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NOTE

TRIVIALLY PERFECT GRAPHS*

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An undirected graph is *trivially perfect* if for every induced subgraph the stability number equals the number of (maximal) cliques. We characterize the trivially perfect graphs as a proper subclass of the triangulated graphs (thus disproving a claim of Buneman [3]), and we relate them to some well-known classes of perfect graphs.

Let $m(G)$ denote the number of *cliques* (maximal complete subgraphs) of an undirected graph G and let $\alpha(G)$ be the *stability number*, that is the cardinality of the largest set of pairwise nonadjacent vertices. Clearly,

$$\alpha(G) \leq m(G) \tag{1}$$

since there must be $\alpha(G)$ distinct cliques containing the members of a maximum stable set.

A graph is *triangulated* if every simple cycle of length >3 has a chord. Buneman [3, p. 210] stated falsely that equality holds in (1) for triangulated graphs. For example, equality is not even true for trees. Fulkerson and Gross [6, p. 852] have proved the following for a graph with n vertices:

Theorem 1. *If G is a triangulated graph, then $m(G) \leq n$.*

This bound is tight if one considers the graph with no edges.

We may well ask, for which graphs is there equality in (1)? Unfortunately, we cannot expect to discover much about the structure of such graphs. Indeed, let G be any undirected graph with cliques $C_1, C_2, \ldots, C_m$; add new vertices $x_1, x_2, \ldots, x_m$ and connect x_i with each vertex of C_i to form an augmented graph H. Clearly, $\alpha(H) = m(H) = m$. For this reason, we shall add a hereditary condition.

An undirected graph $G = (V, E)$ is said to be *trivially perfect* if for each $A \subseteq V$, the induced subgraph G_A of G satisfies $\alpha(G_A) = m(G_A)$. This name was chosen since it is trivial to show that such a graph is perfect. A graph $G = (V, E)$ is *perfect* if for each $A \subseteq V$, the stability number $\alpha(G_A)$ equals the least number of cliques

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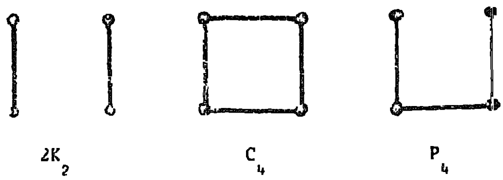

Fig. 1

of G_A whose union covers V (see [1, 2, 9, 10]). The next theorem characterizes trivially perfect graphs.

Theorem 2. *A graph $G=(V,E)$ is trivially perfect if and only if it contains no induced subgraph isomorphic to C_4 or P_4 (see Fig. 1).*

Proof. ($\Leftarrow$) Let S be a maximum stable set of G_A, and suppose that there is a vertex s in S which contained in two distinct cliques X and Y. Then there exist vertices $x\in X$ and $y\in Y$ such that $xy\notin E$. Hence, $|S|>1$.

Let $u\in S-\{s\}$. If $xu\in E$ (resp. $yu\in E$), then $G_{\{y,s,x,u\}}$ (resp. $G_{\{x,s,y,u\}}$) would be isomorphic to either C_4 or P_4. Therefore, $xu\notin E$ and $yu\notin E$ which implies that $\{x,y\}\cup(S-\{s\})$ is a stable set larger than S, a contradiction.

($\Rightarrow$) Since $\alpha(G_4)=\alpha(P_4)=2$ and $m(C_4)=4$ and $m(P_4)=3$, the implication follows.

Remark 3. Every trivially perfect graph is triangulated, but not conversely.

Wolk's [11] characterization of graphs which admit a transitive orientation whose Hasse diagram is a rooted tree yields the next result.

Corollary 4. *A connected graph is trivially perfect if and only if it is the comparability graph of a rooted tree.*

Unlike the general case of perfect graphs [9, 10], the complement of a trivially perfect graph may not itself be trivially perfect. The following characterization is immediate.

Corollary 5. *Let $\bar{G}$ denote the complement of an undirected graph G. Then G and $\bar{G}$ are both trivially perfect iff G contains no induced subgraph isomorphic to C_4, P_4 or $2K_2$ (see Fig. 1).*

A graph satisfying Corollary 5 is a *threshold graph*. By definition, an n-vertex graph G is *threshold* if there exists a hyperplane in $\mathbb{R}^n$ separating the characteristic vectors of the stable sets of G from the characteristic vectors of the nonstable sets. Threshold graphs were introduced by Chvátal and Hammer [4, 5] who gave the forbidden subgraph characterization. See also Henderson and Zalcstein [8] [illegible] [7].

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