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Lattice Homomorphisms of Non-periodic Groups

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1. INTRODUCTION

In a group G the set of all subgroups, partially ordered by inclusion, is a complete algebraic lattice which we denote by $l(G)$. A map τ from $l(G)$ to a complete lattice $\mathscr{L}$ is called a *complete lattice homomorphism* (or a *complete l-homomorphism*) if for all non-empty subsets $\mathscr{S}$ of $l(G)$ we have

$$\left(\bigcap_{X\in\mathscr{S}} X\right)^{\tau} = \bigwedge_{X\in\mathscr{S}} X^{\tau} \quad \text{and} \quad \langle X \mid X\in\mathscr{S}\rangle^{\tau} = \bigvee_{X\in\mathscr{S}} X^{\tau}. \qquad (\dagger)$$

Usually we shall write simply $\tau\colon G\to\mathscr{L}$ to denote the map τ and speak of a complete l-homomorphism from G to $\mathscr{L}$. We call τ *trivial* if all subgroups of G have the same image under τ; and we call τ *proper* if τ is not trivial and not injective. If (†) holds for all finite subsets $\mathscr{S}$, then τ is called a *lattice homomorphism* (or *l-homomorphism*). An l-homomorphism from G to the lattice $l(\bar{G})$ of a group $\bar{G}$ is called a *projectivity* if it is a bijection. Of course, a projectivity is always complete. A projectivity $\sigma\colon G\to\bar{G}$ is said to be *index-preserving* if, for $K\leqslant H\leqslant G$ with $|H:K|$ finite,

$$|H:K| = |H^{\sigma}:K^{\sigma}|.$$

The existence of either a proper l-homomorphism from a finite group G to some lattice or a non-index-preserving projectivity of G imposes severe restrictions on the structure of G (see [9]). In this work we consider com-

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plete l-homomorphisms of non-periodic groups. Our aim is to give some general conditions satisfied by a group $\bar{G}$ which guarantee that every non-trivial complete l-homomorphism of $\bar{G}$ is injective and every complete l-epimorphism from a group G to $\bar{G}$ ($\neq 1$) is necessarily an index-preserving projectivity.

To be able to state our main results we need some notation and definitions. If $H \leqslant G$, then $[G/H]$ denotes the lattice of all subgroups of G containing H; and $[G/H]$ is said to be *non-periodic* if there is an element $g \in G$ such that $|\langle g \rangle : H \cap \langle g \rangle|$ is infinite. We write $H \leqslant_d G$ if H is a Dedekind subgroup of G (see [17]). Suppose that $H \leqslant_d G$. If for all $g \in G$ and subgroups K such that

$$H \leqslant K \leqslant \langle H, g \rangle = L,$$

say,

$$|L : K| \text{ is finite if and only if } [L/K] \text{ is finite},$$

then we write $H \leqslant_D G$ and say that *H is D-embedded in G.* A group G is called *modular* if $l(G)$ is a modular lattice. Suppose that a group G has an ascending normal series whose factors are locally finite or abelian. Then we define the *Hirsch length* $h(G)$ of G to be the sum of the torsion-free ranks of the abelian factors. Thus $h(G)$ is an invariant of G. Our main results are then contained in

THEOREM A. *Let σ: $G \to \bar{G}$ ($\neq 1$) be a complete l-epimorphism of groups, $\mathscr{L}$ a complete lattice and τ: $\bar{G} \to \mathscr{L}$ a non-trivial complete l-homomorphism. Suppose that $\bar{H} \leqslant_D \bar{G}$ and that $\bar{H}$ has an ascending normal series with factors locally finite or abelian. If*

(i) *$h(\bar{H}) \geqslant 3$ or*

(ii) *$h(\bar{H}) = 2$ and either $\bar{H}$ is modular or $[\bar{G}/\bar{H}]$ is non-periodic, then*

(a) *σ is an index-preserving projectivity and*

(b) *τ is injective.*

An idea used by Ivanov [5] for handling certain infinite systems of algebraic equations will play an important role in our argument (see Section 3). Several known criteria for l-homomorphisms to be injective or for projectivities to be index-preserving will be used many times and for convenience we list them here.

1.1. *Let τ: $G \to \mathscr{L}$ be an l-homomorphism and let H be a subgroup of G. If τ is injective on the intervals $[\langle H, g \rangle / H]$, for all $g \in G$, then τ is injective on $[G/H]$* [15, Proposition 2.1].

1.2. *Let σ: $G \to \bar{G}$ be a projectivity. If σ is index-preserving on all the cyclic subgroups of G, then σ is index-preserving on G* [12, Corollary 3].

1.3. *Let $N = N_0 \leqslant N_1 \leqslant \cdots \leqslant N_n = G$ be a finite chain of subgroups of G with each $N_i \leqslant_d G$ and let τ: $G \to \mathscr{L}$ be an l-homomorphism. If τ is injective on all intervals $[N_{i+1}/N_i]$, then τ is injective on $[G/N]$* [15, Corollary 2.2(i)].

1.4. *Let σ: $G \to \bar{G}$ be a projectivity and let N be a quasinormal subgroup of G. If σ is index-preserving on N and on all cyclic intervals $[\langle g \rangle N/N]$ of $[G/N]$, then σ is index-preserving on G* [13, Theorem 2.7].

1.5. *Let τ: $G \to \mathscr{L}$ be a complete l-epimorphism, where G is a non-periodic group and $\mathscr{L}$ is a non-trivial complete lattice, and let H be a locally finite Dedekind subgroup of G. Then $\tau|_H$ is injective. Moreover, if $\mathscr{L}$ is the lattice of a group, then $\tau|_H$ is index-preserving* [15, Proposition 3.1]).

1.6. *Let τ: $G \to \mathscr{L}$ be a complete l-epimorphism, where G is a non-periodic locally polycyclic group and $\mathscr{L}$ is a non-trivial complete lattice. Then*

(i) *τ is injective on all periodic subgroups of G; and*

(ii) *τ is injective on G if G contains two elements a, b of infinite order with $\langle a \rangle \cap \langle b \rangle = 1$.*

Moreover, if $\mathscr{L}$ is the lattice of a group, then τ is index-preserving on the periodic subgroups of G; and τ is index-preserving on G if the hypothesis of (ii) *holds* [15, Proposition 3.2].

We define classes Γ, Ω of groups as follows. A group G belongs to Γ if every non-trivial complete l-homomorphism τ: $G \to \mathscr{L}$ ($\mathscr{L}$ a complete lattice) is injective; and a group G belongs to Ω if every projectivity σ: $G \to \bar{G}$ is index-preserving. Then, by 1.1 and 1.2,

$$G \in \Gamma (G \in \Omega) \text{ if and only if for each } g \in G \text{ the restriction } \tau|_{\langle g \rangle} \ (\sigma|_{\langle g \rangle}) \text{ is injective (index-preserving).} \tag{1}$$

Easy consequences are

$$\text{L}\Gamma = \Gamma, \qquad \text{L}\Omega = \Omega. \tag{2}$$

(For the definition of L and other closure operations, see [7].)

When dealing with a non-trivial complete l-homomorphism τ of a non-periodic group G, it is important to recall the useful fact that

$$\text{the lower kernel of } \tau \text{ is } 1. \tag{3}$$

(See [9, Theorem 5, p. 63].) In particular τ cannot be trivial on the non-periodic sections of G. A consequence of this is the following. Let Γ_1 be the class of non-periodic groups in Γ. Then

$$\acute{P}\Gamma_1 = \Gamma_1.$$

For, let τ: $G \to \mathscr{L}$ be a non-trivial complete l-homomorphism and let $\{X_\alpha \mid \alpha \leqslant \beta\}$ be an ascending Γ_1-series of G. Suppose for a contradiction that τ is not injective and choose α minimal such that $\tau|_{X_\alpha}$ is not injective. Thus $\alpha \geqslant 1$ (assuming $X_0 = 1$) and α cannot be a limit ordinal by (1). Therefore τ is injective on $X_{\alpha-1}$ and on $X_\alpha / X_{\alpha-1}$ (being non-trivial on this quotient). Then τ is injective on X_α, by 1.3, a contradiction. Similarly,

$$\acute{P}\Omega = \Omega,$$

using 1.4.

Let $G_1 = N_1 N_2$ with $N_i \lhd G$, $N_i \in \Gamma_1$ $(i = 1, 2)$ and let τ: $G \to \mathscr{L}$ be a non-trivial complete l-homomorphism. Then τ is injective on each N_i, hence on G/N_i and therefore on G, again by 1.3. Similarly, using 1.4, we find that $G \in \Omega$ if N_1, $N_2 \in \Omega$. Combining these results with (2), we obtain

$$N\Gamma_1 = \Gamma_1, \qquad N\Omega = \Omega.$$

Next we point out that *if H, K are non-periodic groups, then the direct product $G = H \times K$ belongs to Γ_1*. For, let τ be a non-trivial complete l-homomorphism of G. Every periodic element x of H belongs to a non-periodic abelian subgroup of G; and every non-periodic element x of H belongs to an abelian subgroup of G of Hirsch length 2. Thus, by 1.6, τ is injective on $\langle x \rangle$ in both cases and hence $\tau|_H$ is injective, by 1.1. Similarly $\tau|_K$ is injective. Therefore τ is injective, by 1.3, and so $G \in \Gamma_1$. In the same way we find that $G \in \Omega$.

A few words about cyclic subgroups will be appropriate. First if τ: $G \to \bar{G}$ is an l-epimorphism, then τ maps cyclic subgroups of G to cyclic subgroups of $\bar{G}$. Moreover, if X is an infinite cyclic subgroup of G and τ is complete, then either X^τ is an infinite cyclic subgroup of $\bar{G}$ or τ is trivial, by (3). Conversely, if τ is complete and $\bar{X}$ is a cyclic subgroup of $\bar{G}$, then there is a cyclic subgroup X of G such that $X^\tau = \bar{X}$ (see [9, p. 60, 61]).

The organization of the remaining sections is as follows. Section 2 contains preliminary results of a fairly general nature relating to complete l-homomorphisms and projectivities. Then in Section 3 we consider a critical case of Theorem A(b); and in Section 4 we do the same for Theorem A(a). Section 5 contains applications of the preceding results, leading to the proof of Theorem A. Finally in Section 6 we establish some examples (Theorems B and C) which indicate the necessity of the hypotheses in Theorem A.

Further notation. If $\tau: G \to \mathscr{L}$ is a complete l-epimorphism and $L \in \mathscr{L}$, then L^{τ^*} is the maximal subgroup of G which maps under τ to L. If $\mathscr{L}$ is the lattice of a group $\bar{G}$, then we write $H^\tau = \bar{H}$ for all $H \leqslant G$. For any group G, $\mathbf{P}(G)$ denotes the maximal normal periodic subgroup of G. By [13, Proposition 1.12], $\mathbf{P}(G)$ is also the join of all the periodic Dedekind subgroups of G. The class of groups which possess an ascending normal series with the factors abelian or locally finite will be denoted by $\mathfrak{X}$. The derived length of a soluble group G is denoted by $d(G)$; and $C_G(H)$ is the centralizer in a group G of a subgroup H. Also H^G is the normal closure of H in G and H_G is the intersection of the conjugates of H in G. The centre of G is denoted by $Z(G)$. The subgroup of G generated by the elements of infinite order is denoted by $a(G)$; and if $A \leqslant G$, then

$$a_G(A) = \langle g \mid g \in G, |\langle g \rangle : \langle g \rangle \cap A| = \infty \rangle.$$

The multiplicative group of non-zero complex numbers is written as $\mathbb{C}^*$ and $\mathbb{Z}_+$ denotes the positive integers. The set of positive prime divisors of $n \in \mathbb{Z}$ is denoted by $\pi(n)$.

2. Preliminary Results

We collect together various results about complete l-epimorphisms between groups. Throughout the section:

> *G and $\bar{G}$ are groups and $\sigma: G \to \bar{G}$ denotes a complete l-epimorphism from G to $\bar{G}$.*

Lemma 2.1. *Let $\bar{N} \lhd \bar{G}$, $N = \bar{N}^{\sigma^*}$, $T = N_G$, and $M = N^G$. Then $\bar{T} \lhd \bar{G}$ and $\bar{M} \lhd \bar{G}$.*

Proof. Let $L = (\bar{T}^{\bar{G}})^{\sigma^*}$. Then $T \leqslant L \leqslant N$. By [15, Theorem 2.12], $L \lhd G$ and so $T = L$, i.e., $\bar{T} = \bar{L} \lhd \bar{G}$. Now let

$$U = (\bar{M}_{\bar{G}})^{\sigma^*}.$$

Then $N \leqslant U$ and, by [15, Theorem 2.9], $U \lhd G$. Thus $M \leqslant U$ and so $\bar{M} = \bar{M}_{\bar{G}} \lhd \bar{G}$. ∎

Regarding periodic radicals we have

Lemma 2.2. *If $\bar{G} \neq 1$, then $\mathbf{P}^{\sigma^*}(\bar{G}) = \mathbf{P}(G)$.*

Proof. Clearly we may assume that G is not periodic. Also

$$\mathbf{P}^\sigma(G) \leqslant_d \bar{G}$$

and therefore $\mathbf{P}^\sigma(G) \leqslant \mathbf{P}(\bar{G})$. Hence

$$\mathbf{P}(G) \leqslant \mathbf{P}^{\sigma^*}(\bar{G}). \tag{1}$$

Conversely, $\mathbf{P}^{\sigma^*}(\bar{G})$ is periodic. Let g be an element of infinite order in G. Then

$$\langle g, \mathbf{P}^{\sigma^*}(\bar{G}) \rangle^\sigma = \langle \bar{g}, \mathbf{P}(\bar{G}) \rangle,$$

where $\langle \bar{g} \rangle = \langle g \rangle^\sigma$. So $\bar{g}$ has infinite order. Choose

$$x \in \langle g, \mathbf{P}^{\sigma^*}(\bar{G}) \rangle = T,$$

say, with x of finite order. Then $\langle x \rangle^\sigma \leqslant \mathbf{P}(\bar{G})$; hence

$$x \in \mathbf{P}^{\sigma^*}(\bar{G})$$

and therefore $\mathbf{P}^{\sigma^*}(\bar{G}) = \mathbf{P}(T) \lhd T$. It follows that

$$\mathbf{P}^{\sigma^*}(\bar{G}) \leqslant \mathbf{P}(a(G)) \leqslant \mathbf{P}(G).$$

Together with (1) this gives the desired result. ∎

When $\bar{G}$ is not periodic and $\mathbf{P}(\bar{G})$ is locally finite, then we can say much more.

LEMMA 2.3. *Suppose that* $\mathbf{P}(\bar{G}) < \bar{G}$ *and* $\mathbf{P}(\bar{G})$ *is locally finite. Then* σ *induces an index-preserving projectivity from* $\mathbf{P}(G)$ *to* $\mathbf{P}(\bar{G})$. *In particular* $\mathbf{P}(G)$ *is locally finite.*

Proof. Since $\bar{G}$ is not periodic, it follows that G is not periodic and so the lower kernel of σ is 1, by (3) in Section 1. Let

$$\mathbf{P}(\bar{G}) = \bar{K}_1 \times \bar{K}_2 \times \cdots$$

be the *maximal Hall decomposition* of $\mathbf{P}(\bar{G})$, i.e., the orders of the elements of $\bar{K}_i$ ($\neq 1$) are relatively prime to the orders of the elements of $\bar{K}_j$, all $i \neq j$, and, for each i, $\bar{K}_i$ cannot be expressed non-trivially as a direct product of subgroups whose elements have relatively prime orders. Let

$$K_i = \bar{K}_i^{\sigma^*}.$$

By Lemma 2.2, $\mathbf{P}(G) = \mathbf{P}^{\sigma^*}(\bar{G})$ and then, by [15, Proposition 1.4],

$$\mathbf{P}(G) = K_1 \times K_2 \times \cdots$$

is a maximal Hall decomposition of $\mathbf{P}(G)$.

If we can show that $\mathbf{P}(G)$ is locally finite, then 1.5 will complete our argument. Thus it suffices to show that each subgroup K_i is locally finite. Let $\bar{K}=\bar{K}_i$ and $K=K_i$. We distinguish two cases.

(a) *Suppose first that $\bar{K}$ is not locally cyclic.* Assume, for a contradiction, that K is not locally finite. Then there are elements $h_1, ..., h_m$ in K such that

$$T=\langle h_1, ..., h_m\rangle$$

is not finite. Let $\bar{T}=\bar{T}_1\times\cdots\times\bar{T}_n$ be the maximal Hall decomposition of the finite subgroup $\bar{T}$. If

$$T_j=\bar{T}_j^{\sigma^*}\cap T$$

(the full preimage of $\bar{T}_j$ under $\sigma|_T$), then, again by [15, Proposition 1.4], $T=T_1\times\cdots\times T_n$. By Proposition 1.5 of the same paper, for some j, $\bar{T}_j$ must be a cyclic p-group, for some prime p, with T_j infinite. Suppose without loss of generality that $j=1$. We further distinguish two cases.

(i) Suppose that $\bar{K}$ has a maximal p-subgroup which is not locally cyclic. Then $\bar{T}_1$ lies in a finite non-cyclic p-subgroup $\bar{F}$ of $\bar{K}$, and $F=\bar{F}^{\sigma^*}$ is finite, again by [15, Proposition 1.5]. But $F\geqslant T_1$ and T_1 is infinite, a contradiction.

(ii) Now suppose that all the maximal p-subgroups of $\bar{K}$ are locally cyclic. Since we are assuming that $\bar{K}$ is not locally cyclic, it follows that $\bar{K}$ is not a p-group; and there is a maximal p-subgroup $\bar{S}$ of $\bar{K}$ and a p'-element $\bar{y}\in\bar{K}$ such that $[\bar{S}, \bar{y}]\neq 1$. Choose $\bar{z}\in\bar{S}$ such that $[\bar{z}, \bar{y}]\neq 1$. Then the finite subgroup $\langle\bar{T}_1, \bar{z}, \bar{y}\rangle=\bar{F}$, say, is *not* the direct product of a p-group and a p'-group. Let $\bar{S}_1$ be a Sylow p-subgroup of $\bar{F}$ containing $\bar{T}_1$. Thus if

$$\bar{F}=\bar{R}_1\times\cdots\times\bar{R}_s$$

is the maximal Hall decomposition of $\bar{F}$, then $\bar{S}_1\leqslant\bar{R}_1$ (say) and $\bar{R}_1$ cannot be cyclic. Again by [15, Proposition 1.5], $\bar{R}_1^{\sigma^*}$ is finite and contains T_1, a contradiction as before.

(b) *Now suppose that $\bar{K}$ is locally cyclic.* Thus $\bar{K}$ is a p-group, for some prime p. Choose $\bar{P}\leqslant\bar{K}$ with $\bar{P}$ $(\neq 1)$ finite and let $P=\bar{P}^{\sigma^*}$. Also choose $\bar{g}\in\bar{G}$ with $|\bar{g}|$ infinite and $g\in G$ such that $\langle g\rangle^\sigma=\langle\bar{g}\rangle$. So $|g|$ is infinite, $\bar{K}\lhd\bar{G}$, and

$$K=\bar{K}^{\sigma^*}\lhd G.$$

We have $\bar{P}\lhd\bar{G}$ and $\bar{P}=\mathbf{P}\langle\bar{P}, \bar{g}\rangle$. By Lemma 2.2,

$$P=\mathbf{P}\langle P, g\rangle$$

and so $P \lhd \langle P, g\rangle$. Let P_0 be the upper kernel of $\sigma|_P$. Then $P_0 \lhd P$; and P_0 is cyclic of prime power order, by [9, Proposition 3.1, p. 59]. Thus $P_0^{\langle g\rangle}$ is locally finite and then σ is injective and index-preserving on $P_0^{\langle g\rangle}$, by 1.5. Therefore $P_0 = P_0^{\langle g\rangle}$ is a p-group.

Consider the induced l-epimorphism,

$$\langle P, g\rangle/P_0 \to \langle \bar{P}, \bar{g}\rangle/\bar{P} \qquad (\cong C_\chi),$$

in which $P/P_0 \to 1$. The lower kernel of this homomorphism is 1, by (3) in Section 1, and therefore $P = P_0$. Thus K is a locally cyclic p-group and hence locally finite. ∎

We shall need the folowing easy consequence of [9, Theorem 4, p. 61].

LEMMA 2.4. *Suppose that $\bar{G}$ is a non-trivial torsion-free locally cyclic group. Then G is also torsion-free and locally cyclic.*

We pass now to a consideration of preimages of residually finite p-groups.

LEMMA 2.5. *Suppose that $\bar{G}$ is residually a finite p-group (p a prime), but that $\bar{G}$ is not periodic and not locally cyclic. Then*

(i) *for all $g \in G$, $|\langle g\rangle : \langle g^p\rangle| = |\langle g\rangle^\sigma : \langle g^p\rangle^\sigma|$, and*

(ii) *G is residually a finite p-group.*

Also there is a function f such that if $\bar{G}$ is soluble with $d(\bar{G}) = n$, then

(iii) *G is soluble with $d(G) \leqslant f(n)$.*

Proof. (i) Clearly we may assume that $\bar{G}$ is finitely generated; and then there are normal subgroups $\bar{N}_\lambda$ $(\lambda \in \Lambda)$ of $\bar{G}$ with $\bigcap_\lambda \bar{N}_\lambda = 1$ and each $\bar{G}/\bar{N}_\lambda$ a finite p-group of exponent $\geqslant p^2$, not cyclic, and (when $p=2$) not generalized quaternion.

By [15, Corollary 1.3], $N_\lambda = \bar{N}_\lambda^{\sigma^*}$ has finite index in G. Let $T_\lambda = (N_\lambda)_G$. By Lemma 2.1, $\bar{T}_\lambda \lhd \bar{G}$ and so σ induces an l-epimorphism

$$G/T_\lambda \to \bar{G}/\bar{T}_\lambda$$

between finite groups. Here $\bar{T}_\lambda^{\sigma^*}/T_\lambda$ is the lower kernel and normal in G/T_λ; hence $\bar{T}_\lambda^{\sigma^*} = T_\lambda$, i.e.,

$$\textit{the lower kernel is } 1. \tag{2}$$

Let $\bar{S}/\bar{T}_\lambda$ be a Sylow p-subgroup of $\bar{G}/\bar{T}_\lambda$ and put $S = \bar{S}^{\sigma^*}$. By [9, Proposition 3.9, p. 82], S/T_λ is a p-group and non-cyclic since $\bar{S}/\bar{T}_\lambda$ is non-cyclic. Thus if σ does not induce a projectivity from S/T_λ to $\bar{S}/\bar{T}_\lambda$, then S/T_λ must

Since $(m/n)A = A$, it follows that $A \geqslant \mathbb{Z}[1/mn] \lhd G$ and so we may assume that

$$A = \mathbb{Z}[1/mn].$$

We distinguish two possibilities:

Case 1. *Suppose that* $p \nmid mn$. Thus $pA < A$. Let

$$C_{\langle g \rangle}(A/pA) = \langle g^l \rangle.$$

By 1.6, τ is injective on A/pA and then it is easy to see that τ must be injective on

$$A\langle g^l \rangle/(pA)\langle g^{pl} \rangle \cong C_p \times C_p.$$

Hence

$$\langle g^l \rangle^\tau > \langle g^{pl} \rangle^\tau. \tag{2}$$

However, $(l, p) = 1$ and so

$$\begin{aligned} \langle g^l \rangle^\tau = \langle g \rangle^\tau \cap \langle g^l \rangle^\tau &= \langle g^p \rangle^\tau \cap \langle g^l \rangle^\tau \qquad \text{(by (1))} \\ &= (\langle g^p \rangle \cap \langle g^l \rangle)^\tau = \langle g^{pl} \rangle^\tau, \end{aligned}$$

contradicting (2).

Case 2. *Suppose now that* $p \mid mn$. Then $pA = A$. Replacing g by g^{-1} if necessary, we may suppose that $m \geqslant 2$. Thus if $n > 0$, then

$$m^{p-1} + m^{p-2}n + \cdots + n^{p-1} \neq \pm 1. \tag{3}$$

If $n < 0$ and $p \neq 2$, we may replace g by g^2 and then (3) holds. On the other hand, if $p = 2$ and $m + n = \pm 1$, it is easy to see that $m^3 + n^3 \neq \pm 1$ and, replacing g by g^3, again (3) holds. Thus in all cases there is a prime q dividing $m^{p-1} + m^{p-2}n + \cdots + n^{p-1}$.

By Lemma 3.1,

$$q \nmid mn \qquad \text{and} \qquad q \nmid (m-n). \tag{4}$$

Thus $|A/qA| = q$ and A/qA is generated by $1 + qA$. Since

$$\frac{m}{n} - 1 = \frac{m-n}{n} \notin qA$$

(by (4)), we see that

g acts non-trivially on A/qA.

group of $\bar{G}$ containing $\bar{g}$ with $\bar{K}$ not cyclic. Then $\bar{K}$ is residually a finite q-group and applying (i) to $K=\bar{K}^{\sigma^*}$ gives $\langle g\rangle = \langle g^q\rangle$, a contradiction. Therefore (5) is true. It follows from [9, Theorem 3, p. 35], that $G \cong \bar{G}$ and hence G is residually a finite p-group.

Now suppose that $\bar{G}/\bar{N}_\lambda$ is not cyclic for some λ. Then we can find a non-empty subset Λ_1 of Λ such that, for all $\lambda \in \Lambda_1$, $\bar{G}/\bar{N}_\lambda$ is not cyclic, has exponent $\geqslant p^2$, is not generalized quaternion in case $p=2$, and $\bigcap_{\lambda\in\Lambda_1} \bar{N}_\lambda = 1$. Fix $\lambda \in \Lambda_1$ and let $|\bar{G} : \bar{N}_\lambda| = p^\alpha$. Let $N_\lambda = \bar{N}_\lambda^{\sigma^*}$ and $T_\lambda = (N_\lambda)_G$ as in (i). For any $g \in G\backslash N_\lambda$ and $\langle g\rangle^\sigma = \langle \bar{g}\rangle$,

$$\bar{g} \text{ has order } p^\beta \text{ modulo } \bar{N}_\lambda$$

for some $1 \leqslant \beta \leqslant \alpha$. We claim that

$$|\langle g\rangle : \langle g^{p^\beta}\rangle| = p^\beta. \tag{6}$$

For, this is clear if $|g| = \infty$. Thus suppose that $|g|$ is finite. Then $|\bar{g}|$ is finite and so $\bar{g}$ has p-power order. Let $\bar{S}/\bar{T}_\lambda$ be a Sylow p-subgroup of $\bar{G}/\bar{T}_\lambda$ containing $\bar{g}$ and let $S=\bar{S}^{\sigma^*}$. As in (i), S/T_λ is a finite p-group and σ induces a projectivity from S/T_λ to $\bar{S}/\bar{T}_\lambda$. Since p^β divides the order of $\bar{g}$ modulo $\bar{T}_\lambda$, p^β divides the order of g modulo T_λ. Hence $p^\beta \mid |g|$ and (6) follows.

By (i),

$$|\langle g\rangle : \langle g^{p^\beta}\rangle| = |\langle g\rangle^\sigma : \langle g^{p^\beta}\rangle^\sigma|$$

and therefore $g^{p^\beta} \in N_\lambda$. Thus

$$G^{p^\beta} \leqslant N_\lambda$$

and so $G^{p^\beta} \leqslant T_\lambda$. Therefore G/T_λ is a finite p-group and, since $\bigcap_{\lambda\in\Lambda_1} T_\lambda = 1$, G is residually a finite p-group.

(iii) Finally suppose that $\bar{G}$ is soluble with derived length n. Arguing as in (ii) and adopting the notation used there, we have G/T_λ is a finite p-group, all $\lambda \in \Lambda_1$, and

$$\bigcap_{\lambda\in\Lambda_1} T_\lambda = 1.$$

If σ is not injective on G/T_λ, then $d(G/T_\lambda) \leqslant 2$, by [9, Proposition 3.4, p. 70]. Otherwise if σ is injective on G/T_λ, then $d(G/T_\lambda)$ is bounded by a function of n, by [11]. Thus $d(G)$ is bounded by a function of n. ∎

Remark. Under the hypotheses of Lemma 2.5, (i) and (ii) show that σ is injective and index-preserving on the periodic subgroups of G (using 1.1 and 1.2).

Now we impose additional hypotheses on $\bar{G}$. Recall that $\mathfrak{X}$ denotes the class of groups which possess an ascending normal series with the factors abelian or locally finite.

LEMMA 2.6. *Let* $P = \mathbf{P}(G)$, $\bar{P} = \mathbf{P}(\bar{G})$.

(i) *Suppose that* $\bar{G} \in \mathfrak{X}$ *with* $h(\bar{G}) = 1$. *Then* $\sigma|_P$ *is an index-preserving projectivity from* P *to* $\bar{P}$. *Also either* G/P *and* $\bar{G}/\bar{P}$ *are torsion-free locally cyclic groups or* σ *is an index-preserving projectivity and* G/P *and* $\bar{G}/\bar{P}$ *are extensions of torsion-free locally cyclic groups by an involution which acts by inversion. And if* $\bar{G}$ *is soluble, then so is* G *with* $d(G)$ *bounded by a function of* $d(\bar{G})$.

(ii) *If* $\bar{G}$ *is nilpotent with* $h(\bar{G}) \geqslant 2$, *then* σ *is an index-preserving projectivity; and* σ *is induced by an isomorphism from* G *to* $\bar{G}$ *if* $\bar{G}$ *is abelian.*

Proof. (i) The first statement follows from Lemma 2.3. For the rest, by Lemma 2.3 and [11], we may assume that $P = \bar{P} = 1$.

Let $\bar{A}$ be a maximal normal abelian subgroup of $\bar{G}$. Then $\bar{A}$ $(\neq 1)$ is torsion-free and locally cyclic. Also $\bar{G}/\bar{A}$ is locally finite and if $\bar{C} = C_{\bar{G}}(\bar{A})$, then $|\bar{G} : \bar{C}| \leqslant 2$. Moreover, $\bar{C}'$ is locally finite and hence 1. Therefore $\bar{C} = \bar{A}$. If $\bar{A} = \bar{G}$, then G is torsion-free and locally cyclic, by Lemma 2.4.

Thus suppose that $|\bar{G} : \bar{A}| = 2$. If $\bar{g} \in \bar{G} \backslash \bar{A}$, then $\bar{g}$ must act by inversion on $\bar{A}$ and so $|\bar{g}| = 2$. Using a local argument, it follows from Lemma 2.5(i) and (ii) that if $\langle g \rangle^\sigma = \langle \bar{g} \rangle$, then $|g| = 2$. Now we may assume that G is finitely generated, so $\bar{G}$ is an infinite dihedral group. As in the proof of Lemma 2.5(ii), there are subgroups $\bar{T}_\lambda \lhd \bar{G}$ such that $\bar{T}_\lambda < \bar{A}$, $\bar{G}/\bar{T}_\lambda$ is a dihedral 2-group,

$$\bigcap_\lambda \bar{T}_\lambda = 1,$$

and, with $T_\lambda = \bar{T}_\lambda^{\sigma^*}$, we have $T_\lambda \lhd G$, G/T_λ is a finite 2-group, and σ induces an l-epimorphism from G/T_λ to $\bar{G}/\bar{T}_\lambda$.

If σ is not injective on G/T_λ, then G/T_λ must be generalized quaternion, by [9, Proposition 3.4, p. 70], and we obtain a contradiction as in Lemma 2.5(i). Thus σ induces a projectivity from G/T_λ to $\bar{G}/\bar{T}_\lambda$. Since $\bar{G}/\bar{T}_\lambda$ is generated by two involutions, so is G/T_λ; i.e., G/T_λ is a dihedral 2-group, for all λ.

Let $A = \bar{A}^{\sigma^*}$. Then $A \lhd G$, A is abelian, and if $a \in A$, then $aa^g \in T_\lambda$, all λ. Hence $a^g = a^{-1}$. Let $K < H \leqslant A$ with $|H : K| = p$ (prime). Then $\langle H, g \rangle / K$ is dihedral of order $2p$ and σ must induce a projectivity from $\langle H, g \rangle / K$ to $\langle \bar{H}, \bar{g} \rangle / \bar{K}$. Therefore these two quotients are isomorphic and hence $|\bar{H} : \bar{K}| = p$. Thus $\sigma|_A$ is injective and index-preserving and therefore so is σ (by 1.3 and 1.4). Finally, we see that A is infinite cyclic and G is infinite dihedral.

(ii) To show that σ is an index-preserving projectivity, we may assume that $\bar{G}$ is finitely generated (by 1.1 and 1.2). Also, by Lemma 2.3, we may assume that $P = \bar{P} = 1$. Then $\bar{G}$ is residually a finite p-group, for all primes p (see [3]); and, by Lemma 2.5(i), it follows that σ is an index-preserving projectivity.

That σ is induced by an isomorphism when $\bar{G}$ is abelian is a well-known theorem of Baer (see [9, Theorem 3, p. 35]). ∎

A further case when the solubility of G can be deduced from that of $\bar{G}$ is

LEMMA 2.7. *Suppose that $\bar{G}$ is soluble, residually finite, and not periodic. Then G is soluble and $d(G)$ is bounded by a function of $d(\bar{G})$.*

Proof. By Lemma 2.6(i), we may assume that $h(\bar{G}) \geqslant 2$. By hypothesis there are normal subgroups $\bar{N}_\lambda$ $(\lambda \in \Lambda)$ of $\bar{G}$ with $\bar{G}/\bar{N}_\lambda$ finite and $\bigcap_\lambda \bar{N}_\lambda = 1$. Also we may assume that each $\bar{G}/\bar{N}_\lambda$ is not cyclic (by Lemma 2.6(ii)).

Let $\bar{C}_\lambda/\bar{N}_\lambda$ be the intersection of all the maximal cyclic subgroups of $\bar{G}/\bar{N}_\lambda$. Thus $[\bar{G}, \bar{C}_\lambda] \leqslant \bar{N}_\lambda$. Take $C_\lambda = \bar{C}_\lambda^{\sigma^*}$. By [15, Corollary 1.3], $|G : C_\lambda|$ is finite. Let $T_\lambda = (C_\lambda)_G$. Then $\bar{T}_\lambda \lhd \bar{G}$, by Lemma 2.1. Now the lower kernel of $\sigma|_{G/T_\lambda}$ is contained in C_λ/T_λ and is normal in G/T_λ. Therefore the lower kernel is 1. It follows from [9, Proposition 3.8, p. 73; 11] that G/T_λ is soluble with $d(G/T_\lambda)$ bounded by a function f of $d(\bar{G}) = n$, say.

Let $\bar{T} = \bigcap_\lambda \bar{T}_\lambda$ and $\bar{C} = \bigcap_\lambda \bar{C}_\lambda$. So $[\bar{G}, \bar{C}] = 1$ and hence $[\bar{G}, \bar{T}] = 1$. Let $T = \bigcap_\lambda T_\lambda$. Thus G/T is soluble and $d(G) \leqslant f(n)$. If T is not periodic, then $\bar{T}$ is not periodic and Lemma 2.6 applied to $\sigma|_T$ gives T soluble with $d(T)$ bounded. On the other hand, if T is periodic, then $T \leqslant \mathbf{P}(G) = P$, say; and, by Lemma 2.3, $\sigma|_P$ is a projectivity from P to $\mathbf{P}(\bar{G})$, hence T is metabelian. ∎

Recent work (as yet unpublished) by G. Busetto and F. Napolitani shows that $f(n) = 4n$ suffices.

3. LATTICE HOMOMORPHISMS AND INJECTIVITY

In our main results the critical situation is that of an infinite cyclic extension G of an abelian group A and we begin with the case $h(A) = 1$. We need information about the G-action on certain chief factors of prime order lying in A. The following elementary fact will be required.

LEMMA 3.1. *Let m, n be relatively prime integers and p be a prime dividing m. If q is a prime dividing*

$$m^{p-1}+m^{p-2}n+\cdots+n^{p-1},$$

then $q \nmid mn$ and $q \nmid (m-n)$.

Proof. The first statement is clear. Assume, for a contradiction, that $q \mid (m-n)$. Then, for all k, $1 \leqslant k \leqslant p-1$,

$$q \text{ divides } (km^{p-k}n^{k-1}-km^{p-k-1}n^{k}).$$

Adding for all k gives

$$q \text{ divides } (m^{p-1}+m^{p-2}n+\cdots+mn^{p-2}-(p-1)n^{p-1})$$

and therefore by hypothesis $q \mid pn^{p-1}$. Since $p \mid m$ and $q \nmid mn$, $q \neq p$. Then $q \mid n$, a contradiction. ∎

Now we can prove

THEOREM 3.2. *Let G be a group, $A \lhd G$ with A abelian, $h(A)=1$ and G/A infinite cyclic. If τ is a non-trivial complete l-homomorphism from G to some complete lattice, then $\tau|_{G/A}$ is injective.*

Proof. By (3) in Section 1, the lower kernel of τ is 1 and so $\tau|_{G/A}$ is non-trivial. In particular, if T is the torsion subgroup of A, then $\tau|_{G/T}$ is non-trivial and so we may assume that $T=1$.

Suppose, for a contradiction, that $\tau|_{G/A}$ is not injective. For some element $g \in G$ we have

$$G = A \rtimes \langle g \rangle$$

and without loss of generality we may assume that

$$\langle g \rangle^{\tau} = \langle g^{p} \rangle^{\tau} \tag{1}$$

for some prime p. Identify A with an additive subgroup of $\mathbb{Q}$ containing 1. The conjugation action of g on A is multiplication by some rational m/n with $(m,n)=1$. Suppose that $m/n = \pm 1$ and let $a \in A$, $a \neq 0$. Then $H=\langle a, g \rangle$ is metacyclic with two independent elements of infinite order and $\tau|_{H}$ is non-trivial. Thus $\tau|_{H}$ is injective, by 1.6, contradicting (1). Therefore we may assume that

$$m/n \neq \pm 1.$$

Since $(m/n)A = A$, it follows that $A \geqslant \mathbb{Z}[1/mn] \lhd G$ and so we may assume that

$$A = \mathbb{Z}[1/mn].$$

We distinguish two possibilities:

Case 1. *Suppose that* $p \nmid mn$. Thus $pA < A$. Let

$$C_{\langle g \rangle}(A/pA) = \langle g^l \rangle.$$

By 1.6, τ is injective on A/pA and then it is easy to see that τ must be injective on

$$A\langle g^l \rangle/(pA)\langle g^{pl} \rangle \cong C_p \times C_p.$$

Hence

$$\langle g^l \rangle^\tau > \langle g^{pl} \rangle^\tau. \tag{2}$$

However, $(l, p) = 1$ and so

$$\begin{aligned}\langle g^l \rangle^\tau &= \langle g \rangle^\tau \cap \langle g^l \rangle^\tau = \langle g^p \rangle^\tau \cap \langle g^l \rangle^\tau \qquad \text{(by (1))}\\ &= (\langle g^p \rangle \cap \langle g^l \rangle)^\tau = \langle g^{pl} \rangle^\tau,\end{aligned}$$

contradicting (2).

Case 2. *Suppose now that* $p \mid mn$. Then $pA = A$. Replacing g by g^{-1} if necessary, we may suppose that $m \geqslant 2$. Thus if $n > 0$, then

$$m^{p-1} + m^{p-2}n + \cdots + n^{p-1} \neq \pm 1. \tag{3}$$

If $n < 0$ and $p \neq 2$, we may replace g by g^2 and then (3) holds. On the other hand, if $p = 2$ and $m + n = \pm 1$, it is easy to see that $m^3 + n^3 \neq \pm 1$ and, replacing g by g^3, again (3) holds. Thus in all cases there is a prime q dividing $m^{p-1} + m^{p-2}n + \cdots + n^{p-1}$.

By Lemma 3.1,

$$q \nmid mn \qquad \text{and} \qquad q \nmid (m - n). \tag{4}$$

Thus $|A/qA| = q$ and A/qA is generated by $1 + qA$. Since

$$\frac{m}{n} - 1 = \frac{m-n}{n} \notin qA$$

(by (4)), we see that

g acts non-trivially on A/qA.

However, $(m/n)^p - 1 = (m^p - n^p)/n^p \in qA$ and hence g^p acts trivially on A/qA. Therefore $(qA)\langle g^p \rangle \lhd G$ and $G/(qA)\langle g^p \rangle$ is non-abelian of order pq. A non-trivial l-homomorphism of such a group is injective; and τ is injective on A/qA, by 1.6. Hence $\langle g \rangle^\tau > \langle g^p \rangle^\tau$, contradicting (1). ▌

In the situation of Theorem 3.2 we can show that, in fact, distinct subgroups of G with the same image under τ must lie in A. This will follow from

LEMMA 3.3. *Let $N \lhd G$ with G/N ($\neq 1$) torsion-free and let τ be a complete l-homomorphism from G to some complete lattice such that $\tau|_{G/N}$ is injective. If $X \neq Y$ are subgroups of G with $X^\tau = Y^\tau$, then $\langle X, Y \rangle \leqslant N$.*

Proof. Without loss of generality we may assume that $X < Y$ and, by choosing X to be the minimal preimage of X^τ under τ, we have $X \lhd Y$, by [9, Theorem 1, p. 58]. Now $(YN)^\tau = (XN)^\tau$ and so $XN = YN$ by hypothesis. Therefore $X \cap N < Y \cap N$. Suppose, for a contradiction, that $Y \nleqslant N$. Then $X \nleqslant N$, otherwise $X^\tau \leqslant N^\tau$, whereas $Y^\tau \nleqslant N^\tau$ by hypothesis.

Choose $x \in X \backslash N$ and $y \in (Y \cap N) \backslash X$. Then

$$N_1 = X \cap N \lhd \langle x \rangle (\langle X, y \rangle \cap N) = G_1,$$

say, and

$$N_2 = \langle X, y \rangle \cap N \lhd G_1.$$

Using bars to denote factors modulo N_1, we have

$$\bar{G}_1 = \bar{N}_2 \rtimes \overline{\langle x \rangle},$$

$\bar{N}_2$ ($\neq 1$) is cyclic, and $\overline{\langle x \rangle} \cong C_\infty$. Moreover, by hypothesis, τ is injective on $\overline{\langle x \rangle}$ and so τ is injective on $\bar{N}_2$, by 1.6. Therefore,

$$(X \cap N)^\tau = (Y \cap N)^\tau \geqslant N_2^\tau > N_1^\tau = (X \cap N)^\tau,$$

giving the required contradiction. ▌

Combining Theorem 3.2 and Lemma 3.3 gives

COROLLARY 3.4. *Assume the hypotheses of Theorem 3.2 and let $X \neq Y$ be subgroups of G with $X^\tau = Y^\tau$. Then $\langle X, Y \rangle \leqslant A$.*

In Section 6 we shall construct examples which show that, under the hypotheses of Theorem 3.2, τ need not be injective on A. However, when the torsion-free rank of A is at least 2 (i.e., when $h(A) \geqslant 2$), then τ has to be injective. In order to prove this, we need a result of Ivanov [5, Lemma 5]:

LEMMA 3.5. *Let $\xi \in \mathbb{C}$, $\xi \neq 0$, and let p be a prime. Then there is a positive integer k, not divisible by p, such that*

$$1 + \xi^k + \xi^{2k} + \cdots + \xi^{(p-1)k}$$

is not invertible in $\mathbb{Z}[\xi, \xi^{-1}]$.

Then we have

THEOREM 3.6. *Let G be a group, $A \lhd G$, with A abelian, $h(A) \geqslant 2$ and G/A infinite cyclic. If τ is a non-trivial complete l-homomorphism from G to some complete lattice, then τ is injective.*

Proof. By (3) in Section 1, the lower kernel of τ is 1. Thus $\tau|_A$ is injective, by 1.6. Therefore, using 1.3, we may assume that A is torsion-free. Assume, for a contradiction, that τ is not injective. Then, by 1.1, there is an element $x \in G$, $x \neq 1$, and a prime p such that

$$\langle x \rangle^\tau = \langle x^p \rangle^\tau. \tag{5}$$

Thus $x \notin A$ and so $\langle x \rangle \cap A = 1$.

Let $x(\neq 1)$ be any element of G satisfying (5). If k is an integer not divisible by p, then (as in the argument of case 1 of Theorem 3.2)

$$\langle x^k \rangle^\tau = \langle x^{pk} \rangle^\tau. \tag{6}$$

Let $a \in A$. We claim that

$$\langle xa \rangle^\tau = \langle (xa)^p \rangle^\tau. \tag{7}$$

For,

$$\begin{aligned}\langle xa \rangle^\tau &= \langle x, A \rangle^\tau \cap \langle xa \rangle^\tau = \langle x^p, A \rangle^\tau \cap \langle xa \rangle^\tau \\ &= (\langle x^p, A \rangle \cap \langle xa \rangle)^\tau = \langle (xa)^p \rangle^\tau \qquad \text{(by (5))}.\end{aligned}$$

Also we have

$$\langle a \rangle^{\langle x \rangle} = \langle a \rangle^{\langle x^p \rangle}. \tag{8}$$

To see this,

$$\begin{aligned}(\langle a \rangle^{\langle x \rangle})^\tau &\leqslant \langle a, x \rangle^\tau \cap A^\tau = \langle a, x^p \rangle^\tau \cap A^\tau \\ &= (\langle a, x^p \rangle \cap A)^\tau = (\langle a \rangle^{\langle x^p \rangle})^\tau \qquad \text{(by (5))}.\end{aligned}$$

Therefore, since $\tau|_A$ is injective,

$$\langle a \rangle^{\langle x \rangle} \leqslant \langle a \rangle^{\langle x^p \rangle}$$

and (8) follows.

Now suppose that $a \neq 1$ and let k be an integer not divisible by p. Then

$$(x^k a)^p = x^{pk} c_k,$$

where

$$c_k = a^{1 + x^k + x^{2k} + \cdots + x^{(p-1)k}}.$$

Let

$$A_k = \langle a \rangle^{\langle x^k \rangle}.$$

We claim that

$$A_k = \langle c_k \rangle^{\langle x^{pk} \rangle}. \tag{9}$$

For,

$$\begin{aligned}\langle x^k, x^k a \rangle^\tau &= \langle \langle x^{pk} \rangle^\tau, \langle (x^k a)^p \rangle^\tau \rangle \\ &= \langle x^{pk}, x^{pk} c_k \rangle^\tau = \langle x^{pk}, c_k \rangle^\tau \qquad \text{(by (6) and (7))}.\end{aligned}$$

Intersecting with A_k^τ gives

$$A_k^\tau = \langle x^{pk}, c_k \rangle^\tau \cap A_k^\tau = (\langle x^{pk}, c_k \rangle \cap A_k)^\tau$$

and, hence,

$$A_k = \langle x^{pk}, c_k \rangle \cap A_k = \langle c_k \rangle^{\langle x^{pk} \rangle},$$

giving (9).

Next we show that

$$h(A_1) \text{ is finite.} \tag{10}$$

For, by (8), there is a non-zero polynomial φ over $\mathbb{Z}$ such that

$$a^{\varphi(x)} = 1.$$

Let φ have degree m and let

$$A_0 = \langle a^{x^i} \mid |i| \leqslant m - 1 \rangle.$$

Then A_0 is finitely generated and A_1/A_0 is periodic. Hence (10) is true.

For any integer k, let

$$q_k(t) = 1 + t^k + t^{2k} + \cdots + t^{(p-1)k}$$

(t indeterminate) and from now on view A_1 as an $\langle x \rangle$-module. Then A_1 embeds in

$$V = \mathbb{C} \otimes_{\mathbb{Z}} A_1,$$

a finite dimensional $\mathbb{C}$-space, by (10). Let γ denote the non-singular linear map induced by x on V. If $p \nmid k$, then, by (9), there is an element

$$f_k(t) \in \mathbb{Z}[t, t^{-1}]$$

such that

$$a = c_k \cdot f_k(\gamma^{pk}) = a \cdot q_k(\gamma) f_k(\gamma^{pk}).$$

Therefore $q_k(\gamma) f_k(\gamma^{pk})$ is the identity map on V. Thus if ξ is an eigenvalue of γ, then we have

$$q_k(\xi) f_k(\xi^{pk}) = 1,$$

i.e.,

$$q_k(\xi)^{-1} \in \mathbb{Z}[\xi, \xi^{-1}]$$

for all k not divisible by p, contradicting Lemma 3.5. ∎

We conclude this section with a generalization of Ivanov's main result in [5].

THEOREM 3.7. *Let G be a non-periodic group and let H be a proper, non-trivial subgroup of G such that the map*

$$X \mapsto H \cap X$$

(for all $X \leqslant G$) is an l-endomorphism of G. If $H \in L\mathfrak{X}$, then all the periodic elements of G lie in H and form a subgroup and $h(H) = 1$. Moreover, if $G \in L\mathfrak{X}$ and H is torsion-free, then G is locally cyclic.

Remark. A subgroup of a group G has been called *dually standard* (d.s.) if the map $X \mapsto H \cap X$ (all $X \leqslant G$) is an l-endomorphism of G. Ivanov established the special case of Theorem 3.7 when G is locally soluble and torsion-free.

Proof of Theorem 3.7. Let τ be the complete l-epimorphism defined by $X \mapsto H \cap X$. Then H is the upper kernel of τ and so $H \lhd G$, by [9, Theorem 1, p. 58]. Since the lower kernel of τ is 1 (by (3) in Section 1), we must have G/H periodic and hence H is not periodic. Then, by Lemma 2.3, τ is injective on $\mathbf{P}(G)$, therefore $\mathbf{P}(G) \leqslant H$ and so $\mathbf{P}(G) = \mathbf{P}(H)$. Thus we may assume that $\mathbf{P}(G) = 1$.

If $K \leqslant G$, then clearly $H \cap K$ is a d.s. subgroup of K. Therefore, from now on, we may assume that G is finitely generated and G/H is finite. Thus H is finitely generated and (assuming that $H \in L\mathfrak{X}$) we see that $G \in \mathfrak{X}$. It suffices to show that G is cyclic.

Now H contains a torsion-free abelian subgroup A ($\neq 1$) with $A \lhd G$. It follows, from 1.6, that the periodic elements of G must lie in H. Choose $g \in G \setminus H$. By Theorems 3.2 and 3.6, $\langle g, A \rangle / A$ is finite. Thus $h(A) = 1$, by 1.6. Since H/A is a d.s. subgroup of G/A, it follows from what we have already proved that H/A is periodic. Therefore $h(H) = 1$ and then G is an infinite cyclic group, by Corollary 2.6(i). ▌

If H is not torsion-free, then G need not be locally cyclic. For, let

$$G = \langle a \rangle \times \langle b \rangle,$$

where $|a| = \infty$ and $|b| = p$ (prime); and let

$$H = \langle a^q \rangle \times \langle b \rangle,$$

where q is a prime different from p. Then H *is a d.s. subgroup of* G. For, if U, $V \leqslant G$, we must show that

$$H \cap UV = (H \cap U)(H \cap V).$$

This is clear if U or V lies in H. On the other hand, if $W \nleqslant H$, then $|W/W^q| = q$ and so $H \cap W = W^q$. Thus our claim follows.

4. Index-Preserving Projectivities

Suppose that the group G has a normal abelian subgroup A with G/A infinite cyclic. If τ is a non-trivial complete l-homomorphism of G, then we have seen that $\tau|_{G/A}$ is injective if $h(A) = 1$ (Theorem 3.2) and τ is injective if $h(A) \geqslant 2$ (Theorem 3.6). Again these will be the critical cases in Theorem A when trying to show certain projectivities are index-preserving. Indeed the principal results of this section should be compared with Theorems 3.2 and 3.6.

Theorem 4.1. *Let G be a group, $A \lhd G$, A abelian with $h(A) \geqslant 1$ and G/A infinite cyclic. Then every non-trivial complete l-epimorphism from G to a group $\bar{G}$ is index-preserving on G/A.*

In order to prove this theorem we need a result about algebraic numbers bearing the flavour of Lemma 3.5. This in turn requires Lemma 8 of [5] which we now state.

Lemma 4.2. *Let $\mathbb{Z}_+$ denote the additive semigroup of positive integers. Suppose that $\varphi_1, ..., \varphi_s$ are homomorphisms from $\mathbb{Z}_+$ to the multiplicative* *group E^* of some field E and let $z \in \mathbb{Z}_+$. Suppose also that there are elements $c_1, ..., c_s \in E^*$ such that*

$$c_1 \varphi_1(x) + \cdots + c_s \varphi_s(x) = 0$$

for all $x \in \mathbb{Z}_+$ with $(x, z) = 1$. Then there exists a partition

$$\{1, ..., s\} = P_1 \cup \cdots \cup P_w$$

with P_i non-empty $(1 \leqslant i \leqslant w)$ such that

(i) $\sum_{j \in P_i} c_j \varphi_j(x) = 0$, *for all x relatively prime to z; and*

(ii) *if $j, k \in P_i$ (any i), then there is a zth root ε of 1 in E such that*

$$\varphi_j(x) = \varphi_k(x) \cdot \varepsilon^x$$

for all $x \in \mathbb{Z}_+$.

In this situation we say that φ_j and φ_k *differ by a zth root of* 1. Then we can prove

LEMMA 4.3. *Suppose that ξ, λ are non-periodic elements of $\mathbb{C}^*$ with*

$$\mathbb{Z}[\xi, \xi^{-1}] = \mathbb{Z}[\lambda, \lambda^{-1}] = S,$$

say. Let π be a non-identity element of the restricted symmetric group on the set of all primes and let

$$k \mapsto k'$$

be the unique extension of π to an automorphism of the multiplicative semigroup $\mathbb{Z}_+$. For $k, l \in \mathbb{Z}_+$, write

$$\eta_{k,l} = \frac{1 + \xi^k + \xi^{2k} + \cdots + \xi^{(l-1)k}}{1 + \lambda^{k'} + \lambda^{2k'} + \cdots + \lambda^{(l'-1)k'}}.$$

(Note that numerator and denominator here are non-zero.) Then there are positive integers k, l such that either $\eta_{k,l} \notin S$ or $\eta_{k,l}^{-1} \notin S$.

Proof. Suppose, for a contradiction, that the lemma is false. Then, for all $k, l \in \mathbb{Z}_+$,

$$\eta_{k,l}, \eta_{k,l}^{-1} \in S. \tag{1}$$

First we show that

$$\xi \text{ and } \lambda \text{ are algebraic numbers.} \tag{2}$$

Clearly $\mathbb{Q}(\xi)=\mathbb{Q}(\lambda)$ and so ξ is algebraic if and only if λ is algebraic. Suppose, for a contradiction, that ξ and λ are transcendental. Then

$$\xi=\frac{a\lambda+b}{c\lambda+d}, \tag{3}$$

where a, b, c, $d\in\mathbb{Z}$. (See [10, Section 63, p. 198].) By hypothesis there are prime numbers $p<q$ such that $p'=q$. Taking $k=k'=1$ and $l=p$, $l'=q$, we have

$$1+\xi+\xi^2+\cdots+\xi^{p-1}=(1+\lambda+\lambda^2+\cdots+\lambda^{q-1})\eta_{1,p}. \tag{4}$$

Also $\eta_{1,p}=g(\lambda)/\lambda^s$, where $g(\lambda)\in\mathbb{Z}[\lambda]$ and $s\geqslant 0$. Substituting for ξ from (3) in (4), we obtain

$$\begin{aligned}[]&[(c\lambda+d)^{p-1}+(a\lambda+b)(c\lambda+d)^{p-2}+\cdots+(a\lambda+b)^{p-1}]\lambda^s\\ &\quad=(c\lambda+d)^{p-1}(1+\lambda+\lambda^2+\cdots+\lambda^{q-1})\,g(\lambda).\end{aligned} \tag{5}$$

Thus $s\neq 0$. If $s\geqslant 1$, we may assume that λ does not divide $g(\lambda)$ and therefore $s\leqslant p-1$. But then the two sides of (5) do not have the same degree. Thus (2) follows.

Let $K_0=\mathbb{Q}(\xi)$. By [1, Theorem 20.14, p. 130], there is a finite extension field of K_0, say K, with ring I of integers such that every element ζ of K_0 can be written as a quotient ζ_1/ζ_2 with ζ_1, ζ_2 relatively prime integers of I. Thus

$$\xi=\xi_1/\xi_2 \qquad\text{and}\qquad \lambda=\lambda_1/\lambda_2.$$

Let

$$\gamma_{kl}=\xi_1^{(l-1)k}+\xi_1^{(l-2)k}\xi_2^k+\cdots+\xi_2^{(l-1)k} \tag{6}$$

and

$$\delta_{k'l'}=\lambda_1^{(l'-1)k'}+\lambda_1^{(l'-2)k'}\lambda_2^{k'}+\cdots+\lambda_2^{(l'-1)k'}.$$

Then γ_{kl} and $\delta_{k'l'}$ are non-zero elements of I; and

$$1+\xi^k+\xi^{2k}+\cdots+\xi^{(l-1)k}=\gamma_{kl}/\xi_2^{(l-1)k},$$

$$1+\lambda^{k'}+\lambda^{2k'}+\cdots+\lambda^{(l'-1)k'}=\delta_{k'l'}/\lambda_2^{(l'-1)k'}.$$

We claim that, for all k, $l\in\mathbb{Z}_+$,

$$\gamma_{kl}\text{ and }\xi_1\xi_2\text{ are relatively prime.} \tag{7}$$

For, if this is not the case, then there is a prime ideal P of I containing γ_{kl}

and $\xi_1\xi_2$. Thus P contains ξ_1 or ξ_2 and suppose, without loss of generality, that $\xi_1 \in P$. Then from (6) it follows that $\xi_2 \in P$, contradicting the fact that ξ_1 and ξ_2 are relatively prime. Therefore (7) is true and similarly

$$\delta_{k'l'} \text{ and } \lambda_1\lambda_2 \text{ are relatively prime,} \tag{8}$$

for all $k', l' \in \mathbb{Z}_+$.

Next we prove that

$$\gamma_{kl}/\delta_{k'l'} \text{ is a unit of } I, \tag{9}$$

for all $k, l \in \mathbb{Z}_+$. For, we can write

$$I\gamma_{kl}\delta_{k'l'}^{-1} = P_1 \cdots P_r Q_1^{-1} \cdots Q_s^{-1}, \tag{10}$$

where P_i, Q_j are prime ideals of I, $P_i \neq Q_j$ for all i, j, and

$$\gamma_{kl} \in \bigcap_{i=1}^{r} P_i, \quad \delta_{k'l'} \in \bigcap_{j=1}^{s} Q_j.$$

By (1),

$$\eta_{k,l} = \gamma_{kl}\lambda_2^{(l'-1)k'}/\delta_{k'l'}\xi_2^{(l-1)k} \in S = \mathbb{Z}[\lambda, \lambda^{-1}].$$

Therefore $\eta_{k,l} = f(\lambda_1, \lambda_2)/\lambda_1^m\lambda_2^n$, for some $f(\lambda_1, \lambda_2) \in \mathbb{Z}[\lambda_1, \lambda_2]$, $m, n \geqslant 0$. Then from (10),

$$P_1 \cdots P_r \lambda_1^m \lambda_2^{(l'-1)k'+n} = Q_1 \cdots Q_s \xi_2^{(l-1)k} f(\lambda_1, \lambda_2).$$

If $s \geqslant 1$, then each Q_j contains $\lambda_1\lambda_2$ along with $\delta_{k'l'}$, contradicting (8). Therefore $s = 0$. Similarly, since

$$\eta_{k,l}^{-1} \in S = \mathbb{Z}[\xi, \xi^{-1}],$$

we have $r = 0$. Thus (9) follows from (10).

The major step now is to show that

$$\xi_1, \xi_2, \lambda_1 \text{ and } \lambda_2 \text{ are units of } I. \tag{11}$$

Then it will follow that ξ and λ are also units of I. To prove (11), let E be the normal closure of K over $\mathbb{Q}$. By (9), $\gamma_{kl}/\delta_{k'l'}$ is a unit in the ring of integers of E, for all $k, l \in \mathbb{Z}_+$. Hence, writing N for the norm of $E/\mathbb{Q}$, we have

$$N(\gamma_{kl}) = \pm N(\delta_{k'l'}). \tag{12}$$

Let $\Gamma = \{\tau_1, ..., \tau_t\}$ be the Galois group of $E/\mathbb{Q}$ and write

$$\tau_i(\xi_1) = \xi_{1i}, \qquad \tau_i(\xi_2) = \xi_{2i}.$$

Then from (6)

$$N(\gamma_{kl}) = \prod_{i=1}^{l} \tau_i\left(\sum_{j=1}^{l} \xi_1^{(l-j)k}\xi_2^{(j-1)k}\right) = \sum_{j=1}^{l} \tilde{\varphi}_j(k),$$

where

$$\tilde{\varphi}_j(k) = \prod_{i=1}^{l} \xi_{1i}^{(l-j)k}\xi_{2i}^{(j-1)k} = N(\xi_1)^{(l-j)k}\, N(\xi_2)^{(j-1)k} \in \mathbb{Z}.$$

Similarly, with $\tau_i(\lambda_1) = \lambda_{1i}$ and $\tau_i(\lambda_2) = \lambda_{2i}$,

$$N(\delta_{k'l'}) = \sum_{j=1}^{l'} \tilde{\psi}_j(k'),$$

where

$$\tilde{\psi}_j(k') = \prod_{i=1}^{l} \lambda_{1i}^{(l'-j)k'}\lambda_{2i}^{(j-1)k'} = N(\lambda_1)^{(l'-j)k'}\, N(\lambda_2)^{(j-1)k'} \in \mathbb{Z}.$$

(Observe that l can be arbitrary, but the maps $\tilde{\varphi}_j$, $\tilde{\psi}_j$ depend on l, l', respectively.)

Let k_0 be the smallest even positive integer such that $k_0' = k_0$ and let z be the product of the primes p for which $p' \neq p$. For all positive integers k relatively prime to z, we have $(k_0 k)' = k_0 k$. For these values of k, define

$$\varphi_j(k) = \tilde{\varphi}_j(k_0 k) = \tilde{\varphi}_j(k)^{k_0}, \qquad 1 \leqslant j \leqslant l,$$
$$\psi_j(k) = \tilde{\psi}_j(k_0 k) = \tilde{\psi}_j(k)^{k_0}, \qquad 1 \leqslant j \leqslant l'.$$

Then

$$\varphi_j(k), \psi_j(k) \in \mathbb{Z}_+. \tag{13}$$

Therefore, from (12),

$$\sum_{j=1}^{l} \varphi_j(k) = \sum_{j=1}^{l'} \psi_j(k),$$

for all k relatively prime to z.

Now choose $l > l'$. Then, by Lemma 4.2, there are integers j_1, j_2, $1 \leqslant j_1 \neq j_2 \leqslant l$, such that φ_{j_1} and φ_{j_2} differ by a zth root of 1. Thus, by (13),

$$\varphi_{j_1}(1) = \varphi_{j_2}(1).$$

Therefore,

$$N(\xi_1)^{(l-j_1)k_0}\, N(\xi_2)^{(j_1-1)k_0} = N(\xi_1)^{(l-j_2)k_0}\, N(\xi_2)^{(j_2-1)k_0};$$

i.e.,

$$N(\xi_1)^{(j_2-j_1)k_0} = N(\xi_2)^{(j_2-j_1)k_0}$$

and hence

$$N(\xi_1) = \pm N(\xi_2). \tag{14}$$

Similarly choosing $l < l'$, we obtain

$$N(\lambda_1) = \pm N(\lambda_2). \tag{15}$$

Again by Lemma 4.2 there are integers j_1, j_2, $1 \leqslant j_1 \leqslant l$, $1 \leqslant j_2 \leqslant l'$, such that φ_{j_1} and ψ_{j_2} differ by a zth root of 1. Thus, from (13), we have

$$\varphi_{j_1}(1) = \psi_{j_2}(1);$$

and (14) and (15) then give

$$N(\xi_1)^{(l-1)k_0} = N(\lambda_1)^{(l'-1)k_0}.$$

Choosing $l = l' \neq 1$, we get $N(\xi_1) = \pm N(\lambda_1)$. Then choosing $l \neq l'$, it follows that

$$N(\xi_1) = \pm 1 = N(\lambda_1);$$

and therefore, by (14) and (15),

$$N(\xi_2) = \pm 1 = N(\lambda_2).$$

Thus (11) is proved and so

$$\xi, \lambda \text{ are units of } I.$$

Finally we see now that S is a free additive group of finite rank (equal to the degree of ξ over $\mathbb{Z}$). Let p be a prime such that $p' = q \neq p$ and let $\bar{S} = S/pS$, a finite ring of characteristic p. By our initial assumption (1),

$$\bar{\eta}_{k,l} \text{ is an invertible element of } \bar{S}. \tag{16}$$

But there are integers $n_1, n_2 \in \mathbb{Z}_+$ such that

$$\bar{\xi}^{n_1} = \bar{\lambda}^{n'_2} = \bar{1}.$$

Choose $k = n_1 n_2$. Then

$$k' = n'_1 n'_2$$

and taking $l = p$ (and hence $l' = q$), we have

$$\bar{\eta}_{k,l} = p\bar{1}/q\bar{1} = \bar{0},$$

contradicting (16). This concludes the proof of Lemma 4.3. ∎

Now we can prove the main result of this section.

Proof of Theorem 4.1. We have $A \lhd G$ with A abelian, $h(A) \geqslant 1$, and G/A infinite cyclic. We must show that any complete l-epimorphism σ from G to a group $\bar{G}$ ($\neq 1$) is index-preserving on G/A. By Theorems 3.2 and 3.6, $\sigma|_{G/A}$ is injective.

Let bars denote images of subgroups of G under σ. If B is any subgroup of A which is normal in G, then $\bar{B} \lhd \bar{G}$, by [13, Proposition 1.6]. Thus we may assume that A is torsion-free and so $\bar{A}$ is also torsion-free and therefore abelian ([6; see also 9, Proposition 1.12, p. 19]). Then $h(A) = h(\bar{A})$ and $\bar{A} \lhd \bar{G}$. Also, by Theorem 3.6, σ is a projectivity if $h(A) \geqslant 2$; and in this case $\sigma|_A$ is induced by an isomorphism $A \to \bar{A}$, by Baer's theorem (see [9, Theorem 3, p. 35]). Let

$$G = A \rtimes \langle g \rangle \qquad \text{and} \qquad \bar{G} = \bar{A} \rtimes \langle \bar{g} \rangle,$$

where $\langle \bar{g} \rangle = \langle g \rangle^{\sigma}$. We may assume that A is a cyclic $\langle g \rangle$-module, generated by a_0, say. Suppose that $h(A)$ is infinite. Then the elements

$$a_0^{g^i} \qquad (i \in \mathbb{Z})$$

are independent and so A is free abelian. Since, for any prime p, G/A^p is residually finite [4], there is a normal subgroup V of G with $A^p < V < A$ and A/V finite. Then $\bar{V} \lhd \bar{G}$ and, since G/V is polycyclic, σ preserves p-indices in G/V [13, Corollary 2.10(ii)]. Thus σ is index-preserving on G/A, as required.

From now on we may assume that

$$h(A) \text{ is finite}$$

and, by induction on $h(A)$, that

A is rationally irreducible.

Then *A is divisible by only finitely many primes.* (See, for example, the argument in [4, pp. 596, 597].) On the other hand,

if σ is p-singular on G/A, then A is p-divisible.

For, if $A^p < A$, then A/A^p is finite and non-trivial and so σ is injective on G/A^p (by 1.5) and preserves p-indices on G/A^p [13, Corollary 2.10(ii)], hence on G/A. In particular $\sigma|_{G/A}$ *is singular for at most finitely many primes.*

Denote the g-action on A by θ. Then we may assume that, for all $t \in \mathbb{Z}$, $t \neq 0$,

$$(1-\theta^t)|_A \text{ is injective.} \tag{17}$$

For, if not, there exists an element $a \in A$, $a \neq 1$, such that

$$\langle g^t, a \rangle = \langle g^t \rangle \times \langle a \rangle \cong C_\infty \times C_\infty$$

and then, by 1.6, σ is index-preserving on $\langle g^t, a \rangle$, therefore on $\langle g \rangle$ and hence on G/A.

For each $k \in \mathbb{Z}_+$, there exists a unique $k' \in \mathbb{Z}_+$ such that

$$\langle g^k \rangle^\sigma = \langle \bar{g}^{k'} \rangle.$$

The map $k \mapsto k'$ is an automorphism of the multiplicative semigroup $\mathbb{Z}_+$ induced by a permuation of the positive primes with support equal to the (finite) set of primes for which $\sigma|_{G/A}$ is singular. (See [2].) Choose any $a \in A$ and $k \in \mathbb{Z}_+$. Then

$$\langle g^k a \rangle^\sigma = \langle \bar{g}^{k'} \bar{b} \rangle$$

for some $\bar{b} \in \bar{A}$. Now, with $\langle a \rangle^\sigma = \langle \bar{a} \rangle$,

$$\begin{aligned} \langle \bar{g}^{k'}, \bar{a} \rangle &= \langle g^k, a \rangle^\sigma = \langle g^k, g^k a \rangle^\sigma \\ &= \langle \bar{g}^{k'}, \bar{g}^{k'} \bar{b} \rangle = \langle \bar{g}^{k'}, \bar{b} \rangle. \end{aligned}$$

Therefore,

$$\langle \bar{g}, \bar{a} \rangle = \langle \bar{g}, \bar{b} \rangle. \tag{18}$$

Denote the $\bar{g}$-action on $\bar{A}$ by $\bar{\theta}$. Choose $l \in \mathbb{Z}_+$ and let

$$H = \langle g, (g^k a)^l \rangle = \langle g, a(1 + \theta^k + \theta^{2k} + \cdots + \theta^{(l-1)k}) \rangle. \tag{19}$$

So

$$H^\sigma = \langle \bar{g}, (\bar{g}^{k'} \bar{b})^{l'} \rangle = \langle \bar{g}, \bar{b}(1 + \bar{\theta}^{k'} + \bar{\theta}^{2k'} + \cdots + \bar{\theta}^{(l'-1)k'}) \rangle. \tag{20}$$

We distinguish the cases $h(A) = 1$ and $h(A) \geqslant 2$:

Case 1. Suppose that $h(A) = 1$. We may consider A and $\bar{A}$ embedded (as additive subgroups) in $\mathbb{Q}$ with $1 \in A \cap \bar{A}$. Then θ is multiplication in A by a rational number m/n and $\bar{\theta}$ is multiplication in $\bar{A}$ by a rational m_1/n_1. We suppose that $(m, n) = (m_1, n_1) = 1$. By (17),

$$m/n \neq \pm 1.$$

Clearly $mnA = A$ and $m_1n_1\bar{A} = \bar{A}$. Since $U = \mathbb{Z}[1/mn] \leqslant A$ and $U \lhd G$, we may assume that $A = U$.

Let $\bar{B} \leqslant \bar{A}$ with $\bar{B} \lhd \bar{G}$. If $B = \bar{B}\sigma^*$, then we find $B \leqslant A$ and

$$(\langle g, B \rangle \cap A)\sigma = \bar{B}.$$

Hence $B = \langle g, B \rangle \cap A \lhd G$.

We claim that

$$\pi(mn) = \pi(m_1 n_1). \tag{21}$$

For, let $B \lhd G$, $B \leqslant A$. Then $\bar{B} \lhd \bar{G}$ and so $m_1n_1\bar{B} = \bar{B}$. Also, for any prime p,

$$p\bar{B} = \bar{B} \text{ if and only if } pB = B. \tag{22}$$

For, if $pB < B$, then σ is index-preserving on B/pB and if $p\bar{B} < \bar{B}$, then σ is index-preserving on $B/B \cap (p\bar{B})\sigma^*$, whose image under σ is $\bar{B}/p\bar{B}$ (both by 1.6). Therefore (22) follows and by choosing $B = A$, we see that $\pi(m_1n_1) \subseteq \pi(mn)$. Conversely, let $p \in \pi(mn)$ and put $\bar{B} = \mathbb{Z}[1/m_1n_1] \leqslant \bar{A}$. Then $\bar{B} \lhd \bar{G}$, $B = \bar{B}\sigma^* \lhd G$, and $B \leqslant A$. Thus $pB = B$ and so $p \mid m_1n_1$, by (22). Hence (21) holds.

Observe that if a prime $p \nmid mn$, then it follows easily that $\sigma|_A$ preserves all p-indices (using 1.5). Now suppose that some p-index (p prime) in A maps under σ to a q-index (q prime, $q \neq p$) in $\bar{A}$. Then, by above, $p \mid mn$. Also $q \mid mn$. For otherwise $\sigma|_A$ preserves q-indices, while there exists $a \in A$ ($a \neq 0$) such that

$$\langle pa \rangle \sigma = \langle qa \rangle \sigma = q(\langle a \rangle \sigma),$$

giving $\langle a \rangle \sigma = q(\langle a \rangle \sigma)$, a contradiction.

Now, by (19),

$$\begin{aligned} H^\sigma &= \langle \bar{g}, \langle a(1 + \theta^k + \theta^{2k} + \cdots + \theta^{(l-1)k}) \rangle \sigma \rangle \\ &= \left\langle \bar{g}, \left\langle a \left(\frac{n^{(l-1)k} + n^{(l-2)k}m^k + \cdots + m^{(l-1)k}}{n^{(l-1)k}} \right) \right\rangle \sigma \right\rangle \\ &= \left\langle \bar{g}, \bar{a} \left(\frac{n^{(l-1)k} + n^{(l-2)k}m^k + \cdots + m^{(l-1)k}}{r} \right) \right\rangle, \end{aligned} \tag{23}$$

where r is a $\pi(mn)$-number. Notice that the numerator in the inner bracket of (23) is relatively prime to mn and $\sigma|_A$ preserves such indices. Also, from (18), $\bar{b} = \bar{a}u$, where u is a unit in $S = \mathbb{Z}[1/mn]$ and hence

$$\begin{aligned} &\bar{b}(1 + \bar{\theta}^{k'} + \bar{\theta}^{2k'} + \cdots + \bar{\theta}^{(l'-1)k'}) \\ &\quad = \bar{a}(1 + (m_1/n_1)^{k'} + (m_1/n_1)^{2k'} + \cdots + (m_1/n_1)^{(l'-1)k'})u. \end{aligned}$$

Substituting in (20), comparing with (23) and putting $\bar{a}=1$, we obtain

$$\begin{aligned}(1+(m/n)^k+(m/n)^{2k}+\cdots+(m/n)^{(l-1)k})S\\ =(1+(m_1/n_1)^{k'}+(m_1/n_1)^{2k'}+\cdots+(m_1/n_1)^{(l'-1)k'})S.\end{aligned}$$

Then to avoid contradicting Lemma 4.3 (with $\xi=m/n$, $\lambda=m_1/n_1$), we must have $k=k'$, for all $k\in\mathbb{Z}_+$ and hence $\sigma|_{G/A}$ is index-preserving.

Case 2. Suppose now that $h(A)\geqslant 2$. Then $\sigma|_A$ is induced by an isomorphism $A\to\bar{A}$ which we also denote by θ. From (19), $H^\sigma=\langle\bar{g},\bar{a}\sigma^{-1}(1+\theta^k+\theta^{2k}+\cdots+\theta^{(l-1)k})\sigma\rangle$. Then from (18) and (20) we obtain

$$\begin{aligned}\bar{a}\sigma^{-1}(1+\theta^k+\theta^{2k}+\cdots+\theta^{(l-1)k})\sigma\cdot\mathbb{Z}\langle\bar{g}\rangle\\ =\bar{a}(1+\bar{\theta}^{k'}+\bar{\theta}^{2k'}+\cdots+\bar{\theta}^{(l'-1)k'})\cdot\mathbb{Z}\langle\bar{g}\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\bar{a}(1+\bar{\varphi}^k+\bar{\varphi}^{2k}+\cdots+\bar{\varphi}^{(l-1)k})\,\mathbb{Z}\langle\bar{g}\rangle\\ =\bar{a}(1+\bar{\theta}^{k'}+\bar{\theta}^{2k'}+\cdots+\bar{\theta}^{(l'-1)k'})\,\mathbb{Z}\langle\bar{g}\rangle,\end{aligned}\tag{24}$$

where $\bar{\varphi}=\sigma^{-1}\theta\sigma$. Also the isomorphism $\sigma\colon A\to\bar{A}$ extends to a $\mathbb{C}$-isomorphism (again denoted by σ) from

$$V=A\otimes_{\mathbb{Z}}\mathbb{C}\qquad\text{to}\qquad\bar{V}=\bar{A}\otimes_{\mathbb{Z}}\mathbb{C},$$

and we write $\bar{v}=v\sigma$, all $v\in V$. Moreover, V, $\bar{V}$ become $\mathbb{C}\langle g\rangle$, $\mathbb{C}\langle\bar{g}\rangle$-spaces, respectively, and we continue to denote the g, $\bar{g}$-actions on V, $\bar{V}$ by θ, $\bar{\theta}$.

Then we claim that *Eq.* (24) *holds for all* $\bar{a}\in\bar{V}$. To see this, let $\bar{v}\in\bar{V}$, $\bar{v}\neq 0$, and let Γ be a basis for $\mathbb{C}$ over $\mathbb{Q}$. Then

$$\bar{v}=\sum_{\gamma\in\Gamma}\bar{v}_\gamma\otimes\gamma,$$

where $v_\gamma\in A\otimes_{\mathbb{Z}}\mathbb{Q}$. Choose $n\in\mathbb{Z}_+$ such that, for all $\gamma\in\Gamma$, $nv_\gamma\in A$. (Here we identify A with $A\otimes 1$ and $\bar{A}$ with $\bar{A}\otimes 1$.) Since $\bar{v}\neq 0$, it follows that $v\neq 0$ and we can find γ such that $v_\gamma\neq 0$. Recall that A is rationally irreducible as $\mathbb{Z}\langle g\rangle$-module and hence each non-zero element a of A generates $A\otimes_{\mathbb{Z}}\mathbb{Q}$ as $\mathbb{Q}\langle g\rangle$-module. Therefore the annihilator of a in $\mathbb{Q}\langle g\rangle$ coincides with the annihilator of $A\otimes_{\mathbb{Z}}\mathbb{Q}$ in $\mathbb{Q}\langle g\rangle$. It follows that $nv\mapsto nv_\gamma$ defines a $\mathbb{Z}\langle g\rangle$-isomorphism

$$(nv)\,\mathbb{Z}\langle g\rangle\to(nv_\gamma)\,\mathbb{Z}\langle g\rangle$$

and hence

$$\langle nv, g\rangle \cong \langle nv_\gamma, g\rangle \tag{25}$$

in the split extension $V \rtimes \langle g\rangle$. Also σ restricts to a projectivity

$$\langle nv_\gamma, g\rangle \to \langle n\bar{v}_\gamma, \bar{g}\rangle. \tag{26}$$

Similarly, $n\bar{v}_\gamma \mapsto n\bar{v}$ defines a $\mathbb{Z}\langle \bar{g}\rangle$-isomorphism from $(n\bar{v}_\gamma)\,\mathbb{Z}\langle \bar{g}\rangle$ to $(n\bar{v})\,\mathbb{Z}\langle \bar{g}\rangle$. Thus

$$\langle n\bar{v}_\gamma, \bar{g}\rangle \cong \langle n\bar{v}, \bar{g}\rangle \tag{27}$$

in $\bar{V} \rtimes \langle \bar{g}\rangle$. Combining (25), (26), and (27), we obtain a projectivity

$$\langle nv, g\rangle \to \langle n\bar{v}, \bar{g}\rangle.$$

Therefore the argument establishing (24) shows that (24) remains valid with $\bar{a}$ replaced by $n\bar{v}$, and then, dividing by n, also with $\bar{a}$ replaced by $\bar{v}$.

Now let $v \in V$. Then

v is an eigenvector for θ if and only if $\bar{v}$ is an eigenvector for $\bar{\theta}$; moreover, if ξ, λ are eigenvalues corresponding to eigenvectors v, $\bar{v}$, respectively, then $\mathbb{Z}[\xi, \xi^{-1}] = \mathbb{Z}[\lambda, \lambda^{-1}]$. (28)

For, take $k = 1$, $l = 2$ in (24). Thus, for all $\bar{v} \in \bar{V}$,

$$\bar{v}(1 + \bar{\varphi})\,\mathbb{Z}\langle \bar{g}\rangle = \bar{v}(1 + \bar{\theta} + \bar{\theta}^2 + \cdots + \bar{\theta}^{p-1})\,\mathbb{Z}\langle \bar{g}\rangle. \tag{29}$$

Let $\bar{v}$ be an eigenvector for $\bar{\theta}$ with corresponding eigenvalue λ. From (29) we see that

$$\bar{v}\bar{\varphi} \in \mathbb{Z}[\lambda, \lambda^{-1}]\,\bar{v}$$

and so $\bar{v}$ is an eigenvector for $\bar{\varphi}$ with eigenvalue $\xi \in \mathbb{Z}[\lambda, \lambda^{-1}]$. Then $\bar{v}\bar{\varphi} = \xi\bar{v}$, i.e., $v\theta = \xi v$ and v is an eigenvector for θ. Replacing g, $\bar{g}$ throughout by g^{-1}, $\bar{g}^{-1}$, respectively, we obtain, similarly, $\xi^{-1} \in \mathbb{Z}[\lambda, \lambda^{-1}]$. Thus each eigenvector for $\bar{\theta}$ is the image under σ of an eigenvector for θ and in the above notation

$$\mathbb{Z}[\xi, \xi^{-1}] \subseteq \mathbb{Z}[\lambda, \lambda^{-1}].$$

By an analogous argument, interchanging σ and σ^{-1}, we have the reverse inclusion and so (28) holds.

Finally, let v ($\in V$) be an eigenvector for θ with eigenvalue ξ. Then $\bar{v}$ is an eigenvector for $\bar{\varphi}$ and $\bar{\theta}$ with corresponding eigenvalues ξ and λ (say), respectively; and $\mathbb{Z}[\xi, \xi^{-1}] = \mathbb{Z}[\lambda, \lambda^{-1}] = S$, say. Taking $\bar{a} = \bar{v}$ in (24), we obtain

$$(1 + \xi^k + \xi^{2k} + \cdots + \xi^{(l-1)k})S = (1 + \lambda^{k'} + \lambda^{2k'} + \cdots + \lambda^{(l'-1)k'})S.$$

Note that ξ and λ are not periodic, by (17) and its analogue for $\bar{\theta}$ and $\bar{A}$. Thus, as in Case 1, in order to avoid contradicting Lemma 4.3, we must have $\sigma|_{G/A}$ index-preserving.

This completes the proof of Theorem 4.1. ∎

COROLLARY 4.4. *Let $A \lhd G$ with A abelian, $h(A) \geq 2$ and G/A infinite cyclic. Then every non-trivial complete l-epimorphism σ: $G \to \bar{G}$ is an index-preserving projectivity.*

Proof. By Theorem 3.6, σ is a projectivity. Also, by 1.6, $\sigma|_A$ is index-preserving; and, by Theorem 4.1, $\sigma|_{G/A}$ is index-preserving. The result follows, by 1.4. ∎

5. APPLICATIONS

First we require two technical results about groups with a normal torsion-free locally cyclic subgroup.

LEMMA 5.1. *Let $A \lhd G$ with A torsion-free and locally cyclic and let $C = C_G(A)$. Suppose that $h(G/C) \geq 2$. If τ: $G \to \mathscr{L}$ is a non-trivial complete l-homomorphism, then $\tau|_A$ is injective.*

Proof. Consider A embedded in $\mathbb{Q}$ and suppose, for a contradiction, that the lemma is false. Thus there are subgroups

$$Y < X \leq A$$

with $|X : Y| = p$, a prime, and $X^\tau = Y^\tau$. Let

$$H = \{u/v \mid u/v \in A, (v, p) = 1\}.$$

Since $h(G/C) \geq 2$, it is easy to see that there is an element $g \in G \backslash C$ such that the conjugation action of g on A is multiplication by m/n, where $(m, n) = 1$ and $p \nmid mn$. Then $H \lhd \langle A, g \rangle$ and A/H is a p-group. By 1.5, $\tau|_{A/H}$ is injective and hence $X + H = Y + H$. Therefore $|X \cap H : Y \cap H| = p$ and since $(X \cap H)^\tau = (Y \cap H)^\tau$, we may assume that $Y < X \leq H$. Clearly

$$\bigcap_{i \geq 0} p^i H = 0.$$

Choose i maximal such that $p^iH \geqslant X$. Then $p^{i+1}H \ngeqslant X$, but

$$p^{i+1}H \geqslant pX = Y.$$

Thus $(p^iH)^\tau = (p^{i+1}H)^\tau$. Again by 1.5, τ is injective on $H/p^{i+1}H$, giving the required contradiction. ▌

In a similar vein we have

LEMMA 5.2. *Let $A \lhd G$ with A torsion-free and locally cyclic and let $C = C_G(A)$. Suppose that $h(G/C) \geqslant 2$ and let $\sigma\colon G \to \bar{G}$ $(\neq 1)$ be a complete l-epimorphism. Then $\sigma|_A$ is index-preserving.*

Proof. Again consider A embedded in $\mathbb{Q}$ and let p be any prime. Take H and g as in Lemma 5.1. Then $H \lhd \langle H, g\rangle$ and $pH < H$. Hence, by 1.5,

$$|H^\sigma : (pH)^\sigma| = p.$$

Since $\sigma|_A$ is injective (by Lemma 5.1) it follows easily that $\sigma|_A$ preserves p-indices, all p, and so $\sigma|_A$ is index-preserving. ▌

Now we apply results from Section 3 in order to analyse a situation which will be critical in proving Theorem A.

LEMMA 5.3. *Let $\bar{G} = \langle \bar{g}, \bar{A}\rangle$ with $\bar{A} \lhd \bar{G}$, $\bar{A}$ abelian, and let $\mathscr{L}$ be a non-trivial complete lattice. Let $\sigma\colon G \to \bar{G}$ be a complete l-epimorphism with $G = \langle g, A\rangle$, where*

$$A = \bar{A}^{\sigma *} \qquad \textit{and} \qquad \langle g\rangle^\sigma = \langle \bar{g}\rangle;$$

and let $\tau\colon \bar{G} \to \mathscr{L}$ be a non-trivial complete l-homomorphism:

(i) *Suppose that $h(\bar{A}) = 1$ and $\langle \bar{g}\rangle \cap \bar{A}$ is finite. Then*

(a) *$\sigma|_{\langle g\rangle}$ is index-preserving; and if $|\langle \bar{g}\rangle : \langle \bar{g}\rangle \cap \bar{A}| = \infty$, then $A \lhd G$; and*

(b) *$\tau|_{\langle \bar{g}\rangle}$ is injective.*

(ii) *Suppose that $h(\bar{A}) \geqslant 2$. Then*

(a) *σ is an index-preserving projectivity; and*

(b) *τ is injective.*

Proof. By Lemma 2.6, A is soluble (assuming $h(\bar{A}) \geqslant 1$), $A/\mathbf{P}(A)$ is abelian and $h(A) = h(\bar{A})$. Also A is abelian if $h(\bar{A}) \geqslant 2$. Moreover, by a local argument and Lemma 2.7, G is soluble. We distinguish two cases.

Case 1. Suppose that $|\langle \bar{g}\rangle : \langle \bar{g}\rangle \cap \bar{A}|$ is finite.

(i) We have $h(\bar{A})=1$ with $\langle \bar{g}\rangle$ finite. Take $\bar{a}\in\bar{A}$ with $|\bar{a}|$ infinite and write $\bar{K}=\langle \bar{g},\bar{a}\rangle$, $K=\langle g,a\rangle$, where $\langle a\rangle^{\sigma}=\langle \bar{a}\rangle$. In order to prove (a), we may assume, by Lemma 2.3, that $\mathbf{P}(K)=\mathbf{P}(\bar{K})=1$. Then, by Lemma 2.6(i), $\sigma|_{\langle g\rangle}$ is index-preserving. The truth of (b) follows from 1.6(i).

(ii) Now $h(\bar{A})\geqslant 2$ and we may assume that G and $\bar{G}$ are finitely generated. Then $\bar{G}$ is polycyclic. It is easy to see that there exists $\bar{B}\lhd\bar{G}$, $\bar{B}\leqslant\bar{A}$, $\bar{G}/\bar{B}$ finite and with the non-trivial Sylow subgroups of $\bar{G}/\bar{B}$ neither cyclic nor generalized quaternion. Thus the intersection of the maximal cyclic subgroups of $\bar{G}/\bar{B}$ is trivial and then, by [15, Corollary 1.3], $B=\bar{B}^{\sigma^*}$ has finite index in G. Therefore $|G:A|$ is finite and so G is polycyclic. Hence (a) and (b) follow from 1.6.

Case 2. *Suppose that* $|\langle \bar{g}\rangle:\langle \bar{g}\rangle\cap\bar{A}|=\infty$. By 1.5 and Lemma 2.3, we may assume that $\mathbf{P}(G)=\mathbf{P}(\bar{G})=1$. So G and $\bar{G}$ are torsion-free and A is abelian. Let $x=g^n$, for some integer n. If $h(A)=1$, then

$$h(\overline{A^x})=h(A^x)=1 \qquad \text{and} \qquad \overline{A^x} \text{ is abelian.}$$

If $h(A)\geqslant 2$, then $h(A^x)\geqslant 2$ and, by 1.6(ii), σ is injective on A^x, whence $\overline{A^x}$ is abelian and $h(\overline{A^x})=h(\bar{A})$.

Suppose that $A\neq A^x$. If $A^x<A$, then $A^{x^{-1}}>A$ and so $\overline{A^{x^{-1}}}>\bar{A}$, giving $h(\overline{A^{x^{-1}}})>h(\bar{A})$. But this contradicts the above (with x^{-1} for x). Therefore,

$$\bar{B}=\langle \bar{A},\overline{A^x}\rangle>\bar{A} \qquad \text{and} \qquad \bar{D}=\bar{A}\cap\overline{A^x}<\bar{A}. \tag{1}$$

Moreover, since $\bar{B}/\bar{A}\cong\overline{A^x}/\bar{D}$, we have $\overline{A^x}/\bar{D}\cong C_\infty$. Now without loss we may assume that $\bar{A}$ is a cyclic $\langle \bar{g}\rangle$-module, generated by $\bar{a}$, say. Thus if $h(\bar{A})=\infty$, then $\bar{A}$ is free abelian with basis $\{\bar{a}^{\bar{g}^i}\mid i\in\mathbb{Z}\}$. But then $h(\bar{D})=\infty$ and $\bar{D}$ is centralized by some non-trivial power of $\bar{g}$, a contradiction. Therefore $h(\bar{A})$ is finite and then

$$h(\bar{A}/\bar{D})=1.$$

Thus $h(\bar{B}/\bar{D})=2$. Let $\bar{C}=C_{\bar{G}}(\bar{A})$.

(i)(a) By Theorem 4.1, it is sufficient to show that $A\lhd G$. We distinguish three possibilities:

$$(1)\ \bar{C}=\bar{G}; \qquad (2)\ \bar{A}<\bar{C}<\bar{G}; \qquad (3)\ \bar{C}=\bar{A}.$$

(1) Here $\bar{G}$ is abelian and so G is abelian, by Lemma 2.6(ii).

(2) Now $\bar{C}=\langle \bar{g}^2\rangle\times\bar{A}$ has Hirsch length 2 and so, replacing $\bar{A}$ by $\bar{C}$, it follows from Case 1(ii) that σ is an index-preserving projectivity. Then $A\lhd G$, by [13, Proposition 1.6].

(3) Assume that $\bar{A}=\bar{C}$. Let $E\ (\neq 1)$ be a normal abelian subgroup of G. Then E is torsion-free, $\bar{E}\ (\neq 1)$ is abelian and $\bar{E}\leqslant_d \bar{G}$. If $E\cap A=1$, it follows that $\bar{E}\cap\bar{A}=1$ and $\bar{E}\lhd\overline{EA}$, by [8, Theorem 2.1]. Thus $\bar{E}\leqslant\bar{C}$, a contradiction. Therefore $E\cap A\neq 1$ and so

$$1\neq\bar{E}\cap\bar{A}\leqslant Z(\langle\bar{A},\bar{E}\rangle).$$

Since the $\bar{g}$-action on $\bar{A}\ (\leqslant\mathbb{Q})$ is multiplication by a rational $(\neq\pm 1)$, we must have $\bar{E}\leqslant\bar{A}$. Then $\bar{E}\lhd\bar{G}$ (*loc. cit.*) and $E\leqslant A$. Now σ induces a complete l-epimorphism from G/E to $\bar{G}/\bar{E}$ and, by Lemma 2.2,

$$\mathbf{P}(G/E)=(\bar{A}/\bar{E})^{\sigma^*}=A/E\lhd G/E.$$

Hence $A\lhd G$.

(i)(b) This follows from Theorem 3.2.

(ii)(a) If $A\lhd G$, then σ is an index-preserving projectivity, by Corollary 4.4. Suppose therefore that

$$A\neq A^{g^n},$$

for some integer n. Referring to (1), $\bar{A}/\bar{D}\lhd\bar{B}/\bar{D}$ with $h(\bar{A}/\bar{D})=1$. Let $x=g^n$, $B=\langle A, A^x\rangle$, $D=A\cap A^x$. Since A is abelian, $D\lhd B$; and (i)(a) applied to σ: $B/D\to\bar{B}/\bar{D}$ gives $A\lhd B$. Therefore

$$A\lhd A^G\lhd G.$$

Now $g^n\in A^G$, for some $n\geqslant 1$, and so A^G is generated by finitely many conjugates of A. Therefore A^G is nilpotent. Let

$$Z=Z(A^G)\lhd A^G\lhd G.$$

By 1.6(ii), σ is injective on A^G. Suppose that $Z\cap\langle g\rangle=1$. Then, by Theorems 3.2 and 3.6, σ is injective on $\langle g\rangle$ and therefore σ is a projectivity (1.1). But then $A\lhd G$, by [13, Proposition 1.6], a contradiction. Therefore

$$Z\cap\langle g\rangle\neq 1$$

and $\langle Z, g\rangle$ is locally polycyclic.

If $h(Z)\geqslant 2$, then $\sigma|_{\langle g\rangle}$ is injective, by 1.6(ii), and we obtain a contradiction as before. We are left with $h(Z)=1$. Since $h(A)\geqslant 2$, G/Z is not periodic. Since G is locally polycyclic, σ is injective on all the periodic subgroups of G/Z, by 1.6(i). Hence $\sigma|_{\langle g\rangle}$ is injective, leading to $A\lhd G$ yet again, a contradiction.

(ii)(b) This follows from Theorem 3.6. ∎

Finally, before proving Theorem A, we need

LEMMA 5.4. *Let σ: $G \to \bar{G}$ be a complete l-epimorphism of groups and let $\bar{T}$, $\bar{K}$, and $\bar{M} = \langle \bar{T}, \bar{K} \rangle$ be subgroups of $\bar{G}$, $\bar{g} \in \bar{K}$, and $\bar{P} = \mathbf{P}(\bar{T})$. Suppose that $\bar{P} \leqslant \bar{K}$, $\bar{K} \in \mathfrak{X}$, $h(\bar{K}) \geqslant 2$ and that there exists $\bar{A} \lhd \bar{M}$ with $\bar{P} < \bar{A} \leqslant \bar{T} \cap \bar{K}$ and $\bar{A}/\bar{P}$ abelian (and therefore torsion-free) with $h(\bar{A}/\bar{P}) = 1$. Let*

$$\bar{C} = C_{\bar{T}}(\bar{A}/\bar{P})$$

and assume that either $h(\bar{T}/\bar{C}) \geqslant 2$ or $\bar{C}/\bar{A}$ is not periodic. Let $g \in G$ such that $\langle g \rangle^\sigma = \langle \bar{g} \rangle$. Then $\sigma|_{\langle g \rangle}$ is an index-preserving projectivity from $\langle g \rangle$ to $\langle \bar{g} \rangle$.

Proof. Put $A = \bar{A}^{\sigma^*}$, $T = \bar{T}^{\sigma^*}$, $K = \bar{K}^{\sigma^*}$, $P = \bar{P}^{\sigma^*}$, $B = \langle g, A \rangle$, and $\bar{B} = \langle \bar{g}, \bar{A} \rangle$. Since $\bar{P} \leqslant \bar{K} \in \mathfrak{X}$, $\bar{P}$ is locally finite. Also $\mathbf{P}(\bar{A}) = \bar{P}$ and so, by Lemma 2.2, $P = \mathbf{P}(A)$; and P is locally finite, by Lemma 2.3. By Lemma 2.4, A/P is torsion-free abelian with $h(A/P) = 1$. Moreover, it follows from 1.5 that, for any $x \in G$,

$$\sigma|_{P^x} \text{ is an index-preserving projectivity.} \tag{2}$$

Let $M = \bar{M}^{\sigma^*}$. Consider an element $x \in M$ such that

$$|\langle x \rangle : A \cap \langle x \rangle| = \infty.$$

If $\langle \bar{x} \rangle = \langle x \rangle^\sigma$, then $\bar{A} \cap \langle \bar{x} \rangle = 1$ and $\mathbf{P}(\langle \bar{A}, \bar{x} \rangle) = \bar{P}$. Again by Lemma 2.2, $\mathbf{P}(\langle A, x \rangle) = P$. Thus we may apply Lemma 5.3(i)(a) to σ: $\langle A, x \rangle / P \to \langle \bar{A}, \bar{x} \rangle / \bar{P}$ and deduce that $A \lhd \langle A, x \rangle$. It follows that

$$A \lneqq a_T(A) \qquad \text{and} \qquad A \ntrianglelefteq a_K(A). \tag{3}$$

Let $E = a_T(A)$ and $D = C_E(A/P)$. For $x \in G$, $\bar{x} \in \bar{G}$ with $\langle x \rangle^\sigma = \langle \bar{x} \rangle$, we have $A \cap \langle x \rangle = 1$ if and only if $\bar{A} \cap \langle \bar{x} \rangle = 1$. Therefore,

$$E^\sigma = \bar{E} = a_{\bar{T}}(\bar{A}) \qquad \text{and} \qquad (a_K(A))^\sigma = a_{\bar{K}}(\bar{A}).$$

Moreover, the interval $[T/E]$ is periodic. We claim that

$$\text{either } h(E/D) \geqslant 2 \text{ or } D/A \text{ is non-periodic.} \tag{4}$$

For, if D/A is periodic, then E/D is abelian and not periodic, by (3). Thus $\bar{D} \lhd \bar{E}$, by [15, Proposition 3.4], and $\bar{E}/\bar{D}$ is a modular group. If $\bar{C}/\bar{A}$ is not periodic, then D/A is not periodic, by Lemma 2.6(ii), a contradiction. Therefore $\bar{C}/\bar{A}$ is periodic. Thus, by hypothesis $h(\bar{T}/\bar{C}) \geqslant 2$. Also, since $[T/E]$ is periodic, we have $[\bar{T}/\bar{E}]$ periodic and so $h(\bar{E}/\bar{E} \cap \bar{C}) \geqslant 2$. But it is easy to see, from Lemma 2.6(i), that $\bar{D} \leqslant \bar{C}$, and therefore $\bar{E}/\bar{D}$ is abelian, by [9, Theorem 16, p. 20]. Then $h(E/D) \geqslant 2$, by Lemma 2.6(ii), and (4) follows.

Now we can show that, for all integers i,

$$\sigma \text{ is an index-preserving projectivity on } A^{g^i}. \tag{5}$$

For, if $h(E/D) \geqslant 2$, then this follows from (2) and Lemma 5.2. In the other case, by (4), there is an element $c \in D^{g^i}$ such that

$$\langle c, A^{g^i} \rangle / P^{g^i} \text{ is abelian of Hirsch length 2.}$$

Then (5) follows from (2) and 1.6.

By Lemma 5.3(i)(a), we may suppose that $|\langle \bar{g} \rangle : \bar{A} \cap \langle \bar{g} \rangle|$ is finite. Consider $x \in G$ such that $|\langle x \rangle : A \cap \langle x \rangle| = \infty$. Then $|\langle x \rangle^\sigma| = \infty$ and $\bar{A} \cap \langle x \rangle^\sigma = 1$. Thus $\bar{B} \cap \langle x \rangle^\sigma = 1$ and so $B \cap \langle x \rangle = 1$. Therefore, for each integer i,

$$1 = B^{g^i} \cap \langle x \rangle^{g^i} = B \cap \langle x \rangle^{g^i} = A \cap \langle x \rangle^{g^i}.$$

Hence, for each i and $x \in G$,

$$|\langle x \rangle : A \cap \langle x \rangle| = \infty \text{ if and only if } |\langle x \rangle^{g^i} : A \cap \langle x \rangle^{g^i}| = \infty.$$

Then, using (3),

$$A^{g^i} \lhd a_{K^{g^i}}(A^{g^i}) = a_K(A) \lhd \langle a_K(A), g \rangle = L, \tag{6}$$

say. Thus, by [8, Theorem 2.1],

$$(A^{g^i})^\sigma \lhd a_K(\bar{A}) \leqslant \bar{L}(= L^\sigma). \tag{7}$$

From (6) we obtain $A^{\langle g \rangle} \lhd L$ and, using (5) and 1.4, we see that $\sigma|_{A^{\langle g \rangle}}$ is an index-preserving projectivity. Also, from (7),

$$(A^{\langle g \rangle})^\sigma \lhd a_{\bar{K}}(\bar{A})$$

and hence $(A^{\langle g \rangle})^\sigma \lhd \bar{L}$.

Now consider the induced l-epimorphism

$$\sigma: L/A^{\langle g \rangle} \to \bar{L}/(A^{\langle g \rangle})^\sigma.$$

The subgroup $\mathbf{P}(\bar{L}/(A^{\langle g \rangle})^\sigma) = \bar{F}/(A^{\langle g \rangle})^\sigma$ (say) is locally finite, since $\bar{L} \in \mathfrak{X}$. Let

$$F/A^{\langle g \rangle} = \mathbf{P}(L/A^{\langle g \rangle}).$$

Then, from Lemma 2.3 (observing that $\bar{L}/\bar{F}$ is not periodic), we deduce that $\sigma|_F$ is an index-preserving projectivity from F to $\bar{F}$. Finally, $\bar{L}/\bar{F}$ has a non-trivial torsion-free abelian normal subgroup and hence, by Lemma 5.3, $\sigma|_{\langle g, F \rangle}$ is index-preserving. Thus $\sigma|_{\langle g \rangle}$ is index-preserving, as required. ∎

We come now to our main result.

Proof of Theorem A. By 1.1 and 1.2, it suffices to show that σ restricted to each cyclic subgroup of G is an index-preserving projectivity and τ restricted to each cyclic subgroup of $\bar{G}$ is injective. Thus let $g \in G$, $\langle g \rangle^\sigma = \langle \bar{g} \rangle$ and $\bar{K} = \langle \bar{H}, \bar{g} \rangle$, $K = \bar{K}^{\sigma^*}$. By hypothesis,

$$[\bar{K}/\bar{H}] \cong [\langle \bar{g} \rangle / \bar{H} \cap \langle \bar{g} \rangle].$$

Therefore $\bar{K} \in \mathfrak{X}$, for if $[\bar{K}/\bar{H}]$ is not finite, then $\bar{H} \lhd \bar{K}$, by [8, Theorem 2.1].

Let $\bar{P} = \mathbf{P}(\bar{K})$ and suppose first that $\bar{K}/\bar{P}$ has a normal abelian subgroup $\bar{A}/\bar{P}$ of Hirsch length $\geqslant 2$. (This is certainly the case if $h(\bar{H}) \geqslant 2$ and $\bar{H}$ is modular, for then $\bar{H}$ is abelian, by [9, Theorem 16, p. 20].) Let $P = \mathbf{P}(K)$. Then $\sigma|_P$ is an index-preserving projectivity from P to $\bar{P}$, by Lemma 2.3; and, by Lemma 5.3(ii)(a), σ induces an index-preserving projectivity from $\langle g, P \rangle / P$ to $\langle \bar{g}, \bar{P} \rangle / \bar{P}$. Therefore $\sigma|_{\langle g \rangle}$ is an index-preserving projectivity from $\langle g \rangle$ to $\langle \bar{g} \rangle$, as required. Similarly, by Lemma 5.3(ii)(b), τ is injective on $\langle \bar{g}, \bar{P} \rangle / \bar{P}$. By 1.5, $\tau|_{\bar{P}}$ is injective. Thus $\tau|_{\langle \bar{g} \rangle}$ is injective.

From now on we may assume that the non-trivial normal abelian subgroups of $\bar{K}/\bar{P}$ have Hirsch length 1. We distinguish two cases:

(i) *Suppose that* $h(\bar{K}) \geqslant 3$. In Lemma 5.4, take $\bar{T} = \bar{K}$ and let $\bar{A}/\bar{P}$ be a normal abelian (torsion-free) subgroup of $\bar{K}/\bar{P}$ with $h(\bar{A}/\bar{P}) = 1$. Then the hypotheses of Lemma 5.4 are satisfied and we conclude that $\sigma|_{\langle g \rangle}$ is an index-preserving projectivity.

With regard to τ, assume without loss of generality that $\bar{P} = 1$ (since $\tau|_{\bar{P}}$ is injective, by 1.5), and let $\bar{C} = C_{\bar{K}}(\bar{A})$. We claim that

$$\tau|_{\bar{A}} \text{ is injective.} \tag{8}$$

For, either $h(\bar{K}/\bar{C}) \geqslant 2$ and then (8) follows from Lemma 5.1; or $\bar{C}/\bar{A}$ is not periodic and so there exists $\bar{c} \in \bar{C}$ such that $\langle \bar{c}, \bar{A} \rangle$ is abelian with Hirsch length 2; then (8) follows from 1.6(ii). Now if $\langle \bar{g} \rangle \cap \bar{A}$ is finite, then $\tau|_{\langle \bar{g} \rangle}$ is injective, by Lemma 5.3(i)(b). If, on the other hand, $\langle \bar{g} \rangle \cap \bar{A}$ is infinite, let $\bar{F}/\bar{A} = \mathbf{P}(\bar{K}/\bar{A})$. By Lemma 5.3(i)(b) or (ii)(b), τ is injective on $\langle \bar{g}, \bar{F} \rangle / \bar{F}$. But τ is injective on $\bar{F}/\bar{A}$, by 1.5, and so τ is injective on $\langle \bar{g}, \bar{F} \rangle$, by (8). Therefore $\tau|_{\langle \bar{g} \rangle}$ is injective.

(ii) *Suppose now that* $h(\bar{K}) = 2$. Then $h(\bar{H}) = 2$ and $\bar{H} \leqslant_D \bar{G}$ implies that $|\bar{K} : \bar{H}|$ is finite. We may replace $\bar{H}$ by $\bar{H}^{\bar{K}}$, for, by [14, Theorem 3.2], $\bar{H}^{\bar{K}} \leqslant_D \bar{G}$. Thus $\bar{H} \lhd \bar{K}$. Since we are left with the case when $[\bar{G}/\bar{H}]$ is not periodic, there is an element $\bar{y} \in \bar{G}$ such that

$$|\langle \bar{y} \rangle : \bar{H} \cap \langle \bar{y} \rangle| = \infty.$$

Let $\bar{T} = \langle \bar{H}, \bar{y} \rangle$. Then

$$\bar{H} \lhd \bar{T} \in \mathfrak{X} \qquad \text{and} \qquad h(\bar{T}) = 3.$$

Let $\bar{M} = \langle \bar{T}, \bar{K} \rangle$. Thus $\bar{H} \lhd \bar{M}$. Our previous notation for $\mathbf{P}(\bar{K})$ will not appear again and so let

$$\bar{P} = \mathbf{P}(\bar{T}) = \mathbf{P}(\bar{H}) \lhd \bar{M}$$

and let $\bar{A}/\bar{P}$ be a non-trivial abelian normal subgroup of $\bar{H}/\bar{P}$. Then

$$\bar{A}^{\bar{M}}/\bar{P} \leqslant \bar{H}/\bar{P}$$

and $\bar{A}^{\bar{M}}/\bar{P}$ is locally nilpotent, torsion-free, and has Hirsch length $\leqslant 2$. Thus $\bar{A}^{\bar{M}}/\bar{P}$ is abelian and hence has Hirsch length 1. Therefore we may assume that $\bar{A} \lhd \bar{M}$ and $h(\bar{A}) = 1$.

Applying Lemma 5.4 to this situation, we find that $\sigma|_{\langle g \rangle}$ is index-preserving.

Finally, arguing with $\bar{T}$ in place of $\bar{K}$ in case (i), we find $\tau|_{\bar{A}}$ is injective and then $\tau|_{\langle g \rangle}$ is injective as in that case. ∎

6. Critical Examples

Let G be a non-periodic soluble group, $\mathscr{L}$ a complete lattice and τ: $G \to \mathscr{L}$ a proper complete l-homomorphism. By Theorem A we must have $h(G) \leqslant 2$. We proceed to construct a metabelian group G with $h(G) = 2$ and a proper complete l-epimorphism τ from G to G.

Let $G = A \rtimes \langle g \rangle$ with $\langle g \rangle$ infinite and A torsion-free abelian of rank 1. We can identify A with an additive subgroup of $\mathbb{Q}$ and the action of g on A with multiplication by a rational m/n, $(m, n) = 1$. Let τ: $G \to \mathscr{L}$ be a proper complete l-homomorphism. Then there are subgroups X, Y of G with $Y < X$ and $X^\tau = Y^\tau$. By Theorem 3.2, τ is injective on G/A and then, by Lemma 3.3, $X \leqslant A$. Without loss of generality we may assume that $|X : Y| = p$, a prime. By the argument in the proof of Lemma 5.1, $p \mid mn$. Hence $m/n \neq \pm 1$ and $pA = A$.

Before defining τ, an easy technical result will be useful.

Lemma 6.1. *Let S be an additive subgroup of $\mathbb{Q}$ and let p be a prime. Suppose that S contains rationals $u/p^\alpha v$, $p \nmid u$, for each integer $\alpha \geqslant 0$. Then $pS = S$.*

Proof. Let $u/v \in S$ and write $v = p^\beta v_1$, $\beta \geqslant 0$, $p \nmid v_1$. By hypothesis there exists $u_1/p^{\beta+1} \in S$, $u_1 \in \mathbb{Z}$, $p \nmid u_1$. Thus $uu_1/p^{\beta+1} \in S$. Also there are integers λ, μ such that

$$\lambda p + \mu u_1 v_1 = 1$$

and so $u/pv = (\lambda p + \mu u_1 v_1) u/pv = \lambda u/v + \mu u_1 u/p^{\beta+1}$ belongs to S, as required. ∎

Now let G have the structure described above with $Y < X \leqslant A$, $|X : Y| = p$ (prime), $p \mid mn$, and $pA = A$. We define a map τ: $l(G) \to l(G)$ as follows:

$$U^\tau = \begin{cases} U & \text{if} \quad U \nleqslant A, \\ pU & \text{if} \quad U \leqslant A \text{ and } U \cap X \nleqslant Y, \\ U & \text{if} \quad U \leqslant A \text{ and } U \cap X \leqslant Y. \end{cases}$$

We claim that

$$\tau \textit{ is a proper complete l-epimorphism.} \tag{1}$$

To see this, let $\{U_\lambda \mid \lambda \in \Lambda\}$ be a set of subgroups of G. We distinguish various cases.

(i) *Suppose that* $U_\lambda \nleqslant A$, *for all* $\lambda \in \Lambda$. Then

$$\langle U_\lambda \mid \lambda \in \Lambda \rangle^\tau = \langle U_\lambda \mid \lambda \in \Lambda \rangle = \langle U_\lambda^\tau \mid \lambda \in \Lambda \rangle$$

and

$$\left(\bigcap_\lambda U_\lambda\right)^\tau = \bigcap_\lambda U_\lambda^\tau \text{ if } I = \bigcap_\lambda U_\lambda \nleqslant A.$$

Thus suppose that $I \leqslant A$. Then

$$U_\lambda \cap A \lhd U_\lambda = \langle g^{\alpha_\lambda} a_\lambda, U_\lambda \cap A \rangle$$

with $\alpha_\lambda \neq 0$ and $a_\lambda \in A$. Hence $U_\lambda \cap A$ is invariant under multiplication by $(m/n)^{\alpha_\lambda}$ and so $p(U_\lambda \cap A) = U_\lambda \cap A$. Therefore, $pI = I$ and

$$\left(\bigcap_\lambda U_\lambda\right)^\tau = \bigcap_\lambda U_\lambda^\tau.$$

(ii) *Suppose that* $U_\lambda \leqslant A$ *and* $U_\lambda \cap X \nleqslant Y$, *for all* $\lambda \in \Lambda$. Then each $X/U_\lambda \cap X$ is a p'-group. If $\bigcap_\lambda (U_\lambda \cap X) \neq 0$, then $X/\bigcap_\lambda (U_\lambda \cap X)$ is a p'-group and so

$$\left(\bigcap_\lambda U_\lambda\right) \cap X = \bigcap_\lambda (U_\lambda \cap X) \nleqslant Y.$$

Therefore $(\bigcap_\lambda U_\lambda)^\tau = p(\bigcap_\lambda U_\lambda) = \bigcap_\lambda pU_\lambda = \bigcap_\lambda U_\lambda^\tau$. On the other hand, if $\bigcap_\lambda (U_\lambda \cap X) = 0$, then $\bigcap_\lambda U_\lambda = 0$ and

$$0 = \left(\bigcap_\lambda U_\lambda\right)^\tau \leqslant \bigcap_\lambda U_\lambda^\tau = \bigcap_\lambda pU_\lambda \leqslant \bigcap_\lambda U_\lambda = 0.$$

In both cases, $\langle U_\lambda \mid \lambda \in \Lambda \rangle \cap X \nleqslant Y$ and hence

$$\langle U_\lambda \mid \lambda \in \Lambda \rangle^\tau = p\langle U_\lambda \mid \lambda \in \Lambda \rangle = \langle pU_\lambda \mid \lambda \in \Lambda \rangle = \langle U_\lambda^\tau \mid \lambda \in \Lambda \rangle.$$

(iii) *Suppose that* $U_\lambda \leqslant A$ *and* $U_\lambda \cap X \leqslant Y$, *for all* $\lambda \in \Lambda$. We have

$$\left(\bigcap_\lambda U_\lambda\right) \cap X \leqslant Y \qquad \text{and so} \qquad \left(\bigcap_\lambda U_\lambda\right)^\tau = \bigcap_\lambda U_\lambda = \bigcap_\lambda U_\lambda^\tau;$$

and

$$\langle U_\lambda \mid \lambda \in \Lambda \rangle \cap X = \langle U_\lambda \cap X \mid \lambda \in \Lambda \rangle \leqslant Y$$

gives

$$\langle U_\lambda \mid \lambda \in \Lambda \rangle^\tau = \langle U_\lambda^\tau \mid \lambda \in \Lambda \rangle.$$

(iv) *Suppose that* $U_\lambda \leqslant A$, *for all* $\lambda \in \Lambda$. Let $U_\lambda \cap X \nleqslant Y$ for $\lambda \in \Lambda_1 \neq \varnothing$ and $U_\lambda \cap X \leqslant Y$ for $\lambda \in \Lambda_2 \neq \varnothing$. So Λ is the disjoint union of Λ_1 and Λ_2. Put

$$U = \bigcap_{\lambda \in \Lambda_1} U_\lambda, \qquad V = \bigcap_{\lambda \in \Lambda_2} U_\lambda.$$

Using (ii) and (iii), we have

$$\bigcap_{\lambda \in \Lambda} U_\lambda^\tau = \left(\bigcap_{\lambda \in \Lambda_1} U_\lambda^\tau\right) \cap \left(\bigcap_{\lambda \in \Lambda_2} U_\lambda^\tau\right) = U^\tau \cap V^\tau = U^\tau \cap V.$$

If $U \cap X \leqslant Y$, then $U^\tau = U$ and $(U \cap V)^\tau = U \cap V$, as required. If $U \cap X \nleqslant Y$, then $U^\tau = pU$. Since $(U \cap V)^\tau = U \cap V$, we have to show that

$$pU \cap V = U \cap V. \tag{2}$$

We may assume that $V \neq 0$. Since $U \cap X \nleqslant Y$,

$$X/Y \cong (U \cap X)/(U \cap Y)$$

and, hence, $p(U \cap X) = U \cap Y = pU \cap Y$. Then $V \cap X \leqslant Y$ implies $V \cap X = V \cap Y$ and, therefore,

$$U \cap V \cap X = U \cap V \cap Y = pU \cap V \cap Y = N,$$

say. Now $(U \cap V)/(pU \cap V)$ and X/Y are p-sections of the finite cyclic group $(U \cap V)X/N$ and so we must have

$$pU \cap V = U \cap V.$$

Thus (2) follows. Also we have shown that

$$\tau|_A \text{ preserves intersections.} \tag{3}$$

Regarding joins, now let $U = \langle U_\lambda \mid \lambda \in \Lambda_1 \rangle$, $V = \langle U_\lambda \mid \lambda \in \Lambda_2 \rangle$. Then

$$U \cap X \nleqslant Y \qquad \text{and} \qquad V \cap X \leqslant Y.$$

Therefore $U^\tau = pU$, $V^\tau = V$, and $\langle U, V \rangle^\tau = p\langle U, V \rangle$. We claim that

$$\langle pU, V \rangle = p\langle U, V \rangle. \tag{4}$$

For, if $pU = U$, then A/U is a p'-group and so $A/\langle U, V \rangle$ is a p'-group. Therefore,

$$A/\langle U, V \rangle \cong pA/p\langle U, V \rangle = A/p\langle U, V \rangle$$

is a p'-group and hence $p\langle U, V \rangle = \langle U, V \rangle$. Thus

$$\langle U, V \rangle \geqslant \langle pU, V \rangle \geqslant p\langle U, V \rangle = \langle U, V \rangle$$

and (4) holds. On the other hand, if $pU < U$, since $U \cap V = pU \cap V$ (see the derivation of (2) above), we have

$$\langle U, V \rangle > \langle pU, V \rangle \geqslant p\langle U, V \rangle.$$

Then $|\langle U, V \rangle : p\langle U, V \rangle| = p$ implies (4) holds again. Now

$$\begin{aligned} \langle U_\lambda \mid \lambda \in \Lambda \rangle^\tau = \langle U, V \rangle^\tau &= \langle U^\tau, V^\tau \rangle \qquad \text{(by (4))} \\ &= \langle \langle U_\lambda^\tau \mid \lambda \in \Lambda_1 \rangle, \langle U_\lambda^\tau \mid \lambda \in \Lambda_2 \rangle \rangle, \end{aligned}$$

by (ii) and (iii). It follows that $\tau|_A$ is a complete l-endomorphism of A.

(v) *Suppose that the U_λ are arbitrary.* Define

$$\Lambda_1 = \{\lambda \mid U_\lambda \nleqslant A\}, \qquad \Lambda_2 = \{\lambda \mid U_\lambda \leqslant A\}.$$

By (i) and (iv), we may assume that $\Lambda_1 \neq \varnothing \neq \Lambda_2$. Let

$$U = \langle U_\lambda \mid \lambda \in \Lambda_1 \rangle, \qquad V = \langle U_\lambda \mid \lambda \in \Lambda_2 \rangle.$$

Then $U \nleqslant A$ and $V \leqslant A$. By (i) and (iv),

$$\langle U_\lambda^\tau \mid \lambda \in \Lambda \rangle = \langle \langle U_\lambda^\tau \mid \lambda \in \Lambda_1 \rangle, \langle U_\lambda^\tau \mid \lambda \in \Lambda_2 \rangle \rangle$$
$$= \langle U^\tau, V^\tau \rangle = \langle U, V^\tau \rangle.$$

Since $\langle U, V \rangle^\tau = \langle U, V \rangle$, we have to show that

$$\langle U, V \rangle = \langle U, V^\tau \rangle. \tag{5}$$

If $V \cap X \leqslant Y$, then $V^\tau = V$ and (5) holds. If $V \cap X \nleqslant Y$, then $V^\tau = pV$ and

$$\langle U, pV \rangle = U(pV)^U = \langle U, p(V^U) \rangle.$$

But, by Lemma 6.1, $p(V^U) = V^U$ and so again (5) holds. Thus τ *preserves joins*.

Regarding intersections, we show first that, for any $U \nleqslant A$ and $V \leqslant A$,

$$(U \cap V)^\tau = U^\tau \cap V^\tau. \tag{6}$$

If $V \cap X \leqslant Y$, then $U \cap V \cap X \leqslant Y$ and (6) holds. If $V \cap X \nleqslant Y$, then $V^\tau = pV$. Thus

$$(U \cap V)^\tau = ((U \cap A) \cap V)^\tau = (U \cap A)^\tau \cap V^\tau,$$

by (iv). Since $p(U \cap A) = U \cap A$ (by Lemma 6.1),

$$(U \cap A)^\tau = U \cap A$$

and so

$$(U \cap V)^\tau = U \cap A \cap V^\tau = U \cap V^\tau = U^\tau \cap V^\tau$$

and (6) holds.

Finally,

$$\left(\bigcap_{\lambda \in \Lambda} U_\lambda \right)^\tau = \left(\left(\bigcap_{\lambda \in \Lambda_1} U_\lambda \right) \cap \left(\bigcap_{\lambda \in \Lambda_2} U_\lambda \right) \right)^\tau$$
$$= \left(\bigcap_{\lambda \in \Lambda_1} U_\lambda \right)^\tau \cap \left(\bigcap_{\lambda \in \Lambda_2} U_\lambda \right)^\tau \qquad \text{(by (3) or (6))}.$$

Then, by case (i) and (3),

$$\left(\bigcap_{\lambda \in \Lambda} U_\lambda \right)^\tau = \left(\bigcap_{\lambda \in \Lambda_1} U_\lambda^\tau \right) \cap \left(\bigcap_{\lambda \in \Lambda_2} U_\lambda^\tau \right) = \bigcap_{\lambda \in \Lambda} U_\lambda^\tau,$$

as required.

We have shown that τ is a complete l-endomorphism of G and τ is proper since $X^\tau = Y^\tau$. Moreover, τ is surjective. For, let $H \leqslant G$. If $H \nleqslant A$, then $H^\tau = H$. If $H \leqslant A$ and $H \cap X \leqslant Y$, then $H^\tau = H$; and if $H \cap X \nleqslant Y$, then $p^{-1}H \cap X \nleqslant Y$ and so $(p^{-1}H)^\tau = H$. (Note that A is p-divisible, by Lemma 6.1.)

Now (1) has been established. Summing up, we have proved

THEOREM B. *Let $G = A \rtimes \langle g \rangle$ with $0 \neq A \leqslant \mathbb{Q}$ and $\langle g \rangle$ infinite. Let g act on A by conjugation as multiplication by the rational m/n, $(m, n) = 1$, and let $Y < X \leqslant A$ with $|X : Y| = p$, a prime.*

(i) *If $p \mid mn$, then there exists a complete l-epimorphism τ: $G \to G$ with $X^\tau = Y^\tau$.*

(ii) *If $p \nmid mn$, then there does not exist any non-trivial complete l-homomorphism τ of G with $X^\tau = Y^\tau$.*

If G is a non-periodic soluble group and σ: $G \to \bar{G}$ is a non-index-preserving projectivity, then by Theorem A, $h(G) \leqslant 2$. We show finally that there exists a metabelian torsion-free group G with $h(G) = 2$ and an autoprojectivity of G which is *not* index-preserving.

Let $G = A \rtimes \langle g \rangle$, where $\mathbb{Z} \leqslant A \leqslant \mathbb{Q}$, $|g| = \infty$, and the conjugation action of g on A is multiplication by m/n, $(m, n) = 1$. Let σ: $G \to G$ be an autoprojectivity and suppose that there is an element $x \in G$ and a prime p such that some p-index in $\langle x \rangle$ is mapped under σ to a q-index, where q is a prime different from p. By 1.6, we know that $m/n \neq \pm 1$; by Theorem 4.1, $x \in A$; and, by the argument of Lemma 5.2, p and q belong to the set π of prime divisors of mn.

Now let ρ ($\neq 1$) be a permutation of the set of all primes with the support of ρ contained in π. Consider the map σ: $A \to A$ defined as follows:

$$\text{if } r/s = (-1)^\varepsilon \prod_i p_i^{\alpha_i} \in A, \text{ then } (r/s)^\sigma = (-1)^\varepsilon \prod_i (\rho(p_i))^{\alpha_i}.$$

From [2] (see Section 4), we know that σ is a bijection which induces an autoprojectivity of A, singular for the primes permuted by ρ. Denote this projectivity also by σ. We extend σ to G by defining

$$U^\sigma = U, \qquad \text{for all} \quad U \nleqslant A.$$

Clearly σ is a bijection from $l(G)$ to $l(G)$. We claim that σ and σ^{-1} preserve inclusions (and then σ is an autoprojectivity of G.) For, let $V < U \leqslant G$. If $V \nleqslant A$ or $U \leqslant A$, then the conclusion is immediate. Thus assume that $V \leqslant A$ and $U \nleqslant A$. We may suppose that $V \neq 0$ and so

$$0 \neq V \leqslant U \cap A \lhd U.$$

Therefore $U \cap A$ is divisible by all primes in π and so $(U \cap A)^\sigma = U \cap A$. Hence

$$V^\sigma \leqslant (U \cap A)^\sigma = U \cap A \leqslant U = U^\sigma.$$

Similarly, σ^{-1} preserves inclusions. Summing up, we have proved

THEOREM C. *Let $G = A \rtimes \langle g \rangle$ with $\mathbb{Z} \leqslant A \leqslant \mathbb{Q}$ and $\langle g \rangle$ infinite cyclic. Suppose that the conjugation action of g on A is multiplication by m/n, $(m, n) = 1$. Let π be the set of prime divisors of mn.*

(i) *If p, q are distinct primes in π, then there exists an autoprojectivity σ of G and an element x (necessarily in A) such that*

$$|\langle x \rangle^\sigma : \langle px \rangle^\sigma| = q.$$

(ii) *If $p \notin \pi$, then each autoprojectivity of G preserves the p-indices.*

REFERENCES

1. C. W. CURTIS AND I. REINER, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1962.
2. E. GASPERINI AND C. METELLI, On projectivities of abelian groups of torsion-free rank one, *Boll. Un. Mat. Ital. A* (6) **3** (1984), 363–371.
3. K. W. GRUENBERG, Residual properties of infinite soluble groups, *Proc. London Math. Soc.* (3) **7** (1957), 29–62.
4. P. HALL, On the finiteness of certain soluble groups, *Proc. London Math. Soc.* (3) **9** (1959), 595–622.
5. S. G. IVANOV, *L*-homomorphisms of locally soluble torsion-free groups, *Mat. Zametki* **37**, No. 5 (1985), 627–635.
6. K. IWASAWA, On the structure of infinite *M*-groups, *Japan J. Math.* **18** (1943), 709–728.
7. D. J. S. ROBINSON, "Finiteness Conditions and Generalized Soluble Groups," Springer-Verlag, Berlin, 1972.
8. S. E. STONEHEWER, Modular subgroup structure in infinite groups, *Proc. London Math. Soc.* (3) **32** (1976), 63–100.
9. M. SUZUKI, "Structure of a Group and the Structure of Its Lattice of Subgroups," Springer-Verlag, Berlin, 1956.
10. B. L. VAN DER WAERDEN, "Modern Algebra," Vol. I, Ungar, New York, 1949.
11. B. V. YAKOVLEV, Lattice isomorphisms of soluble groups, *Algebra i Logika* **9** (1970), 349–369.
12. G. ZACHER, Una caratterizzazione reticolare della finitezza dell'indice di un sottogruppo in un gruppo, *Rend. Accad. Naz. Lincei* **69** (1980), 317–323.
13. G. ZACHER, Sulle imagini dei sottogruppi normali nelle proiettività, *Rend. Sem. Mat. Univ. Padova* **67** (1982), 39–74.
14. G. ZACHER, Una relazione di normalità sul reticolo dei sottogruppi di un gruppo, *Ann. Mat. Pura Appl.* **131** (1982), 57–73.
15. G. ZACHER, Sottogruppi normali e *r*-omomorfismi completi tra gruppi, *Ann. Mat. Pura Appl.* **139** (1985), 83–106.
16. G. ZAPPA, Una soluzione del problema 61 di Birkhoff, *Period. Mat.* **46** (1968), 395–398.
17. H. ZASSENHAUS, "The Theory of Groups," Chelsea, New York, 1958.