Eigenvalues of finite graphs

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Received 20 December 1988
Revised 20 November 1990

Abstract


In this paper we present some questions and answers about the set of eigenvalues of graphs. The questions sound like 'Is a given number x an eigenvalue of some graph?' and the answers like 'no' or 'there is a nonnull integer n such that nx is the eigenvalue of some graph'.

Introduction

There are many studies about graphs with eigenvalues in given intervals, and properties related to eigenvalues [3, 4]. We are interested here in the question

What reals are eigenvalues of finite graphs?

and some related problems.

The restriction to finite graphs is unavoidable, since every complex number is an eigenvalue of the bidirectional ray:

Proposition 1. The eigenvalues of the adjacency matrix of a finite graph are totally real algebraic integers.

Proof. Since the entries of the matrix are integers, its eigenvalues are algebraic integers. Since the matrix is symmetric, its eigenvalues are real. Let a be an eigenvalue; a is a real algebraic integer, and its conjugates also are real, since they also are roots of the characteristic polynomial of the matrix. Thus, a is totally real. □
Examples. The roots of the irreducible polynomial \( X^3 - 3X + 1 \), namely, \( 2 \cos (2\pi/9) \), \( 2 \cos (4\pi/9) \) and \( 2 \cos (8\pi/9) \), are totally real. Indeed, they are eigenvalues of the 9-cycle.

The roots of the irreducible polynomial \( X^3 - X + 1 \) are not all real. Hence, the real root \( x \) of this polynomial is not totally real and, therefore, no finite graph has \( x \) as eigenvalue.

Thus, we are led to the following problem.

Main question. Is every totally real algebraic integer an eigenvalue of some finite graph?

Theorem 1. For every totally real algebraic integer \( x \), there exists a nonnull integer \( N \) such that \( Nx \) is an eigenvalue of some finite graph.

Let us recall some results from [3].

Lemma 1. All rational integers are eigenvalues of finite graphs.

Proof. The eigenvalues of \( K_{n,n} \) are \( n, -n \) and also 0 if \( n \geq 2 \) [3, 2.6.n° 5].

Lemma 2 (Čvetkovic et al. [3, Theorem 2.23]). If \( x \) is an eigenvalue of a graph \( G \) and \( y \) is an eigenvalue of a graph \( H \), then \( xy \) is an eigenvalue of the product \( G \times H \) and \( x + y \) is an eigenvalue of the cartesian sum \( G + H \).

Lemma 3. If \( n \) and \( n' \) are integers such that \( nx \) and \( n'x \) are eigenvalues of finite graphs, then their greatest common divisor \( n' \) has the same property.

Proof. It is a mere corollary of Lemmas 1 and 2, owing to Bezout’s theorem [1, Theorem 23, p. 117].

Lemma 4 (Čvetkovic et al. [3, Theorem 0.12]). If \( x \) is an eigenvalue of the matrix with positive integer entries \( m_{ij} \), satisfying the following semi-symmetry conditions,

- for every list of indices \( i_1, i_2, \ldots, i_k \) the products
  
  \[ m_{i_1i_2} m_{i_2i_3} \cdots m_{i_ki_1} \quad \text{and} \quad m_{i_1i_2} m_{i_2i_3} \cdots m_{i_ki_1} \]

  are equal,
  
  - if \( m_{ij} = 0 \) then also \( m_{ji} = 0 \),

then \( x \) is an eigenvalue of the symmetric matrix where each entry \( m_{ii} \) is replaced with a block \( M_{ii} \) that is the adjacency matrix of a \( m_{ii} \)-regular graph on \( n_i \) vertices, each entry \( m_{ij} \) is replaced by a block \( M_{ij} \) with \( n_i \) lines and \( n_j \) columns, such that every row contains \( m_{ij} \) entries equal to 1 and every column contains \( m_{ji} \) entries equal to 1, all other entries are null. Of course, it is required that no \( n_i \) is null and that, for all pairs \( i, j \), the relations \( m_{ji} n_i = m_{ij} n_j \) and \( M_{ij} = M_{ji} \) hold.
Examples. Since $\sqrt{6}$ is an eigenvalue of
\[
\begin{bmatrix}
0 & 3 \\
2 & 0
\end{bmatrix},
\]
it is an eigenvalue of $K_{2,3}$.

Since $(1 + \sqrt{13})/2$ is an eigenvalue of
\[
\begin{bmatrix}
1 & 3 \\
1 & 0
\end{bmatrix},
\]
it is an eigenvalue of the tree shown in Fig. 1.

We now use a property related to orthogonal polynomials [7]. Let us call monic a polynomial with leading coefficient 1, as in [2, p. 10].

Lemma 5. If $P_0$ is a real monic polynomial with degree $n + 1$ having $n + 1$ distinct roots, there exist sequences of
- real monic polynomials $P_i$, $0 \leq i \leq n + 1$,
- real numbers $a_i$, $0 \leq i \leq n$,
- positive real numbers $b_i$, $0 < i \leq n$,

such that $P_i$ has degree $n + 1 - i$ and $P_{i-1} + b_i P_{i+1} = (X - a_i) P_i$.

Proof. It is sufficient to choose $P_1$ such that it has $n$ real roots interlaced with the roots of $P_0$. Then the polynomial $P_2$ is defined as the remainder when $P_0$ is divided by $P_1$ and it is easy to check that $P_2(x_i) P_2(x_{i+1}) < 0$ for any two consecutive roots $x_i, x_{i+1}$ of $P_1$. Thus, $P_2$ has $n - 1$ roots, that are interlaced with the roots of $P_1$ and its degree must be $n - 1$. Hence, $a_0$ and $b_1$ are defined, and $P_0(x_n) P_2(x_n) < 0$ (with $x_n$ the largest root of $P_1$), hence the sign of $b_1$. An easy induction achieves the proof. \(\square\)

Calculations of the same kind show that the condition on $P_1$ is also necessary. This idea leads to fine results like Sturm’s theorem [6, Theorem 8.9].

Remarks. The derivative $P'_0$ is a convenient $P_1$ (up to division by the degree $n + 1$ since $P_1$ has to be monic).

Since the roots of the convenient polynomials $P_i$ are located in open intervals, these polynomials constitute an open set of monic polynomial of degree $n$.

Proof of Theorem 1. Let $x$ be a totally real algebraic integer. Let $P_0$ be the minimal polynomial of $x$ (i.e. the monic polynomial with integer coefficients of smallest positive
degree such that \( P_0(x) = 0 \) and \( P_1 \) a monic polynomial with rational coefficients satisfying the conditions in Lemma 5. Then \( x \) is an eigenvalue of the tridiagonal matrix \( M \) such that \( m_{ii} = a_i \) and \( m_{i-1,i} = b_i \) are defined according to Lemma 5, and \( m_{i,i-1} = 1 \) and all other entries are 0. Let \( N \) be the (positive) least common multiple of the denominators of the \( a_i \)’s and \( b_i \)’s. Then \( Nx \) is an eigenvalue of the matrix \( NM \), that has integer entries. If the diagonal entries of \( NM \) are positive, then Lemma 4 provides a graph having \( Nx \) as eigenvalue. If not, let \( A \) be the smallest diagonal entry of \( NM \). Then \( Nx - A \) is an eigenvalue of \( NM - AI \), and of a graph \( G \) provided by Lemma 4, and \( Nx \) is an eigenvalue of \( G + K_{-A,-A} \).

Thus, we have found a multiplier \( N \) such that \( Nx \) belongs to the set of eigenvalues of graphs.

Of course, other sequences of polynomials are convenient and may provide other multipliers \( m_i \).

If the greatest common divisor of these multipliers is \( d \), then \( dx \) is itself an eigenvalue of a graph (Lemma 3).

**Examples.**
- The irreducible polynomial \( X^3 - 3X + 1 \) allows the sequence \( P_1 = X^2 - 2, P_2 = X - 1, P_3 = 1 \), with \( a_0 = 0, a_1 = -1, a_2 = 1, b_0 = b_1 = 1 \). Its roots are eigenvalues of the matrix

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & 1
\end{bmatrix}.
\]

The roots with 1 added are eigenvalues of the graph shown in Fig. 2. The roots are eigenvalues of the sum with \( K_2 \) (Fig. 3).

- The irreducible polynomial \( X^3 - 7X - 7 \) allows the sequences

\[
P_1 = X^2 - X - 4, \quad P_2 = X + 3/2, \quad a_0 = -1, \quad a_1 = 5/2, \quad a_2 = -3/2,
\]

\[
b_1 = 2, \quad b_2 = 1/4,
\]

\[
P_1 = X^2 - 2, \quad P_2 = X + 7/5, \quad a_0 = 0, \quad a_1 = 7/5, \quad a_2 = -7/5,
\]

\[
b_1 = 5, \quad b_2 = 1/25.
\]
Let us build a graph having the roots of $X^3 - 7X - 7$ among its eigenvalues. If $x$ is a root of the polynomial, $x$ is an eigenvalue of the matrices
\[
\begin{pmatrix}
-1 & 2 & 0 \\
1 & 5/2 & 1/4 \\
0 & 1 & -3/2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 5 & 0 \\
1 & 7/5 & 1/25 \\
0 & 1 & -7/5
\end{pmatrix}.
\]
Thus, $2x$ is an eigenvalue of the matrices
\[
\begin{pmatrix}
-2 & 4 & 0 \\
2 & 5 & 1/2 \\
0 & 2 & -3
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-2 & 4 & 0 \\
2 & 5 & 1 \\
0 & 1 & -3
\end{pmatrix},
\]
and $2x + 3$ is an eigenvalue of the graph $G_1$ with 30 vertices, summarized in Fig. 4. Similarly, $5x + 7$ is an eigenvalue of the graph $G_2$ with 88 vertices, summarized in Fig. 5.

Then $-4x - 6$ is an eigenvalue of the graph on 120 vertices $G_3 = G_1 \times K_{2,2}$ and $x + 1$ is an eigenvalue of $G_3 + G_2$ (10,560 vertices), and $x$ is an eigenvalue of $G_3 + G_2 + K_2$, that has 21,120 vertices.

We now give an upper bound on the best multiplier that depends only on the degree $n$ in terms of $p$-valuations. Let us recall this notion. If $p$ is a prime, the $p$-valuation $v_p(N)$ of an integer $N$ is the highest exponent $v$ such that $p^v$ divides $N$.

**Theorem 2.** If $x$ is a totally real algebraic integer with degree $n \geq 2$, the smallest positive multiplier $m$ such that $mx$ is an eigenvalue of some finite graph satisfies, for all primes $p$,

\[
v_p(m) \leq \left\lfloor \frac{v_p((n-2)!)}{2} \right\rfloor.
\]

Let us begin with a (somewhat stronger) local version of Theorem 2.
Lemma 6. Let \( P \) be a monic polynomial with integer coefficients, degree \( n \) and \( n \) distinct real roots. Let \( p \) be a prime. There exists a matrix \( M_p \) with rational entries such that

- the characteristic polynomial of \( M_p \) is \( P \),
- the denominators of the entries have a small \( p \)-valuation, that is, at most \( \lceil v_p((n-2)!)/2 \rceil \),
- the entries off the diagonal are \( \geq 0 \),
- the semi-symmetry holds.

Proof. We work in the local ring \( \mathbb{Z}(p) \) of rationals with denominator prime to \( p \). We choose some power \( q \) of \( p \) larger than \( n-2 \). We choose \( a_1 < a_2 < \cdots < a_{n-1} \) in \( \mathbb{Z}(p) \), interlacing the roots of \( P \), and such that \( a_i \equiv i \mod q \). This is possible since \( i + q\mathbb{Z}(p) \) is a dense subset of the reals. We then choose \( m_{ii} = a_i \) for \( 1 \leq i \leq n-1 \), and \( m_{00} \) to make the trace of \( M_p \) equal to the sum of the roots of \( P \).

Let us set

\[
  w = \left\lfloor \frac{v_p((n-2)!)}{2} \right\rfloor.
\]

The other entries \( m_{i0} \) are \( 1/p^w \), the entries \( m_{0i} \) are

\[
  m_{0i} = p^w \frac{P(a_i)}{\prod_{j \neq i} (a_i - a_j)}
\]

and all other entries are 0.

Since the denominator of \( a_i \) is prime to \( p \), the denominator of \( P(a_i) \) also is. Each factor \( a_i - a_j \) is congruent to \( i - j \) in \( \mathbb{Z}(p) \). The product \( \prod_{j \neq i} (a_i - a_j) \), thus, has the same \( p \)-valuation as \( \prod_{j \neq i} (i - j) = (-1)^{n-i-1}(n-i-1)! (i-1)! \); its \( p \)-valuation is at most \( v_p((n-2)!)/2 \). Hence, the \( p \)-valuations of the denominators of the entries do not exceed \( w \).

The entries \( m_{0i} \) are positive because of the interlacing. It is easy to check that the characteristic polynomial \( \det(XI - M_p) \) is

\[
  (X - m_{00}) \prod_i (X - a_i) + \sum_i \left( P(a_i) \prod_{j \neq i} \frac{X - a_j}{a_i - a_j} \right).
\]

It has the same two leading coefficients and the same values on the \( a_i \)'s as \( P \); hence, it is \( P \).

The reader may recognize a Lagrange interpolation [6, 3.2]. The same kind of matrices is used by Krakowski [5, Lemma 3.1, p. 229].

Proof of Theorem 2. Using Lemma 6 for each prime \( p \), we obtain a multiplier \( m_p \), the least common multiple of the denominators of the entries of \( M_p \), and its \( p \)-valuation \( v_p(m_p) \) is at most

\[
  w_p = \left\lfloor \frac{v_p((n-2)!)/2}{2} \right\rfloor.
\]

since all these denominators have \( p \)-valuation at most \( w_p \).
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Of course, \( m_p \) may be much larger than \( p^{\nu_p} \).

Using Lemma 4, we can build a graph \( G_p \) having \( m_p \) among its eigenvalues.

We introduce now the greatest common divisor \( m \) of all multipliers \( m_p \). This \( m \) is also a linear combination with integer coefficients of a finite number of these \( m_p \)'s, since the ring \( \mathbb{Z} \) is a principal ideal domain. Thus, \( m \) is itself a multiplier (Lemma 2) and we can build a graph having \( mx \) among its eigenvalues.

We now derive an upper bound for \( m \), from upper bounds for its primary factors.

For each prime \( \pi \) the \( \pi \)-valuation of \( m \) is \( \inf_{p \text{ prime}} v_\pi(m_p) \), that is, at most \( w_\pi \) since, already, \( v_\pi(m_p) \leq w_\pi \).

**Examples.** If \( p > n - 2 \) then \( m \) is prime to \( p \).

The upper bounds for \( m \) given by Theorem 2 are displayed in the Table 1.

**Theorem 3.** If \( x \) is a totally real algebraic integer with degree at most 4, then \( x \) is an eigenvalue of some finite graph.

**Proof.** Obvious from Theorem 2 for degrees 2 and 3. The case of degree 4 requires the following result.

**Lemma 7.** Let \( P \) be a monic polynomial of degree 4, with integer coefficients. If \( P \) modulo 2 has a root in \( \mathbb{Z}_{(2)} \), then there is a matrix \( M_2 \) with entries in \( \mathbb{Z}_{(2)} \) satisfying the other conditions of Lemma 6.

**Proof of Lemma 7.** If \( P(0) \) is even, we choose \( a_2 \) odd and \( a_1 \) and \( a_3 \) even, with \( a_1 - a_3 \) not a multiple of 4 in \( \mathbb{Z}_{(2)} \). If \( P(1) \) is even, we choose \( a_2 \) even and \( a_1 \) and \( a_3 \) odd, with \( a_1 - a_3 \) not a multiple of 4 in \( \mathbb{Z}_{(2)} \). Then we take the off-diagonal \( m_{01} = 1 \) and

\[
m_{0i} = \frac{-P(a_i)}{\prod_{j \neq i}(a_i - a_j)}.
\]

They all are in \( \mathbb{Z}_{(2)} \).

**Proof of Theorem 3 (conclusion).** We have to get rid off the denominator 2 when Lemma 7 did not help. The polynomial \( P \) modulo 2 is then among the following four:

\[
X^4 + X^3 + X^2 + X + 1, \quad X^4 + X^3 + 1, \quad X^4 + X^2 + 1, \quad X^4 + X + 1.
\]

<table>
<thead>
<tr>
<th>Degree ( n )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m ) divides</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>60</td>
<td>60</td>
<td>420</td>
<td>1680</td>
</tr>
</tbody>
</table>
They are obtained (modulo 2) with the matrices
\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}.
\]

We will now use Lemma 5 as in the proof of Theorem 1, taking care that the coefficients of the polynomials \( P_i \) are in \( \mathbb{Z}_{(2)} \), that is, with odd denominators, and that \( a_i \) and \( b_i \) also are in \( \mathbb{Z}(,\) ,

To build a convenient sequence, it is sufficient to choose a monic polynomial \( P_1 \)
with coefficients in \( \mathbb{Z}_{(2)} \), and with roots interlaced with the roots of \( P \), and with \( P_1 \) modulo 2 equal to
\[
X^3 + X^2, \quad X^3 + X, \quad X^3 \quad \text{and} \quad X^3 + X,
\]
respectively. This is possible since \( \mathbb{Z}_{(2)} \) is dense in \( \mathbb{R} \). By induction on \( i \), this will ensure that \( a_{i-1} \) is in \( \mathbb{Z}_{(2)} \), that \( b_i \) is odd (in other words, invertible) in \( \mathbb{Z}_{(2)} \), and that the polynomial \( P_{i+1} \) has its coefficients in \( Z_{(2)} \), allowing a computation modulo 2 at each step.

**Example.** The irreducible polynomial \( X^4 - 4X^2 - X + 1 \) has roots interlaced with \(-1, 0, 1\) and \( P_1 = X^3 - X \) has coefficients of adequate parity. The corresponding matrix is
\[
\begin{pmatrix}
0 & 3 & 0 & 0 \\
1 & 1/3 & 5/9 & 0 \\
0 & 1 & -2/15 & 9/25 \\
0 & 0 & 1 & -1/5
\end{pmatrix}.
\]
The same separating set yields, with the method of Lemma 6, the matrix
\[
\begin{pmatrix}
0 & 1/2 & 1 & 3/2 \\
1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]
with the same characteristic polynomial.

**Related problems**

We now ask similar questions about the various kinds of graphs, and answer or partially answer some of them.
Proposition 2. For every totally imaginary algebraic integer $x$, there exists a nonnull integer $N$ such that $Nx$ is an eigenvalue of an antisymmetric graph (the corresponding adjacency matrix is defined by $m_{ij} = 1$ and $m_{ji} = -1$ if $ij$ is an arc in $G$, the other $m_{ij}$'s are 0).

Proof. If $x$ is a totally imaginary algebraic integer, then $xi$ is a totally real algebraic integer; thus, $mx$ is an eigenvalue of some graph $G$ for some nonnull integer $m$. If $M$ is the adjacency matrix of $G$, then $mx$ is an eigenvalue of the skew-symmetric matrix

$$
\begin{bmatrix}
0 & M \\
-M & 0
\end{bmatrix}.
$$

Question 1. Is every totally imaginary algebraic integer an eigenvalue of some finite antisymmetric graph?

The argument above shows that this question receives a positive answer if the main question about totally real algebraic integer does.

Question 2. Is every totally real algebraic integer an eigenvalue of some signed graph (with $m_{ij} = m_{ji} = 1$ or $-1$ if $\{i, j\}$ is an edge with the same sign, the other $m_{ij}$ being 0)?

This question was reduced to the one with no signs by a referee (who receives here due thanks). He noted that the eigenvalues $\lambda$ of $B - C$ with eigenvector $H$ is among the eigenvalues of the matrix with blocks

$$
\begin{bmatrix}
B & C \\
C & B
\end{bmatrix}
$$

since

$$
\begin{bmatrix}
B & C \\
C & B
\end{bmatrix}
\begin{bmatrix}
H \\
-H
\end{bmatrix} = \lambda
\begin{bmatrix}
H \\
-H
\end{bmatrix}.
$$

Question 3. Is every algebraic integer an eigenvalue of some antisymmetric directed graph (with adjacency matrix described by the rules $m_{ij} = 1$ if $ij$ is an arc of $G$, the other $m_{ij}$'s being null)?

Answer. Let $P = X^n + a_1 X^{n-1} + \cdots + a_n$ be the minimal polynomial of $x$. If the degree is $\geq 2$, we remark that $(X^2 - a_1 X - a_2 + a_1^2) \cdot P$ has a companion matrix with integer entries, with null diagonal and no pair $i, j$ of indices such that $m_{ij} m_{ji} \neq 0$, with the roots of $P$ among its eigenvalues. The trick in Question 2 gives then a matrix with the same properties and no negative entry. Replacing entries by blocks (as in Lemma 4) gives then a 0–1 matrix that represents a directed graph.
If the degree is 1, we can use the companion matrix of $X^3 - a_1^3$.

**Remark.** As pointed by another referee, the trick of Question 2 and Krakowski's theorem [5] are sufficient to give another proof of Theorem 1.

**References**