Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 984141, 14 pages
doi:10.1155/2010/984141

Research Article

Some Nonlinear Integral Inequalities in Two Independent Variables

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Received 14 January 2010; Accepted 29 May 2010

Academic Editor: Binggen Zhang

We investigate some new nonlinear integral inequalities in two independent variables. The inequalities given here can be used as tools in the qualitative theory of certain nonlinear partial differential equations.

1. Introduction

It is well known that the integral inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential equations. In the past few years, a number of integral inequalities had been established by many scholars, which are motivated by certain applications. For details, we refer to literatures [1–10] and the references therein. In this paper we investigate some new nonlinear integral inequalities in two independent variables, which can be used as tools in the qualitative theory of certain partial differential equations.

2. Main Results

In what follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$ is the given subset of $\mathbb{R}$. The first-order partial derivatives of a $z(x, y)$ defined for $x, y \in \mathbb{R}$ with respect to x and y are denoted by $z_x(x, y)$, and $z_y(x, y)$ respectively. Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals involved exist on the respective domains of their definitions, $C(M, S)$ denotes the class of all continuous functions defined on set M with range in the set S, p and q are constants, and $p \geq 1$, $0 < q \leq p$.

We firstly introduce two lemmas, which are useful in our main results.

Lemma 2.1 (Bernoulli's inequality [11]). *Let* $0 < \alpha \leq 1$ *and* $x > -1$. *Then* $(1+x)^{\alpha} \leq 1 + \alpha x$.

Lemma 2.2 (see [7]). *Let* $u(t), a(t),$ *and* $b(t)$ *be nonnegative and continuous functions defined for* $t \in \mathbb{R}_{+}$.

(i) *Assume that* $a(t)$ *is nondecreasing for* $t \in \mathbb{R}_{+}$. *If*

$$u(t) \leq a(t) + \int_0^t b(s)u(s)\mathrm{d}s, \tag{2.1}$$

for $t \in \mathbb{R}_{+}$, *then*

$$u(t) \leq a(t)\exp\left(\int_0^t b(s)\mathrm{d}s\right), \tag{2.2}$$

for $t \in \mathbb{R}_{+}$.

(ii) *Assume that* $a(t)$ *is nonincreasing for* $t \in \mathbb{R}_{+}$. *If*

$$u(t) \leq a(t) + \int_t^{\infty} b(s)u(s)\mathrm{d}s, \tag{2.3}$$

for $t \in \mathbb{R}_{+}$, *then*

$$u(t) \leq a(t)\exp\left(\int_t^{\infty} b(s)\mathrm{d}s\right), \tag{2.4}$$

for $t \in \mathbb{R}_{+}$.

Next, we establish our main results.

Theorem 2.3. *Let* $u(x,y), a(x,y), b(x,y), f(x,y), g(x,y) \in C(\mathbb{R}_{+}^{2}, \mathbb{R}_{+})$ *and* $a(x,y) > 0$.

(i) *If*

$$u^p(x,y) \leq a(x,y) + b(x,y)\int_0^x \int_y^{\infty} \left[f(s,t)u^p(s,t) + g(s,t)u^q(s,t)\right]\mathrm{d}t\,\mathrm{d}s, \quad x,y \in \mathbb{R}_{+}, \tag{E1}$$

then

$$u(x,y) \leq a^{1/p}(x,y) + \frac{1}{p}a^{(1/p)-1}(x,y)b(x,y)h(x,y)\exp\left(\int_0^x \int_y^{\infty} F(s,t)\mathrm{d}t\,\mathrm{d}s\right), \quad x,y \in \mathbb{R}_{+}, \tag{2.5}$$

where

$$h(x,y) = \int_0^x \int_y^\infty \left[f(s,t)a(s,t) + g(s,t)a^{q/p}(s,t)\right] \mathrm{d}t\, \mathrm{d}s, \tag{2.6}$$

$$F(x,y) = b(x,y)\left(f(x,y) + \frac{q}{p}a^{(q/p)-1}(x,y)g(x,y)\right). \tag{2.7}$$

(ii) *If*

$$u^p(x,y) \le a(x,y) + b(x,y)\int_x^\infty \int_y^\infty \left[f(s,t)u^p(s,t) + g(s,t)u^q(s,t)\right] \mathrm{d}t\, \mathrm{d}s, \quad x,y \in \mathbb{R}_+, \tag{$E'1$}$$

then

$$u(x,y) \le a^{1/p}(x,y) + \frac{1}{p}a^{(1/p)-1}(x,y)b(x,y)\overline{h}(x,y)\exp\left(\int_x^\infty \int_y^\infty F(s,t)\mathrm{d}t\, \mathrm{d}s\right), \quad x,y \in \mathbb{R}_+, \tag{2.5$'$}$$

where

$$\overline{h}(x,y) = \int_x^\infty \int_y^\infty \left[f(s,t)a(s,t) + g(s,t)a^{q/p}(s,t)\right] \mathrm{d}t\, \mathrm{d}s, \tag{2.6$'$}$$

and $F(x,y)$ is defined by (2.7).

Proof. We only give the proof of (i). The proof of (ii) can be completed by following the proof of (i).

(i) Define a function $z(x,y)$ by

$$z(x,y) = \int_0^x \int_y^\infty \left[f(s,t)u^p(s,t) + g(s,t)u^q(s,t)\right] \mathrm{d}t\, \mathrm{d}s, \quad x,y \in \mathbb{R}_+. \tag{2.8}$$

Then $(E1)$ can be restated as

$$u^p(x,y) \le a(x,y) + b(x,y)z(x,y) = a(x,y)\left(1 + \frac{b(x,y)z(x,y)}{a(x,y)}\right). \tag{2.9}$$

Using Lemma 2.1, from (2.9), we easily obtain

$$u(x,y) \le a^{1/p}(x,y) + \frac{1}{p}a^{(1/p)-1}(x,y)b(x,y)z(x,y), \tag{2.10}$$

$$u^q(x,y) \le a^{q/p}(x,y) + \frac{q}{p}a^{(q/p)-1}(x,y)b(x,y)z(x,y). \tag{2.11}$$

Combining (2.8), (2.9), and (2.11), we have

$$\begin{aligned} z(x,y) &\le \int_0^x \int_y^\infty \Big[f(s,t)(a(s,t)+b(s,t)z(s,t)) \\ &\quad + g(s,t)\Big(a^{q/p}(s,t) + \frac{q}{p} a^{(q/p)-1}(s,t) b(s,t) z(s,t) \Big) \Big] \mathrm{d}t\, \mathrm{d}s \\ &= h(x,y) + \int_0^x \int_y^\infty F(s,t) z(s,t) \mathrm{d}t\, \mathrm{d}s, \quad x,y \in \mathbb{R}_+, \end{aligned} \tag{2.12}$$

where $h(x,y)$ and $F(x,y)$ are defined by (2.6) and (2.7), respectively. Obviously, $h(x,y)$ is nonnegative, continuous, nondecreasing in x, and nonincreasing in y for $x,y \in \mathbb{R}_+$.

Firstly, we assume that $h(x,y) > 0$ for $x,y \in \mathbb{R}_+$. From (2.12), we easily observe that

$$\frac{z(x,y)}{h(x,y)} \le 1 + \int_0^x \int_y^\infty \frac{F(s,t)z(s,t)}{h(s,t)} \mathrm{d}t\, \mathrm{d}s. \tag{2.13}$$

Letting

$$v(x,y) = 1 + \int_0^x \int_y^\infty \frac{F(s,t)z(s,t)}{h(s,t)} \mathrm{d}t\, \mathrm{d}s, \tag{2.14}$$

we easily see that $v(x,y)$ is nonincreasing in $y, y \in \mathbb{R}_+$, and

$$z(x,y) \le h(x,y)v(x,y), \quad x,y \in \mathbb{R}_+. \tag{2.15}$$

Therefore,

$$\begin{aligned} v_x(x,y) &= \int_y^\infty \frac{F(x,t)z(x,t)}{h(x,t)} \mathrm{d}t \\ &\le \int_y^\infty F(x,t)v(x,t)\mathrm{d}t \\ &\le v(x,y) \int_y^\infty F(x,t)\mathrm{d}t. \end{aligned} \tag{2.16}$$

Treating y, $y \in \mathbb{R}_+$, fixed in (2.16), dividing both sides of (2.16) by $v(x,y)$, setting $x = s$, and integrating the resulting inequality from 0 to $x, x \in \mathbb{R}_+$, we have

$$v(x,y) \le \exp\left(\int_0^x \int_y^\infty F(s,t)\, \mathrm{d}t\, \mathrm{d}s \right). \tag{2.17}$$

It follows from (2.15) and (2.17) that

$$z(x,y) \le h(x,y)\exp\left(\int_0^x \int_y^\infty F(s,t)\,dt\,ds\right). \tag{2.18}$$

Therefore, the desired inequality (2.5) follows from (2.10) and (2.18).

If $h(x,y)$ is nonnegative, we carry out the above procedure with $h(x,y)+\varepsilon$ instead of $h(x,y)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.5). This completes the proof. □

Theorem 2.4. *Assume that* $u(x,y), a(x,y), b(x,y), f(x,y), g(x,y) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, *and* $L \in C(\mathbb{R}_+^3, \mathbb{R}_+)$. *Let* $a(x,y) > 0$, *and*

$$0 \le L(s,t,u) - L(s,t,v) \le K(s,t,v)(u-v), \tag{2.19}$$

for $u \ge v \ge 0$, *where* $K \in C(\mathbb{R}_+^3, \mathbb{R}_+)$.

(i) *If*

$$u^p(x,y) \le a(x,y) + b(x,y)\int_0^x \int_y^\infty \left[f(s,t)u^q(s,t) + L(s,t,u(s,t))\right]dt\,ds, \quad x,y \in \mathbb{R}_+, \tag{$E2$}$$

then

$$u(x,y) \le a^{1/p}(x,y) + \frac{1}{p}a^{(1/p)-1}(x,y)b(x,y)G(x,y)\exp\left(\int_0^x \int_y^\infty H(s,t)dt\,ds\right), \quad x,y \in \mathbb{R}_+, \tag{2.20}$$

where

$$G(x,y) = \int_0^x \int_y^\infty \left[f(s,t)a^{q/p}(s,t) + L\left(s,t,a^{1/p}(s,t)\right)\right]dt\,ds, \tag{2.21}$$

$$H(x,y) = \left[\frac{q}{p}f(x,y)a^{(q/p)-1}(x,y) + K\left(x,y,a^{1/p}(x,y)\right)\frac{1}{p}a^{(1/p)-1}(x,y)\right]b(x,y). \tag{2.22}$$

(ii) *If*

$$u^p(x,y) \le a(x,y) + b(x,y)\int_x^\infty \int_y^\infty \left[f(s,t)u^q(s,t) + L(s,t,u(s,t))\right]dt\,ds, \quad x,y \in \mathbb{R}_+, \tag{$E'2$}$$

then

$$u(x,y) \le a^{1/p}(x,y) + \frac{1}{p} a^{(1/p)-1}(x,y) b(x,y) \overline{G}(x,y) \times \exp\left(\int_x^\infty \int_y^\infty H(s,t) \mathrm{d}t\, \mathrm{d}s \right), \quad x,y \in \mathbb{R}_+, \tag{2.20$'$}$$

where

$$\overline{G}(x,y) = \int_x^\infty \int_y^\infty \left[f(s,t) a^{q/p}(s,t) + L\left(s,t,a^{1/p}(s,t)\right) \right] \mathrm{d}t\, \mathrm{d}s, \tag{2.21$'$}$$

and $H(x,y)$ is defined by (2.22).

Proof. We only prove the part (i). The proof of (ii) can be completed by following the proof of (i).

(i) Define a function $z(x,y)$ by

$$z(x,y) = \int_0^x \int_y^\infty \left[f(s,t) u^q(s,t) + L(s,t,u(s,t)) \right] \mathrm{d}t\, \mathrm{d}s. \tag{2.23}$$

Then, as in the proof of Theorem 2.3, we obtain (2.9)–(2.11). Therefore, we have

$$\int_0^x \int_y^\infty f(s,t) u^q(s,t) \mathrm{d}t\, \mathrm{d}s \le \int_0^x \int_y^\infty f(s,t) \left(a^{q/p}(s,t) + \frac{q}{p} a^{(q/p)-1}(s,t) b(s,t) z(s,t) \right) \mathrm{d}t\, \mathrm{d}s, \tag{2.24}$$

$$\begin{aligned} \int_0^x \int_y^\infty L(s,t,u(s,t)) \mathrm{d}t\, \mathrm{d}s \le& \int_0^x \int_y^\infty \left\{ L\left(s,t,a^{1/p}(s,t) + \frac{1}{p} a^{(1/p)-1}(s,t) b(s,t) z(s,t) \right) \right. \\ & \left. - L\left(s,t,a^{1/p}(s,t)\right) + L\left(s,t,a^{1/p}(s,t)\right) \right\} \mathrm{d}t\, \mathrm{d}s \\ \le& \int_0^x \int_y^\infty L\left(s,t,a^{1/p}(s,t)\right) \mathrm{d}t\, \mathrm{d}s \\ &+ \int_0^x \int_y^\infty K\left(s,t,a^{1/p}(s,t)\right) \frac{1}{p} a^{(1/p)-1}(s,t) b(s,t) z(s,t) \mathrm{d}t\, \mathrm{d}s. \end{aligned} \tag{2.25}$$

It follows from (2.23)–(2.25) that

$$\begin{aligned} z(x,y) &\le \int_0^x\int_y^\infty \Big[f(s,t)a^{q/p}(s,t) + L\Big(s,t,a^{1/p}(s,t)\Big)\Big]\mathrm{d}t\,\mathrm{d}s \\ &\quad + \int_0^x\int_y^\infty \Big[\frac{q}{p}f(s,t)a^{(q/p)-1}(s,t)b(s,t) \\ &\qquad + K\Big(s,t,a^{1/p}(s,t)\Big)\frac{1}{p}a^{(1/p)-1}(s,t)b(s,t)\Big]z(s,t)\mathrm{d}t\,\mathrm{d}s \\ &= G(x,y) + \int_0^x\int_y^\infty H(s,t)z(s,t)\mathrm{d}t\,\mathrm{d}s, \end{aligned} \tag{2.26}$$

where $G(x,y)$ and $H(x,y)$ are defined by (2.21) and (2.22), respectively.

It is obvious that $G(x,y)$ is nonnegative, continuous, nondecreasing in x, and nonincreasing in y for $x,y \in \mathbb{R}_+$. By following the proof of Theorem 2.3, from (2.26), we have

$$z(x,y) \le G(x,y)\exp\left(\int_0^x\int_y^\infty H(s,t)\,\mathrm{d}t\,\mathrm{d}s\right). \tag{2.27}$$

Combining (2.10) and (2.27), we obtain the desired inequality (2.20). The proof is complete. □

Theorem 2.5. *Let $a(x,y), u(x,y), L(s,t,u)$, and $K(s,t,u)$ be the same as in Theorem 2.4, and $r(x,y) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$.*

(i) *Assume that $a(x,y)$ is nondecreasing in x, $x \in \mathbb{R}_+$, and the condition* (2.19) *holds. If*

$$u^p(x,y) \le a(x,y) + \int_0^x r(s,y)u^p(s,y)\mathrm{d}s + \int_0^x\int_y^\infty L(s,t,u^q(s,t))\mathrm{d}t\,\mathrm{d}s, \quad x,y \in \mathbb{R}_+, \tag{E3}$$

then

$$\begin{aligned} u(x,y) &\le B^{1/p}(x,y) \\ &\quad \times\left[a^{1/p}(x,y) + \frac{1}{p}a^{(1/p)-1}(x,y)J(x,y)\exp\left(\int_0^x\int_y^\infty M(s,t)\mathrm{d}t\,\mathrm{d}s\right)\right], \quad x,y \in \mathbb{R}_+, \end{aligned} \tag{2.28}$$

where

$$B(x,y) = \exp\left(\int_0^x r(s,y)\mathrm{d}s\right), \tag{2.29}$$

$$J(x,y) = \int_0^x \int_y^\infty L\left(s,t,B^{q/p}(s,t)a^{q/p}(s,t)\right)\mathrm{d}t\,\mathrm{d}s, \tag{2.30}$$

$$M(x,y) = K\left(x,y,B^{q/p}(x,y)a^{q/p}(x,y)\right)B^{q/p}(x,y)\frac{q}{p}a^{(q/p)-1}(x,y). \tag{2.31}$$

(ii) *Assume that* $a(x,y)$ *is nonincreasing in* x, $x \in \mathbb{R}_+$, *and the condition* (2.19) *holds. If*

$$u^p(x,y) \le a(x,y) + \int_x^\infty r(s,y)u^p(s,y)\mathrm{d}s + \int_x^\infty \int_y^\infty L(s,t,u^q(s,t))\mathrm{d}t\,\mathrm{d}s, \quad x,y \in \mathbb{R}_+, \tag{$E'3$}$$

then

$$u(x,y) \le \widetilde{B}^{1/p}(x,y) \times \left[a^{1/p}(x,y) + \frac{1}{p}a^{(1/p)-1}(x,y)\widetilde{J}(x,y)\exp\left(\int_x^\infty \int_y^\infty \widetilde{M}(s,t)\,\mathrm{d}t\,\mathrm{d}s\right)\right], \quad x,y \in \mathbb{R}_+, \tag{2.28$'$}$$

where

$$\widetilde{B}(x,y) = \exp\left(\int_x^\infty r(s,y)\mathrm{d}s\right), \tag{2.29$'$}$$

$$\widetilde{J}(x,y) = \int_x^\infty \int_y^\infty L\left(s,t,\widetilde{B}^{q/p}(s,t)a^{q/p}(s,t)\right)\mathrm{d}t\,\mathrm{d}s, \tag{2.30$'$}$$

$$\widetilde{M}(x,y) = K\left(x,y,\widetilde{B}^{q/p}(x,y)a^{q/p}(x,y)\right)\widetilde{B}^{q/p}(x,y)\frac{q}{p}a^{(q/p)-1}(x,y). \tag{2.31$'$}$$

Proof. (i) Define a function $z(x,y)$ by

$$z(x,y) = a(x,y) + v(x,y), \tag{2.32}$$

where

$$v(x,y) = \int_0^x \int_y^\infty L(s,t,u^q(s,t))\mathrm{d}t\,\mathrm{d}s. \tag{2.33}$$

Then ($E3$) can be restated as

$$u^p(x,y) \le z(x,y) + \int_0^x r(s,y)u^p(s,y)\mathrm{d}s. \tag{2.34}$$

Noting the assumption that $a(x,y)$ is nondecreasing in x, $x \in \mathbb{R}_+$, we easily see that $z(x,y)$ is a nonnegative and nondecreasing function in $x, x \in \mathbb{R}_+$. Therefore, treating $y, y \in \mathbb{R}_+$, fixed in (**??**) and using part (i) of Lemma 2.2 to (**??**), we get

$$u^p(x,y) \le B(x,y)z(x,y). \tag{2.35}$$

that is,

$$u^p(x,y) \le B(x,y)(a(x,y)+v(x,y)) = B(x,y)a(x,y)\left(1+\frac{v(x,y)}{a(x,y)}\right), \tag{2.36}$$

where $B(x,y)$ is defined by (2.29). Using Lemma 2.1, from (2.35) we have

$$u(x,y) \le B^{1/p}(x,y)\left(a^{1/p}(x,y)+\frac{1}{p}a^{(1/p)-1}(x,y)v(x,y)\right), \tag{2.37}$$

$$u^q(x,y) \le B^{q/p}(x,y)\left(a^{q/p}(x,y)+\frac{q}{p}a^{(q/p)-1}(x,y)v(x,y)\right). \tag{2.38}$$

Combining (2.33) and (2.37), and noting the hypotheses (2.19), we obtain

$$\begin{aligned} v(x,y) \le \int_0^x\int_y^\infty \Big\{ & L\left(s,t,B^{q/p}(s,t)\left(a^{q/p}(s,t)+\frac{q}{p}a^{(q/p)-1}(s,t)v(s,t)\right)\right) \\ & -L\left(s,t,B^{q/p}(s,t)a^{q/p}(s,t)\right)+L\left(s,t,B^{q/p}(s,t)a^{q/p}(s,t)\right)\Big\}\mathrm{d}t\,\mathrm{d}s \\ \le J(x,y) & +\int_0^x\int_y^\infty M(s,t)v(s,t)\mathrm{d}t\,\mathrm{d}s, \end{aligned} \tag{2.39}$$

where $J(x,y)$ and $M(x,y)$ are defined by (2.30) and (2.31), respectively.

It is obvious that $J(x,y)$ is nonnegative, continuous, nondecreasing in x and nonincreasing in y for $x, y \in \mathbb{R}_+$. By following the proof of Theorem 2.3, from (2.38), we obtain

$$v(x,y) \le J(x,y)\exp\left(\int_0^x\int_y^\infty M(s,t)\mathrm{d}t\,\mathrm{d}s\right), \quad x,y \in \mathbb{R}_+. \tag{2.40}$$

Obviously, the desired inequality (2.28) follows from (2.36) and (2.39).

(ii) Noting the assumption that $a(x,y)$ is nonincreasing in $x, x \in \mathbb{R}_+$, and using the part (ii) of Lemma 2.2, we can complete the proof by following the proof of (i) with suitable changes. Therefore, the details are omitted here. □

By using the ideas of the proofs of Theorems 2.5 and 2.3, we easily prove the following theorem.

Theorem 2.6. *Let* $a(x,y), u(x,y), r(x,y), f(x,y), g(x,y) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, *and* $a(x,y) > 0$.

(i) *Assume that* $a(x,y)$ *is nondecreasing in* x, $x \in \mathbb{R}_+$. *If*

$$\begin{aligned} u^p(x,y) \le a(x,y) &+ \int_0^x r(s,y)u^p(s,y)\mathrm{d}s \\ &+ \int_0^x \int_y^\infty \left[f(s,t)u^p(s,t) + g(s,t)u^q(s,t)\right]\mathrm{d}t\,\mathrm{d}s, \quad x,y \in \mathbb{R}_+, \end{aligned} \tag{$E4$}$$

then

$$\begin{aligned} u(x,y) &\le B^{1/p}(x,y) \\ &\times \left[a^{1/p}(x,y) + \frac{1}{p}a^{(1/p)-1}(x,y)H(x,y)\exp\left(\int_0^x \int_y^\infty P(s,t)\mathrm{d}t\,\mathrm{d}s\right)\right], \quad x,y \in \mathbb{R}_+, \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} H(x,y) &= \int_0^x \int_y^\infty \left[f(s,t)B(s,t)a(s,t) + g(s,t)B^{q/p}(s,t)a^{q/p}(s,t)\right]\mathrm{d}t\,\mathrm{d}s, \\ P(x,y) &= f(x,y)B(x,y) + g(x,y)B^{q/p}(x,y)\frac{q}{p}a^{(q/p)-1}(x,y), \end{aligned} \tag{2.42}$$

and $B(x,y)$ *is defined by* (2.29).

(ii) *Assume that* $a(x,y)$ *is nonincreasing in* x, $x \in \mathbb{R}_+$. *If*

$$\begin{aligned} u^p(x,y) \le a(x,y) &+ \int_x^\infty r(s,y)u^p(s,y)\mathrm{d}s \\ &+ \int_x^\infty \int_y^\infty \left[f(s,t)u^p(s,t) + g(s,t)u^q(s,t)\right]\mathrm{d}t\,\mathrm{d}s, \quad x,y \in \mathbb{R}_+, \end{aligned} \tag{$E'4$}$$

then

$$\begin{aligned} u(x,y) &\le \widetilde{B}^{1/p}(x,y) \\ &\times \left[a^{1/p}(x,y) + \frac{1}{p}a^{(1/p)-1}(x,y)\widetilde{H}(x,y)\exp\left(\int_x^\infty \int_y^\infty \widetilde{P}(s,t)\mathrm{d}t\,\mathrm{d}s\right)\right], \quad x,y \in \mathbb{R}_+, \end{aligned} \tag{2.40$'$}$$

where

$$\widetilde{H}(x,y) = \int_x^\infty \int_y^\infty \left[f(s,t)\widetilde{B}(s,t)a(s,t) + g(s,t)\widetilde{B}^{q/p}(s,t)a^{q/p}(s,t)\right]\mathrm{d}t\,\mathrm{d}s,$$
$$\widetilde{P}(x,y) = f(x,y)\widetilde{B}(x,y) + g(x,y)\widetilde{B}^{q/p}(x,y)\frac{q}{p}a^{(q/p)-1}(x,y), \tag{2.41'}$$

and $\widetilde{B}(x,y)$ *is defined by* (25′).

Remark 2.7. Noting that p and q are constants, and $p \geq 1, 0 < q \leq p$, we can obtain many special integral inequalities by using our main results. For example, let $p = 1, q = 1/4$, and $p = q = 2$, respectively; from Theorem 2.3, we obtain the following corollaries.

Corollary 2.8. *Let* $u(x,y), a(x,y), b(x,y), f(x,y), g(x,y) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ *and* $a(x,y) > 0$.

(i) *If*

$$u(x,y) \leq a(x,y) + b(x,y)\int_0^x \int_y^\infty \left[f(s,t)u(s,t) + g(s,t)u^{1/4}(s,t)\right]\mathrm{d}t\,\mathrm{d}s, \quad x,y \in \mathbb{R}_+, \tag{E5}$$

then

$$u(x,y) \leq a(x,y) + b(x,y)\widehat{h}(x,y)\exp\left(\int_0^x \int_y^\infty \widehat{F}(s,t)\mathrm{d}t\,\mathrm{d}s\right), \quad x,y \in \mathbb{R}_+, \tag{2.43}$$

where

$$\widehat{h}(x,y) = \int_0^x \int_y^\infty \left[f(s,t)a(s,t) + g(s,t)a^{1/4}(s,t)\right]\mathrm{d}t\,\mathrm{d}s, \tag{2.44}$$

$$\widehat{F}(x,y) = b(x,y)\left(f(x,y) + \frac{1}{4}a^{-3/4}(x,y)g(x,y)\right). \tag{2.45}$$

(ii) *If*

$$u(x,y) \leq a(x,y) + b(x,y)\int_x^\infty \int_y^\infty \left[f(s,t)u(s,t) + g(s,t)u^{1/4}(s,t)\right]\mathrm{d}t\,\mathrm{d}s, \quad x,y \in \mathbb{R}_+, \tag{E'5}$$

then

$$u(x,y) \leq a(x,y) + b(x,y)\widetilde{h}(x,y)\exp\left(\int_0^x \int_y^\infty \widehat{F}(s,t)\mathrm{d}t\,\mathrm{d}s\right), \quad x,y \in \mathbb{R}_+, \tag{2.42'}$$

where

$$\widetilde{h}(x,y) = \int_x^\infty \int_y^\infty \left[f(s,t)a(s,t) + g(s,t)a^{1/4}(s,t)\right] \mathrm{d}t\,\mathrm{d}s, \tag{2.43$'$}$$

and $\widehat{F}(x,y)$ *is defined by* (2.44).

Corollary 2.9. *Let* $u(x,y), a(x,y), b(x,y), g(x,y) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ *and* $a(x,y) > 0$.

(i) *If*

$$u^2(x,y) \le a(x,y) + b(x,y) \int_0^x \int_y^\infty g(s,t)u^2(s,t)\mathrm{d}t\,\mathrm{d}s, \quad x,y \in \mathbb{R}_+, \tag{$E6$}$$

then

$$u(x,y) \le a^{1/2}(x,y) + \frac{1}{2}a^{-1/2}(x,y)b(x,y)m(x,y) \times \exp\left(\int_0^x \int_y^\infty b(s,t)g(s,t)\mathrm{d}t\,\mathrm{d}s\right), \quad x,y \in \mathbb{R}_+, \tag{2.46}$$

where

$$m(x,y) = \int_0^x \int_y^\infty g(s,t)a(s,t)\mathrm{d}t\,\mathrm{d}s. \tag{2.47}$$

(ii) *If*

$$u^2(x,y) \le a(x,y) + b(x,y) \int_x^\infty \int_y^\infty g(s,t)u^2(s,t)\mathrm{d}t\,\mathrm{d}s, \quad x,y \in \mathbb{R}_+, \tag{$E'6$}$$

then

$$u(x,y) \le a^{1/2}(x,y) + \frac{1}{2}a^{-1/2}(x,y)b(x,y)\widetilde{m}(x,y) \times \exp\left(\int_0^x \int_y^\infty b(s,t)g(s,t)\mathrm{d}t\,\mathrm{d}s\right), \quad x,y \in \mathbb{R}_+, \tag{2.45$'$}$$

where

$$\widetilde{m}(x,y) = \int_x^\infty \int_y^\infty g(s,t)a(s,t)\mathrm{d}t\,\mathrm{d}s. \tag{2.46$'$}$$

Remark 2.10. If we add $a(x,y)>0$ to the assumptions of [7, Theorems 2.2–2.4], then we easily see that [7, Theorems 2.2–2.4] are special cases of Theorems 2.3, 2.5, and 2.6, respectively. Therefore, our paper gives some extensions of the results of [7] in a sense.

3. An Application

In this section, using Theorem 2.3, we obtain the bound on the solution of a nonlinear differential equation.

Example 3.1. Consider the partial differential equation:

$$pu^{p-1}(x,y)u_{xy}(x,y)+p(p-1)u^{p-2}(x,y)u_x(x,y)u_y(x,y)=h(x,y,u(x,y))+r(x,y),$$
$$u(x,\infty)=\sigma(x),\qquad u(\infty,y)=\tau(y),\qquad u(\infty,\infty)=d, \tag{3.1}$$

where $h\in C(\mathbb{R}_+^2\times\mathbb{R},\mathbb{R}), r\in C(\mathbb{R}_+^2,\mathbb{R}),\sigma,\tau\in C(\mathbb{R}_+,\mathbb{R})$, and d is a real constant, and $p\geq 1$ is a constant.

Suppose that

$$|h(x,y,u(x,y))|\leq f(x,y)|u(x,y)|^p+g(x,y)|u(x,y)|^q,$$
$$\left|\sigma(x)+\tau(y)-d+\int_x^\infty\int_y^\infty r(s,t)\mathrm{d}t\,\mathrm{d}s\right|\leq a(x,y), \tag{3.2}$$

where $a(x,y),f(x,y),g(x,y)\in C(\mathbb{R}_+^2,\mathbb{R}_+)$ and $a(x,y)>0$ for $x,y\in\mathbb{R}_+$, and $0<q\leq p$ is a constant. Let $u(x,y)$ be a solution of (3.1) for $x,y\in\mathbb{R}_+$; then

$$|u(x,y)|\leq a^{1/p}(x,y)+\frac{1}{p}a^{(1/p)-1}(x,y)E(x,y)\exp\left(\int_x^\infty\int_y^\infty W(s,t)\mathrm{d}t\,\mathrm{d}s\right),\quad x,y\in\mathbb{R}_+, \tag{3.3}$$

where

$$E(x,y)=\int_x^\infty\int_y^\infty\left[f(s,t)a(s,t)+g(s,t)a^{q/p}(s,t)\right]\mathrm{d}t\,\mathrm{d}s,$$
$$W(x,y)=f(x,y)+\frac{q}{p}a^{(q/p)-1}(x,y)g(x,y). \tag{3.4}$$

In fact, if $u(x,y)$ is a solution of (3.1), then it can be written as (see [1, page 80])

$$[u(x,y)]^p=\sigma(x)+\tau(y)-d+\int_x^\infty\int_y^\infty[h(s,t,u(s,t))+r(s,t)]\mathrm{d}t\,\mathrm{d}s, \tag{3.5}$$

for $x,y\in\mathbb{R}_+$.

It follows from (3.2) and (3.5) that

$$|u(x,y)|^p \le a(x,y) + \int_x^\infty \int_y^\infty \left[f(s,t)|u(s,t)|^p + g(s,t)|u(s,t)|^q \right] \mathrm{d}t\, \mathrm{d}s. \tag{3.6}$$

Now, a suitable application of part (ii) of Theorem 2.3 to (3.6) yields the required estimate in (3.3).

Acknowledgment

This work is supported by the National Natural Science Foundation of China (10971018), the Natural Science Foundation of Shandong Province (ZR2009AM005), China Postdoctoral Science Foundation Funded Project (20080440633), Shanghai Postdoctoral Scientific Program (09R21415200), the Project of Science and Technology of the Education Department of Shandong Province (J08LI52), and the Doctoral Foundation of Binzhou University (2006Y01).

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