained in both cases are included in Table 4.

To illustrate the performance of the visco-hyperelastic behaviour of ligaments and the importance of the strain rates during their movement, a model of the human anterior cruciate ligament (ACL) was constructed to simulate its behaviour under a physiological anterior tibial displacement, see Fig. 4. The surface geometries of femur and tibia were reconstructed from a set of Computer Tomography scans, while for the ACL, MRI (Magnetic Resonance Images) were used (Peña et al., 2006). Bones were assumed to be rigid in this model, and a hexahedral block-structured mesh of the ACL was built. Three different strain rates were applied: low (0.012% s\(^{-1}\)), moderate (1% s\(^{-1}\)) and high (50% s\(^{-1}\)) that correspond to quasi-static, physiological and non-physiological strain rates. Boundary conditions were defined as follows. Ligaments were attached to bone by establishing the final row of elements at their proximal and distal ends to be composed of the same material than the nearby bone (Gardiner and Weiss, 2003). The motion of each bone was controlled by the six degrees of freedom of its reference node. In the analyses, tibia and fibula remained fixed. The position at full extension served as the reference initial configuration. An anterior load of 134 N was applied to the femur. In this example we did not consider initial strains (Peña et al., in press).

If the motion of the tibia relative to the femur is slow, corresponding to a low strain rate for the ACL, the contribution of the ACL is minimal. If this motion is fast, the contribution of the ACL to knee stability increases. This is clear from the stiffer stress–strain curves at higher strain rates (Fig. 5). Due to this stiffening

<table>
<thead>
<tr>
<th>Table 4</th>
<th>ACL material parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>( C_2 )</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0</td>
</tr>
<tr>
<td>( \gamma_{11} ) Anisotropic model</td>
<td>( \tau_{11}(s) )</td>
</tr>
<tr>
<td>0.99</td>
<td>0.15</td>
</tr>
<tr>
<td>( \gamma_{11} ) Isotropic model</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Fig. 4. Finite element model of the human ACL.
effect, the maximal principal stress becomes much higher especially located in the central part of the ligament and in the tibial insertion (Fig. 6). The maximal principal stress of 8.51 MPa obtained in the tibial insertion for the higher load rate is due to the stress concentration induced by the sudden change of material properties. Besides this effect, the maximal principal stress predicted for the higher load rate was closed to the load rupture estimated in 7 MPa by Pioletti et al. (1998). On the contrary, under quasi-static and physiological strain rates the maximal principal stress of 2.87 MPa is far from the ultimate stress. It is well-known that damage in mature ligaments occurs usually at high load rates; on the contrary bone–ligament insertions are usually ruptured under lower load rates (Woo and Young, 1991).

Using isotropic visco-hyperelastic properties, it is not possible to fit the experimental data for low strain rates with the same accuracy than for high strain rates. As show in Fig. 5b, the fitted curve under low strain rates was stiffer than the actual experimental data. This provokes an overestimation of the stress in the finite element simulation for both low and moderate strain rates, see Fig. 7a. On the contrary, the results for high strain rates are similar to the anisotropic case.

5.4. Patellar tendon graft after initial prestress

Another clinical application of the model proposed herein is the evolution along time of the initial prestress in bone-patellar tendon-bone grafts. Surgical reconstruction of the ACL is a common practice to treat the disability or chronic instability of knees due to ACL insufficiency (Beynnon et al., 2002). The bone-patellar

![Experimental results obtained by Pioletti et al. (1998) and theoretical stress–strain curves at different rates of elongation for the human ACL.](image-url)
tendon-bone autograft remains a common practice due to its high ultimate strength and stiffness that allows for a more predictable restoration of the knee stability (Fu et al., 1994; Noyes and Barrer-Westin, 2001; Adam et al., 2002; Beynnon et al., 2002; Fox et al., 2002). Before graft fixation, an initial pretension is applied. This initial tension applied to the replacing graft significantly alters the joint kinematics. This prestress helps to provide joint stability, but a very high pretension produces an important additional stress in the graft during the knee movement. This may cause problems in revascularization and remodelling during the postoperative healing process (Cyril and Douglas, 1997; Kampen et al., 1998; Tohyama and Yasuda, 1998). Viscoelasticity decreases the tension imposed during surgery until getting the final value after reaching equilibrium. The decrease of this initial stress can compromise the joint stability, affecting the postoperative results. So, it is desirable to estimate the minimal initial stress needed to maintain joint stability after relaxation. In this example, we study the evolution of the initial stress in the graft.

An idealized 10-mm wide graft was modelled. The 3D finite element model of the graft and bone plugs is shown in Fig. 8a. In order to keep the correct geometry of the joint and of the tunnels and simultaneously impose the initial prestress on the graft we needed to model the plugs. These plugs were modelled as elastic with a very high stiffness in comparison with that of the graft in order to reduce the numerical difficulties that appear at the connections between the plugs and the femoral and tibial tunnels. The constitutive law of the graft tendon was the same of the ligament (Weiss and Gardiner, 2001) with the initial elastic stored-energy function (34), while bone plugs were considered to behave as a linearly elastic and isotropic material with an elastic modulus of \( E = 14,220 \) MPa and a Poisson ratio of \( \nu = 0.3 \) (Jacobs, 1994). The elastic and viscoelastic parameters of the graft were obtained fitting the stress-curved obtained by Pioletti et al. (1998) from the human patellar tendon (PT). These parameters are included in Table 5.
Displacements were applied to the femoral bone plug up to an initial stress of 2.93 MPa (Fig. 8b) corresponding to a pretension of about 60 N (Peña et al., 2005) and then fixed, while the tibial bone plug remained always fixed. After that, the relaxation process of the graft was computed until thermodynamic equilibrium (Fig. 8c).

Fig. 9 illustrates the evolution of the initial prestress with time. As can be observed, the initial value at time $t = 0^+$ decreased very fast at the beginning of the relaxation process. This result shows that tension within the PT graft is reduced shortly after the fixation. For $t = 1000$ s the stress decreased a 32.5%. Grant et al. (1994) showed a reduction of 30% in the graft load when tensioned up to a strain of 2.5% after 10 min, we tensioned up to a strain of 2.4%. To minimize the stress relaxation response, preconditioning of the graft is usually recommended.

5.5. Patellar tendon graft under cyclic load

The amount of tension in the graft is therefore influenced by cyclic preconditioning (multiple cycles of elongation performed prior to fixation). Preconditioning limits the most pronounced stress-relaxation effects and provides a uniform strain history. Biomechanical testing protocols include preconditioning (cyclic or static stretching of the graft prior to implantation) to ensure that ligaments have a uniform strain history and decrease relaxation of the initial stress in the graft (Fu et al., 1994).

In this example we consider the numerical simulation of a cyclic pretension of the patellar tendon graft explained in the previous section. At time $t = 0$ an axial load of 60 N was applied, following a triangular time-history with amplitude of 30 N and time period of 1 s.

The applied axial load versus the axial stretch is plotted in Fig. 10a. The stress–strain responses to loading and unloading are different. The axial stretch varies with respect to the axial load followed by an elliptically shaped hysteresis loop until the steady state is reached after 15 s. This behaviour is typically observed in ligaments, where stress softening effects occur during the first few load cycles (Woo et al., 1999). Finally, Fig. 10b shows the evolution of the stretch $\lambda$ with time. A phase-shift between pressure and the stretch period of 0.03 s can be observed.

<table>
<thead>
<tr>
<th>Table 5</th>
<th>PT material parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>2.7</td>
</tr>
<tr>
<td>$C_2$</td>
<td>0.0</td>
</tr>
<tr>
<td>$C_3$</td>
<td>15.3146</td>
</tr>
<tr>
<td>$C_4$</td>
<td>107.473</td>
</tr>
<tr>
<td>$D$</td>
<td>0.004938</td>
</tr>
<tr>
<td>$\gamma_{11}$</td>
<td>0.55</td>
</tr>
<tr>
<td>$\tau_{11}(s)$</td>
<td>10</td>
</tr>
<tr>
<td>$\gamma_{12}$</td>
<td>0.55</td>
</tr>
<tr>
<td>$\tau_{12}(s)$</td>
<td>10</td>
</tr>
<tr>
<td>$\gamma_{14}$</td>
<td>0.35</td>
</tr>
<tr>
<td>$\tau_{14}(s)$</td>
<td>150</td>
</tr>
</tbody>
</table>
The relaxation example of the previous section was applied after cycling. Fig. 9 shows the evolution of the initial prestress without (previous example) and with previous cyclic load. The tension within the PT graft with cyclic load was reduced only 24.1%, in agreement with clinical recommendations (Fu et al., 1994).

6. Conclusions

We have presented an anisotropic Kelvin–Voigt-type visco-hyperelastic constitutive model capable to model fiber-reinforced composite materials undergoing finite strains as ligaments. The structural model was formulated by employing Simo’s constitutive framework based on irreversible thermodynamics with internal variables (Simo, 1987), where we have considered different viscoelastic behaviour for the matrix material and the different families of fibers. Motivated by Holzapfel and Gasser (2001) and in order to consider different contributions of the matrix and fibers on the non-equilibrium part, we considered the internal variables to correspond to separated contributions of the matrix and fibers. A numerical integration procedure that is second-order accurate and takes place entirely in the reference configuration was used; this fact implies that incremental objectivity is trivially satisfied. To our knowledge, the anisotropic visco-hyperelastic model based on a local additive decomposition of the stress tensor into initial and non-equilibrium parts (Kelvin–Voigt generalized model) and the explicit expression for the stress and fourth-order elasticity tensor in the spatial
description for time-dependent finite strains in fibered materials have not been recorded previously in the literature. The model provides a very efficient tool to determine realistic predictions of stress, strain and strain rates distributions in ligaments with different viscoelastic behaviour of collagen tissue and ground substance. To the authors knowledge, this is the first Kelvin–Voigt-type visco-hyperelastic constitutive model capable to model fiber-reinforced composite materials undergoing finite strains as ligaments. The main disadvantage of the model is that the evolution equations are linear.

Several numerical examples under finite strains have been presented in order to illustrate the good performance and the physical mechanisms inherent to the constitutive model. We apply the model to simulate the viscoelastic behaviour of ligaments in some clinical applications. The examples demonstrated the relaxation, rate strain dependency and cyclic loading response of the ligaments. The clinical relevance of the anisotropic visco-hyperelastic properties of the ligament and the need of using different viscoelastic properties for the matrix and fibers are also pointed out.

Acknowledgements

The authors gratefully acknowledge the research support of the Spanish Ministry of Science and Technology through the research project DPI2003-09110-C02-01, DPI2004-07410-C03-01 and the Spanish Ministry of Health through the National Network IM3 (Molecular and Multimodal Medical Imaging, Associated Partner, 300++, 2003–2005).

Appendix

In order to apply the model presented herein, it is necessary to know the stress response and elastic tangent modulus of the hyperelastic material in separated form for each invariant. In this appendix we provide explicit expressions for the initial stress and elasticity tensors of the anisotropic hyperelastic behaviour depending of the defined invariants. We only present the particularized form of the expressions here used, with $I_5$ omitted, as usually employed for ligaments.

Using the decoupled form of the free-energy function

$$\Psi = \Psi_{\text{vol}}(J) + \Psi_{\text{iso}}(\overline{C}, m_0 \otimes m_0) = \Psi_{\text{vol}}(J) + \Psi_{\text{iso}}(I_1, I_2, I_4)$$

with $I_1$, $I_2$ and $I_4$ defined in (5). The stress response is obtained from the derivatives of the stored-energy function

$$S = 2 \frac{\partial \Psi}{\partial \overline{C}} = JpC^{-1} + 2J^{-\frac{1}{2}} \left[ \left( \frac{\partial \Psi_{\text{iso}}^{I_1}}{\partial I_1} + I_1 \frac{\partial \Psi_{\text{iso}}^{I_2}}{\partial I_2} \right) I - \frac{\partial \Psi_{\text{iso}}^{I_2}}{\partial I_2} C + I_4 \frac{\partial \Psi_{\text{iso}}^{I_4}}{\partial I_4} m_0 \otimes m_0 ight. \\
- \frac{1}{3} \left( \frac{\partial \Psi_{\text{iso}}^{I_2 I_1}}{\partial I_1} + 2 \frac{\partial \Psi_{\text{iso}}^{I_2 I_2}}{\partial I_2} + \frac{\partial \Psi_{\text{iso}}^{I_4 I_1}}{\partial I_1} \right) C^{-1} \right]$$

with $p$ the hydrostatic pressure defined in (7)
The Cauchy stress tensor $\sigma$ is $1/J$ times the push-forward of $S$ ($\sigma = J^{-1} \chi_*(S)$). From (A.2), we obtain

$$\sigma = pI + 2 \frac{1}{J} \left( \frac{\partial \Psi_{\text{iso}}^{I_1}}{\partial I_1} + \frac{\partial \Psi_{\text{iso}}^{I_2}}{\partial I_2} \right) b + \frac{\partial \Psi_{\text{iso}}^{I_2}}{\partial I_2} b^2 + I_4 \frac{\partial \Psi_{\text{iso}}^{I_4}}{\partial I_4} m \otimes m - \frac{1}{3} \left( \frac{\partial \Psi_{\text{iso}}^{I_2 I_1}}{\partial I_1} + 2 \frac{\partial \Psi_{\text{iso}}^{I_2 I_2}}{\partial I_2} + \frac{\partial \Psi_{\text{iso}}^{I_4 I_1}}{\partial I_1} \right) I$$

Applying the definition (8) in (A.2), we can establish the elasticity tensor in the material description:

$$\mathbf{C} = 2 \frac{\partial S(C)}{\partial C} = \mathbf{C}_{\text{vol}} + \mathbf{C}_{\text{iso}} = 2 \frac{\partial S_{\text{vol}}}{\partial C} + 2 \frac{\partial S_{\text{iso}}}{\partial C}$$

The elasticity tensor in the spatial description is defined as the push-forward of $\mathbf{C}$

$$\mathbf{c} = \mathbf{c}_{\text{vol}} + \mathbf{c}_{\text{iso}}$$
where  is defined as
\[ c_{\text{vol}} = p (1 \otimes I - 2I) \] (A.6)

with  the fourth-order identity tensor,  defined in (7) and  as
\[
c_{\text{iso}} = -\frac{2}{3} (\text{dev}\sigma \otimes I + I \otimes \text{dev}\sigma) - \frac{4}{3J} [\Psi + \Psi_{T1}I] - \Psi_{T2}(T_1^2 - 2T_2) + \Psi_{T4}(I - \frac{1}{3}I \otimes I) \\
+ 4[(\Psi_{T1} + 2\Psi_{T2}I_1 + \Psi_2 + \Psi_{T4}I_1) \otimes b - (\Psi_{T1} + \Psi_{T2}I_1)(b \otimes b + b^2 \otimes b) + \Psi_{T4}b^2 \otimes b) \\
- \Psi_{T2}b \otimes b + (\Psi_{T4} + \Psi_{T2}I_1)T_4(b \otimes m \otimes m + m \otimes m \otimes b) - \Psi_{T2}I_4(b \otimes m \otimes m + m \otimes m \otimes b) \\
+ \Psi_{T4}I_4(m \otimes m \otimes m \otimes m)] - \frac{4}{3} [\Psi_{T1}I + \Psi_{T2}I_1 + \Psi_{T2}I_2 + \Psi_{T4}I_1 + 2\Psi_{T4}I_2 + \Psi_{T4}I_4] \\
+ \Psi_{T4}I_4b - (\Psi_{T1}I_1 + 2\Psi_{T2}I_2 + \Psi_2 + \Psi_{T4}I_4) \otimes b + (\Psi_{T4}I_1 + 2\Psi_{T4}I_2 + \Psi_{T4}I_4)b - (\Psi_{T1}I_1 \\
+ 2\Psi_{T2}I_2 + \Psi_2 + \Psi_{T4}I_4) \otimes b + (\Psi_{T4}I_1 + 2\Psi_{T4}I_2 + \Psi_{T4}I_4) \otimes m] + \frac{4}{3} [(\Psi_{T1} - \Psi_{T2}I_1 \\
+ 2\Psi_{T2}I_2 + 4\Psi_{T4}I_2 + 4\Psi_{T4}I_2I_2 + 2\Psi_{T4}I_4I_4 + 4\Psi_{T4}I_4I_4) \otimes I \otimes I] \] (A.7)

In order to clarify the used subscript nomenclature, we replace  for . Note that expressions (A.2) and (A.7) permit to easily compute  for each invariant in the expressions (24) and (31).

References


