NOTES ON TOLERANCE RELATIONS OF LATTICES:
A CONJECTURE OF R.N. McKenzie*

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1. Introduction

A tolerance relation $T$ on a lattice $L$ is defined as a reflexive and symmetric binary relation having the substitution property. A maximal $T$-connected subset of $L$ is a $T$-block. The quotient lattice $L/T$ consists of the $T$-blocks with the natural ordering.

If $V$ and $W$ are lattice varieties, their product $V \circ W$ consists of all lattices $L$ for which there is a congruence relation $\Theta$ satisfying: (i) all $\Theta$-classes of $L$ are in $V$; (ii) $L/\Theta$ is in $W$. In general, $V \circ W$ is not a variety; however, $H(V \circ W)$ (the class of all homomorphic images of members of $V \circ W$) always is.

If $L$ is in $V \circ W$ (established by $\Theta$), then $L/\Phi$ is a typical member of $H(V \circ W)$. On $L/\Phi$, $\Theta/\Phi$ is a tolerance relation. The following theorem was conjectured by R.N. McKenzie: a lattice $K$ belongs to the variety generated by $V \circ W$ iff there is a tolerance relation $T$ on $K$ satisfying: (i) all $T$-classes of $L$ are in $V$; (ii) $L/T$ is in $W$.

In this paper we disprove this conjecture:

**Theorem.** The lattice $F$ of Fig. 1 is in $H(M_3 \circ D)$. However, there is no $A$ in $M_3 \circ D$ with congruence $\Theta$ establishing this such that $F$ can be represented as $A/\Phi$ and $T = \Theta/\Phi$ satisfies (i) all $T$-classes of $A$ are in $M_3$; (ii) $A/T$ is in $D$.

In this theorem, $M_3$ is the variety generated by the modular lattice $M_3$ and $D$ is the variety of all distributive lattices. The Theorem holds for some varieties other than $M_3$, see Section 5.

In Section 2 we introduce a new lattice construction, called **hinged-product** which

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we shall utilize to construct an interesting lattice. Since we hope that this construction will find other applications as well, we develop it in some detail.

In Section 3 we introduce the lattice $F$ of Fig. 1, and using a hinged-product (see Figs 2 and 3), we show that $F$ is in $\mathbf{M}_3 \circ \mathbf{D}$.

In Section 4 we investigate the tolerance relations of $F$; they form a lattice shown in Fig. 6.

Finally, in Section 5, we prove the Theorem.

For the basic concepts and unexplained notations, the reader is referred to [1].

2. Hinged-products

We start with a definition:

**Definition 1.** We are given a family, $L_i$, $i \in I$, of lattices; in each lattice $L_i$, we are given three elements, the hinge: $l_i \leq m_i \leq u_i$. The *hinged-product* $H = H(L_i, l_i, m_i, u_i)$ consists of the following subsets of the direct product of the $L_i$, $i \in I$:

(i) the ideal $l(H)$, the direct product of $(l_i)$, $i \in I$;
(ii) the dual ideal $u(H)$, the direct product of $(u_i)$, $i \in I$;
(iii) for every $i \in I$, the *ith frame*, $f_i(H)$, consisting of all elements whose $j$th coordinate is $m_j$ for all $j \neq i$.

$H$ is partially ordered componentwise.

Observe that these sets may not be disjoint: for instance, if $m_i = l_i$, then $l(H) \cap f_i(H)$ is non-empty; if $m_i = u_i$, then $u(H) \cap f_i(H)$ is non-empty. The element $\mu$ whose $i$th coordinate is $u_i$ for all $i \in I$ belongs to all $f_j(H)$, $j \in I$. 
Lemma 2. $H$ is a lattice.

Proof. Let $\alpha, \beta \in H$. Then the componentwise join, $\alpha \lor \beta$, is always in $H$ (hence it is also the join in $H$) with one possible exception: $\alpha \in f_i(H)$ and $\beta \in f_j(H)$, $i \neq j$. Let $\alpha = \langle a_k \rangle$, $\beta = \langle b_k \rangle$; now if $a_i \lor m_i > m_i$ and $b_j \lor m_j > m_j$, then $\alpha \lor \beta$ has no upper bound in $l(H)$, in any $f_k(H)$, and it has a least upper bound, namely, $\alpha \lor \beta \lor \mu$, in $u(H)$. Hence, $\alpha$ and $\beta$ have a least upper bound in $H$, namely, $\alpha \lor \beta \lor \mu$. We can argue the meets dually. This completes the proof of Lemma 2. $\Box$

We shall continue to denote componentwise join and meet with $\lor$ and $\land$; the join and meet in $H$ will be denoted by $\vee$ and $\wedge$, respectively.

It is not very easy to visualize $H$. The $i$th frame, $f_i(H)$, is isomorphic to $L_i$. The $f_i(H)$ are glued together at the hinges: $l_i, m_i, u_i$. The glued frames are completed into a lattice by $l(H)$ and $u(H)$. The example of Section 3 may help illuminate this point.

If we have homomorphisms $\phi_i : L_i \to L'_i$, then under certain conditions these homomorphisms have a joint extension from the hinged-product $H$ to the hinged-product $H'$:

Lemma 3. Let $H = H(L_i, l_i, m_i, u_i)$ and $H' = H(L'_i, l'_i, m'_i, u'_i)$ be hinged-products with the same index set $I$. Let $\phi_i : L_i \to L'_i$ be homomorphisms for $i \in I$. Let us assume that $\phi_i(l_i) = l'_i$, $\phi_i(m_i) = m'_i$, and $\phi_i(u_i) = u'_i$. If (i) or (ii) below holds, then the restriction $\phi$ of the product of the homomorphisms $\phi_i$, $i \in I$, to $H$ is a homomorphism of $H$ into $H'$.

(i) For all $i \in I$, $x_i \lor m_i > m_i$ implies that $\phi(x_i) \lor m'_i > m'_i$; and the dual condition for $\land$.

(ii) For all $i \in I$, $\phi(l_i) = \phi(u_i)$.

Proof. Under the first condition, whenever $\alpha \lor \beta \neq \alpha \lor \beta$, then $\phi(\alpha) \lor \phi(\beta) \neq \phi(\alpha) \lor \phi(\beta)$; therefore, $\alpha \lor \beta = \alpha \lor \beta \lor \mu$ and $\phi(\alpha) \lor \phi(\beta) = \phi(\alpha) \lor \phi(\beta) \lor \phi(\mu)$. It now follows that $\phi(\alpha) \lor \phi(\beta) = \phi(\alpha \lor \beta)$. The argument for the meet is dual.
Under the second condition, \( \phi(m_i) = \phi(u_i) \), so \( \phi(\alpha \lor \phi(\beta)) = \phi(\alpha \lor \beta) \) is obvious.

\[ \square \]

Note that a necessary and sufficient condition for \( \phi \) to be a homomorphism can be easily formulated. We only need the sufficient conditions of Lemma 3 in this paper.

3. \( F \) is in \( H(M_3 \circ D) \)

We start with the lattice of Fig. 2. We take three copies of \( L, L_1, L_2, \) and \( L_3 \). The elements will be denoted accordingly: \( u_1, u_2, \) and so on. Let \( A \) denote the hinged-product (power) of \( L_1, L_2, \) and \( L_3 \). The diagram of \( A \) is given in Fig. 3; \( l(H) = \{0\}, u(H) = [\mu] \).

Now consider the congruence relation \( \Theta(0,\mu) \) of \( L \), where \( 0 = \langle l_1, l_2, l_3 \rangle \) and \( \mu = \langle m_1, m_2, m_3 \rangle = \langle u_1, u_2, u_3 \rangle \). The natural homomorphism of \( L \) onto \( L/\Theta(0,\mu) \) obviously satisfies (ii) of Lemma 3, hence \( A \) has a natural homomorphism onto the appropriate hinged-product of three copies of \( L/\Theta(0,\mu) \); this new lattice is isomorphic to \( (C_2)^3 \).

There are eight \( \Theta \)-classes: four are isomorphic to \( M_3 \), three to \( (C_2)^2 \), and one to \( (C_2)^3 \). Since the congruence classes of \( \Theta \) are either distributive or isomorphic to \( M_3 \), and \( A/\Theta \) is distributive, we conclude that \( A \) belongs to \( M_3 \circ D \).
Next consider the congruence $\Theta(\mu, k)$ on $L$, where $k = \langle k_1, k_2, k_3 \rangle$. $L/\Theta(\mu, k)$ is the lattice of Fig. 4. The hinged cube of that lattice is isomorphic to the lattice $F$ of Fig. 1. Hence $F \in \mathbb{H}(M_3 \circ D)$. This proves the first sentence of the Theorem.

4. The tolerance relations of $F$

Fig. 5 represents $F$ with the tolerance relation $T = \Theta/\Theta(0.\mu)$. $T$ is a natural tolerance on $F$; unfortunately, $F/T$ is isomorphic to $M_3$, and it is not distributive.

This makes the Theorem plausible. If there are $A$, $\Theta$, $\Phi$ such that $A \in M_3 \circ D$ is established by $\Theta$, and $T = \Theta/\Phi$ satisfies that (i) all $T$-classes of $A$ are in $M_3$; (ii) $A/T$ is in $D$, then $A$ must be the lattice of Fig. 3 (or a fatter version with larger distributive classes), and $\Theta$ and $\Phi$ must be as in Section 3.

In Section 5 we shall prove this. As a first step, we have to describe all the tolerance relations on $F$. 
Lemma 4. The lattice $F$ has nine tolerance relations; they form a lattice as shown in Fig. 6. The tolerance $T_0$ is depicted in Fig. 5, and the tolerance $T_1$ in Fig. 7.

Proof. Since $F$ is a finite, simple, modular lattice, any two prime intervals of $F$ are projective. Therefore, $F$ has a unique minimal proper tolerance relation, $T_0$, generated by any prime interval. Moreover, if any two distinct elements of a sublattice isomorphic to $M_3$ are collapsed by a tolerance relation, then the whole sublattice is collapsed. It follows immediately that $T_0$ is as described in Fig. 5.

Next consider the tolerance relation $T_1$ of Fig. 7. Symmetrically, we can define $T_2$, and $T_3$. We shall prove that any proper tolerance relation is either a $T_i$ or of the form $T_i \lor T_j$, $i \neq j$. 
So let $T$ be a tolerance relation, $T > T_0$. Then there are $x, y \in F, x < y$, such that $x = y (T)$ but not modulo $T_0$.

**Claim.** $0 = d_i (T)$ for some $i$.

**Proof.** Up to symmetry, there are three cases to consider:

- **Case 1.** $a_1 \leq x < d_1$ and $d_1 \lor d_2 \leq y$. Then
  \[ b_2 = b_2 \land (d_1 \lor d_2) \equiv b_2 \land x = 0 \quad (T) . \]
  Similarly, $c_2 \equiv 0 (T)$, so $d_2 \equiv 0 (T)$, as claimed.
- **Case 2.** $a_1 = x$ and $d_1 \leq y$. Then
  \[ 0 = 0 \lor 0 = (x \land b_3) \lor (x \land c_3) = (y \land b_3) \lor (y \land c_3) = b_3 \lor c_3 = d_3 \quad (T) \]
  as claimed.
- **Case 3.** $x = 0$ and $b_3 \leq y$. Then
  \[ 0 = 0 \lor 0 = x \lor x = (x \land b_3) \lor ((x \land a_3) \lor a_2) \land a_1 \equiv (y \land b_3) \lor (((y \land a_3) \lor a_2) \land a_1) = b_3 \lor a_1 = d_3 \quad (T) \]
  as claimed. This completes the proof of the Claim. \(\Box\)

Now we complete the proof of Lemma 4. The relation $d_i \equiv 0$ obviously generates the tolerance $T_i$. Similarly we get $T_2$, and $T_3$.

Finally, let $T$ be a tolerance relation satisfying $T > T_i$. Then we must have $x = y (T), x < y$, such that $[x, y]$ properly contains a $T_i$-block. We distinguish four cases according to which $T_i$-block $[x, y]$ contains.

- **Case 1.** The block $[u, 1]$. $x < u$ and $1 \leq y$ imply that, say, $x \leq a_2$ and $y = 1$. Thus $a_2 = 1 (T)$. Hence,
  \[ 0 = 0 \lor 0 = (c_1 \land a_2) \lor (c_3 \land a_2) \equiv (c_1 \land 1) \lor (c_3 \land 1) = d_1 \lor d_3 \quad (T) . \]
- **Case 2.** The block $[0, d_1]$. $x \leq 0$ and $d_1 \leq y$ imply that $x = 0$ and, say, $d_1 \lor d_3 \leq y$. Thus $0 = d_1 \lor d_3 \quad (T)$. Hence,
  \[ a_2 = b_2 \land c_2 = (0 \lor b_2) \land (0 \lor c_2) \equiv ((d_1 \lor d_3) \lor b_2) \land ((d_1 \lor d_3) \lor c_2) = 1 \quad (T) . \]
- **Case 3.** The block $[a_2, d_1 \lor d_3]$. $x \leq a_2$ and $d_1 \lor d_3 \leq y$, but not both $x = a_2$ and $d_1 \lor d_3 = y$. Hence either $x = 0$ and $d_1 \lor d_3 \leq y$, in which case we proceed as in Case 2, or $x \leq a_2$ and $y = 1$, and we proceed as in Case 1.
- **Case 4.** The block $[a_3, d_1 \lor d_2]$. We proceed, by symmetry, as in Case 3.

Thus in all four cases we have $0 = d_1 \lor d_3 \quad (T)$ and $a_2 = 1 (T)$ or $0 = d_1 \lor d_2 \quad (T)$ and $a_3 = 1 (T)$. These, obviously, describe the tolerances $T_1 \lor T_3$ and $T_1 \lor T_2$, respectively.

Since $T_i \lor T_j, i \neq j$ are maximal tolerances, the proof of Lemma 4 is complete. \(\Box\)
5. The proof of the Theorem

Let us assume that there is an $A$ in $M_3 \circ D$ with congruence $\Theta$ establishing this such that $F$ can be represented as $A/\Phi$ and $T = \Theta/\Phi$ satisfies (i) all $T$-classes of $A$ in $M_3$; (ii) $A/T$ is in $D$.

It is easy to see that the lattice $A/\Theta \wedge \Phi$ and the congruences $\Phi/\Theta \wedge \Phi$ and $\Theta/\Theta \wedge \Phi$ satisfy the same conditions, and the new congruences are disjoint. In other words, we can assume that $\Theta \wedge \Phi = \omega$.

An element $x$ of $F$ is represented as a $\Phi$ congruence class, $C(x)$. $A/T$ is distributive; since $A/T_0$ is isomorphic to $M_3$, $T > T_0$. Thus by the Claim in the proof of Lemma 4, we can assume that $\Theta = d_3 (T)$.

Hence there are elements $0' \in C(0)$ and $d_3' \in C(d_3)$ satisfying $0' = d_3'$ ($\Theta$). We can obviously assume that $0' < d_3'$. Substituting an arbitrary $a_3' \in C(a_3)$ by $(0' \vee a_3') \wedge d_3'$, we obtain $a_3' \in C(a_3)$ satisfying $0' < a_3' < d_3'$. Similarly, we can choose $u' \in C(u)$, $b_3' \in C(b_3)$, and $c_3' \in C(c_3)$ satisfying $u', b_3', c_3' \in [a_3', d_3']$ and elements $a_3' \in C(a_3)$, $\; a_2' \in C(a_2)$ in $[0', u']$.

Now it is easy to see that the elements $0'$, $a_1'$, $a_2'$, $a_3'$, $u'$, $b_3'$, $c_3'$, and $d_3'$ form a sublattice of $A$ isomorphic to the interval $[0, d_3]$ of $F$. Indeed, the map $\phi : x \rightarrow x'$ is obviously one-to-one. Since $0' = d_3'$ ($\Theta$), all these elements belong to the same $\Theta$ class. We have to show that the $\vee$ and $\wedge$ work properly. As an example, let us show that $a_1' \vee a_2' = u'$. Indeed, $a_1' \vee a_2' = u'$ ($\Theta$) since all these elements are in the same $\Theta$ class. On the other hand, $a_1' \vee a_2' = u'$ ($\Phi$) since both $a_1' \vee a_2'$ and $u'$ map onto $u$. Therefore, $a_1' \vee a_2' = u'$ ($\Theta \wedge \Phi$). Since $\Theta \wedge \Phi = \omega$, we conclude that $a_1' \vee a_2' = u'$ ($\omega$), that is, $a_1' \vee a_2' = u'$, as claimed.

Since every $\Theta$ class is in $M_3$, we get that the interval $[0, d_3]$ of $F$ is in $M_3$, an obvious contradiction which proves the Theorem.

It is obvious from the proof, that the Theorem holds for any lattice variety $V$ in place of $M_3$ that does not contain the interval $[0, d_3]$ of $F$. The most general such variety is $M_\omega$, the lattice variety generated by $M_\omega$, the modular lattice of length two with countably infinite atoms.

References