Two results on powers of 2 in Waring–Goldbach problem

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ABSTRACT

In this paper, it is proved that every sufficiently large odd integer is a sum of a prime, four cubes of primes and 106 powers of 2. What is more, every sufficiently large even integer is a sum of two squares of primes, four cubes of primes and 211 powers of 2.

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1. Introduction

Linnik [12] proved under GRH in 1951, and two years later unconditionally [13] that every sufficiently large even integer can be written as the sum of two primes and $K_1$ powers of 2, where $K_1$ is an unspecified absolute constant, that is

$$N = p_1 + p_2 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{K_1}}. \quad (1.1)$$

The Goldbach conjecture implies clearly $K_1 = 0$. An explicit value for the number $K_1$ of powers of 2 was first established by Liu, Liu and Wang [19], who found that $K_1 = 54,000$ is acceptable. The original value for the number $K_1$ was subsequently improved by Li [8], Wang [32], Li [9] and Heath-Brown and Puchta [3]. Assuming GRH, the corresponding value was known due to Liu, Liu and Wang [18], Liu, Liu and Wang [20], Wang [32], Heath-Brown and Puchta [3] and Pintz and Ruzsa [28].
In 1999, Liu, Liu and Zhan [21] proved that every large even integer $N$ can be written as a sum of four squares of primes and $K_2$ powers of 2,

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{y_1} + 2^{y_2} + \cdots + 2^{y_{K_2}}. \quad (1.2)$$

The value for the number $K_2$ was subsequently determined by Liu and Liu [15], Liu and Lü [22], Li [10].

Liu and Liu [17] proved that every large even integer $N$ can be written as a sum of eight cubes of primes and $K_3$ powers of 2,

$$N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{y_1} + 2^{y_2} + \cdots + 2^{y_{K_3}}. \quad (1.3)$$

The acceptable value for the number $K_3$ was determined by the authors [24].

As a hybrid problem of (1.1) and (1.2), Liu, Liu and Zhan [21], among other important results, proved that every large odd integer $N$ can be written as a sum of one prime, two squares of primes and $K_4$ powers of 2, namely

$$N = p_1 + p_2^2 + p_3^2 + 2^{y_1} + 2^{y_2} + \cdots + 2^{y_{K_4}}. \quad (1.4)$$

Liu [14], Li [11] and Lü and Sun [26] gave the acceptable values for the number $K_4$.

In this paper, we consider the hybrid problem of (1.1) and (1.3),

$$N = p_1 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + 2^{y_1} + 2^{y_2} + \cdots + 2^{y_{K_5}}, \quad (1.5)$$

by giving the following result.

**Theorem 1.1.** Every sufficiently large odd integer is a sum of a prime, four cubes of primes and 106 powers of 2.

Similarly, we can construct the hybrid problem of (1.2) and (1.3),

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + 2^{y_1} + 2^{y_2} + \cdots + 2^{y_{K_6}}. \quad (1.6)$$

**Theorem 1.2.** Every sufficiently large even integer is a sum of two squares of primes, four cubes of primes and 211 powers of 2.

The following estimate, obtained by Brun's sieve, also plays an important role in those problems. The result refers to the generalized twin-prime problem.

**Lemma 1.3.** Let $N$ be sufficiently large and $h$ be any positive integer. There exists a constant $C^*$ such that

$$R(h) = \#\{h = p_1 - p_2: p_i \leq N\} < (1 + o(1))C^* \cdot 2C_0 f(h) \frac{N}{\log^2 N} \quad (1.7)$$

where

$$\left\{ \begin{array}{l}
    f(h) = \sum_{\substack{d|h \\
        \mu(d) \neq 0 \}} k(d), \\
    k(d) = \prod_{\substack{p|d \\
        p > 2}} \frac{1}{p - 2}, \\
    C_0 = \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^2}\right) = 0.66016\ldots
\end{array} \right. \quad (1.8)$$
We note that Hardy and Littlewood [2] conjectured that if \( h \) is fixed and \( N \to \infty \) then

\[
R(h) \sim 2C_0 f(h) \frac{N}{\log^2 N}.
\]  

(1.9)

An extension of the conjecture (1.9) would imply

Conjecture 1.4. (1.7) holds with \( C^* = 1 \).

Assuming Conjecture 1.4, one can improve the value for the number \( K_1 \) (Pintz [27]) and the value for the number \( K_4 \) (the authors [25]). In this paper, we also give the improvement of the value for the number \( K_5 \) if assume Conjecture 1.4.

Theorem 1.5. Suppose Conjecture 1.4 holds. Every sufficiently large odd integer is a sum of a prime, four cubes of primes and 74 powers of 2.

Notation. As usual, \( \varphi(n) \) and \( \Lambda(n) \) denote the Euler totient function and the von Mangoldt function, respectively. We write \( N \) for a large integer, and \( L = \log_2 N \). Further, \( r \sim R \) means \( R < r \leq 2R \), and \( A \asymp B \) means \( c_1 A \leq B \leq c_2 A \). If there is no ambiguity, we express \( \frac{a}{b} + \theta \) as \( a/b + \theta \) or \( \theta + a/b \). The same convention will be applied for quotients. The letters \( \varepsilon \) and \( A \) denote positive constants, which are arbitrarily small and arbitrarily large, respectively.

2. Outline of the method

Here we give an outline for the proof of Theorem 1.1. In order to apply the circle method, we set

\[
P = N^{1/9 - 2\delta}, \quad Q = N^{8/9 + \varepsilon}.
\]  

(2.1)

By Dirichlet’s lemma [31, Lemma 2.1], each \( \alpha \in [1/Q, 1 + 1/Q] \) may be written in the form

\[
\alpha = a/q + \lambda, \quad |\lambda| \leq 1/qQ,
\]  

(2.2)

for some integers \( a, q \) with \( 1 \leq a \leq q \leq Q \) and \( (a, q) = 1 \). Denote by \( \mathcal{M}(a, q) \) the set of \( \alpha \) satisfying (2.2), and define the major arcs \( \mathcal{M} \) and the minor arcs \( C(\mathcal{M}) \) as follows:

\[
\mathcal{M} := \bigcup_{1 \leq q \leq P} \bigcup_{1 \leq a \leq q (a, q) = 1} \mathcal{M}(a, q), \quad C(\mathcal{M}) = \left[ \frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus \mathcal{M}.
\]  

(2.3)

It follows from \( 2P \leq Q \) that the major arcs \( \mathcal{M}(a, q) \) are mutually disjoint. As the values in [30], let \( \delta = 10^{-4} \), and

\[
U = \left( \frac{N}{16(1 + \delta)} \right)^{1/3}, \quad V = U^{5/6}.
\]  

(2.4)

As usual in the circle method, let

\[
f(\alpha) = \sum_{p \leq N} (\log p)e(p\alpha), \quad g(\alpha) = \sum_{p^2 \leq N} (\log p)e(p^2\alpha).
\]  

(2.5)
\[
S(\alpha) = \sum_{p \sim U} (\log p)e(p^2\alpha), \quad T(\alpha) = \sum_{p \sim V} (\log p)e(p^2\alpha).
\]

(2.6)

\[
G(\alpha) = \sum_{2^i \leq N} e(2^i\alpha) = \sum_{\nu \leq L} e(2^\nu\alpha).
\]

(2.7)

The circle method, in the form we require in here, begin with the observation that

\[
R_1(N) = \sum_{\substack{N = p_1 + p_2^3 + \cdots + p_5^3 \nu_{12} + \cdots + 2^k \nu_{12} \leq N, \ p_1 \sim U, \ p_2 \sim U, \ p_3 \sim V, \ p_4 \sim V, \ p_5 \sim V}} \prod_{i=1}^{5} (\log p_i).
\]

(2.8)

Then \(R_1(N)\) can be written as

\[
R_1(N) = \int_0^1 f(\alpha) S_2(\alpha) T^2(\alpha) G^k(\alpha) e(-N\alpha) \, d\alpha
\]

(2.9)

Similarly,

\[
R_2(N) = \sum_{\substack{N = p_1^2 + p_2^3 + \cdots + p_6^3 \nu_{12} + \cdots + 2^k \nu_{12} \leq N, \ p_1 \sim U, \ p_2 \sim U, \ p_3 \sim V, \ p_4 \sim V, \ p_5 \sim V, \ p_6 \sim V}} \prod_{i=1}^{6} (\log p_i).
\]

(2.10)

Then \(R_2(N)\) can be written as

\[
R_2(N) = \int_0^1 g(\alpha) S_2(\alpha) T^2(\alpha) G^k(\alpha) e(-N\alpha) \, d\alpha
\]

(2.11)

To handle the integral on the major arcs, we shall state the following lemmas.

**Lemma 2.1.** Let \(\mathcal{M}\) be as in (2.3), with \(P\) and \(Q\) determined by (2.1). Then for \(N/2 \leq n \leq N\), we have

\[
\int_{\mathcal{M}} f(\alpha) S_2(\alpha) T^2(\alpha) e(-n\alpha) \, d\alpha = \frac{1}{34} \mathcal{S}_1(n) J_1(n) + O\left(N^{\frac{1}{3}} L^{-1}\right).
\]

Here \(\mathcal{S}_1(n)\) is singular series, which is defined as (3.9) and satisfies \(\mathcal{S}_1(n) \gg 1\) for \(n \equiv 1 \pmod{2}\). \(J_1(n)\) is defined as

\[
J_1(n) := \sum_{\substack{m_1 + \cdots + m_5 = n \leq \frac{U^3}{2}, \ m_2, m_3 \leq 8U^3, \ m_4, m_5 \leq 8V^3}} (m_2 \cdots m_5)^{-2/3}.
\]
and satisfies
\[ N^{\frac{11}{3}} \ll J_1(n) \ll N^{\frac{11}{3}}. \]

**Lemma 2.2.** Let \( M \) be as in (2.3), with \( P \) and \( Q \) determined by (2.1). Then for \( N/2 \leq n \leq N \), we have
\[
\int_{\mathcal{M}} g^2(\alpha) S^2(\alpha) T^2(\alpha) e(-n\alpha) \, d\alpha = \frac{1}{2^2 \cdot 3^4} \Theta_2(n) J_2(n) + O\left( N^{\frac{11}{3}} L^{-1} \right).
\]

Here \( S_2(n) \) is singular series, which is defined as (3.9) and satisfies \( S_2(n) \gg 1 \) for \( n \equiv 0 \pmod{2} \). \( J_2(n) \) is defined as
\[
J_2(n) := \sum_{m_1 + \cdots + m_6 = n} (m_1 \cdot m_2)^{-1/2} (m_3 \cdots m_6)^{-2/3},
\]
and satisfies
\[ N^{\frac{11}{3}} \ll J_2(n) \ll N^{\frac{11}{3}}. \]

The proofs of Lemmas 2.1 and 2.2 are usually important and standard to handle enlarged major arcs in the circle method. The detailed discussion can be found in many papers (see [16,23], etc.).

In this paper, it is important to decide the constants in the \( \gg \) and \( \ll \) symbols. We suppose that \( S_1(n) > C_1 \), \( J_1(n) > C_2 N^{\frac{11}{3}} \), \( S_2(n) > C_3 \), and \( J_2(n) > C_4 N^{\frac{11}{3}} \), and the values of \( C_1 \), \( C_2 \), \( C_3 \) and \( C_4 \) will be determined in the following parts.

**Lemma 2.3.** Let \( f(\alpha) \) and \( G(\alpha) \) be as in (2.5) and (2.7). Then
\[
\int_{C(M)} \left| f(\alpha)G(\alpha) \right|^2 \, d\alpha \lesssim (12.322645 + o(1)) C_0 NL^2.
\]

If we assume Conjecture 1.4, we have
\[
\int_{C(M)} \left| f(\alpha)G(\alpha) \right|^2 \, d\alpha \lesssim (1.0500639 + o(1)) C_0 NL^2,
\]
where \( C_0 \) is defined in (1.8).

**Proof.** This lemma is actually Lemma 10 in [3]. By Lemma 2’ of [27], we can replace (41) of [3] by \( C_2 \leq 1.93657 \), and by the result of Wu [33] we can replace (32) of [3] by 7.8209. Furthermore, by Conjecture 1.4, we can replace this number by 2. Then by the proof of Lemma 9 of [3] this lemma follows. \( \square \)

**Lemma 2.4.** Let \( g(\alpha) \) and \( G(\alpha) \) be as in (2.5) and (2.7). Then
\[
\int_0^1 \left| g(\alpha)G(\alpha) \right|^4 \, d\alpha \lesssim c_1 \frac{\pi^2}{16} NL^4,
\]
where
\[ c_1 \leq \left( \frac{32^4 \cdot 101 \cdot 1.620767}{3} + \frac{8 \cdot \log^2 2}{\pi^2} \right) \cdot (1 + \varepsilon)^9 \]

with arbitrarily small positive constant \( \varepsilon > 0 \).

**Proof.** The first version of this lemma was established in Liu and Liu [15]. Then the constant was subsequently refined in [22] and [10]. □

A crucial step in bounding the contributions of minor arcs is an upper bound for the number of solutions of the equation
\[ n = p_1^3 + \cdots + p_4^3 - p_5^3 - \cdots - p_8^3, \quad 0 \leq |n| \leq N. \tag{2.12} \]

We quote the following lemma.

**Lemma 2.5.** Let \( n \equiv 0 \pmod{2} \) be an integer, and \( \rho(n) \) the number of representations of \( n \) in the form (2.12), and subject to
\[ p_1, p_2, p_5, p_6 \sim U, \quad p_3, p_4, p_7, p_8 \sim V. \]

Then we have
\[ \rho(n) \leq b U V^4 (\log N)^{-8}, \tag{2.13} \]

with \( b = 268096 \).

The inequality (2.13) is (2.7) in Ren [29] by sieve methods, and the value of \( b \) is determined in Ren [30].

It is easy to change Lemma 2.5 into the following form.

**Lemma 2.6.** Let \( S(\alpha) \) and \( T(\alpha) \) be as in (2.6). Then
\[ \int_0^1 \left| S(\alpha) T(\alpha) \right|^4 d\alpha \leq 0.35917897 N^{11}. \]

On the minor arcs, we also need estimates for the measure of the set \( \mathcal{E}_\lambda \), where
\[ \mathcal{E}_\lambda = \{ \alpha \in (0, 1]: \left| G(\alpha) \right| \geq \lambda L \}. \tag{2.14} \]

The following lemma is due to Heath-Brown and Puchta [3].

**Lemma 2.7.** Let
\[ G_h(\alpha) = \sum_{0 \leq n \leq h-1} e(\alpha 2^n), \]
and

\[ F(\xi, h) = \frac{1}{2^h} \sum_{r=0}^{2^h-1} \exp\left[ \xi \Re\left(G_h\left(\frac{r}{2^h}\right)\right)\right]. \]

Then

\[ \text{meas}(E_\lambda) \leq N^{-E(\lambda)}, \]

where

\[ E(\lambda) = \frac{\xi \lambda}{\log 2} - \frac{\log F(\xi, h)}{h \log 2} - \frac{\varepsilon}{\log 2} \]

holds true for any \( h \in \mathbb{N} \), any \( \xi > 0 \) and \( \varepsilon > 0 \).

On the minor arcs, the results on exponential sums over primes will also be applied. Lemma 2.8 is due to Vinogradov, we can see the proof in [31]. Lemma 2.9 is Theorem 3 of [7] for \( k = 2, 3 \).

**Lemma 2.8 (Vinogradov).** Let

\[ S_1(x, \alpha) = \sum_{p \sim x} (\log p)e(p\alpha) \]

where \( \alpha = a/q + \lambda \) subject to \( 1 \leq a \leq q \leq N \), \((a, q) = 1\), and \( |\lambda| \leq 1/q^2 \). Then

\[ S_1(x, \alpha) \ll (Nq^{-1/2} + N^{4/5} + N^{1/2}q^{1/2})L^c. \]

**Lemma 2.9 (Kumchev).** Let

\[ S_k(x, \alpha) = \sum_{p \sim x} (\log p)e(p^k\alpha) \]

where \( \alpha = a/q + \lambda \) subject to \( 1 \leq a \leq q \), \((a, q) = 1\), and \( |\lambda| \leq 1/q Q \), with

\[ Q = \begin{cases} x^{3/2}, & \text{if } k = 2, \\ x^{12/7}, & \text{if } k = 3. \end{cases} \]

Then

\[ S_k(x, \alpha) \ll x^{1-\varepsilon+\varepsilon} + \frac{q^\varepsilon xL^c}{\sqrt{q(1+|\lambda|x^k)}}, \]

\[ Q = \begin{cases} 1/8, & \text{if } k = 2, \\ 1/14, & \text{if } k = 3. \end{cases} \]

We give the value of \( \mathcal{S}_1(n) \) in Section 3, and complete the proof of Theorem 1.1 in Section 4. In Sections 5 and 6, we give the value of \( \mathcal{S}_2(n) \) and the proof of Theorem 1.2, respectively.
3. The value of $S_1(n)$

For $\chi \mod q$, define

$$C_1(\chi, a) := \sum_{h=1}^{q} \overline{\chi}(h)e\left(\frac{ah}{q}\right), \quad C_1(q, a) := C_1(\chi^0, a),$$  

(3.1)

$$C_2(\chi, a) := \sum_{h=1}^{q} \overline{\chi}(h)e\left(\frac{ah^2}{q}\right), \quad C_2(q, a) := C_2(\chi^0, a),$$  

(3.2)

$$C_3(\chi, a) := \sum_{h=1}^{q} \overline{\chi}(h)e\left(\frac{ah^3}{q}\right), \quad C_3(q, a) := C_3(\chi^0, a),$$  

(3.3)

where $C_1(q, a)$ is the Ramanujan sum and $C_1(q, a) = \mu(q)$, if $(a, q) = 1$. If $\chi_1, \chi_2, \ldots, \chi_6$ are characters mod $q$, then we write

$$B_1(n, q; \chi_1, \ldots, \chi_5) := \sum_{\substack{a = 1 \atop (a, q) = 1}}^{q} C_1(\chi_1, a)C_2(\chi_2, a)\ldots C_3(\chi_5, a)e\left(\frac{-an}{q}\right),$$  

(3.4)

and

$$B_2(n, q; \chi_1, \ldots, \chi_6) := \sum_{\substack{a = 1 \atop (a, q) = 1}}^{q} C_2(\chi_1, a)C_2(\chi_2, a)C_3(\chi_3, a)\ldots C_3(\chi_6, a)e\left(\frac{-an}{q}\right),$$  

(3.5)

$$B_1(n, q) := B_1(n, q; \chi^0, \ldots, \chi^0),$$  

(3.6)

and

$$B_2(n, q) := B_2(n, q; \chi^0, \ldots, \chi^0),$$  

(3.7)

$$A_1(n, q) := \frac{B_1(n, q)}{\varphi^5(q)}, \quad A_2(n, q) := \frac{B_2(n, q)}{\varphi^6(q)},$$  

(3.8)

$$\mathcal{S}_1(n) = \sum_{q=1}^{\infty} A_1(n, q), \quad \mathcal{S}_2(n) = \sum_{q=1}^{\infty} A_2(n, q).$$  

(3.9)

Hence,

$$A_1(n, q) = \frac{\mu(q)}{\varphi^5(q)} \sum_{\substack{a = 1 \atop (a, q) = 1}}^{q} C_3^4(q, a)e\left(\frac{-an}{q}\right),$$  

(3.10)

and for $k \geq 2$,

$$A_1(n, p^k) = 0.$$
Since \( A_1(n, q) \) is multiplicative, we have

\[
\mathcal{S}_1(n) = \prod_{p \geq 2} (1 + A_1(n, p)).
\]  

(3.11)

Let \( A_1(n, q) \) be defined as in (3.10). We will compute \( A_1(n, p) \) for different \( p \) in (3.11).

For \( p = 2 \), one has

\[
1 + A_1(n, 2) = \begin{cases} 
0, & n \equiv 0 \, (\text{mod} \, 2), \\
2, & n \not\equiv 0 \, (\text{mod} \, 2), 
\end{cases}
\]  

(3.12)

by direct calculation.

For \( p = 3 \),

\[
C_3(3, a) = \sum_{h=1}^{2} e\left(\frac{ah}{3}\right) = e\left(\frac{a}{3}\right) + e\left(-\frac{a}{3}\right) = 2\cos\frac{2\pi a}{3},
\]

so,

\[
A_1(n, 3) = -\frac{1}{\varphi^5(3)} \sum_{a=1}^{2} \left(2\cos\frac{2\pi a}{3}\right)^4 e\left(-\frac{an}{3}\right)
\]

\[
= -\frac{1}{2^5} \left(e\left(-\frac{n}{3}\right) + e\left(-\frac{2n}{3}\right)\right) = -\frac{1}{2^4} \cos\frac{2\pi n}{3}.
\]

Thus,

\[
A_1(n, 3) = \begin{cases} 
-1/2^4, & n \equiv 0 \, (\text{mod} \, 3), \\
1/2^5, & n \not\equiv 0 \, (\text{mod} \, 3), 
\end{cases}
\]  

(3.13)

we can get

\[
1 + A_1(n, 3) > 1 - 1/2^4 = 0.9375.
\]  

(3.14)

For \( p \geq 5 \), if \( p \equiv 2 \, (\text{mod} \, 3) \) and \( (p, a) = 1 \), we have \( C_3(p, a) = -1 \). So,

\[
B_1(n, p) = \sum_{a=1}^{p-1} C_3(p, a) e\left(-\frac{an}{p}\right) = -\sum_{a=1}^{p-1} e\left(-\frac{an}{p}\right) = \begin{cases} 
-(p - 1), & p \mid n, \\
1, & p \nmid n.
\end{cases}
\]

Thus,

\[
1 + A_1(n, p) = 1 + \frac{B_1(n, p)}{\varphi^5(p)} = \begin{cases} 
1 - \frac{1}{(p-1)^4}, & p \mid n, \\
1 + \frac{1}{(p-1)^4}, & p \nmid n.
\end{cases}
\]  

(3.15)

For \( p \equiv 1 \, (\text{mod} \, 3) \), when \( p \geq 5 \). First, when \( p = 7 \),

\[
C_3(7, a) = \sum_{h=1}^{6} e\left(\frac{ah^3}{7}\right) = 3\left(e\left(\frac{a}{7}\right) + e\left(-\frac{a}{7}\right)\right) = 6\cos\frac{2\pi a}{7}.
\]
so,

\[ A_1(n, 7) = -\frac{1}{\varphi^5(7)} \sum_{a=1}^{6} \left( 6 \cos \frac{2\pi a}{7} \right)^4 e\left(-\frac{an}{7}\right) \]

\[ = -\frac{1}{6} \sum_{a=1}^{6} \left( \cos \frac{2\pi a}{7} \right)^4 \cos \frac{2\pi an}{7}. \]

For different \( n \), \( A(n, 7) \) will take 4 different values, and we can get that

\[ A_1(n, 7) > -0.27083334. \]

Thus we can get

\[ 1 + A_1(n, 7) > 1 - 0.27083334 = 0.72916666. \] (3.16)

For \( p \geq 13 \) and \( p \equiv 1 \pmod{3} \), noting that the elementary estimate of \( C_3(p, a) \),

\[ |C_3(p, a)| \leq 2\sqrt{p} + 1, \]

we can get that

\[ |B_1(n, p)| = \left| \sum_{a=1}^{p-1} C_4^4(p, a)e\left(-\frac{an}{p}\right) \right| \leq (2\sqrt{p} + 1)^4(p - 1). \]

Thus

\[ 1 + A_1(n, p) > 1 - \frac{(2\sqrt{p} + 1)^4}{(p - 1)^4}. \] (3.17)

Thus we have

\[ \prod_{p \geq 5} \{1 + A_1(n, p)\} \geq \{1 + A_1(n, 7)\} \prod_{p \equiv 1 \pmod{3}} \left( 1 - \frac{(2\sqrt{p} + 1)^4}{(p - 1)^4} \right) \]

\[ \times \prod_{p \equiv 2 \pmod{3}} \left( 1 - \frac{1}{(p - 1)^4} \right) \prod_{p \equiv 5} \left( 1 + \frac{1}{(p - 1)^5} \right) \]

\[ \geq \{1 + A(n, 7)\} \prod_{p \equiv 1 \pmod{3}} \left( 1 - \frac{(2\sqrt{p} + 1)^4}{(p - 1)^4} \right) \]

\[ \times \prod_{p \equiv 2 \pmod{3}} \left( 1 - \frac{1}{(p - 1)^4} \right) \cdot \prod_{p \equiv 5 \pmod{3}} \left( 1 - \frac{1}{(p - 1)^5} \right). \] (3.18)
To estimate the products above, we apply the elementary inequality
\[(1 + x)^a < 1 + ax - \frac{a(a - 1)}{2}x^2 \text{ if } a > 2, \quad -1 < x < 0.\]

For \(p \geq 1138\) and \(p \equiv 1 \pmod{3}\), we have
\[1 - \frac{(2\sqrt{p} + 1)^4}{(p-1)^4} \geq \left(1 - \frac{1}{(p-1)^2}\right)^{17}.\]

Thus we have
\[
\prod_{p \geq 5} \left\{1 + A_1(n, p)\right\} \geq \left\{1 + A_1(n, 7)\right\} \prod_{13 \leq p \leq 1137 \atop p \equiv 1 \pmod{3}} \left(1 - \frac{1}{(p-1)^2}\right)^{17} \prod_{p \equiv 5 \pmod{3}} \left(1 - \frac{1}{(p-1)^4}\right)
\]
\[\times \prod_{p \equiv 1 \pmod{3}} \left\{1 - \frac{1}{(p-1)^2}\right\}^{-17} \prod_{13 \leq p \leq 1137 \atop p \equiv 1 \pmod{3}} \left(1 - \frac{(2\sqrt{p} + 1)^4}{(p-1)^4}\right) \left(1 - \frac{1}{(p-1)^2}\right)^{-17}
\]
\[\times \prod_{5 \leq p \leq 1137 \atop p \equiv 2 \pmod{3}} \left\{1 - \frac{1}{(p-1)^4}\right\} \left(1 - \frac{1}{(p-1)^2}\right)^{-17}
\]
\[\times \prod_{p \equiv 3 \pmod{3}} \left(1 - \frac{1}{(p-1)^2}\right)^{17}
\]
\[
\geq 0.72916666 \times \left(\frac{48}{35}\right)^{17} \times 0.829697661 \times 4.225188575 \times 0.66016^{17}
\]
\[
\geq 0.4715997367, \quad (3.19)
\]

where we have used \(\prod_{p \equiv 3}(1 - (p-1)^{-2}) = 0.66016\ldots\) (see [1]).

This, in combination with (3.11) (3.12) and (3.14), ensures that we can take
\[S_1(n) > 0.8842495063, \quad (3.20)\]
when \(n \equiv 1 \pmod{2}\).

### 4. Proof of Theorem 1.1

We need the following four lemmas.
Lemma 4.1. Let $\Xi(N, k) = \{(1 - \delta)N \leq n \leq N: n = N - 2^{\nu_1} - \cdots - 2^{\nu_k}\}$, with $k \geq 2$. Then for $N \equiv 1 \pmod{2}$,
\[
\sum_{\substack{n \in \Xi(N, k) \\ n \equiv 1 \pmod{2}}} 1 \geq (1 - \varepsilon)(\log_2 N)^k.
\]

**Proof.** The proof of Lemma 4.1 is straightforward, so we omit the detail. □

Lemma 4.2. For $(1 - \delta)N \leq n \leq N$, we have $J_1(n) > C_2N^{11/9}$, with
\[C_2 = 2.7335671.\]

**Proof.** The domain of the second sum $J_1(n)$ can be written as
\[\mathcal{D} = \{(m_1, \ldots, m_5): m_1 \leq N, U^3 < m_2, m_3 \leq 8U^3, V^3 < m_4, m_5 \leq 8V^3\},\]
with $m_1 = n - m_2 - \cdots - m_5$. We can deduce from $(1 - \delta)N < n \leq N$ and (2.4) that
\[1 \leq m_1 = n - m_2 - \cdots - m_5 \leq N.\]
The sum $J_1(n)$ is
\[
J_1(n) \geq \sum_{U^3 < m_2, m_3 \leq 8U^3} m_2^{-2/3} \sum_{V^3 < m_4, m_5 \leq 8V^3} m_3^{-2/3} \sum_{V^3 < m_4} m_4^{-2/3} \sum_{V^3 < m_5} m_5^{-2/3} \geq 3^4 \cdot 2^2 V^2 + o(2^2 V^2) \geq 2.7335671N^{11/9},
\]
where in the third inequality, we used the fact that $\sum_{W^3 < m \leq 8W^3} m^{-2/3}$ is well approximated by the corresponding integral $\int_{W^3}^{8W^3} y^{-2/3} dy$. (See also the discussion of §1.5 in [6].) So, we get that $C_2 = 2.7335671$. □

Lemma 4.3. Let $C(M)$ be as in (2.3), with $P$ and $Q$ determined by (2.1), and $f(\alpha)$ and $g(\alpha)$ be as in (2.5), $S(\alpha)$ be as in (2.6). We have
\[
\max_{\alpha \in C(M)} |f(\alpha)| \ll N^{1-1/18+\varepsilon},
\]
\[
\max_{\alpha \in C(M)} |g(\alpha)| \ll N^{1/2-1/18+\varepsilon},
\]
\[
\max_{\alpha \in C(M)} |S(\alpha)| \ll N^{1/3-1/42+\varepsilon}.
\]

**Proof.** We give only the proof of the estimate for $S(\alpha)$, the other two bounds can be proved in similar way using Lemmas 2.8 and 2.9. By Dirichlet’s lemma on rational approximations, each real number
\( \alpha \in \mathcal{C}(\mathcal{M}) \) can be written as \( \alpha = a/q + \lambda \), \((a,q) = 1\) with

\[
1 \leq q \leq Q_0 = N^{4/7}, \quad |\lambda| \leq 1/qQ_0.
\]

If \( q \leq P = N^{1/9 - 2\varepsilon} \), since \( \alpha \in \mathcal{C}(\mathcal{M}) \), we have \(|\lambda| > 1/qQ\); otherwise \( q > P \). In either case, we have

\[
\sqrt{q(1 + |\lambda|U^3)} > \min\left( P^{1/2}, \left( \frac{U^3}{Q} \right)^{1/2} \right) = N^{1/18 - \varepsilon}.
\]

By Lemma 2.9, we have

\[
\max_{\alpha \in \mathcal{C}(\mathcal{M})} |S(\alpha)| \ll N^{1/3 - 1/42 + \varepsilon}.
\]

In order to apply Lemma 2.6 in this paper, we need to find an optimal \( \lambda \) such that \( E(\lambda) > 113/126 \). Thus we have to compute

\[
F(\xi, h) = \frac{1}{2^h} \sum_{r=0}^{2^h-1} \exp \left[ \xi \Re \left( G_h \left( \frac{r}{2^h} \right) \right) \right].
\]

and optimize for \( \xi \) and \( h \). We can take \( \xi = 1.56 \), \( h = 23 \) in Lemma 2.7 to get

**Lemma 4.4.** Let \( E(\lambda) \) be as in Lemma 2.7. Then

\[
E(0.961917) > \frac{113}{126} + 10^{-10}.
\]

Now we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let \( N \equiv 1 \pmod{2} \), \( \mathcal{E}_k \) be as in (2.14) and \( \mathcal{M} \) as in (2.3), with \( P \) and \( Q \) determined by (2.1). Then (2.9) has

\[
R_1(N) = \int_0^1 f(\alpha)S^2(\alpha)T^2(\alpha)G^k(\alpha)e(-N\alpha) \, d\alpha
\]

\[
= \int_{\mathcal{M}} + \int_{\mathcal{C}(\mathcal{M}) \cap \mathcal{E}_k} + \int_{\mathcal{C}(\mathcal{M}) \cap \mathcal{C}(\mathcal{E}_k)}.
\]  

Introducing the notation \( \mathcal{S}(N,k) \) and then applying Lemma 4.1, we see that the first integral on the right-hand side of (4.1) is

\[
= \sum_{n \in \mathcal{S}(N,k)} \int_{\mathcal{M}} f(\alpha)S^2(\alpha)T^2(\alpha)e(-n\alpha) \, d\alpha
\]

\[
= \frac{1}{3^4} \sum_{n \in \mathcal{S}(N,k)} \mathcal{S}_1(n)f_1(n) + O\left(N^{11/9}L^{k-1}\right)
\]

\[
\geq \frac{1}{3^4} C_1 C_2 N^{11/9} \sum_{n \in \mathcal{S}(N,k)} 1 + O\left(N^{11/9}L^{k-1}\right)
\]
\[
\geq \frac{1}{34} C_1 C_2 (1 - \varepsilon) N^{11/9} L^k, \tag{4.2}
\]

where in the last two inequalities we have used Lemmas 4.1, 4.2 and 2.1.

For the second integral in (4.1), using the untrivial estimates for \(f(\alpha)\) and \(S(\alpha)\) (Lemma 4.3) and trivial estimates for \(T(\alpha)\) and \(G(\alpha)\), we have

\[
\int_{C(M) \cap C(E)} \ll N^{-E(\lambda)} N^{11/9} + \frac{1}{L^k} \ll N^{11/9} L^{-k}. \tag{4.3}
\]

On using the definition of \(E_\lambda\) and Lemmas 2.3 and 2.6, the last integral in (4.1) can be estimated as

\[
\int_{C(M)} \leq (\lambda L)^{k-1} \left( \int_{C(M)} |f(\alpha)G(\alpha)|^2 d\alpha \right)^{1/2} \left( \int_0^1 |S(\alpha)T(\alpha)|^4 d\alpha \right)^{1/2} \\
\leq (\lambda L)^{k-1} 1.7093540374 N^{11/9} L \\
\leq \lambda^{k-1} 1.7093540374 N^{11/9} L^k. \tag{4.4}
\]

Combining this with (4.2), (4.3) and (4.1), we get

\[
R_1(N) \geq (0.029841424 - 1.7093540374 \lambda^{k-1})(1 - \varepsilon) N^{11/9} L^k.
\]

When \(k \geq 106\) and \(\varepsilon = 10^{-10}\), we have

\[
R_1(N) > 0,
\]

for sufficiently large odd integer \(N\). This completes the proof of Theorem 1.1. Assuming Conjecture 1.4,

\[
R_1(N) \geq (0.029841424 - 0.498985490 \lambda^{k-1})(1 - \varepsilon) N^{11/9} L^k.
\]

When \(k \geq 74\) and \(\varepsilon = 10^{-10}\), we have

\[
R_1(N) > 0,
\]

for sufficiently large odd integer \(N\). This completes the proof of Theorem 1.5. \(\square\)

5. The value of \(S_2(n)\)

As the discussion in Section 3, we have

\[
A_2(n, q) = \frac{1}{\varphi(q)} \sum_{\substack{a=1 \to q \equiv \varphi(q) \equiv 1 \mod q}} C_2(q, a) C_3(q, a) e\left(-\frac{an}{q}\right), \tag{5.1}
\]

and for \(k \geq 2\),

\[
A_2(n, p^k) = 0.
\]
by Lemma 8.3 in [4]. Since \( A_2(n, q) \) is multiplicative, we have

\[
\mathcal{G}_2(n) = \prod_{p \geq 2} \left( 1 + A_2(n, p) \right).
\] (5.2)

We will compute \( A(n, p) \) for different \( p \) in (5.2).

For \( p = 2 \), one has

\[
1 + A_2(n, 2) = \begin{cases} 
2, & n \equiv 0 \pmod{2}, \\
0, & n \not\equiv 0 \pmod{2},
\end{cases}
\] (5.3)

by direct calculation.

For \( p = 3 \),

\[
C_2(3, a) = \sum_{h=1}^{2} e\left( \frac{ah^2}{3} \right) = e\left( \frac{a}{3} \right) + e\left( \frac{4a}{3} \right) = 2e\left( \frac{a}{3} \right),
\]

\[
C_3(3, a) = \sum_{h=1}^{2} e\left( \frac{ah^3}{3} \right) = e\left( \frac{a}{3} \right) + e\left( -\frac{a}{3} \right) = 2 \cos \frac{2\pi a}{3},
\]

so,

\[
A_2(n, 3) = \frac{1}{\phi^6(3)} \sum_{a=1}^{2} \left( 2e\left( \frac{a}{3} \right) \right)^2 \left( 2 \cos \frac{2\pi a}{3} \right)^4 e\left( -\frac{an}{3} \right)
\]

\[
= \frac{1}{2^4} \left( e\left( \frac{2}{3} - \frac{n}{3} \right) + e\left( \frac{4}{3} - \frac{2n}{3} \right) \right).
\]

Thus,

\[
A_2(n, 3) = \begin{cases} 
1/2^3, & n \equiv 2 \pmod{3}, \\
-1/2^4, & n \not\equiv 2 \pmod{3},
\end{cases}
\]

we can get

\[
1 + A_2(n, 3) > 1 - 1/2^4 = 0.9375.
\] (5.4)

For \( p \geq 5 \), if \( p \equiv 2 \pmod{3} \) and \( (p, a) = 1 \), we have \( C_3(p, a) = -1 \). So,

\[
B_2(n, p) = \sum_{a=1}^{p-1} C_2^2(p, a) C_3^4(p, a) e\left( -\frac{an}{p} \right)
\]

\[
= \sum_{a=1}^{p-1} C_2^2(p, a) e\left( -\frac{an}{p} \right).
\] (5.5)

We will also use the notation \( S(q, a) \) introduced by

\[
S(q, a) = \sum_{h=1}^{q} e\left( \frac{ah^2}{q} \right).
\]
By Theorem 7.5.4 in [5], we have for $p \geq 5$,

$$C_2(p, a) = S(p, a) - 1 = \chi(a)S(p, 1) - 1, \quad (5.6)$$

where $\chi$ is Legendre symbol $(\frac{a}{p})$. Inserting this into (5.5), one sees that

$$B_2(n, p) = S^2(p, 1)c_p(-n) - 2S(p, 1)G(\chi, -n) + c_p(-n),$$

where

$$G(\chi, n) = \sum_{m=1}^{q} \chi(m)e\left(\frac{nm}{q}\right)$$

and $c_q(n)$ is the Ramanujan sum. Using the well-known formulae (see Theorems 7.5.5 and 7.58 in [5])

$$S(p, 1) = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \pmod{4}, \\ i\sqrt{p}, & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

and

$$|G(\chi, n)| = \begin{cases} \sqrt{p}, & \text{if } p \nmid n, \\ 0, & \text{if } p \mid n, \end{cases} \quad c_p(n) = \begin{cases} -1, & \text{if } p \nmid n, \\ p - 1, & \text{if } p \mid n, \end{cases}$$

one obtains

$$B_2(n, p) \geq \begin{cases} -3p - 1, & \text{if } p \nmid n, p \equiv 1 \pmod{4}, \\ -p - 1, & \text{if } p \nmid n, p \equiv 3 \pmod{4}, \\ p^2 - 1, & \text{if } p \mid n, p \equiv 1 \pmod{4}, \\ -p^2 + 2p - 1, & \text{if } p \mid n, p \equiv 3 \pmod{4}. \end{cases}$$

Thus,

$$1 + A_2(n, p) = 1 + \frac{B_2(n, p)}{\varphi^6(p)} \geq 1 - \frac{1}{(p - 1)^4}. \quad (5.7)$$

Let $p \equiv 1 \pmod{3}$, when $p \geq 5$. First, when $p = 7$,

$$C_2(7, a) = \sum_{h=1}^{6} e\left(\frac{ah^2}{7}\right) = 2\left(e\left(\frac{a}{7}\right) + e\left(\frac{2a}{7}\right) + e\left(\frac{4a}{7}\right)\right),$$

$$C_3(7, a) = \sum_{h=1}^{6} e\left(\frac{ah^3}{7}\right) = 3\left(e\left(\frac{a}{7}\right) + e\left(-\frac{a}{7}\right)\right) = 6\cos\frac{2\pi a}{7},$$

so,

$$A_2(n, 7) = \frac{1}{\varphi^6(7)} \sum_{a=1}^{6} \left[2\left(e\left(\frac{a}{7}\right) + e\left(\frac{2a}{7}\right) + e\left(\frac{4a}{7}\right)\right)\right]^2 \times \left(6\cos\frac{2\pi a}{7}\right)^4 e\left(-\frac{an}{7}\right).$$
For different \( n \), \( A_2(n, 7) \) will take 7 different values, and we can get that

\[ A_2(n, 7) > -0.31944445. \]

Thus we can get

\[ 1 + A_2(n, 7) > 1 - 0.31944445 = 0.68055555. \]  \hspace{1cm} (5.8)

For \( p \geq 13 \) and \( p \equiv 1 \) (mod 3), noting that the elementary estimate of \( C_3(p, a) \),

\[ |C_3(p, a)| \leq 2\sqrt{p} + 1, \]

and using (5.6), one has

\[ |C_2^2(p, a)| \leq (\sqrt{p} + 1)^2. \]

We can get that

\[ |B_2(n, p)| = \left| \sum_{a=1}^{p-1} C_2^2(p, a)C_3^4(p, a)e\left( -\frac{an}{p} \right) \right| \leq (\sqrt{p} + 1)^2(2\sqrt{p} + 1)^4(p - 1). \]

Thus

\[ 1 + A_2(n, p) > 1 - \frac{(\sqrt{p} + 1)^2(2\sqrt{p} + 1)^4}{(p - 1)^5}. \]  \hspace{1cm} (5.9)

Thus we have

\[ \prod_{p \geq 5} \{1 + A_2(n, p)\} \geq \{1 + A_2(n, 7)\} \prod_{p \equiv 1 \ (\text{mod 3})}^{p \geq 13} \left( 1 - \frac{(\sqrt{p} + 1)^2(2\sqrt{p} + 1)^4}{(p - 1)^5} \right) \]

\[ \times \prod_{p \equiv 2 \ (\text{mod 3})}^{p \geq 5} \left( 1 - \frac{1}{(p - 1)^4} \right). \]  \hspace{1cm} (5.10)

To estimate the products above, we apply the elementary inequality

\[ (1 + x)^a < 1 + ax - \frac{a(a - 1)}{2} x^2 \quad \text{if} \ a > 2, \ -1 < x < 0. \]

For \( p \geq 4404 \) and \( p \equiv 1 \) (mod 3), we have

\[ 1 - \frac{(\sqrt{p} + 1)^2(2\sqrt{p} + 1)^4}{(p - 1)^5} \geq \left( 1 - \frac{1}{(p - 1)^2} \right)^{17}. \]
Thus we have

\[
\prod_{p \geq 5} \left\{1 + A_2(n, p)\right\} \geq \left\{1 + A_2(n, 7)\right\} \prod_{13 \leq p \leq 4403, \atop p \equiv 1 \pmod{3}} \left(1 - \frac{(\sqrt{p} + 1)^2(2\sqrt{p} + 1)^4}{(p - 1)^5}\right) \prod_{p \equiv 5 \pmod{3}} \left(1 - \frac{1}{(p - 1)^4}\right) \prod_{p \equiv 1 \pmod{3}} \left(1 - \frac{1}{(p - 1)^2}\right)^{17} \\
= \left\{1 + A_2(n, 7)\right\} \prod_{p = 3, 7} \left(1 - \frac{1}{(p - 1)^2}\right)^{-17} \prod_{13 \leq p \leq 4403, \atop p \equiv 1 \pmod{3}} \left\{\left(1 - \frac{(\sqrt{p} + 1)^2(2\sqrt{p} + 1)^4}{(p - 1)^5}\right)\left(1 - \frac{1}{(p - 1)^2}\right)^{-17}\right\} \\
\times \prod_{5 \leq p \leq 4403, \atop p \equiv 2 \pmod{3}} \left\{\left(1 - \frac{1}{(p - 1)^4}\right)\left(1 - \frac{1}{(p - 1)^2}\right)^{-17}\right\} \\
\times \prod_{p \geq 3} \left(1 - \frac{1}{(p - 1)^2}\right)^{17} \\
\geq 0.68055555 \times \left(\frac{48}{35}\right)^{17} \times 0.595556242 \times 4.228306878 \times 0.66016^{17} \\
\geq 0.3161794567, \quad (5.11)
\]

where we have used \(\prod_{p \geq 3} (1 - (p - 1)^{-2}) = 0.66016\ldots\) (see [1]).

This, in combination with (5.1), (5.2) and (5.3), ensures that we can take

\[\Theta_2(n) > 0.592836481, \quad (5.12)\]

when \(n \equiv 0 \pmod{2}\).

6. Proof of Theorem 1.2

We need the following three lemmas.

**Lemma 6.1.** Let \(\Xi(N, k) = \{(1 - \delta)N \leq n \leq N: n = N - 2^{n_1} - \cdots - 2^{n_k}\},\) with \(k \geq 2\). Then for \(N \equiv 0 \pmod{2}\),

\[
\sum_{\substack{n \in \Xi(N, k) \\
\text{mod } 0 \pmod{2}}} 1 \geq (1 - \varepsilon)(\log_2 N)^k.
\]

**Proof.** The proof of Lemma 6.1 is straightforward, so we omit the detail. \(\square\)

**Lemma 6.2.** For \((1 - \delta)N \leq n \leq N\), we have \(J_2(n) > C_4N^{11/9}\), with

\[C_4 = 1.923892477.\]
Proof. The domain of the second sum $J_2(n)$ can be written as

$$\mathcal{D} = \{(m_1, \ldots, m_6): m_1, m_2 \leq N, U^3 < m_3, m_4 \leq 8U^3, V^3 < m_5, m_6 \leq 8V^3\}.$$  

with $m_1 = n - m_2 - \cdots - m_6$. To bound this sum from below, if we define the set

$$\mathcal{D}^* = \{(m_2, \ldots, m_6): m_1 \leq 0.9N, m_2 \leq 0.25N, U^3 < m_3, m_4 \leq 6U^3, V^3 < m_5, m_6 \leq 8V^3\},$$

with the numbers $(0.9, 0.25, 6, 8)$ in $\mathcal{D}^*$ are determined by a PC for a larger number of $C_4$. We can deduce from $(1 - \delta)N < n \leq N$ and (2.4) that

$$1 \leq m_1 = n - m_2 - \cdots - m_6 \leq 0.9N.$$  

Thus $\mathcal{D}^*$ is a subset of $\mathcal{D}$, and consequently, the sum $J_2(n)$ is

$$J_2(n) \geq \sum_{m_2 \leq 0.25N, U^3 < m_3, m_4 \leq 6U^3, V^3 < m_5, m_6 \leq 8V^3} (m_1m_2)^{-1/2}(m_3 \ldots m_6)^{-2/3}$$

$$\geq (0.9N)^{-1/2} \sum_{1 < m_2 \leq 0.25N} m_2^{-1/2}$$

$$\times \sum_{U^3 < m_3 \leq 6U^3} m_3^{-2/3} \sum_{U^3 < m_4 \leq 6U^3} m_4^{-2/3} \sum_{V^3 < m_5 \leq 8V^3} m_5^{-2/3} \sum_{V^3 < m_6 \leq 8V^3} m_6^{-2/3}$$

$$\geq (0.9)^{-1/2} \cdot 2 \cdot (0.25)^{1/2} \cdot 3^4 (6^{1/3} - 1)^2 U^2 V^2 + o(U^2 V^2)$$

$$\geq 1.923892477N^{11/9}.$$  

We also used the fact that $\sum_{W^3 < m \leq 8W^3} m^{-2/3}$ is well approximated by the corresponding integral $\int_{W^3}^{8W^3} y^{-2/3} dy$. So, we get that $C_4 = 1.923892477$. □

In order to apply Lemma 2.7 in this paper, we need to find an optimal $\lambda$ such that $E(\lambda) > 53/63$. Thus we have to compute

$$F(\xi, h) = \frac{1}{2^h} \sum_{r=0}^{2^h-1} \exp\left[\xi \Re\left(G_h\left(\frac{r}{2^h}\right)\right)\right],$$

and optimize for $\xi$ and $h$. We can take $\xi = 1.40$, $h = 23$ in Lemma 2.7 to get

Lemma 6.3. Let $E(\lambda)$ be as in Lemma 2.7. Then

$$E(0.935746) > \frac{53}{63} + 10^{-10}.$$  

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $N \equiv 0 \pmod{2}$, $\mathcal{E}_\lambda$ be as in (2.14) and $\mathcal{M}$ as in (2.3), with $P$ and $Q$ determined by (2.1). Then (2.10) has
\[ R_2(N) = \int_0^1 g^2(\alpha) S^2(\alpha) T^2(\alpha) C^k(\alpha) e(-N\alpha) \, d\alpha \]
\[ = \int_{\mathcal{M}} + \int_{\mathcal{C}(\mathcal{M}) \cap \mathcal{E}_{\lambda}} + \int_{\mathcal{C}(\mathcal{M}) \cap \mathcal{C}(\mathcal{E}_{\lambda})} . \]  
(6.1)

Introducing the notation \( \Xi(N, k) \) and then applying Lemma 6.1, we see that the first integral on the right-hand side of (6.1) is
\[ = \sum_{n \in \Xi(N, k)} \int_{\mathcal{M}} g^2(\alpha) S^2(\alpha) T^2(\alpha) e(-n\alpha) \, d\alpha \]
\[ = \frac{1}{2^{2} \cdot 3^{4}} \sum_{n \in \Xi(N, k)} \mathcal{G}_2(n) J_2(n) + O(N^{11/9} L^{k-1}) \]
\[ \geq \frac{1}{2^{2} \cdot 3^{4}} C_3 C_4 N^{11/9} \sum_{n \in \Xi(N, k)} 1 + O(N^{11/9} L^{k-1}) \]
\[ \geq \frac{1}{2^{2} \cdot 3^{4}} C_3 C_4 (1 - \varepsilon) N^{11/9} L^k , \]  
(6.2)

where in the last two inequalities we have used Lemmas 6.1, 6.2 and 2.2.

For the second integral in (6.1), using the untrivial estimates for \( g(\alpha) \) and \( S(\alpha) \) (Lemma 4.3) and trivial estimates for \( T(\alpha) \) and \( G(\alpha) \), we have
\[ \int_{\mathcal{C}(\mathcal{M}) \cap \mathcal{C}(\mathcal{E}_{\lambda})} \ll N^{-E(\lambda)} N^{53/57} N^{11} \ll N^{11} L^{k-1} . \]  
(6.3)

On using the definition of \( \mathcal{E}_{\lambda} \) and Lemmas 2.4 and 2.6, the last integral in (6.1) can be estimated as
\[ \int_{\mathcal{C}(\mathcal{M}) \cap \mathcal{C}(\mathcal{E}_{\lambda})} \leq (\lambda L)^{k-2} \left( \int_0^1 |g(\alpha) G(\alpha)|^4 \, d\alpha \right)^{1/2} \left( \int_0^1 |S(\alpha) T(\alpha)|^4 \, d\alpha \right)^{1/2} \]
\[ \leq (\lambda L)^{k-2} 3560.4561558 (1 + \varepsilon)^{5} N^{11/9} L^2 \]
\[ \leq \lambda^{k-2} 3560.4561558 (1 + \varepsilon)^{5} N^{11/9} L^k . \]  
(6.4)

Combining this with (6.2), (6.3) and (6.1), we get
\[ R_2(N) \geq \left( 0.0035202273 - 3560.4561558 (1 + \varepsilon)^{5} \lambda^{k-2} \right) (1 - \varepsilon) N^{11/9} L^k. \]

When \( k \geq 211 \) and \( \varepsilon = 10^{-10} \), we have
\[ R_2(N) > 0 , \]
for sufficiently large even integer \( N \). This completes the proof of Theorem 1.2. \( \square \)
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References