Existence and uniqueness of periodic solutions
for a kind of first order neutral functional differential equations

Bingwen Liu\textsuperscript{a,*}, Lihong Huang\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Hunan University of Arts and Science, Changde, Hunan 415000, PR China
\textsuperscript{b} College of Mathematics and Econometrics, Hunan University, Changsha 410082, PR China

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Abstract

In this paper, we use the coincidence degree theory to establish new results on the existence and uniqueness of $T$-periodic solutions for the first order neutral functional differential equation of the form

$$\left( x(t) + Bx(t - \delta) \right)' = g_1(t, x(t)) + g_2(t, x(t - \tau)) + p(t).$$

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1. Introduction

Consider the first order neutral functional differential equation (NFDE) of the form

$$\left( x(t) + Bx(t - \delta) \right)' = g_1(t, x(t)) + g_2(t, x(t - \tau)) + p(t), \quad (1.1)$$

where $p : \mathbb{R} \to \mathbb{R}$ and $g_1, g_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\tau, B$ and $\delta$ are constants, $p$ is $T$-periodic, $g_1$ and $g_2$ are $T$-periodic in the first argument, $|B| \neq 1$ and $T > 0$. 

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* Corresponding author.
E-mail address: liubw007@yahoo.com.cn (B. Liu).
Such a kind of NFDE has been used for the study of distributed networks containing lossless transmission lines [6,7]. Hence, in recent years, the problem of the existence of periodic solutions for Eq. (1.1) has been extensively studied in the literature. We refer the reader to [1,3–8,10] and the references cited therein for more details. However, to the best of our knowledge, there exist no results for the existence and uniqueness of periodic solutions of Eq. (1.1).

The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of $T$-periodic solutions of Eq. (1.1). The results of this paper are new and they complement previously known results. An illustrative example is given in Section 4.

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|^k = \left( \int_0^T |x(t)|^k \, dt \right)^{1/k}, \quad |x|_\infty = \max_{t \in [0,T]} |x(t)|.$$  

Let $X = \{x \mid x \in C(R, R), x(t + T) = x(t), \text{ for all } t \in R\}$ be a Banach space with the norm $\|x\|_X = |x|_\infty$. Define linear operators $A$ and $L$ in the following form:

$$A : X \to X, \quad (Ax)(t) = x(t) + Bx(t - \delta)$$

and

$$L : D(L) \subset X \to X, \quad Lx = (Ax)', \quad (1.2)$$

where $D(L) = \{x \mid x \in X, x' \in C(R, R)\}$.

We also define a nonlinear operator $N : X \to X$ by setting

$$Nx = g_1(t, x(t)) + g_2(t, x(t - \tau)) + p(t). \quad (1.3)$$

By Hale’s terminology [4], a solution $u(t)$ of Eq. (1.1) is that $u \in C(R, R)$ such that $Au \in C^1(R, R)$ and Eq. (1.1) is satisfied on $R$. In general, $u \notin C^1(R, R)$. But from [8, Lemma 1], in view of $|B| \neq 1$, it is easy to see that $(Ax)' = Ax'$. So a $T$-periodic solution $u(t)$ of Eq. (1.1) must be such that $u \in C^1(R, R)$. Meanwhile, according to [8, Lemma 1], we can easily get that $\ker L = R$, and $\im L = \{x \mid x \in X, \int_0^T x(s) \, ds = 0\}$. Therefore, the operator $L$ is a Fredholm operator with index zero. Define the continuous projectors $P : X \to \ker L$ and $Q : X \to X/\im L$ by setting

$$Px(t) = \frac{1}{T} \int_0^T x(s) \, ds \quad \text{and} \quad Qx(t) = \frac{1}{T} \int_0^T x(s) \, ds.$$ 

Hence, $\im P = \ker L$ and $\ker Q = \im L$. Set $L_P = L|_{D(L)\cap \ker P}$, then $L_P$ has continuous inverse $L_P^{-1}$ defined by

$$L_P^{-1}y(t) = A^{-1}\left( \frac{1}{T} \int_0^T sy(s) \, ds + \int_0^T y(s) \, ds \right). \quad (1.4)$$

Therefore, it is easy to see from (1.3) and (1.4) that $N$ is $L$-compact on $\overline{\Omega}$, where $\Omega$ is an open bounded set in $X$. 
2. Preliminary results

In view of (1.2) and (1.3), the operator equation

\[ Lx = \lambda Nx \]

is equivalent to the following equation:

\[ x'(t) + Bx'(t - \delta) = \lambda \left[ g_1(t, x(t)) + g_2(t, x(t - \tau)) + p(t) \right], \tag{2.1}_\lambda \]

where \( \lambda \in (0, 1) \).

For convenience of use, we introduce the continuation theorem [3] as follows.

**Lemma 2.1.** Let \( X \) be a Banach space. Suppose that \( L : D(L) \subset X \rightarrow X \) is a Fredholm operator with index zero and \( N : \Omega \rightarrow X \) is \( L \)-compact on \( \Omega \), where \( \Omega \) is an open bounded subset of \( X \). Moreover, assume that all the following conditions are satisfied:

1. \( Lx \neq \lambda Nx \), for all \( x \in \partial \Omega \cap D(L) \), \( \lambda \in (0, 1) \);
2. \( Nx \notin \text{Im} L \), for all \( x \in \partial \Omega \cap \text{Ker} L \);
3. the Brower degree \( \deg \{ QN, \Omega \cap \text{Ker} L, 0 \} \neq 0 \).

Then equation \( Lx = Nx \) has at least one solution on \( \Omega \cap D(L) \).

The following lemmas will be useful to prove our main results in Section 3.

**Lemma 2.2.** Let \( x(t) \in X \cap C^1(R, R) \). Suppose that there exists a constant \( D \geq 0 \) such that

\[ |x(\tau_0)| \leq D, \quad \tau_0 \in [0, T]. \tag{2.2} \]

Then

\[ |x|_2 \leq \frac{T}{\pi} |x'|_2 + \sqrt{T} D. \tag{2.3} \]

**Proof.** Let

\[ y(t) = \begin{cases} x(t + \tau_0 - T) - x(\tau_0), & T - \tau_0 \leq t \leq T, \\ x(t + \tau_0) - x(\tau_0), & 0 \leq t < T - \tau_0. \end{cases} \]

Then

\[ y(0) = y(T) = 0 \quad \text{and} \quad y'(t) = x'(t + \tau_0) \quad \text{for all } t \in [0, T]. \tag{2.4} \]

Thus, by Theorem 225 in [2], (2.4) implies that

\[ |y|_2 \leq \frac{T}{\pi} |y'|_2. \tag{2.5} \]

In view of the inequality of Minkowski, we have

\[ |x|^2 = \int_{\tau_0}^{T} |x(t)|^2 \, dt + \int_{0}^{\tau_0} |x(t)|^2 \, dt. \]
Combining (2.5) and (2.6), we obtain
\[
|x_2| \leq |y_2| + \sqrt{T}D \leq \frac{T}{\pi} |y'_2| + \sqrt{T}D = \frac{T}{\pi} |x'_2| + \sqrt{T}D.
\]
This completes the proof of Lemma 2.2. \(\square\)

**Lemma 2.3.** Assume that the following conditions are satisfied:

(A0) one of the following conditions holds:

1. \(g_i(t, u_1) - g_i(t, u_2))(u_1 - u_2) > 0\), for \(i = 1, 2, u_i \in R, \forall t \in R\) and \(u_1 \neq u_2\),
2. \(g_i(t, u_1) - g_i(t, u_2))(u_1 - u_2) < 0\), for \(i = 1, 2, u_i \in R, \forall t \in R\) and \(u_1 \neq u_2\);

(\(\overline{A}_0\)) one of the following conditions holds:

1. there exist constants \(b_1\) and \(b_2\) such that \(b_1 + b_2 < \frac{T}{\pi} (1 - |B|)\), and
   \[|g_i(t, u_1) - g_i(t, u_2)| \leq b_1|u_1 - u_2|, \quad \text{for } i = 1, 2, u_i \in R, \forall t \in R,\]
2. there exist constants \(b_1\) and \(b_2\) such that \(b_1 + b_2 < \frac{T}{\pi} (|B| - 1)\), and
   \[|g_i(t, u_1) - g_i(t, u_2)| \leq b_1|u_1 - u_2|, \quad \text{for } i = 1, 2, u_i \in R, \forall t \in R.
\]

Then Eq. (1.1) has at most one \(T\)-periodic solution.

**Proof.** Suppose that \(x_1(t)\) and \(x_2(t)\) are two \(T\)-periodic solutions of Eq. (1.1). Then, we have
\[
\left(x_1(t) + Bx_1(t - \delta)\right)' - g_1(t, x_1(t)) - g_2(t, x_1(t - \tau)) = p(t)
\]
and
\[
\left(x_2(t) + Bx_2(t - \delta)\right)' - g_1(t, x_2(t)) - g_2(t, x_2(t - \tau)) = p(t).
\]
This implies that
\[
\left((x_1(t) - x_2(t)) + B(x_1(t - \delta) - x_2(t - \delta))\right)' - (g_1(t, x_1(t)) - g_1(t, x_2(t)))
- (g_2(t, x_1(t - \tau)) - g_2(t, x_2(t - \tau))) = 0. \tag{2.7}
\]
Set \(Z(t) = x_1(t) - x_2(t)\). Then, from (2.7), we obtain
\[
Z'(t) + BZ'(t - \delta) - (g_1(t, x_1(t)) - g_1(t, x_2(t)))
- (g_2(t, x_1(t - \tau)) - g_2(t, x_2(t - \tau))) = 0. \tag{2.8}
\]
Thus, integrating (2.8) from 0 to \(T\), we have
\[
\int_0^T [(g_1(t, x_1(t)) - g_1(t, x_2(t))) + (g_2(t, x_1(t - \tau)) - g_2(t, x_2(t - \tau)))] dt = 0.
\]
Therefore, in view of integral mean value theorem, it follows that there exists a constant \( \gamma \in [0, T] \) such that
\[
(g_1(\gamma, x_1(\gamma)) - g_2(\gamma, x_2(\gamma))) + (g_2(\gamma, x_1(\gamma - \tau)) - g_2(\gamma, x_2(\gamma - \tau))) = 0.
\] (2.9)

From (A_0), (2.9) implies that
\[
(x_1(\gamma) - x_2(\gamma))(x_1(\gamma - \tau) - x_2(\gamma - \tau)) \leq 0.
\]
Since \( Z(t) = x_1(t) - x_2(t) \) is a continuous function on \( \mathbb{R} \), it follows that there exists a constant \( \xi \in \mathbb{R} \) such that
\[
Z(\xi) = 0.
\] (2.10)

Let \( \xi = nT + \tilde{\gamma} \), where \( \tilde{\gamma} \in [0, T] \) and \( n \) is an integer. Then, (2.10) implies that there exists a constant \( \tilde{\gamma} \in [0, T] \) such that
\[
Z(\tilde{\gamma}) = 0.
\] (2.11)

Then, from Lemma 2.2, using Schwarz inequality and the following relation:
\[
|Z(t)| = \left| Z(\tilde{\gamma}) + \int_{\tilde{\gamma}}^{t} Z'(s) ds \right| \leq \int_{0}^{T} \left| Z'(s) \right| ds,
\]
we obtain
\[
|Z|_\infty \leq \sqrt{T} \| Z' \|_2 \quad \text{and} \quad |Z|_2 \leq \frac{T}{\pi} \| Z' \|_2.
\] (2.12)

Now, we consider two cases.

**Case (i).** If (A_0)(1) holds, multiplying both sides of (2.8) by \( Z'(t) \) and then integrating them from 0 to \( T \), using (2.12) and Schwarz inequality, we have
\[
|Z'|^2 = \int_{0}^{T} \left| Z'(t) \right|^2 dt
\]
\[
= -B \int_{0}^{T} Z'(t)Z'(t - \delta) dt + \int_{0}^{T} (g_1(t, x_1(t)) - g_1(t, x_2(t)))Z'(t) dt
\]
\[
+ \int_{0}^{T} (g_2(t, x_1(t - \tau)) - g_2(t, x_2(t - \tau)))Z'(t) dt
\]
\[
\leq |B||Z'|^2 + b_1 \int_{0}^{T} \left| x_1(t) - x_2(t) \right| \left| Z'(t) \right| dt
\]
\[
+ b_2 \int_{0}^{T} \left| x_1(t - \tau) - x_2(t - \tau) \right| \left| Z'(t) \right| dt
\]
\[
\leq |B||Z'|^2 + b_1 |Z|_2 |Z'|_2 + b_2 |Z|_2 |Z'|_2
\]
\[ \leq \left( |B| + (b_1 + b_2) \frac{T}{\pi} \right) |Z'|_2^2. \]  

(2.13)

From (2.11) and \((\overline{A}_0)(1)\), (2.13) implies that

\[ Z(t) \equiv Z'(t) \equiv 0, \quad \text{for all } t \in \mathbb{R}. \]

Hence, \(x_1(t) \equiv x_2(t)\), for all \(t \in \mathbb{R}\). Therefore, Eq. (1.1) has at most one \(T\)-periodic solution.

**Case (ii).** If \((\overline{A}_0)(2)\) holds, multiplying both sides of (2.8) by \(Z'(t-\delta)\) and then integrating them from 0 to \(T\), using (2.12) and Schwarz inequality, we have

\[
|B||Z'|_2^2 = \left| \int_0^T B|Z'(t-\delta)|^2 \, dt \right| \\
= \left| - \int_0^T Z(t)Z'(t-\delta) \, dt + \int_0^T \left( g_1(t,x_1(t)) - g_2(t,x_2(t)) \right) Z'(t-\delta) \, dt \\
+ \int_0^T \left( g_2(t,x_1(t-\tau)) - g_2(t,x_2(t-\tau)) \right) Z'(t-\delta) \, dt \right| \\
\leq |Z'|_2^2 + b_1 \int_0^T \left|x_1(t) - x_2(t)\right| \left|Z'(t-\delta)\right| \, dt \\
+ b_2 \int_0^T \left|x_1(t-\tau) - x_2(t-\tau)\right| \left|Z'(t-\delta)\right| \, dt \\
\leq |Z'|_2^2 + b_1 |Z|_2 |Z'|_2 + b_2 |Z|_2 |Z'|_2 \\
\leq \left( 1 + (b_1 + b_2) \frac{T}{\pi} \right) |Z'|_2^2. 
\]

(2.14)

Then using the methods similar to those used in Case (i), from (2.11), (2.14) and \((\overline{A}_0)(2)\), we can conclude that Eq. (1.1) has at most one \(T\)-periodic solution. The proof of Lemma 2.3 is now complete. \(\Box\)

**Lemma 2.4.** Assume that \((A_0)\) holds and there exists a constant \(d > 0\) such that one of the following conditions holds:

(A1) \(x(g_1(t,x) + g_2(t,x) + p(t)) > 0\), for all \(t \in \mathbb{R}\), \(|x| \geq d\);  
(A2) \(x(g_1(t,x) + g_2(t,x) + p(t)) < 0\), for all \(t \in \mathbb{R}\), \(|x| \geq d\).

If \(x(t)\) is a \(T\)-periodic solution of (2.1)_\lambda, then

\[ |x|_\infty \leq d + \sqrt{T}|x'|_2. \]

(2.15)
Proof. Let $x(t)$ be a $T$-periodic solution of $(2.1)_\lambda$. Then, integrating $(2.1)_\lambda$ from $0$ to $T$, we have

$$
\int_0^T \left[ g_1(t, x(t)) + g_2(t, x(t - \tau)) + p(t) \right] dt = 0.
$$

This implies that there exists a constant $t_1 \in \mathbb{R}$ such that

$$
g_1(t_1, x(t_1)) + g_2(t_1, x(t_1 - \tau)) + p(t_1) = 0. \tag{2.16}
$$

Now, we shall show that the following claim is true.

Claim. If $x(t)$ is a $T$-periodic solution of $(2.1)_\lambda$, then there exists a constant $t_2 \in \mathbb{R}$ such that

$$
|x(t_2)| \leq d. \tag{2.17}
$$

Assume, by way of contradiction, that (2.17) does not hold. Then

$$
|x(t)| > d, \text{ for all } t \in \mathbb{R}, \tag{2.18}
$$

which, together with $(A_1)$, $(A_2)$ and (2.16), implies that one of the following relations holds:

1. $x(t_1) > x(t_1 - \tau) > d$; \hspace{1cm} (2.19)
2. $x(t_1 - \tau) > x(t_1) > d$; \hspace{1cm} (2.20)
3. $x(t_1) < x(t_1 - \tau) < -d$; \hspace{1cm} (2.21)
4. $x(t_1 - \tau) < x(t_1) < -d$. \hspace{1cm} (2.22)

If (2.19) holds, in view of $(A_0)(1)$, $(A_0)(2)$, $(A_1)$ and $(A_2)$, we shall consider four cases as follows:

Case (i). If $(A_1)$ and $(A_0)(1)$ hold, according to (2.19), we obtain

$$
0 < g_1(t_1, x(t_1 - \tau)) + g_2(t_1, x(t_1 - \tau)) + p(t_1) < g_1(t_1, x(t_1)) + g_2(t_1, x(t_1 - \tau)) + p(t_1),
$$

which contradicts (2.16). This contradiction implies that (2.17) holds.

Case (ii). If $(A_1)$ and $(A_0)(2)$ hold, according to (2.19), we obtain

$$
0 < g_1(t_1, x(t_1)) + g_2(t_1, x(t_1)) + p(t_1) < g_1(t_1, x(t_1)) + g_2(t_1, x(t_1 - \tau)) + p(t_1),
$$

which contradicts (2.16). This contradiction implies that (2.17) holds.

Case (iii). If $(A_2)$ and $(A_0)(1)$ hold, according to (2.19), we obtain

$$
g_1(t_1, x(t_1)) + g_2(t_1, x(t_1 - \tau)) + p(t_1) < g_1(t_1, x(t_1)) + g_2(t_1, x(t_1)) + p(t_1) < 0,
$$

which contradicts (2.16). This contradiction implies that (2.17) holds.
Case (iv). If (A₂) and (A₀)(2) hold, according to (2.19), we obtain
\[
g_1(t_1, x(t_1)) + g_2(t_1, x(t_1 - \tau)) + p(t_1) < g_1(t_1, x(t_1 - \tau)) + g_2(t_1, x(t_1 - \tau)) + p(t_1) < 0,
\]
which contradicts (2.16). This contradiction implies that (2.17) holds.

If (2.20) (or (2.21), or (2.22)) holds, using the methods similar to those used in cases (i)–(iv), we can show that (2.17) holds. Therefore, the above claim is valid.

Let \( t_2 = mT + t_0 \), where \( t_0 \in [0, T] \) and \( m \) is an integer. Then, using Schwarz inequality and the following relation:
\[
||x(t)|| = ||x(t_0) + \int_{t_0}^{t} x'(s) \, ds|| \leq d + \int_{0}^{T} |x'(s)| \, ds, \quad t \in [0, T],
\]
we obtain
\[
|x|_\infty = \max_{t \in [0, T]} |x(t)| \leq d + \sqrt{T} |x'|_2.
\]
This completes the proof of Lemma 2.4. \( \square \)

3. Main results

**Theorem 3.1.** Let (A₀) and (\( \overline{A}_0 \)) hold. Assume that either the condition (A₁) or the condition (A₂) is satisfied. Then Eq. (1.1) has a unique \( T \)-periodic solution.

**Proof.** From Lemma 2.3, together with (A₀) and (\( \overline{A}_0 \)), it follows easily that Eq. (1.1) has at most one \( T \)-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that Eq. (1.1) has at least one \( T \)-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible \( T \)-periodic solutions of Eq. (2.1) are bounded.

Let \( x(t) \) be a \( T \)-periodic solution of Eq. (2.1). In view of (\( \overline{A}_0 \))(1) and (\( \overline{A}_0 \))(2), we shall consider two cases as follows.

Case (i). If (\( \overline{A}_0 \))(1) holds, multiplying both sides of (2.1) by \( x'(t) \) and then integrating them from 0 to \( T \), we have from (2.3), (\( \overline{A}_0 \))(1) and the inequality of Schwarz that
\[
|x'|^2 = \int_{0}^{T} |x'(t)|^2 \, dt
\]

\[
= - \int_{0}^{T} Bx'(t - \delta)x'(t) \, dt + \lambda \int_{0}^{T} g_1(t, x(t))x'(t) \, dt + \lambda \int_{0}^{T} g_2(t, x(t - \tau))x'(t) \, dt
\]

\[
+ \lambda \int_{0}^{T} p(t)x'(t) \, dt
\]

\[
\leq |B||x'|^2 + |p|_2|x'|^2 + \lambda \int_{0}^{T} (g_1(t, x(t)) - g_1(t, 0))x'(t) \, dt + \lambda \int_{0}^{T} g_1(t, 0)x'(t) \, dt
\]
\[ + \lambda \int_0^T (g_2(t, x(t - \tau)) - g_2(t, 0)) x'(t) \, dt + \lambda \int_0^T g_2(t, 0) x'(t) \, dt \]
\[ \leq |B||x'|^2 + \int_0^T |x'(t)| \, dt + \max_{t \in [0,T]} \int_0^T \left| g_1(t, 0) \right| \, dt \]
\[ + b_1 \int_0^T |x'(t)| \, dt + \max_{t \in [0,T]} \int_0^T \left| g_2(t, 0) \right| \, dt \]
\[ \leq |B||x'|^2 + \int_0^T |x'(t)| \, dt + \max_{t \in [0,T]} \left| g_1(t, 0) + \max_{t \in [0,T]} \left| g_2(t, 0) \right| \right| \sqrt{T} |x'(t)|_2 \]
\[ \leq \left( |B| + (b_1 + b_2) \frac{T}{\pi} \right) |x'|_2 \]
\[ + \left( |p|_2 + \max_{t \in [0,T]} |g_1(t, 0)| + \max_{t \in [0,T]} |g_2(t, 0)| \right) \sqrt{T} |x'|_2. \]  
(3.1)

Now, let
\[ D_1 = \frac{|p|_2 + \max_{t \in [0,T]} |g_1(t, 0)| + \max_{t \in [0,T]} |g_2(t, 0)|}{1 - |B| - (b_1 + b_2) \frac{T}{\pi}} \sqrt{T}. \]

In view of (2.15) and (3.1), we obtain
\[ |x'|_2 \leq D_1, \quad |x|_\infty \leq d + \sqrt{T} D_1. \]  
(3.2)

**Case (ii).** If \((\overline{A}_0)(2)\) holds, multiplying both sides of (2.1)_\lambda by \(x'(t - \delta)\) and then integrating them from 0 to \(T\), we have from (2.3), \((\overline{A}_0)(2)\) and the inequality of Schwarz that
\[ |B||x'|^2 = \int_0^T B |x'(t - \delta)|^2 \, dt \]
\[ = \int_0^T x'(t - \delta) x'(t) \, dt + \lambda \int_0^T g_1(t, x(t)) x'(t - \delta) \, dt \]
\[ + \lambda \int_0^T g_2(t, x(t - \tau)) x'(t - \delta) \, dt + \lambda \int_0^T p(t) x'(t - \delta) \, dt \]
\[ \leq |x'|_2^2 + |p|_2 |x'|_2 + \lambda \int_0^T (g_1(t, x(t)) - g_1(t, 0)) x'(t - \delta) \, dt \]
\[ + \lambda \int_0^T g_1(t, 0) x'(t - \delta) \, dt. \]
\[ \lambda \left| \int_0^T \left( g_2(t, x(t - \tau)) - g_2(t, 0) \right) \cdot x'(t - \delta) \, dt \right| + \lambda \left| \int_0^T g_2(t, 0) x'(t - \delta) \, dt \right| \]
\[ \leq |x'|^2 + |p| |x'| + b_1 \int_0^T |x(t)||x'(t - \delta)| \, dt + \max_{t \in [0, T]} |g_1(t, 0)| \int_0^T |x'(t - \delta)| \, dt \]
\[ + b_2 \int_0^T |x(t - \tau)| \cdot |x'(t - \delta)| \, dt + \max_{t \in [0, T]} |g_2(t, 0)| \int_0^T |x'(t - \delta)| \, dt \]
\[ \leq |x'|^2 + |p| |x'| + (b_1 + b_2) |x|_2^2 \]
\[ + \left( \max_{t \in [0, T]} |g_1(t, 0)| + \max_{t \in [0, T]} |g_2(t, 0)| \right) \sqrt{T} |x'(t)|_2 \]
\[ \leq \left(1 + (b_1 + b_2) \frac{T}{\pi}\right) |x'|_2^2 + \left[ |p| + \left( \max_{t \in [0, T]} |g_1(t, 0)| + \max_{t \in [0, T]} |g_2(t, 0)| \right) \right] \sqrt{T} |x'|_2. \tag{3.3} \]

Now, let
\[ \bar{D}_1 = \frac{|p| + \left( \max_{t \in [0, T]} |g_1(t, 0)| + \max_{t \in [0, T]} |g_2(t, 0)| \right) \sqrt{T}}{|B| - 1 - (b_1 + b_2) \frac{T}{\pi}}. \]

In view of (2.15) and (3.3), we obtain
\[ |x'|_2 \leq \bar{D}_1, \quad |x|_\infty \leq d + \sqrt{T} \bar{D}_1. \tag{3.4} \]

If \( x \in \Omega_1 = \{ x \mid x \in \text{Ker } L \cap X \text{ and } N x \in \text{Im } L \} \), then there exists a constant \( M_1 \) such that
\[ x(t) \equiv M_1 \quad \text{and} \quad \int_0^T \left[ g_1(t, M_1) + g_2(t, M_1) + p(t) \right] \, dt = 0. \tag{3.5} \]

Thus,
\[ |x(t)| \equiv |M_1| < d, \quad \text{for all } x(t) \in \Omega_1. \tag{3.6} \]

Let \( M = (D_1 + \bar{D}_1) \sqrt{T} + d + 1 \). Set\n\[ \Omega = \{ x \mid x \in X, \ |x|_\infty < M \}. \]

It is easy to see from (1.3) and (1.4) that \( N \) is \( L \)-compact on \( \bar{D} \). We have from (3.5), (3.6) and the fact \( M > \max\{D_1 \sqrt{T} + d, \ \bar{D}_1 \sqrt{T} + d, \ d \} \) that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, define continuous functions \( H_1(x, \mu) \) and \( H_2(x, \mu) \) by setting
\[ H_1(x, \mu) = (1 - \mu)x + \mu \cdot \frac{1}{T} \int_0^T \left[ g_1(t, x) + g_2(t, x) + p(t) \right] \, dt, \quad \mu \in [0, 1], \]
\[ H_2(x, \mu) = -(1 - \mu)x + \mu \cdot \frac{1}{T} \int_0^T \left[ g_1(t, x) + g_2(t, x) + p(t) \right] \, dt, \quad \mu \in [0, 1]. \]
If (A₁) holds, then
\[ xH_{1}(x, \mu) \neq 0 \quad \text{for all} \quad x \in \partial \Omega \cap \text{Ker } L. \]

Hence, using the homotopy invariance theorem, we have
\[
\deg\{QN, \Omega \cap \text{Ker } L, 0\} = \deg\left\{ \frac{1}{T} \int_{0}^{T} \left[ g_{1}(t, x) + g_{2}(t, x) + p(t) \right] dt, \Omega \cap \text{Ker } L, 0 \right\} \\
= \deg\{x, \Omega \cap \text{Ker } L, 0\} \neq 0.
\]

If (A₂) holds, then
\[ xH_{2}(x, \mu) \neq 0 \quad \text{for all} \quad x \in \partial \Omega \cap \text{Ker } L. \]

Hence, using the homotopy invariance theorem, we obtain
\[
\deg\{QN, \Omega \cap \text{Ker } L, 0\} = \deg\left\{ \frac{1}{T} \int_{0}^{T} \left[ g_{1}(t, x) + g_{2}(t, x) + p(t) \right] dt, \Omega \cap \text{Ker } L, 0 \right\} \\
= \deg\{-x, \Omega \cap \text{Ker } L, 0\} \neq 0.
\]

In view of all the discussions above, we conclude from Lemma 2.1 that Theorem 3.1 is proved. □

4. Example and remark

Example 4.1. The first order NFDE
\[
\left( x(t) + \frac{1}{8} x(t - \delta) \right)' = -\frac{1}{8\pi} x + \frac{1}{4\pi} \left[ 1 - x \left( t - \frac{3\pi}{2} \right) \right] + e^{\cos^{2}t} \tag{4.1}
\]
has a unique $2\pi$-periodic solution.

Proof. From (4.1), we have $B = \frac{1}{8}$, $g_{1}(x) = -\frac{1}{8\pi} x$, $g(x(t - \tau)) = \frac{1}{4\pi} [1 - x(t - \frac{3\pi}{2})]$ and $p(t) = e^{\cos^{2}t}$. Then, $b_{1} = \frac{1}{8\pi}$, $b_{2} = \frac{1}{4\pi}$. It is straightforward to check that all the conditions needed in Theorem 3.1 are satisfied. Therefore, Eq. (4.1) has a unique $2\pi$-periodic solution. □

Remark 4.1. Equation (4.1) is a very simple version of first order NFDE. Since $B \neq 0$, all the results in [1–11] and the references therein cannot be applicable to Eq. (4.1) to obtain the existence and uniqueness of $2\pi$-periodic solutions. This implies that the results of this paper are essentially new.

References