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PDE and ODE limit problems for $p(x)$-Laplacian parabolic equations

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ABSTRACT

In this work we prove continuity of solutions with respect to initial conditions and parameters and we prove upper semicontinuity of a family of global attractors for problems of the form

$$u_t - \operatorname{div}\left(D^\lambda |\nabla u_\lambda|^{p(x)-2} \nabla u_\lambda\right) = B(u_\lambda)$$

in a bounded smooth domain Ω in $\mathbb{R}^N$.

1. Introduction

During the last ten years, function spaces with variable exponent have attracted a lot of interest of mathematicians around the world. A lot of researchers had spend some efforts to obtain results on variable exponent spaces and specially on $p(x)$-Laplacian elliptic problems (see, for example, [1–26] and references therein). According to the works mentioned above the theory of problems with variable exponent spaces has application in electrorheological fluids, thermo-rheological fluids, image restoration, image process and nonlinear elasticity theory.

We give in this work a contribution on $p(x)$-Laplacian parabolic problems. Let us consider the problem

$$\begin{cases} \dfrac{\partial u_\lambda}{\partial t}(t) - \operatorname{div}\left(D^\lambda |\nabla u_\lambda|^{p(x)-2} \nabla u_\lambda\right) = B\left(u_\lambda(t)\right), & t > 0, \\ u_\lambda(0) = u_{0\lambda}, \end{cases} \tag{1}$$

under Dirichlet homogeneous boundary conditions, where $u_{0\lambda} \in H := L^2(\Omega)$, Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 1$, $p(x) \in C(\bar{\Omega})$, $p^- := \operatorname{ess\,inf} p > 2$, $B : H \to H$ is a globally Lipschitz map with Lipschitz constant $L \geqslant 0$, $D^\lambda \in L^\infty(\Omega)$, $0 < \beta \leqslant D^\lambda(x) \leqslant M < \infty$ a.e. in Ω, $\lambda \in [0, \lambda_0]$ and $D^\lambda \to D^{\lambda_1}$ in $L^\infty(\Omega)$ as $\lambda \to \lambda_1 := 0$.

Analogously to [27] we can consider, for each $\lambda \in [0, \lambda_0]$ the operator $A_1^{D_\lambda} : W_0^{1,p(x)}(\Omega) \to [W_0^{1,p(x)}(\Omega)]^*$ given by

$$A_1^{D_\lambda} u(v) := \int_\Omega D^\lambda(x) \left|\nabla u(x)\right|^{p(x)-2} \nabla u(x) \cdot \nabla v(x)\, dx$$

having the following properties:

(i) $A_1^{D_\lambda}$ is monotone;

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(ii) $A_1^{D_\lambda}$ is coercive;
(iii) $A_1^{D_\lambda}$ is hemicontinuous;
(iv) $A_1^{D_\lambda}: X \to X^*$, with domain $X := W_0^{1,p(x)}(\Omega)$ is maximal monotone and $A_1^{D_\lambda}(X) = X^*$;
(v) The operator $A_H^{D_\lambda}$, the realization of $A_1^{D_\lambda}$ at $H = L^2(\Omega)$ given by

$$\begin{cases} \mathcal{D}\big(A_H^{D_\lambda}\big) := \big\{u \in X;\ A_1^{D_\lambda}(u) \in H\big\}, \\ A_H^{D_\lambda}(u) = A_1^{D_\lambda}(u), \quad \text{if } u \in \mathcal{D}\big(A_H^{D_\lambda}\big) \end{cases}$$

is maximal monotone in H. We denote $A_H^{D_\lambda}(u) = -\operatorname{div}(D^\lambda |\nabla u|^{p(x)-2}\nabla u)$.

Moreover, we have

Lemma 1. *Let $p(x) \in C(\bar{\Omega})$ with $p(x) > 2$ in Ω, $p^- := \operatorname{ess\,inf} p$, $p^+ := \operatorname{ess\,sup} p$ and $\lambda \in [0, \lambda_0]$.*

(i) *If $\|v\|_X := \|\nabla v\|_{p(x)} \leqslant 1$, then $\langle A_1^{D_\lambda} v, v\rangle_{X^*,X} \geqslant \beta \|v\|_X^{p^+}$;*
(ii) *If $\|v\|_X \geqslant 1$, then $\langle A_1^{D^\lambda} v, v\rangle_{X^*,X} \geqslant \beta \|v\|_X^{p^-}$.*

Analogously to [28] it can be proved that the problem (1) determines a continuous semigroup of nonlinear operators $T_\lambda(t): H \to H$, where $T_\lambda(t)u_{0\lambda}$ is the global weak solution of problem (1) beginning at $u_{0\lambda}$, which has a maximal compact invariant global attractor $\mathcal{A}_\lambda$ in H. We observe that in [28] the results were established for $2 + \delta \leqslant p(x) \leqslant 3 - \delta$, $\delta > 0$. The reason for this is that in [27] was proved that $\mathrm{cl}_H(\mathcal{D}(A_H^{D_\lambda})) = H$ for $2 < p(x) \leqslant 3 - \delta$, $\delta > 0$. But, the authors in [29] answer the question raised in [27] and now we know that $\mathrm{cl}_H(\mathcal{D}(A_H^{D_\lambda})) = H$ for $p^- > 2$.

In this work we prove that the family of global attractors $\{\mathcal{A}_\lambda\}_{\lambda \in [0,\lambda_0]}$ is upper semicontinuous at $\lambda_1 := 0$, that means,

$$\sup_{a_\lambda \in \mathcal{A}_\lambda} \operatorname{dist}_H(a_\lambda, \mathcal{A}_{\lambda_1}) \to 0$$

as $\lambda \to \lambda_1 = 0$.

We also consider the case $D^\lambda \equiv D \geqslant 1$ constant, that is, we consider the problem

$$\begin{cases} \dfrac{\partial u^D}{\partial t}(t) - D\Delta_{p(x)}\big(u^D\big) = B\big(u^D(t)\big), \quad t > 0, \\ u^D(0) = u_0^D \in H = L^2(\Omega) \end{cases} \tag{2}$$

under Dirichlet homogeneous boundary conditions, where $\Delta_{p(x)}(v) = \operatorname{div}(|\nabla v|^{p(x)-2}\nabla v)$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, and $B, p(x)$ as before. We intend make the diffusion parameter D goes to infinity and prove that the family of global attractors $\{\mathcal{A}_D\}_{D \geqslant 1}$ for the problem (2) is upper and lower semicontinuous at infinity, i.e.,

$$\max\Big\{\sup_{a^D \in \mathcal{A}_D} \operatorname{dist}_H\big(a^D, \mathcal{A}^\infty\big), \sup_{a \in \mathcal{A}^\infty} \operatorname{dist}_H(a, \mathcal{A}_D)\Big\} \to 0$$

as $D \to +\infty$, when there exists the global attractor of the limit problem, $\mathcal{A}^\infty$, where the limit problem is

$$\begin{cases} \dfrac{du}{dt}(t) = \tilde{B}\big(u(t)\big), \quad t > 0, \\ u(0) = u_0 \in \mathbb{R} \end{cases} \tag{3}$$

with $\tilde{B} := B_{|\mathbb{R}}$ if we identify $\mathbb{R}$ with the constant functions which are in H since Ω is a bounded set.

For the problem (1) we have that the limit problem is also a PDE which has a global attractor and this attractor commands the dynamics of the problem as $\lambda \to 0$. For the problem (2) we have a family of PDE and the limit problem is an ODE. This ODE not necessarily has a global attractor. We give examples where the limit problem has a global attractor and examples where the limit problem doesn't have a global attractor. Finally, we conclude that for globally Lipschitz maps $B: H \to H$ such that the limit problem has a global attractor, the ODE commands the asymptotic dynamics of the problem (2) as $D \to +\infty$.

The problem (2) was solved in [30] for Neumann boundary conditions with $p(x) \equiv p > 2$ constant. In that case the limit problem was also an ODE, but it was showed that this ODE has a global attractor for every globally Lipschitz map.

The paper is organized as follows. In Section 2 we obtain uniform estimates for solutions of (1). In Section 3 we prove continuity with respect to the initial values and parameters and we prove upper semicontinuity of the global attractors for the problem (1). Section 4 is devoted to the case $D \to +\infty$.

2. Uniform estimates

We have the following uniform estimates on the solutions of (1):

Lemma 2. *Let u_λ be a solution of* (1)*. Given $T_0 > 0$, there exists a positive number r_0 such that $\|u_\lambda(t)\|_H \leqslant r_0$, for each $t \geqslant T_0$ and $\lambda \in [0, \lambda_0]$.*

Proof. It is enough consider $u_{0,\lambda} \in \mathcal{D}(A_H^{D_\lambda})$. Let $\tau > 0$, multiplying the equation on (1) by $u_\lambda(\tau)$ we have that

$$\left\langle \frac{d}{dt} u_\lambda(\tau), u_\lambda(\tau) \right\rangle + \left\langle A_1^\lambda\big(u_\lambda(\tau)\big), u_\lambda(\tau)\right\rangle = \left\langle B\big(u_\lambda(\tau)\big), u_\lambda(\tau)\right\rangle.$$

Given $T_0 > 0$, if $\|u_\lambda(\tau)\|_X > 1$ then by Lemma 1, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\|u_\lambda(\tau)\big\|_H^2 &\leqslant -\beta\big\|u_\lambda(\tau)\big\|_X^{p^-} + \big\|B\big(u_\lambda(\tau)\big)\big\|_H\big\|u_\lambda(\tau)\big\|_H \\ &\leqslant -\beta\big\|u_\lambda(\tau)\big\|_X^{p^-} + L\big\|u_\lambda(\tau)\big\|_H^2 + C_0\big\|u_\lambda(\tau)\big\|_H \\ &\leqslant -\beta\big\|u_\lambda(\tau)\big\|_X^{p^-} + C_1\big\|u_\lambda(\tau)\big\|_X^2 + C_2\big\|u_\lambda(\tau)\big\|_X,\end{aligned}$$

where $C_0 = \|B(0)\|_H \geqslant 0$, $C_1 = C_1(L, \sigma) > 0$ and $C_2 = C_2(\sigma) \geqslant 0$, with σ the embedding constant from $X \hookrightarrow H$. We have $C_2 = 0$ if, and only if, $C_0 = 0$.

Now, we consider $\epsilon > 0$ arbitrary, $\alpha := \frac{p^-}{2}$, $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ and $\frac{1}{p^-} + \frac{1}{(p^-)'} = 1$. Then using Young's inequality we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\|u_\lambda(\tau)\big\|_H^2 &\leqslant -\beta\big\|u_\lambda(\tau)\big\|_X^{p^-} + \frac{\epsilon}{\epsilon}C_1\big\|u_\lambda(\tau)\big\|_X^2 + \frac{\epsilon}{\epsilon}C_2\big\|u_\lambda(\tau)\big\|_X \\ &\leqslant -\beta\big\|u_\lambda(\tau)\big\|_X^{p^-} + \frac{1}{\alpha}\epsilon^\alpha\big\|u_\lambda(\tau)\big\|_X^{p^-} + \frac{1}{p^-}\epsilon^{p^-}\big\|u_\lambda(\tau)\big\|_X^{p^-} + \left(\frac{1}{\alpha'}\left(\frac{C_1}{\epsilon}\right)^{\alpha'} + \frac{1}{(p^-)'}\left(\frac{C_2}{\epsilon}\right)^{(p^-)'}\right) \\ &= \left(-\beta + \frac{1}{\alpha}\epsilon^\alpha + \frac{1}{p^-}\epsilon^{p^-}\right)\big\|u_\lambda(\tau)\big\|_X^{p^-} + \left(\frac{1}{\alpha'}\left(\frac{C_1}{\epsilon}\right)^{\alpha'} + \frac{1}{(p^-)'}\left(\frac{C_2}{\epsilon}\right)^{(p^-)'}\right).\end{aligned}$$

Now, choose $\epsilon_0 > 0$ sufficiently small such that $\frac{1}{\alpha}\epsilon_0^\alpha + \frac{1}{p^-}\epsilon_0^{p^-} < \frac{\beta}{2}$ in the case $B(0) \neq 0$ ($C_0 \neq 0$) and for the case $B(0) = 0$, choose $\epsilon_0 > 0$ sufficiently small such that $\frac{1}{\alpha}\epsilon_0^\alpha < \frac{\beta}{2}$. So, in both cases, we obtain

$$\frac{1}{2}\frac{d}{dt}\big\|u_\lambda(\tau)\big\|_H^2 \leqslant -\frac{\beta}{2}\big\|u_\lambda(\tau)\big\|_X^{p^-} + C_3,$$

where $C_3 = C_3(L, \sigma, \epsilon_0) > 0$ is constant. So,

$$\frac{1}{2}\frac{d}{dt}\big\|u_\lambda(\tau)\big\|_H^2 \leqslant -\frac{\beta}{2}\sigma^{-p^-}\big\|u_\lambda(\tau)\big\|_H^{p^-} + C_3.$$

Let $I_\lambda := \{\tau \in (0, \infty);\ \|u_\lambda(\tau)\|_X > 1\}$ and $y_\lambda(\tau) := \|u_\lambda(\tau)\|_H^2$, $y_\lambda : I_\lambda \to \mathbb{R}$ satisfies the differential inequality

$$y_\lambda'(\tau) \leqslant -\beta\sigma^{-p^-}\big[y_\lambda(\tau)\big]^{\frac{p^-}{2}} + 2C_3.$$

Therefore, from Lemma 5.1, p. 163 in [31], we get

$$\big\|u_\lambda(\tau)\big\|_H^2 \leqslant \left(\frac{2C_3\sigma^{p^-}}{\beta}\right)^{2/p^-} + \left[\frac{\beta}{2\sigma^{p^-}}\big(p^- - 2\big)T_0\right]^{\frac{-2}{(p^- - 2)}}, \quad \forall \tau \geqslant T_0.$$

If $\|u_\lambda(\tau)\|_X \leqslant 1$, then $\|u_\lambda(\tau)\|_H \leqslant \sigma\|u_\lambda(\tau)\|_X \leqslant \sigma$. So, taking $K_0 := \{(\frac{2C_3\sigma^{p^-}}{\beta})^{2/p^-} + [\frac{\beta}{2\sigma^{p^-}}(p^- - 2)T_0]^{\frac{-2}{(p^- - 2)}}\}^{1/2}$ and $r_0 := \max\{\sigma, K_0\}$ we obtain

$$\big\|u_\lambda(\tau)\big\|_H \leqslant r_0, \quad \forall \tau \geqslant T_0,\ \lambda \in [0, \lambda_0]. \qquad \square$$

Remark 1. The constant r_0 in the above lemma depend neither on the initial data nor on λ.

Corollary 1. *There exists a bounded set B_0 in H such that $\mathcal{A}_\lambda \subset B_0$ for all $\lambda \in [0, \lambda_0]$.*

Remark 2. If u_λ is a solution of (1), then there exists a positive number $K = K(u_{0\lambda}, T_0)$ such that $\|u_\lambda(t)\|_H \leqslant K$, for all $t \in [0, T_0]$. If the initial values are all in a bounded set of H, then K is uniform on $[0, \lambda_0]$, i.e., we have that $\|u_\lambda(t)\|_H \leqslant K$ for each $\lambda \in [0, \lambda_0]$ and $t \in [0, T_0]$. In this case we can consider $T_0 = 0$ in Lemma 2.

Lemma 3. *Let u_λ be a solution of* (1)*. Given $T_1 > 0$, there exists a positive constant $r_1 > 0$, independent of λ, such that*

$$\|u_\lambda(t)\|_{W_0^{1,p(x)}(\Omega)} < r_1,$$

for every $t \geqslant T_1$ and $\lambda \in [0, \lambda_0]$.

Proof. Let u_λ be a solution of (1) and consider $T_1 > 0$. Take $T_0 \in (0, T_1)$. Denote

$$\varphi^\lambda(v) := \begin{cases} \int_\Omega \frac{D^\lambda(x)}{p(x)} |\nabla v|^{p(x)}\, dx, & v \in X, \\ +\infty, & \text{otherwise}. \end{cases}$$

We have that φ^λ is a convex, proper and lower semicontinuous map, and $A_H^{D_\lambda}$ is the subdifferential $\partial\varphi^\lambda$ of φ^λ (see [29]). Thus

$$\begin{aligned} \frac{d}{dt}\varphi^\lambda\big(u_\lambda(t)\big) &= \left\langle \partial\varphi^\lambda\big(u_\lambda(t)\big), \frac{\partial u_\lambda}{\partial t}(t) \right\rangle \\ &= \left\langle B\big(u_\lambda(t)\big) - \frac{\partial u_\lambda}{\partial t}(t), \frac{\partial u_\lambda}{\partial t}(t) \right\rangle \\ &= \left\langle B\big(u_\lambda(t)\big) - \frac{\partial u_\lambda}{\partial t}(t), \frac{\partial u_\lambda}{\partial t}(t) - B\big(u_\lambda(t)\big) + B\big(u_\lambda(t)\big) \right\rangle \\ &= -\left\| B\big(u_\lambda(t)\big) - \frac{\partial u_\lambda}{\partial t}(t) \right\|_H^2 + \left\langle B\big(u_\lambda(t)\big) - \frac{\partial u_\lambda}{\partial t}(t), B\big(u_\lambda(t)\big) \right\rangle \end{aligned}$$

for a.e. t in $(0, \infty)$. Therefore,

$$\frac{d}{dt}\varphi^\lambda\big(u_\lambda(t)\big) + \frac{1}{2}\left\| B\big(u_\lambda(t)\big) - \frac{\partial u_\lambda}{\partial t}(t) \right\|_H^2 \leqslant \frac{1}{2}\big\| B\big(u_\lambda(t)\big)\big\|_H^2.$$

In particular,

$$\frac{d}{dt}\varphi^\lambda\big(u_\lambda(t)\big) \leqslant \frac{1}{2}\big\| B\big(u_\lambda(t)\big)\big\|_H^2 \leqslant \frac{1}{2}K_1^2, \quad \forall t \geqslant T_0,\ \lambda \in [0, \lambda_0], \tag{4}$$

where $K_1 = K_1(L, r_0) > 0$ is a constant ($r_0 = r_0(T_0)$). By definition of subdifferential we have the following inequality

$$\varphi_\lambda\big(u_\lambda(t)\big) \leqslant \big\langle \partial\varphi^\lambda\big(u_\lambda(t)\big), u_\lambda(t)\big\rangle.$$

Thus

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\big\|u_\lambda(t)\big\|_H^2 + \varphi^\lambda\big(u_\lambda(t)\big) &= \left\langle \frac{\partial}{\partial t}u_\lambda(t), u_\lambda(t) \right\rangle + \varphi^\lambda\big(u_\lambda(t)\big) \\ &\leqslant \left\langle \frac{\partial}{\partial t}u_\lambda(t), u_\lambda(t) \right\rangle + \big\langle \partial\varphi^\lambda\big(u_\lambda(t)\big), u_\lambda(t)\big\rangle \\ &= \big\langle B\big(u_\lambda(t)\big), u_\lambda(t)\big\rangle \\ &\leqslant \big\|B\big(u_\lambda(t)\big)\big\|_H \big\|u_\lambda(t)\big\|_H \leqslant K_1 r_0 \end{aligned} \tag{5}$$

for all $t \geqslant T_0$ and $\lambda \in [0, \lambda_0]$. Let $t \geqslant T_0$ and $r := T_1 - T_0 > 0$. Integrating (5) from t to $t + r$ we obtain that

$$\int_t^{t+r} \varphi^\lambda\big(u_\lambda(\tau)\big)\, d\tau \leqslant \frac{1}{2}\big\|u_\lambda(t)\big\|_H^2 + K_1 r_0 r \leqslant \frac{1}{2}r_0^2 + K_1 r_0 r =: A \tag{6}$$

for all $\lambda \in [0, \lambda_0]$. From (4), (6) and the Uniform Gronwall Lemma [31], we obtain

$$\varphi^\lambda\big(u_\lambda(t + r)\big) \leqslant \frac{A}{r} + \frac{1}{2}K_1^2 r =: \tilde{r}_1,$$

for all $t \geqslant T_0$ and $\lambda \in [0, \lambda_0]$. Therefore,

$$\int_\Omega \frac{D^\lambda(x)}{p(x)} \left|\nabla u_\lambda(t+r,x)\right|^{p(x)} dx \leqslant \tilde{r}_1,$$

for all $t \geqslant T_0$ and $\lambda \in [0,\lambda_0]$. So, considering $\rho(v) := \int_\Omega |v(x)|^{p(x)}\,dx$, we have

$$\frac{\beta}{p^+}\rho\big(\nabla u_\lambda(t+r)\big) = \frac{\beta}{p^+}\int_\Omega \left|\nabla u_\lambda(t+r,x)\right|^{p(x)} dx \leqslant \int_\Omega \frac{D^\lambda(x)}{p(x)}\left|\nabla u_\lambda(t+r,x)\right|^{p(x)} dx \leqslant \tilde{r}_1,$$

for all $t \geqslant T_0$ and $\lambda \in [0,\lambda_0]$. Considering $\tilde{\tilde{r}}_1 := \frac{p^+\tilde{r}_1}{\beta}$ and using Proposition 2.3 in [14] we conclude that

$$\left\|u_\lambda(\tau)\right\|_X^{p^-} = \left\|\nabla u_\lambda(\tau)\right\|_{p(x)}^{p^-} \leqslant \rho\big(\nabla u_\lambda(\tau)\big) \leqslant \tilde{\tilde{r}}_1,$$

for all $\tau \geqslant T_1$ and $\lambda \in [0,\lambda_0]$ if $\|u_\lambda(\tau)\|_X > 1$. So considering $r_1 := \max\{1, [\tilde{\tilde{r}}_1]^{1/p^-}\}$ we can conclude that

$$\left\|u_\lambda(t)\right\|_X \leqslant r_1,$$

for all $t \geqslant T_1$ and $\lambda \in [0,\lambda_0]$. $\square$

Corollary 2.

(a) *There exists a bounded set B_1 in X such that $\mathcal{A}_\lambda \subset B_1$ for all $\lambda \in [0,\lambda_0]$.*
(b) $\mathcal{A} := \overline{\bigcup_{\lambda\in[0,\lambda_0]} \mathcal{A}_\lambda}$ *is a compact subset of H.*

Proposition 1. *Let u_λ be a solution of* (1) *with initial values all in a bounded set of $X = W_0^{1,p(x)}(\Omega)$. Given $T_1 > 0$ there exists a positive constant R such that*

$$\left\|u_\lambda(t)\right\|_X \leqslant R,$$

for all $t \in [0,T_1]$ and $\lambda \in [0,\lambda_0]$. In this case we can consider $T_1 = 0$ in Lemma 3.

Proof. Given $T_1 > 0$, if u_λ is a solution of (1) then multiplying the equation by $\frac{\partial u_\lambda}{\partial t}(t)$ we have that

$$\left\|\frac{\partial u_\lambda}{\partial t}(t)\right\|_H^2 + \left\langle A_1^{D_\lambda}\big(u_\lambda(t)\big), \frac{\partial u_\lambda}{\partial t}(t)\right\rangle = \left\langle B\big(u_\lambda(t)\big), \frac{\partial u_\lambda}{\partial t}(t)\right\rangle.$$

As $\langle A_H^{D_\lambda}(u_\lambda(t)), \frac{\partial u_\lambda}{\partial t}(t)\rangle = \frac{d}{dt}\varphi^\lambda(u_\lambda(t))$ where φ^λ is the function that appears at the proof of Lemma 3, we obtain that

$$\frac{1}{2}\left\|\frac{\partial u_\lambda}{\partial t}(t)\right\|_H^2 + \frac{d}{dt}\varphi^\lambda\big(u_\lambda(t)\big) \leqslant \frac{1}{2}\left\|B\big(u_\lambda(t)\big)\right\|_H^2$$

and then

$$\frac{d}{dt}\varphi^\lambda\big(u_\lambda(t)\big) \leqslant \frac{1}{2}\left\|B\big(u_\lambda(t)\big)\right\|_H^2 \leqslant \frac{1}{2}\big(L\left\|u_\lambda(t)\right\|_H + \left\|B(0)\right\|\big)^2.$$

So, by Remark 2 we conclude that

$$\frac{d}{dt}\varphi^\lambda\big(u_\lambda(t)\big) \leqslant C, \quad \text{for all } t \in [0,T_1],\ \lambda \in [0,\lambda_0],$$

where $C > 0$ is a constant. Therefore, integrating the equation above from 0 to τ, for $\tau \leqslant T_1$, we obtain that

$$\varphi^\lambda\big(u_\lambda(\tau)\big) \leqslant \varphi^\lambda(u_{0\lambda}) + CT_1, \quad \text{for all } \tau \in [0,T_1],\ \lambda \in [0,\lambda_0]. \tag{7}$$

If $\|u_\lambda(\tau)\|_X \leqslant 1$ then

$$\frac{\beta}{p^+}\left\|u_\lambda(\tau)\right\|_X^{p^+} \leqslant \varphi^\lambda\big(u_\lambda(\tau)\big) \tag{8}$$

and if $\|u_\lambda(\tau)\|_X > 1$ then

$$\frac{\beta}{p^+}\left\|u_\lambda(\tau)\right\|_X^{p^-} \leqslant \varphi^\lambda\big(u_\lambda(\tau)\big), \tag{9}$$

and by another side

$$\varphi^\lambda(u_{0\lambda}) \leqslant \begin{cases} \frac{M}{p^-}\|u_{0\lambda}\|_X^{p^-} & \text{if } \|u_{0\lambda}\|_X \leqslant 1, \\ \frac{M}{p^-}\|u_{0\lambda}\|_X^{p^+} & \text{if } \|u_{0\lambda}\|_X > 1. \end{cases} \tag{10}$$

As we have that $\|u_{0\lambda}\|_X \leqslant C_1$ for all $\lambda \in [0, \lambda_0]$ the result follows joining (7), (8), (9) and (10). □

3. Continuity with respect to the initial values and upper semicontinuity of attractors

In this section we prove upper semicontinuity for the family of global attractors of problem (1). The next theorem shows the continuity of the solutions with respect to initial conditions and parameters.

Theorem 1. *For each $\lambda \in [0, \lambda_0]$ let u_λ be a solution of (1) with $u_\lambda(0) = u_{0\lambda}$. If $\{u_{0\lambda}\colon \lambda \in [0, \lambda_0]\}$ is a bounded set in X and $u_{0\lambda} \to u_{0\lambda_1}$ in H as $\lambda \to \lambda_1$, then for each $T > 0$, $u_\lambda \to u_{\lambda_1}$ in $C([0, T]; H)$ as $\lambda \to \lambda_1$.*

Proof. Let $T > 0$ and $t \in [0, T]$. We have

$$\frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - u_{\lambda_1}(t)\|_H^2 + \langle A_1^{D_\lambda}(u_\lambda(t)) - A_1^{D_{\lambda_1}}(u_{\lambda_1}(t)), u_\lambda(t) - u_{\lambda_1}(t)\rangle = \langle B(u_\lambda(t)) - B(u_{\lambda_1}(t)), u_\lambda(t) - u_{\lambda_1}(t)\rangle$$

and

$$\begin{aligned} &\langle A_1^{D_\lambda}(u_\lambda(t)) - A_1^{D_{\lambda_1}}(u_{\lambda_1}(t)), u_\lambda(t) - u_{\lambda_1}(t)\rangle \\ &\quad \geqslant \beta \int_\Omega \gamma_0(x)|\nabla u_\lambda - \nabla u_{\lambda_1}|^{p(x)}\,dx + \int_\Omega (D^\lambda - D^{\lambda_1})|\nabla u_{\lambda_1}|^{p(x)-2}\nabla u_{\lambda_1}\cdot(\nabla u_\lambda - \nabla u_{\lambda_1})\,dx, \end{aligned}$$

where $\gamma_0(x) > 0$ for all $x \in \Omega$ by the Tartar's inequality. So

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - u_{\lambda_1}(t)\|^2 &\leqslant L\|u_\lambda(t) - u_{\lambda_1}(t)\|_H^2 + \|D^\lambda - D^{\lambda_1}\|_{L^\infty(\Omega)}\big[2\rho(\nabla u_{\lambda_1}(t)) + \rho(\nabla u_\lambda(t))\big] \\ &\leqslant L\|u_\lambda(t) - u_{\lambda_1}(t)\|_H^2 + \|D^\lambda - D^{\lambda_1}\|_{L^\infty(\Omega)} W_1, \end{aligned}$$

where $W_1 := 3\max\{r_1^{p^+}, r_1^{p^-}\}$ and r_1 is a constant given by Lemma 3 (see also Proposition 1). Integrating the above inequality from 0 to t, $t \leqslant T$, we obtain that

$$\|u_\lambda(t) - u_{\lambda_1}(t)\|_H^2 \leqslant \|u_{0\lambda} - u_{0\lambda_1}\|_H^2 + 2W_1 T\|D^\lambda - D^{\lambda_1}\| + 2L\int_\Omega \|u_\lambda(\tau) - u_{\lambda_1}(\tau)\|_H^2\,d\tau.$$

By Gronwall–Belman's Lemma we obtain that

$$\|u_\lambda(t) - u_{\lambda_1}(t)\|_H^2 \leqslant \big[\|u_{0\lambda} - u_{0\lambda_1}\|_H^2 + 2W_1 T\|D^\lambda - D^{\lambda_1}\|_{L^\infty(\Omega)}\big]e^{2LT},$$

for all $t \in [0, T]$ and $\lambda \in [0, \lambda_0]$. Therefore, if $u_{0\lambda} \to u_{0\lambda_1}$ in H as $\lambda \to \lambda_1$, then $u_\lambda \to u_{\lambda_1}$ in $C([0, T]; H)$ as $\lambda \to \lambda_1$. □

Corollary 3. *For each $\lambda \in [0, \lambda_0]$ let $u_\lambda(t, u_{0\lambda})$ be a solution of (1) with $u_\lambda(0) = u_{0\lambda}$ and $u_{\lambda_1}(t, u_{0\lambda})$ be the solution of (1) with $\lambda = \lambda_1$ and initial condition $u_{0\lambda}$. If $\{u_{0\lambda}\}_{\lambda \in [0, \lambda_0]}$ is a bounded set in X, then for each $T > 0$,*

$$\|u_\lambda(t, u_{0\lambda}) - u_{\lambda_1}(t, u_{0\lambda})\|_H^2 \leqslant 2W_1 T\|D^\lambda - D^{\lambda_1}\|_{L^\infty(\Omega)} e^{2LT},$$

for all $t \in [0, T]$ and $\lambda \in [0, \lambda_0]$.

Theorem 2. *The family of attractors $\{\mathcal{A}_\lambda\}_{\lambda \in [0, \lambda_0]}$ is upper semicontinuous at $\lambda_1 := 0$, that is, $\sup_{a_\lambda \in \mathcal{A}_\lambda} \operatorname{dist}_H(a_\lambda, \mathcal{A}_{\lambda_1}) \to 0$ as $\lambda \to \lambda_1 = 0$.*

Proof. Let $\{x_\lambda\}_{\lambda \in [0, \lambda_0]}$ be such that $x_\lambda \in \mathcal{A}_\lambda$ for all $\lambda \in [0, \lambda_0]$. By Corollary 1 there exists a bounded set $B_0 \subset H$ such that $\mathcal{A}_\lambda \subset B_0$ for all $\lambda \in [0, \lambda_0]$. Since $\mathcal{A}_{\lambda_1}$ attracts bounded sets of H, for every $\epsilon > 0$, there exists $t_0 = t_0(\epsilon, B_0) > 0$ such that

$$\sup_{z_\lambda \in \mathcal{A}_\lambda,\, \lambda \in [0, \lambda_0]} \operatorname{dist}_H(u_{\lambda_1}(t_0, z_\lambda), \mathcal{A}_{\lambda_1}) \leqslant \frac{\epsilon}{2},$$

where $u_{\lambda_1}(t, z_\lambda)$ is a solution of problem (1) with $\lambda = \lambda_1$ and initial condition z_λ. By the invariance of the global attractors, for all $\lambda \in [0, \lambda_0]$ there exists $\psi_\lambda \in \mathcal{A}_\lambda$ such that $x_\lambda = u_\lambda(t_0, \psi_\lambda)$. Using Corollary 2(a) and Corollary 3 we obtain that there exists $\lambda_2 \in [0, \lambda_0]$ such that

$$\|u_\lambda(t_0, \psi_\lambda) - u_{\lambda_1}(t_0, \psi_\lambda)\|_H < \frac{\epsilon}{2},$$

for every $\lambda \in [0, \lambda_2]$. Thus, for every $\lambda \in [0, \lambda_2]$, we obtain that

$$\begin{aligned}\mathrm{dist}_H\big(u_\lambda(t_0,\psi_\lambda),\mathcal{A}_{\lambda_1}\big) &\leqslant \big\|u_\lambda(t_0,\psi_\lambda)-u_{\lambda_1}(t_0,\psi_\lambda)\big\|_H+\mathrm{dist}_H\big(u_{\lambda_1}(t_0,\psi_\lambda),\mathcal{A}_{\lambda_1}\big)\\ &<\epsilon.\end{aligned}$$

So,

$$\begin{aligned}\mathrm{dist}_H(\mathcal{A}_\lambda,\mathcal{A}_{\lambda_1}) &= \sup_{x_\lambda\in\mathcal{A}_\lambda}\mathrm{dist}_H(x_\lambda,\mathcal{A}_{\lambda_1})\\ &= \sup_{\psi_\lambda\in\mathcal{A}_\lambda}\mathrm{dist}_H\big(u_\lambda(t_0,\psi_\lambda),\mathcal{A}_{\lambda_1}\big)\leqslant\epsilon,\end{aligned}$$

for all $\lambda \in [0, \lambda_2]$. The proof is completed. □

4. The $p(x)$-Laplacian problem with large diffusion

In this section we consider the problem (2) and we study the asymptotic behavior as $D \to \infty$.

The proofs of Lemma 2 and Lemma 3 can be adapted to obtain the following $L^2(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ estimates for the solutions u^D's of the problem (2), uniformly on $D \geqslant 1$.

Lemma 4. *Let u^D be a solution of (2). Given $T_0 > 0$, there exists a positive constant r_0 such that $\|u^D(t)\|_H \leqslant r_0$, for each $t \geqslant T_0$ and $D \geqslant 1$.*

We observe that the constant r_0 in the above lemma depend neither on the initial data nor on D.

Corollary 4. *There exists a bounded set B_0 in H such that $\mathcal{A}_D \subset B_0$ for all $D \geqslant 1$.*

Remark 3. Given $T_0 > 0$, if u^D is a solution of (2), then there exists a positive number $K = K(u_0^D, T_0)$ such that $\|u_\lambda(t)\|_H \leqslant K$, for all $t \in [0, T_0]$. If the initial values are all in a bounded set of H, then K is uniform on D, i.e., we have that $\|u^D(t)\|_H \leqslant K$ for each $D \geqslant 1$ and $t \in [0, T_0]$. In this case we can consider $T_0 = 0$ in Lemma 4.

Lemma 5. *Let u^D be a solution of (2). Given $T_1 > 0$, there exists a positive constant $r_1 > 0$, independent of D, such that*

$$\big\|u^D(t)\big\|_{W_0^{1,p(x)}(\Omega)} < r_1, \quad \forall t \geqslant T_1,\ D \geqslant 1.$$

As an important consequence of Lemma 5 it follows

Corollary 5.

(a) *There exists a bounded set B_1 in X such that $\mathcal{A}_D \subset B_1$ for all $D \geqslant 1$.*
(b) *$\mathcal{A} := \overline{\bigcup_{D\geqslant 1}\mathcal{A}_D}$ is a compact subset of H.*

4.1. The limit problem and convergence properties

Our objective in this section is to prove that the limit problem of problem (2) as D increases to infinity is described by an ordinary differential equation. Firstly we observe that the gradients of the solutions u^D converge in norm to zero as $D \to \infty$, which allows us to guess the limit problem (3).

Lemma 6. *Given $T_1 > 0$, if for each $D \geqslant 1$, u^D is a solution of (2) in $(0, \infty)$, then for each $t \geqslant T_1$, the sequence of real numbers $\{\|\nabla u^D(t)\|_H\}$ has a subsequence $\{\|\nabla u^{D_\ell}(t)\|_H\}$ which converges to zero as $\ell \to \infty$.*

Proof. Let $T > T_1$ and $t \in (T_1, T)$. As u^D is a solution of (2) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\|u^D(t)\big\|_H^2 + D\int_\Omega\big|\nabla u^D(t)\big|^{p(x)}\,dx &= \big\langle B\big(u^D(t)\big),u^D(t)\big\rangle_H\\ &\leqslant L\big\|u^D(t)\big\|_H^2 + C\big\|u^D(t)\big\|_H \leqslant K,\end{aligned} \tag{11}$$

t-a.e. in (T_1, T), where $K > 0$ is a constant which is independent of D by Lemma 4. Integrating (11) from T_1 to T, we obtain

$$\frac{1}{2}\left\|u^D(T)\right\|_H^2 + D\int_{T_1}^{T}\int_{\Omega}\left|\nabla u^D(t)\right|^{p(x)}dx\,dt \leqslant \frac{1}{2}\left\|u^D(T_1)\right\|_H^2 + \int_{T_1}^{T} K\,dt$$
$$\leqslant \frac{1}{2}r_0^2 + KT := K_2(T).$$

In particular

$$D\int_{T_1}^{T}\int_{\Omega}\left|\nabla u^D(t)\right|^{p(x)}dx\,dt \leqslant K_2(T)$$

which implies

$$\int_{T_1}^{T}\int_{\Omega}\left|\nabla u^D(t)\right|^{p(x)}dx\,dt \leqslant \frac{K_2(T)}{D} \to 0 \quad \text{as } D \to \infty.$$

Therefore there is a subsequence $\{\int_\Omega |\nabla u^{D_\ell}(t)|^{p(x)}\,dx\}$ such that

$$\int_{\Omega}\left|\nabla u^{D_\ell}(t)\right|^{p(x)}dx \to 0 \quad \text{as } \ell \to \infty,\ t\text{-a.e. in } (T_1, T)$$

and so there exists a subset $J \subset (T_1, T)$ with Lebesgue measure $m((T_1, T)/J) = 0$ such that

$$\int_{\Omega}\left|\nabla u^{D_\ell}(t)\right|^{p(x)}dx \to 0 \quad \text{as } \ell \to \infty,\ \forall t \in J.$$

Given $t \in (T_1, T)$ we claim that there is at least one $s \in J$ with $s < t$, on the contrary we would have $(T_1, t) \cap J = \emptyset$, so $m((T_1, T)/J) > 0$ which is a contradiction. Now pick one $s \in J$ with $T_1 < s < t$ and let $h = t - s$. Let $\epsilon > 0$ and $\ell_0 = \ell(\epsilon)$ be such that if $\ell > \ell_0$ then

$$\int_{\Omega}\left|\nabla u^{D_\ell}(s)\right|^{p(x)}dx < \frac{p^-}{2p^+}\epsilon.$$

Using

$$\frac{d}{d\tau}\varphi^{D_\ell}\big(u^{D_\ell}(s+\tau)\big) = \left\langle \partial\varphi^{D_\ell}\big(u^{D_\ell}(s+\tau)\big), \frac{d}{d\tau}u^{D_\ell}(s+\tau)\right\rangle, \quad \tau\text{-a.e. in } (0, T),$$

where

$$\varphi^{D_\ell}(v) := \begin{cases} D_\ell \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx, & v \in X, \\ +\infty, & \text{otherwise} \end{cases}$$

we obtain

$$D_\ell\int_{\Omega}\frac{1}{p(x)}\left|\nabla u^{D_\ell}(s+h)\right|^{p(x)}dx - D_\ell\int_{\Omega}\frac{1}{p(x)}\left|\nabla u^{D_\ell}(s)\right|^{p(x)}dx \leqslant \frac{1}{2}K_3h,$$

where K_3 depends on L and, according to Lemma 4, can be uniformly chosen on D_ℓ. Thus,

$$\int_{\Omega}\frac{1}{p(x)}\left|\nabla u^{D_\ell}(s+h)\right|^{p(x)}dx \leqslant \frac{1}{2D_\ell}K_3(T-T_1) + \int_{\Omega}\frac{1}{p(x)}\left|\nabla u^{D_\ell}(s)\right|^{p(x)}dx.$$

Now we choose $\ell_1 = \ell_1(\epsilon)$ sufficiently large such that

$$\frac{1}{2D_\ell}K_3(T-T_1) < \frac{\epsilon}{2p^+},$$

whenever $\ell > \ell_1$ and we consider $\ell_2 = \ell_2(\epsilon) = \max\{\ell_0, \ell_1\}$. For $\ell > \ell_2$ we have

$$\begin{aligned}
\frac{1}{p^+}\int_\Omega \left|\nabla u^{D_\ell}(t)\right|^{p(x)}dx &\leqslant \int_\Omega \frac{1}{p(x)}\left|\nabla u^{D_\ell}(t)\right|^{p(x)}dx \\
&= \int_\Omega \frac{1}{p(x)}\left|\nabla u^{D_\ell}(s+t-s)\right|^{p(x)}dx \\
&\leqslant \frac{1}{2D_\ell}K_3(T-T_1) + \int_\Omega \frac{1}{p(x)}\left|\nabla u^{D_\ell}(s)\right|^{p(x)}dx \\
&\leqslant \frac{1}{2D_\ell}K_3(T-T_1) + \frac{1}{p^-}\int_\Omega \left|\nabla u^{D_\ell}(s)\right|^{p(x)}dx \\
&< \frac{\epsilon}{2p^+} + \frac{1}{p^-}\frac{p^-}{2p^+}\epsilon = \frac{1}{p^+}\epsilon.
\end{aligned}$$

Thus, if $\ell > \ell_2$

$$\int_\Omega \left|\nabla u^{D_\ell}(t)\right|^{p(x)}dx < \epsilon$$

and then using Proposition 2.4 in [14] we obtain $\|\nabla u^{D_\ell}(t)\|_{p(x)} \to 0$ as $\ell \to \infty$. As $p(x) > 2$, $\|\nabla u^{D_\ell}(t)\|_H \leqslant C\|\nabla u^{D_\ell}(t)\|_{p(x)}$. So, $\|\nabla u^{D_\ell}(t)\|_H \to 0$ as $\ell \to \infty$. $\square$

As we can see the above lemma is telling us that Eq. (3) is a good candidate for the limit problem. Since $\tilde{B}:\mathbb{R}\to\mathbb{R}$ is a globally Lipschitz map, from Picard's Theorem we can conclude that the problem (3) has a unique global solution. Moreover

Theorem 3. *The problem* (3) *defines a semigroup of class $\mathcal{K}$.*

Proof. We define $S(t):\mathbb{R}\to\mathbb{R}$ by $S(t)v_0 = v(t)$ with v being the unique global solution of the problem (3) with $v(0) = v_0$. It is easy to see that $S(t)$ verifies the semigroup properties.

We will show that $S(t)$ is of class $\mathcal{K}$. In fact, multiplying the equation in (3) by $u(t)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\left|u(t)\right|^2 \leqslant \left(L + \frac{|\tilde{B}(0)|}{2}\right)\left|u(t)\right|^2 + \frac{|\tilde{B}(0)|}{2}. \tag{12}$$

Let $T > 0$ fixed, integrating (12) from 0 to τ, $\tau < T$, we obtain

$$\left|u(\tau)\right|^2 \leqslant \left|u(0)\right|^2 + \left|\tilde{B}(0)\right|T + \int_0^\tau \left(2L + \left|\tilde{B}(0)\right|\right)\left|u(s)\right|^2 ds.$$

So, by Gronwall–Bellman's Lemma it follows that

$$\left|u(\tau)\right|^2 \leqslant \left(|u_0|^2 + \left|\tilde{B}(0)\right|T\right)e^{(2L+|\tilde{B}(0)|)T}, \quad \forall \tau \leqslant T.$$

Thus, we conclude that for each $t > 0$, $S(t)$ maps bounded sets into bounded sets. As a result we conclude that for each $t > 0$ the operator $S(t):\mathbb{R}\to\mathbb{R}$ is compact. $\square$

The next result guarantees that (3) is in fact the limit problem for (2), as $D \to \infty$.

Theorem 4. *For each $D \geqslant 1$, let u^D be a solution of* (2) *with $u^D(0) = u_0^D$ and let u be a solution of* (3) *with $u(0) = u_0$. If $u_0^D \to u_0$ in H as $D \to \infty$, then for each $T > 0$, $u^D \to u$ in $C([0,T];H)$ as $D \to +\infty$.*

Proof. Let $T > 0$ be fixed and suppose that $u_0^D \to u_0$ in H as $D \to \infty$. Subtracting the two equations in (2) and (3) and making the inner product with $u^D - u$ we obtain

$$\left\langle u_t^D - u_t, u^D - u\right\rangle_H + D\int_\Omega \left|\nabla u^D(x)\right|^{p(x)}dx = \left\langle B\left(u^D\right) - B(u), u^D - u\right\rangle_H.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\left\|u^D - u\right\|_H^2 \leqslant L\left\|u^D - u\right\|_H^2, \quad \text{a.e. in } (0,T).$$

Integrating from 0 to t, $t \leqslant T$, we obtain

$$\|u^D(t) - u(t)\|_H^2 \leqslant \|u_0^D - u_0\|_H^2 + \int_0^t 2L\|u^D(s) - u(s)\|_H^2\, ds.$$

So, by Gronwall–Bellman's Lemma we obtain

$$\|u^D(t) - u(t)\|_H^2 \leqslant \|u_0^D - u_0\|_H^2 e^{2LT}, \quad \forall t \in [0, T].$$

Therefore $u^D \to u$ in $C([0,T]; H)$ as $D \to +\infty$, whenever $u_0^D \to u_0$ in H as $D \to \infty$. □

Observe that the semigroup of class $\mathcal{K}$ defined by the problem (3) is not necessarily B-dissipative. For example, if $B: H \to H$ is given by $B(u) = \alpha u$, with α a positive real number, then $\tilde{B}: \mathbb{R} \to \mathbb{R}$ is given by $\tilde{B}(u) = \alpha u$ and so the solution of (3) is $u(t) = u_0 e^{\alpha t}$ and $|u(t)| \to \infty$ as $t \to \infty$. So, it doesn't exist a global B-attractor for the problem (3).

4.2. Continuity of attractors

If the semigroup defined by the limit problem (3) is B-dissipative then, Theorem 2.2 and Proposition 2.2 in [32], guarantee that the semigroup $S(t)$ has a maximal compact invariant global B-attractor $\mathcal{A}^\infty$, given as the union of all bounded complete trajectories in $\mathbb{R}$. The next two examples provide situations where the semigroup defined by the limit problem (3) is B-dissipative.

Example 1. If $B: H \to H$ is given by $B(u) = \beta u$, where β is a negative real number, then $\tilde{B}: \mathbb{R} \to \mathbb{R}$ is given by $\tilde{B}(u) = \beta u$ and so the solution of (3) is $u(t) = u_0 e^{\beta t}$ and $u(t) \to 0$ as $t \to \infty$. So, the semigroup defined by the limit problem (3) is B-dissipative.

Example 2. Consider $f: \mathbb{R} \to \mathbb{R}$ of class C^1, globally Lipschitz and with an odd number m of zeros $\{u_1, u_2, \ldots, u_m\}$ such that $u_1 < u_2 < \cdots < u_m$, $f'(u_1) < 0$, $f'(u_2) > 0$, $f'(u_3) < 0, \ldots, f'(u_m) < 0$, $f(x) > 0$ for all $x < u_1$ and $f(x) < 0$ for all $x > u_m$. Observe that f of class C^1 and $|f'(u)| \leqslant M$ for all $u \in \mathbb{R}$, where $M > 0$ is a constant, it is a sufficient condition to obtain f globally Lipschitz. In this case, using the theory of ODE, we obtain that $\{u_1, u_2, \ldots, u_m\}$ is a bounded global attractor for the limit problem (3) with $\tilde{B} = f$. Observe that in this case the map $B: H \to H$ of the problem (2) is the Nemytskii's operator associated to f.

Now, we suppose that $B: H \to H$ is globally Lipschitz and it is such that the limit problem (3) has a B-dissipative semigroup. So, let $\mathcal{A}^\infty$ be the maximal compact invariant global B-attractor for (3). In this section we prove not only $u^D \to u$ as $D \to \infty$ but we have also that the family of attractors $\mathcal{A}_D$ behaves continuously as a diffusion parameter increases to infinity. We start introducing the following lemma which guarantees that the relevant elements to describe the asymptotic behavior of these problems are around its own spatial average if D is large enough.

Lemma 7. *If for each $D \geqslant 1$, $u_0^D \in \mathcal{A}_D$ and $u_0 = \lim_{D\to\infty} u_0^D$ in H, then u_0 is a constant function.*

Proof. Using Lemma 5 and Lemma 6 we can borrow the arguments presented in the proof of Lemma 4.1 in [30] and we obtain that $u_0 = \overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0(x)\, dx$. Thus u_0 is a constant function. □

Thus, using Corollary 5(b), Lemma 7 and Theorem 4 we can borrow the arguments presented in the proof of Theorem 4.1 in [30] and we obtain that:

Theorem 5. *The family of global B-attractors $\{\mathcal{A}_D\}_{D\geqslant 1}$ of the problem* (2) *is upper semicontinuous at infinity.*

Remark 4. As the limit problem of (2) is the ODE (3) we obtain that $\mathcal{A}^\infty \subset \mathcal{A}_D$ for all $D \geqslant 1$. Thus, the family of attractors $\{\mathcal{A}_D\}_{D\geqslant 1}$ is also lower semicontinuous at infinity and then we conclude that the family of attractors $\{\mathcal{A}_D\}_{D\geqslant 1}$ is continuous at infinity when the limit problem (3) has a bounded global B-attractor. The lower semicontinuity of the family of global B-attractors $\{\mathcal{A}_\lambda\}_{\lambda\in[0,\lambda_0]}$ associated with the problem (1) remains an open problem.

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