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Type I product systems of Hilbert modules ☆

Stephen D. Barreto,[a] B.V. Rajarama Bhat,[b] Volkmar Liebscher,[c] and Michael Skeide[d,*]

[a] *Department of Mathematics, Padre Conceicao College of Engineering, Verna, Goa 403722, India*
[b] *Statistics and Mathematics Unit, Indian Statistical Institute, R. V. College Post, Bangalore 560059, India*
[c] *Institute of Biomathematics and Biometry, GSF-National Research Centre for Environment and Health, 85764 Neuherberg, München, Germany*
[d] *Dipartimento S.E.G.e S., Facoltà di Economia, Università degli Studi del Molise, Via de Sanctis, 86100 Campobasso, Italy*

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Abstract

Christensen and Evans showed that, in the language of Hilbert modules, a bounded derivation on a von Neumann algebra with values in a two-sided von Neumann module (i.e. a sufficiently closed two-sided Hilbert module) are inner. Then they use this result to show that the generator of a normal uniformly continuous completely positive (CP-) semigroup on a von Neumann algebra decomposes into a (suitably normalized) CP-part and a derivation like part. The backwards implication is left open.

In these notes we show that both statements are equivalent among themselves and equivalent to a third one, namely, that any type I tensor product system of von Neumann modules has a unital central unit. From existence of a central unit we deduce that each such product system is isomorphic to a product system of time ordered Fock modules. We, thus, find the analogue of Arveson's result that type I product systems of Hilbert spaces are symmetric Fock spaces.

On the way to our results we have to develop a number of tools interesting on their own right. Inspired by a very similar notion due to Accardi and Kozyrev, we introduce the notion of semigroups of completely positive definite kernels (CPD-semigroups), being a generalization of both CP-semigroups and Schur semigroups of positive definite $\mathbb{C}$-valued kernels. The structure of a type I system is determined completely by its associated CPD-semigroup

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*Corresponding author.
E-mail addresses: stephen_barreto@yahoo.com (S.D. Barreto), bhat@isibang.ac.in (B.V.R. Bhat), liebscher@gsf.de (V. Liebscher), skeide@math.tu-cottbus.de (M. Skeide).
URLs: http://www.isibang.ac.in/Smubang/BHAT/, http://www.gsf.de/ibb/homepages/liebscher/, http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html.

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and the generator of the CPD-semigroup replaces Arveson's covariance function. As another tool we give a complete characterization of morphisms among product systems of time ordered Fock modules. In particular, the concrete form of the projection endomorphisms allows us to show that subsystems of time ordered systems are again time ordered systems and to find a necessary and sufficient criterion when a given set of units generates the whole system. As a byproduct we find a couple of characterizations of other subclasses of morphisms. We show that the set of contractive positive endomorphisms are order isomorphic to the set of CPD-semigroups dominated by the CPD-semigroup associated with type I system.

1. Introduction

Arveson's tensor product systems of Hilbert spaces [Arv89] (Arveson systems, for short) arise in the theory of E_0-semigroups on $\mathscr{B}(G)$, where G is some Hilbert space. They consist of a family $\mathfrak{H}^{\bar{\otimes}} = (\mathfrak{H}_t)_{t\in\mathbb{R}_+}$ of Hilbert spaces $\mathfrak{H}_t$ such that $\mathfrak{H}_s \bar{\otimes} \mathfrak{H}_t = \mathfrak{H}_{s+t}$ in an associative way (plus some measurability conditions). The most important notion for Arveson systems is that of a unit $h^{\otimes} = (h_t)_{t\in\mathbb{R}_+}$ consisting of vectors $h_t \in \mathfrak{H}_t$ such that $h_s \otimes h_t = h_{s+t}$ (plus some measurability conditions). The most prominent example of such an Arveson system is the symmetric Fock space, more precisely, the family $\Gamma^{\otimes}(H)$ of symmetric Fock spaces $\Gamma(L^2([0,t],H))$ for some Hilbert space H. The units of $\Gamma^{\otimes}(H)$ are precisely the exponential vectors $\psi(\mathbb{I}_{[0,t]}h)$ possibly times a renormalizing factor e^{tc} $(c\in\mathbb{C})$. The symmetric Fock space has the property to be spanned by tensor product of such units. Arveson defines a product system with this property to be a *type I* system and he shows that every type I system is *isomorphic* to $\Gamma^{\otimes}(H)$ for a suitable H.

In these notes we show the analogue result for product systems of Hilbert modules (more precisely, of von Neumann modules). Throughout these notes let $\mathcal{B}$ be a unital C^*-algebra. Product systems of Hilbert $\mathcal{B}$–$\mathcal{B}$-modules were discovered in dilation theory of a completely positive semigroup (a CP-semigroup for short) in [BS00]. Meanwhile, we also have a construction of product systems starting from E_0-semigroups on some algebra $\mathscr{B}^{\mathrm{a}}(E)$ of adjointable operators on a Hilbert module; see [Ske02]. A product system $E^{\odot} = (E_t)_{t\in\mathbb{R}_+}$ consists of (pre-)Hilbert $\mathcal{B}$–$\mathcal{B}$-modules E_t which compose (associatively) as $E_s \odot E_t = E_{s+t}$ under (interior) tensor product of two-sided Hilbert modules, and a unit is a family $\xi^{\odot} = (\xi_t)_{t\in\mathbb{R}_+}$ of elements $\xi_t \in E_t$ which composes as $\xi_s \odot \xi_t = \xi_{s+t}$.

The symmetric Fock space is canonically isomorphic to the time ordered Fock space (i.e. the Guichardet picture). As shown in [BS00] it is this picture which can be generalized to Hilbert modules. The (continuous) units for the time ordered Fock module are considerably more complicated, but still can be computed explicitly (see [LS01]) and generate the time ordered Fock module in a suitable sense.

Now it makes sense to ask, whether all product systems generated by their units are time ordered Fock modules. However, unlike for Hilbert spaces (where strong and weak totality of some subset are the same, so that we do not need to distinguish topologies) in a Hilbert module there are several topologies, and the answer to our question depends very much on the topology in which what the units generate algebraically is closed. As one of our main results, we find an affirmative answer, if we use the strong topology of von Neumann modules (as introduced in [Ske00a]).

The crucial step is to establish the equivalence of the results by Christensen and Evans [CE79] on the generator of a normal uniformly continuous CP-semigroup on a von Neumann algebra and the fact that product systems of von Neumann modules which have a continuous unit always have also a (continuous) central unit (i.e. the members ξ_t of the unit commute with the elements of the algebra). Example 4.2.4 describes a product system of Hilbert modules generated by a single continuous unit, but without any central unit. It cannot be a time ordered Fock module, because these always have a central unit, namely, the vacuum unit. Therefore, we may not hope that our result generalizes to all product systems of Hilbert modules. (We know, however, from [Ske01c] that it generalizes under the assumption of existence of a central unit.)

On our way we have to establish several interesting tools. The main tool in [Arv89] was the so-called *covariance function*, i.e. a conditionally positive definite kernel defined on the set of units of an Arveson system which we obtain by differentiating the semigroup $\langle g_t, g'_t \rangle$ (for some units $g^{\otimes}, g'^{\otimes}$) at $t = 0$. What is the substitute for modules? The matrix elements $\langle \xi_t, \xi'_t \rangle$, in general, will not form a semigroup. However, if we consider instead the mappings $\mathfrak{T}_t^{\xi,\xi'}: b \mapsto \langle \xi_t, b\xi'_t \rangle$, then the definition of units (and the tensor product) is born to make $\mathfrak{T}^{\xi,\xi'} = (\mathfrak{T}_t^{\xi,\xi'})_{t \in \mathbb{R}_+}$ a semigroup. The right notion of positivity for such a kernel is *completely positive definiteness*. The idea to consider semigroups of completely positive definite kernels (CPD-semigroups for short) is inspired very much by a new idea from the paper [AK99] by Accardi and Kozyrev. If a product system is generated by its units, then its structure is determined completely by the structure of its associated CPD-semigroup. The substitute for Arveson's covariance function is just the generator of the CPD-semigroup.

Whereas for Arveson systems the structure of the covariance function is well known and easy to derive, in our case we do not know immediately the form of the generator. Only after passing through the theory it turns out that it has a form which generalizes that of the Christensen–Evans generator of CP- to CPD-semigroups. This drops out immediately, when we know existence of a central unit. In order to derive both existence of a central unit (from [CE79]) and that product systems of von Neumann modules generated by their units are time ordered Fock modules we have to master the problem whether a subsystem of a time ordered system is all, and if not how it looks like. We solve this problem with the help of our second main tool, namely, a complete characterization of morphisms among time ordered systems, in particular, of projection morphisms.

These notes are organized as follows. In Section 2 we start with preliminaries from earlier papers. In Section 2.1 we collect the most important definitions and constructions. In particular, we define von Neumann modules from [Ske00a] which is not a standard definition. In Section 2.2 we recall quickly the *exterior tensor product*. (The extensions to von Neumann modules are not standard, and we need them in Appendix B.) Then we use it to define matrices of Hilbert modules which provide the basic technique to deal with completely positive definite kernels. In Section 2.3 we recall the definition of the time ordered Fock module and repeat its basic properties from [BS00,LS01].

In Section 3 we define completely positive definite kernels and semigroups of such and study their basic properties. We state what we can say about the generator without using product systems. In order to give an impression what we have to expect later on, we discuss in Section 3.5 the CPD-semigroup associated with the time ordered Fock modules and conjecture from its generator a theorem about the form of the generators paralleling the Christensen–Evans form of the generator of a CP-semigroup.

After these lengthy preparations we come to product systems in Section 4. After the definition in Section 4.1 we show in Section 4.2 that with each set of units in product system there is associated a natural CPD-semigroup. We explain that a set of units generates a subsystem and use this to define type I product systems (splitting into several cases depending on several topologies). In Section 4.3 we reverse the direction and starting from a CPD-semigroup we construct a product system, the *GNS-system* of the CPD-semigroup, with a set of units, giving us back the original CPD-semigroup. In the following sections we are interested only in uniformly continuous CPD-semigroups. In Section 4.4 we study in how far continuity properties of the CPD-semigroup are reflected by those of the units in the GNS-system.

While Section 4 was still at a rather general level, in Section 5 we point directly to our main goal. In Section 5.1 we show that existence of a central unit among a continuous set of units assures that the generator of the associated CPD-semigroup has Christensen–Evans form. In Section 5.2 we study morphisms of time ordered Fock modules. In Section 5.3 we use the concrete form of the projection morphisms to provide a criterion which allows to decide, whether a (continuous) set of units generates a time ordered system of von Neumann modules and, if not, how the generated subsystem looks like. The idea taken from Bhat [Bha01] is, roughly speaking, that if the subsystem generated by a set of units is not all, then there should exist a non-trivial projection morphism onto the subsystem. In Section 5.4 we put together our results and those by Christensen–Evans [CE79] and obtain very quickly our main result.

As a bonus we obtain that the result about derivations is equivalent to existence of a central unital unit in the GNS-system of a uniformly continuous normal CPD-semigroup. This raises the question for a direct proof of existence of a central unit, thus, providing a different proof of [CE79]. In Section 6 we outline these and other possible directions for future work on product systems.

In Appendix A we extend the analysis of morphisms from Section 5.2. We describe the order structure of positive morphisms and, in particular, of the contractive morphisms. In Appendix B we follow an idea from [AK99], and encode the information on the GNS-system of a CPD-semigroup into a single CP-semigroup on a (much) bigger algebra. In Appendix C we recall the results from [CE79], but entirely in the language of Hilbert modules which is—and we hope that these notes demonstrate this—much better adapted to problems concerning general von Neumann algebras.

Let us close with some general conventions and a definition. In the course of our investigations it is convenient (and sometimes also necessary) to distinguish pre-Hilbert modules, Hilbert modules (i.e. complete pre-Hilbert modules) and von Neumann modules (i.e. strongly closed submodules of some $\mathscr{B}(G,H)$). Consequently, we have to distinguish clearly the several versions of product systems, tensor products, and so on. Tensor products $\otimes, \odot$ are understood algebraically. If we want to complete, then we write $\bar{\otimes}, \bar{\odot}$. Strong closures (in a space of operators) are indicated by a superscript s. We use the same conventions for direct sums. An exception of this convention are Fock modules, which usually are assumed norm complete, because usually it is not reasonable to consider algebraic versions. Where algebraic Fock modules appear, we indicate them by $\underline{\mathcal{F}}$, $\underline{\mathbb{I}\Gamma}$, and so on. The action of an algebra on a module is always non-degenerate. A representation by operators on a module need not be non-degenerate.

By $\mathfrak{S}(\mathbb{R}_+, B)$ we denote the space of *step functions* on $\mathbb{R}_+$ with values in the normed space B, whereas L^2-spaces of functions with values in a Hilbert module are defined in Section 2.2.

Usually, we are interested in $\mathbb{R}_+$ as indexing set for a semigroup, but sometimes we consider also the discrete case $\mathbb{N}_0$. If we do not distinguish we write $\mathbb{T}$. Throughout the isomorphic lattices $\mathbb{I}_t$ and $\mathbb{J}_t$ are important. Let $t>0$ in $\mathbb{T}$. We define $\mathbb{I}_t$ as the set of all tuples $\{(t_n, \ldots, t_1) \in \mathbb{T}^n : n \in \mathbb{N}, t = t_n > \cdots > t_1 > 0\}$. Clearly, $\mathbb{I}_t$ is a lattice partially ordered by "inclusion" with "union" and "intersection" of tuples being the unique maximum and minimum, respectively. We define $\mathbb{J}_t$ to be the set of all tuples $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{T}^n$ $(n \in \mathbb{N}, t_i > 0)$ having *length*

$$|\mathfrak{t}| := \sum_{i=1}^{n} t_i = t.$$

For two tuples $\mathfrak{s} = (s_m, \ldots, s_1) \in \mathbb{J}_s$ and $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_t$ we define the *joint tuple* $\mathfrak{s} \smile \mathfrak{t} \in \mathbb{J}_{s+t}$ by

$$\mathfrak{s} \smile \mathfrak{t} = ((s_m, \ldots, s_1), (t_n, \ldots, t_1)) = (s_m, \ldots, s_1, t_n, \ldots, t_1).$$

We equip $\mathbb{J}_t$ with a partial order by saying $\mathfrak{t} \geqslant \mathfrak{s} = (s_m, \ldots, s_1)$, if for each j $(1 \leqslant j \leqslant m)$ there are (unique) $\mathfrak{s}_j \in \mathbb{J}_{s_j}$ such that $\mathfrak{t} = \mathfrak{s}_m \smile \cdots \smile \mathfrak{s}_1$. We extend the definitions of $\mathbb{I}_t$ and $\mathbb{J}_t$ to $t = 0$, by setting $\mathbb{I}_0 = \mathbb{J}_0 = \{()\}$, where $()$ is the *empty tuple*. For $\mathfrak{t} \in \mathbb{J}_t$ we put $\mathfrak{t} \smile () = \mathfrak{t} = () \smile \mathfrak{t}$. The mapping $(t_n, \ldots, t_1) \mapsto (\sum_{i=1}^{n} t_i, \ldots, \sum_{i=1}^{1} t_i)$ is an order isomorphism $\mathbb{J}_t \to \mathbb{I}_t$ so that also $\mathbb{J}_t$ is a lattice.

2. Preliminaries

2.1. Von Neumann modules, tensor product and GNS-construction

For basics about Hilbert modules over C^*-algebras we refer the reader to [BS00,Lan95,Pas73,Ske00a]. A complete treatment adapted precisely to our needs with full proofs of all statements can be found in [Ske01a]. We recall only that for us Hilbert $\mathcal{B}$-modules are right $\mathcal{B}$-modules with a (strictly positive) $\mathcal{B}$-valued inner product, right $\mathcal{B}$-linear in its right variable. Hilbert $\mathcal{A}$–$\mathcal{B}$-modules are Hilbert $\mathcal{B}$-modules where $\mathcal{A}$ acts non-degenerately as a C^*-algebra of right module homomorphisms. In particular, if $\mathcal{A}$ is unital, the unit of $\mathcal{A}$ acts as unit. The C^*-algebra of adjointable mappings on a Hilbert module E we denote by $\mathscr{B}^{\mathrm{a}}(E)$. By $\mathscr{B}^{\mathrm{a,bil}}(E)$ we denote the bilinear mappings, which we also call two-sided. Using similar notations also for mappings between Hilbert modules, without mention we identify $E \subset \mathscr{B}^{\mathrm{a}}(\mathcal{B}, E)$ (where $x \in E$ is the mapping $b \mapsto xb$) and $E^* \subset \mathscr{B}^{\mathrm{a}}(E, \mathcal{B})$ (where $x^* : y \mapsto \langle x, y \rangle$ is the adjoint of x). Consequently, xy^* is the rank-one operator $z \mapsto x\langle y, z \rangle$. Recall that by definition Hilbert modules are complete with respect to their norm $\|x\| = \sqrt{\|\langle x, x \rangle\|}$. Otherwise, we speak of pre-Hilbert modules. In this case $\mathscr{B}^{\mathrm{a}}(E)$ is only a pre-C^*-algebra. The strong topology is that of operators on a normed or Banach space. The $*$-strong topology on an involutive space of operators on a normed or Banach space is the topology generated by the strong topology and by the strong topology for the adjoints. (When restricted to bounded subsets of $\mathscr{B}^{\mathrm{a}}(E)$ this is the strict topology; see [Lan95].) Another topology on E is the $\mathcal{B}$-weak topology which is generated by the seminorms $\|\langle x, \bullet \rangle\|$ $(x \in E)$.

The following observation provides a method to establish well definedness of certain operators (defined by giving the values on a generating subset) without showing boundedness. (In fact, it works also for unbounded operators.) It can hardly be overestimated.

2.1.1. Observation. If a $\mathcal{B}$-valued inner product on an $\mathcal{A}$–$\mathcal{B}$-module E fails to be *strictly* positive (i.e. $\langle x, x \rangle = 0$ does not necessarily imply $x = 0$), then by the Cauchy–Schwarz inequality

$$\langle x, y \rangle \langle y, x \rangle \leqslant \|\langle y, y \rangle\| \langle x, x \rangle \tag{2.1.1}$$

we may divide out the submodule $\mathscr{N}_E = \{x \in E : \langle x, x \rangle = 0\}$ of length-zero elements and obtain a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module. It is important to notice that any adjointable operator (bounded or not) on E respects $\mathscr{N}_E$ and, therefore, gives rise to an adjointable operator on $E/\mathscr{N}_E$. As a simple consequence we find that a mapping defined on a subset of E which generates E as right module extends to a well-defined mapping on E, if it is formally adjointable on the generating subset.

2.1.2. Definition. The (interior) tensor product (over $\mathcal{B}$) of the pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module E and the pre-Hilbert $\mathcal{B}$–$\mathcal{C}$-module F is the pre-Hilbert $\mathcal{A}$–$\mathcal{C}$-module $E \odot F = E \otimes F / \mathscr{N}_{E \otimes F}$ where $E \otimes F$ is equipped with inner product defined by setting

$\langle x \otimes y, x' \otimes y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$. If $\mathcal{B}$ is unital, then we identify always $E \odot \mathcal{B}$ and E (via $x \odot b = xb$), and we identify always $\mathcal{B} \odot F$ and F (via $b \odot y = by$). If $\mathcal{B}$ is non-unital, then we may identify at least the completions.

Particularly interesting is the tensor product $H = E \odot G$ of a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module E and a pre-Hilbert space G on which $\mathcal{B}$ is represented non-degenerately (so that G is a pre-Hilbert $\mathcal{B}$–$\mathbb{C}$-module). It follows that H is a pre-Hilbert $\mathcal{A}$–$\mathbb{C}$-module, i.e. a pre-Hilbert space with a representation ρ of $\mathcal{A}$ by (adjointable) operators on H. We refer to ρ as the *Stinespring representation* of $\mathcal{A}$ (associated with E and G); cf. Remark 2.1.5.

To each $x \in E$ we associate an operator $L_x : G \to H, g \mapsto x \odot g$ in $\mathscr{B}^a(G, H)$. We refer to the mapping $\eta : x \mapsto L_x$ as the *Stinespring representation* of E (associated with G). If the representation of $\mathcal{B}$ on G is faithful (hence, isometric), then so is η. More precisely, we find $L_x^* L_y = \langle x, y \rangle \in \mathcal{B} \subset \mathscr{B}^a(G)$. We also have $L_{axb} = \rho(a) L_x b$ so that we may identify E as a *concrete* $\mathcal{A}$–$\mathcal{B}$-submodule of $\mathscr{B}^a(G, H)$.

In particular, if $\mathcal{B}$ is a von Neumann algebra on a Hilbert space G, then we consider E always as a concrete subset of $\mathscr{B}(G, E \bar{\odot} G)$. We say E is a *von Neumann* $\mathcal{B}$-module, if it is strongly closed in $\mathscr{B}(G, E \bar{\odot} G)$. If also $\mathcal{A}$ is a von Neumann algebra, then a *von Neumann $\mathcal{A}$–$\mathcal{B}$-module E* is a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module and a von Neumann $\mathcal{B}$-module such that the Stinespring representation ρ of $\mathcal{A}$ on $E \bar{\odot} G$ is normal.

2.1.3. Remark. The (strong closure of the) tensor product of von Neumann modules is again a von Neumann module. Left multiplication by an element of $\mathcal{A}$ is a strongly continuous operation on E. The $*$-algebra $\mathscr{B}^a(E)$ is a von Neumann subalgebra of $\mathscr{B}(E \bar{\odot} G)$.

One may easily show that if $\mathcal{B} = \mathscr{B}(G)$ then $E = \mathscr{B}(G, H)$ and $\mathscr{B}^a(E) = \mathscr{B}(H)$. If E is a von Neumann $\mathscr{B}(G)$–$\mathscr{B}(G)$-module, then $H = G \bar{\otimes} \mathfrak{H}$ and $E = \mathscr{B}^a(G, G \bar{\otimes} \mathfrak{H}) = \mathscr{B}(G) \bar{\otimes}^s \mathfrak{H}$ where $\mathfrak{H}$ is a Hilbert space, Arveson's Hilbert space of intertwiners of the left and right multiplication. In other words, $\mathfrak{H} = C_{\mathscr{B}(G)}(E)$, where generally $C_{\mathcal{B}}(E) = \{x \in E : bx = xb (b \in \mathcal{B})\}$ is the *$\mathcal{B}$-center* of a $\mathcal{B}$–$\mathcal{B}$-module.

2.1.4. Remark. Von Neumann modules are self-dual. Consequently, each bounded right linear mapping on (or between) von Neumann modules is adjointable and von Neumann modules are *complementary* (i.e. for any von Neumann submodule F of a pre-Hilbert module E there exists a projection $p \in \mathscr{B}^a(E)$ onto F). We refer to [Ske00a,Ske01a] for details.

For any element ξ in a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module E, the mapping $a \mapsto \langle \xi, a\xi \rangle$ is completely positive. (The axioms of Hilbert modules are quasi modelled to have this property.) Conversely, if $T : \mathcal{A} \to \mathcal{B}$ is a completely positive mapping between unital C^*-algebras, then by setting $\langle a \otimes b, a' \otimes b' \rangle = b^* T(a^* a') b'$ we define an inner product on the $\mathcal{A}$–$\mathcal{B}$-module $\mathcal{A} \otimes \mathcal{B}$. Set $E = \mathcal{A} \otimes \mathcal{B} / \mathcal{N}_{\mathcal{A} \otimes \mathcal{B}}$ and $\xi = \mathbf{1} \otimes \mathbf{1} + \mathcal{N}_{\mathcal{A} \otimes \mathcal{B}}$. Then $T(a) = \langle \xi, a\xi \rangle$ and $E = \operatorname{span} \mathcal{A} \xi \mathcal{B}$. We refer to the pair (E, ξ) as the *GNS-construction* for T and to E as the *GNS-module* with *cyclic vector* ξ. The

GNS-construction is determined by the stated properties up to two-sided isomorphism. If T is a normal mapping between von Neumann algebras, then $\bar{E}^s$ is a von Neumann $\mathcal{A}$–$\mathcal{B}$-module.

2.1.5. Remark. Assume that $\mathcal{B}$ is represented faithfully on a (pre-)Hilbert space G and let us construct the Stinespring representation ρ of $\mathcal{A}$ as described above. Then $T(a) = \langle \xi, a\xi \rangle = L_\xi^* L_{a\xi} = L_\xi^* \rho(a) L_\xi$ so that ρ with the *cyclic mapping* $L_\xi \in \mathscr{B}^a(G, H)$, indeed, coincides with the usual Stinespring construction.

The most important advantage of considering GNS-constructions of completely positive mappings instead of Stinespring constructions appears, if we consider compositions.

2.1.6. Example. Let $T: \mathcal{A} \to \mathcal{B}$ and $S: \mathcal{B} \to \mathcal{C}$ be completely positive mappings with GNS-modules E and F and with cyclic vectors ξ and ζ, respectively. Then we have $S \circ T(a) = \langle \xi \odot \zeta, a\xi \odot \zeta \rangle$ (so that $S \circ T$ is completely positive). Let G be the GNS-module of the composition $S \circ T$ with cyclic vector χ. Then the mapping

$$\chi \mapsto \xi \odot \zeta$$

extends (uniquely) as a two-sided isometric homomorphism $G \to E \odot F$. Observe that $E \odot F = \mathsf{span}(\mathcal{A}\xi\mathcal{B} \odot \mathcal{B}\zeta\mathcal{C}) = \mathsf{span}(\mathcal{A}\xi \odot \mathcal{B}\zeta\mathcal{C}) = \mathsf{span}(\mathcal{A}\xi\mathcal{B} \odot \zeta\mathcal{C})$. By the above isometry we may identify G as the submodule $\mathsf{span}(\mathcal{A}\xi \odot \zeta\mathcal{C})$ of $E \odot F$. In other words, inserting a unit $\mathbf{1}$ in $\chi = \xi \odot \zeta$ in between ξ and ζ amounts to an isometry. Varying, instead, $b \in \mathcal{B}$ in $\xi b \odot \zeta = \xi \odot b\zeta$, we obtain a set which generates all of $E \odot F$.

This operation is crucial in the construction of tensor product systems. We explain immediately, why the Stinespring construction cannot do the same job. Suppose that $\mathcal{B}$ and $\mathcal{C}$ are algebras of operators on some pre-Hilbert spaces. Then, unlike the GNS-construction, the knowledge of the Stinespring construction for the mapping T does not help in finding the Stinespring construction for $S \circ T$. What we need is the Stinespring construction for T based on the representation of $\mathcal{B}$ arising from the Stinespring construction for S. The GNS-construction, on the other hand, is *representation free*. It is sufficient to do it once for each completely positive mapping. Yet in other words, we can formulate as follows.

2.1.7. Functoriality. A pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module E is a *functor* sending (non-degenerate) representations of $\mathcal{B}$ on F to (non-degenerate) representations of $\mathcal{A}$ on $E \odot F$, and the composition of two such functors is the tensor product. The Stinespring construction is a dead end for this functoriality.

We close quoting some results about positivity of operators on a pre-Hilbert module.

2.1.8. Definition. We say a linear operator a on a pre-Hilbert $\mathcal{B}$-module E is *positive*, if $\langle x, ax \rangle \geqslant 0$ for all $x \in E$. In this case (by linearity and polarization) a is adjointable.

Of course, $a^* a$ is positive, if a^* exists. The following lemma due to Paschke [Pas73] shows that for $a \in \mathscr{B}^{\mathrm{a}}(E)$ this definition of positivity is compatible with the C^*-algebraic definition. An elegant proof can be found in [Lan95].

2.1.9. Lemma. *Let E be a pre-Hilbert $\mathcal{B}$-module and let a be a bounded $\mathcal{B}$-linear mapping on E. Then the following conditions are equivalent*:

1. *a is positive in the C^*-algebra $\mathscr{B}^{\mathrm{a}}(\bar{E})$.*
2. *a is positive according to Definition* 2.1.8.

Notice that if E is complete, then it is sufficient to require just that a is $\mathcal{B}$-linear, because a is closed and, therefore, bounded. A similar argument allows to generalize a well-known criterion for contractivity to pre-Hilbert modules.

2.1.10. Lemma. *A positive operator a on E is a contraction, if and only if*

$$\langle x, ax \rangle \leqslant \langle x, x \rangle \tag{2.1.2}$$

for all $x \in E$.

Proof. Of course, a positive contraction fulfills (2.1.2). Conversely, let us assume that $a \geqslant 0$ fulfills (2.1.2). By positivity, $(x, y)_a = \langle x, ay \rangle$ is a (semi-)inner product. In particular, by Cauchy–Schwartz inequality (2.1.1) we have $\|(y, x)_a (x, y)_a\| \leqslant \|(x, x)_a\| \, \|(y, y)_a\|$, hence,

$$\|\langle x, ay \rangle\|^2 \leqslant \|\langle x, ax \rangle\| \, \|\langle y, ay \rangle\| \leqslant \|\langle x, x \rangle\| \, \|\langle y, y \rangle\|,$$

i.e. $\|a\| \leqslant 1$. □

2.2. Exterior tensor product and matrices of Hilbert modules

Matrices with entries in a Hilbert module are a crucial tool in these notes. Like L^2-spaces of functions with values in a Hilbert module they can be understood most easily as very particular examples of exterior tensor products. In Appendix B we need the properties of exterior tensor products in full generality.

The *exterior* tensor product is based on the observation that the (vector space) tensor product $E_1 \otimes E_2$ of a pre-Hilbert $\mathcal{B}_i$-modules E_i $(i = 1, 2)$ is a $\mathcal{B}_1 \otimes \mathcal{B}_2$-module in an obvious way. It is not difficult to show that the sesquilinear mapping on

$E_1 \otimes E_2$, defined by setting

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \otimes \langle y, y' \rangle \tag{2.2.1}$$

is positive, i.e. an inner product. It is even more easy (see [Ske98]) to see that it is strictly positive, so that the $E_1 \otimes E_2$ is a pre-Hilbert $\mathcal{B}_1 \otimes \mathcal{B}_2$-module over the pre-$C^*$-algebra $\mathcal{B}_1 \otimes \mathcal{B}_2$ equipped with whatever cross C^*-norm. In practice, we consider only the spatial C^*-norm on the tensor product. Observe that, if we want to complete $\mathcal{B}_1 \otimes \mathcal{B}_2$, then we must, in general, complete also $E_1 \otimes E_2$.

If E_i are pre-Hilbert $\mathcal{A}_i$–$\mathcal{B}_i$-modules, then $E_1 \otimes E_2$ is a pre-Hilbert $\mathcal{A}_1 \otimes \mathcal{A}_2$–$\mathcal{B}_1 \otimes \mathcal{B}_2$-module and the representation of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on $E_1 \otimes E_2$ is a contraction for the spatial norm (hence, for all norms) on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Moreover, if the representations of $\mathcal{A}_i$ on E_i are faithful, then the representation of $\mathcal{A}_1 \otimes \mathcal{A}_2$ is an isometry for the spatial norm. One easily checks the property

$$(E_1 \otimes E_2) \odot (F_1 \otimes F_2) = (E_1 \odot F_1) \otimes (E_2 \odot F_2). \tag{2.2.2}$$

If E_i are von Neumann $\mathcal{A}_i$–$\mathcal{B}_i$-submodules of $\mathscr{B}(G_i, E_i \bar{\odot} G_i)$, then the strong closure of $E_1 \otimes E_2$ in $\mathscr{B}(G_1, E_1 \bar{\odot} G_1) \bar{\otimes}^{\mathrm{s}} \mathscr{B}(G_2, E_2 \bar{\odot} G_2) = \mathscr{B}(G_1 \bar{\otimes} G_2, (E_1 \bar{\odot} G_1) \bar{\otimes} (E_2 \bar{\odot} G_2))$ is a von Neumann $\mathcal{A}_1 \bar{\otimes}^{\mathrm{s}} \mathcal{A}_2$–$\mathcal{B}_1 \bar{\otimes}^{\mathrm{s}} \mathcal{B}_2$-module and the Stinespring representation ρ of $\mathcal{A}_1 \bar{\otimes}^{\mathrm{s}} \mathcal{A}_2$ on $(E_1 \bar{\odot} G_1) \bar{\otimes} (E_2 \bar{\odot} G_2)$ is, indeed, just the tensor product of the Stinespring representations ρ_i of $\mathcal{A}_i$. In particular, we have $\mathscr{B}^{\mathrm{a}}(E_1 \bar{\otimes}^{\mathrm{s}} E_2) = \mathscr{B}^{\mathrm{a}}(E_1) \bar{\otimes}^{\mathrm{s}} \mathscr{B}^{\mathrm{a}}(E_2)$ (as von Neumann algebras). See [Ske01a] for details.

For a Hilbert module E and a measure space M we define $L^2(M, E) = E \bar{\otimes} L^2(M)$. For a von Neumann $\mathcal{A}$–$\mathcal{B}$-module E we define the von Neumann $\mathcal{A} \bar{\otimes}^{\mathrm{s}} \mathscr{B}(L^2(M))$–$\mathcal{B}$-module $L^{2,\mathrm{s}}(M, E) = E \bar{\otimes}^{\mathrm{s}} L^2(M)$.

For some Hilbert spaces G, H the space $\mathscr{B}(G, H)$ is a von Neumann $\mathscr{B}(H)$–$\mathscr{B}(G)$-module with inner product $\langle L, M \rangle = L^* M$ and the obvious module operations. In particular, the $n \times m$-matrices $M_{nm} = \mathscr{B}(\mathbb{C}^m, \mathbb{C}^n)$ are von Neumann M_n–M_m-modules. One easily checks that $M_{n\ell} \odot M_{\ell m} = M_{nm}$ where $X \odot Y = XY$ gives the canonical identification.

By $M_{nm}(E) = E \otimes M_{nm}$ we denote the spaces of $n \times m$-matrices with entries in a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module. By construction $M_{nm}(E)$ is a pre-Hilbert $M_n(\mathcal{A})$–$M_m(\mathcal{B})$-module. It is complete and strongly closed, if and only if E is complete and strongly closed, respectively.

$M_{nm}(E)$ consists of matrices $X = (x_{ki})$ whose inner product is

$$\langle X, Y \rangle_{ij} = \sum_{k=1}^{n} \langle x_{ki}, y_{kj} \rangle.$$

An element of $M_m(\mathcal{B})$ acts from the right on the right index and an element of $M_n(\mathcal{A})$ acts from the left on the left index of X in the usual way. Considering E as pre-Hilbert $\mathscr{B}^{\mathrm{a}}(E)$–$\mathcal{B}$-module and making use of matrix units for $M_n(\mathscr{B}^{\mathrm{a}}(E))$, one easily shows that $\mathscr{B}^{\mathrm{a}}(M_{nm}(E)) = M_n(\mathscr{B}^{\mathrm{a}}(E))$. From (2.2.2) we conclude that

$M_{n\ell}(E) \odot M_{\ell m}(F) = M_{nm}(E \odot F)$ where $(X \odot Y)_{i,j} = \sum_k x_{ik} \odot y_{kj}$ gives the canonical identification. In particular, for square matrices we find $M_n(E) \odot M_n(F) = M_n(E \odot F)$.

Conversely, let E_{nm} be a pre-Hilbert $M_n(\mathcal{A})$–$M_m(\mathcal{B})$-module. For simplicity, assume that $\mathcal{A}, \mathcal{B}$ are unital (otherwise use approximate units) and define Q_i as the matrix in $M_n(\mathcal{A})$ with $\mathbf{1}$ in the ith place in the diagonal. $P_i \in M_m(\mathcal{B})$ is defined analogously. Then all submodules $Q_i E_{nm} P_j$ are isomorphic to the same pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module E and $E_{nm} = M_{nm}(E)$. (Each of these *entries* $Q_i E_{nm} P_j$ takes its $\mathcal{A}$–$\mathcal{B}$-module structure by embedding $\mathcal{A}$ and $\mathcal{B}$ into that unique place in the diagonal of $M_n(\mathcal{A})$ and $M_m(\mathcal{B})$, respectively, where it acts non-trivially. The isomorphism between two entries can be constructed with the help of matrix units in M_n, M_m.)

Special forms are $E^n = M_{n1}(E)$ and $E_n = M_{1n}(E)$. Both consist of elements $X = (x_1, \ldots, x_n)$ $(x_i \in E)$. However, the former is an $M_n(\mathcal{A})$–$\mathcal{B}$-module with inner product $\langle X, Y \rangle = \sum_i \langle x_i, y_i \rangle$ and $\mathscr{B}^{\mathrm{a}}(E^n) = M_n(\mathscr{B}^{\mathrm{a}}(E))$ (it is just the n-fold direct sum over E), whereas, the latter is an $\mathcal{A}$–$M_n(\mathcal{B})$-module with inner product $\langle X, Y \rangle_{i,j} = \langle x_i, y_i \rangle$ and $\mathscr{B}^{\mathrm{a}}(E_n) = \mathscr{B}^{\mathrm{a}}(E)$. Observe that $E_n \odot F^n = E \odot F$, whereas, $E^n \odot F_m = M_{nm}(E \odot F)$.

Let us set $X = (\delta_{ij} x_i) \in M_n(E)$ for some $x_i \in E$ $(i = 1, \ldots, n)$, and Y correspondingly. Then the mapping $T : M_n(\mathcal{A}) \to M_n(\mathcal{B})$, defined by setting $T(A) = \langle X, AY \rangle$ acts matrix-element-wise on A, i.e.

$$(T(A))_{ij} = \langle x_i, a_{ij} y_j \rangle.$$

In particular, if $Y = X$, then T is completely positive. $T(A)$ may be considered as the *Schur product* of the matrix T of mappings $\langle x_i, \bullet y_j \rangle : \mathcal{A} \to \mathcal{B}$ and the matrix A of elements $a_{ij} \in \mathcal{A}$.

If S is another mapping coming in a similar manner from diagonal matrices X', Y' with entries in a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$-module F, then we find as in Example 2.1.6 that the *Schur composition* of $S \circ T$ of the mappings T and S (i.e. the pointwise composition) is given by

$$S \circ T(A) = \langle X \odot X', AY \odot Y' \rangle.$$

This observation is crucial for the analysis of CPD-semigroups in Section 3.

2.3. The time ordered Fock module

2.3.1. Definition. Let $\mathcal{B}$ be a unital C^*-algebra and let E be a (pre-)Hilbert $\mathcal{B}$–$\mathcal{B}$-module. Then the *full Fock module $\mathcal{F}(E)$ over E* is the completion of the pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module

$$\underline{\mathcal{F}}(E) = \bigoplus_{n=0}^{\infty} E^{\odot n},$$

where $E^{\odot 0} = \mathcal{B}$ and $\omega = \mathbf{1} \in \mathcal{B} = E^{\odot 0}$ is the *vacuum*. If $\mathcal{B}$ is a von Neumann algebra, then by $\mathcal{F}^{\mathrm{s}}(E)$ we denote the von Neumann $\mathcal{B}$–$\mathcal{B}$-module obtained by strong closure of $\mathcal{F}(E)$.

2.3.2. Definition. For any contraction $T \in \mathscr{B}^{\mathrm{a,bil}}(E)$ we define its *second quantization*

$$\mathcal{F}(T) = \bigoplus_{n \in \mathbb{N}_0} T^{\odot n} \in \mathscr{B}^{\mathrm{a}}(\mathcal{F}(E)) \quad (T^{\odot 0} = \mathsf{id}).$$

2.3.3. Example. Let F be a two-sided Hilbert module. One of the most important full Fock modules is $\mathcal{F}(L^2(\mathbb{R}, F))$. The time shift $\mathscr{s}_t$ in $\mathscr{B}^{\mathrm{a,bil}}(L^2(\mathbb{R}, F))$ for some Hilbert $\mathcal{B}$–$\mathcal{B}$-module F is defined by setting $[\mathscr{s}_t f](s) = f(s-t)$. The corresponding second quantized time shift $\mathcal{F}(\mathscr{s}_t)$ gives rise to the *time shift automorphism group* $\mathscr{S}$ on $\mathscr{B}^{\mathrm{a}}(\mathcal{F}(L^2(\mathbb{R}, E)))$, defined by setting

$$\mathscr{S}_t(a) = \mathcal{F}(\mathscr{s}_t) a \mathcal{F}(\mathscr{s}_t)^*.$$

$\mathcal{F}(\mathscr{s}_t)$ is $\mathcal{B}$–$\mathcal{B}$-linear so that $\mathscr{S}$ leaves invariant $\mathcal{B} \subset \mathscr{B}^{\mathrm{a}}(\mathcal{F}(L^2(\mathbb{R}, E)))$ and it is strongly continuous.

As the name tells us, the construction of the *time ordered Fock module* is connected with the time structure of its one-particle sector $L^2(\mathbb{R}, F)$. We take this into account by speaking of the time ordered Fock module over F rather than over $L^2(\mathbb{R}, F)$. Additionally, we are interested mainly in the real half-line $\mathbb{R}_+$ and include also this in the definition.

2.3.4. Definition. By Δ_n we denote the indicator function of the subset $\{(t_n, \ldots, t_1) : t_n > \cdots > t_1\}$ of $\mathbb{R}^n$. Let $\mathcal{B}$ be a unital C^*-algebra, let F be a Hilbert $\mathcal{B}$–$\mathcal{B}$-module and set $E = L^2(\mathbb{R}, F)$ and $E_K = L^2(K, F)$ for any measurable subset K of $\mathbb{R}$. Then Δ_n acts as a projection on $E^{\bar{\odot} n} = L^2(\mathbb{R}^n, F^{\bar{\odot} n})$. We call the range of Δ_n applied to $E^{\bar{\odot} n}$ (or some submodule) the *time ordered part* of $E^{\bar{\odot} n}$ (or of this submodule).

The *time ordered Fock module* over F is

$$\mathrm{I\!\Gamma}(F) = \bigoplus_{n=0}^{\infty} \Delta_n E_{\mathbb{R}_+}^{\bar{\odot} n} = \Delta \mathcal{F}(E_{\mathbb{R}_+}) \subset \mathcal{F}(E_{\mathbb{R}_+})$$

where $\Delta = \oplus_{n=0}^{\infty} \Delta_n$ is the projection onto the *time ordered part* of $\mathcal{F}(E)$. The *extended time ordered Fock module* is $\breve{\mathrm{I\!\Gamma}}(F) = \Delta \mathcal{F}(E)$. We use the notations $\mathrm{I\!\Gamma}_t(F) = \Delta \mathcal{F}(E_{[0,t)})$ $(t \geqslant 0)$ and $\mathrm{I\!\Gamma}_K(F) = \Delta \mathcal{F}(E_K)$ (K a measurable subset of $\mathbb{R}$). If $\mathcal{B}$ is a von Neumann algebra on a Hilbert space G, then we indicate the strong closure by $\mathrm{I\!\Gamma}^{\mathrm{s}}(F)$, and so on.

The *algebraic time ordered Fock module* is $\underline{\mathrm{I\!\Gamma}}(F) = \Delta \underline{\mathcal{F}}(\mathfrak{S}(\mathbb{R}_+, F))$ (where $\mathfrak{S}$ denotes the step functions and F maybe only a pre-Hilbert module). Observe that $\underline{\mathrm{I\!\Gamma}}(F)$ is not a subset of $\underline{\mathcal{F}}(\mathfrak{S}(\mathbb{R}_+, F))$ (unless $F \odot F$ is trivial).

Definition 2.3.4 and the factorization in Theorem 2.3.6 are due to [BS00]. The time ordered Fock module is a straightforward generalization to Hilbert modules of the Guichardet picture of the symmetric Fock space [Gui72] and the generalization to the higher-dimensional case discussed by Schürmann [Sch93] and Bhat [Bha98].

2.3.5. Observation. The time shift $\mathscr{S}$ leaves invariant the projection $\Delta \in \mathscr{B}^a(\mathcal{F}(E))$. It follows that $\mathscr{S}$ restricts to an automorphism group on $\mathscr{B}^a(\breve{\mathbb{I}\Gamma}(F))$ and further to an E_0-semigroup $\mathscr{B}^a(\mathbb{I}\Gamma(F))$ (of course, both strongly continuous and normal in the case of von Neumann modules).

The following theorem is the analogue of the well-known factorization $\Gamma(L^2([0, s+t])) = \Gamma(L^2([t, s+t])) \otimes \Gamma(L^2([0, t]))$ of the symmetric Fock space. However, in the theory of product systems, be it of Hilbert spaces in the sense of Arveson [Arv89] or of Hilbert modules in the sense of Section 4 (of which the time ordered Fock modules are to be the most fundamental examples), we put emphasis on the length of intervals rather than on their absolute position on the half line. (We comment on this crucial difference in [BS00, Observation 4.2].) Therefore, we are more interested to write the above factorization in the form $\Gamma(L^2([0, s+t])) = \Gamma(L^2([0, s])) \otimes \Gamma(L^2([0, t]))$, where the first factor has first to be time shifted by t. Adopting this way of thinking (where the time shift is *encoded* in the tensor product) has enormous advantages in many formulae. We will use it consequently throughout. Observe that, contrary to all good manners, we write the future in the first place and the past in the second. This order is forced upon us and, in fact, we will see in Remark 2.3.10 that the order is no longer arbitrary for Hilbert modules.

2.3.6. Theorem (Bhat and Skeide [BS00]). *The mapping u_{st}, defined by setting*

$$[u_{st}(X_s \odot Y_t)](s_m, \ldots, s_1, t_n, \ldots, t_1) = [\mathcal{F}(\mathscr{S}_t) X_s](s_m, \ldots, s_1) \odot Y_t(t_n, \ldots, t_1)$$
$$= X_s(s_m - t, \ldots, s_1 - t) \odot Y_t(t_n, \ldots, t_1), \qquad (2.3.1)$$

$(s + t > s_m \geqslant \cdots \geqslant s_1 \geqslant t > t_n \geqslant \cdots \geqslant t_1 \geqslant 0, X_s \in \Delta_m E_{[0,s]}^{\odot m}, Y_t \in \Delta_n E_{[0,t]}^{\odot n}$ *extends as a two-sided isomorphism* $\mathbb{I}\Gamma_s(F) \odot \mathbb{I}\Gamma_t(F) \to \mathbb{I}\Gamma_{s+t}(F)$. *It extends further to two-sided isomorphisms* $\mathbb{I}\Gamma_s(F) \bar{\odot} \mathbb{I}\Gamma_t(F) \to \mathbb{I}\Gamma_{s+t}(F)$ *and* $\mathbb{I}\Gamma^s_s(F) \bar{\odot}^s \mathbb{I}\Gamma^s_t(F) \to \mathbb{I}\Gamma^s_{s+t}(F)$, *respectively. Moreover,*

$$u_{r(s+t)}(\mathrm{id} \odot u_{st}) = u_{(r+s)t}(u_{rs} \odot \mathrm{id}).$$

2.3.7. Observation. Letting in the preceding computation formally $s \to \infty$, we see that (2.3.1) defines a two-sided isomorphism $u_t : \mathbb{I}\Gamma(F) \odot \mathbb{I}\Gamma_t(F) \to \mathbb{I}\Gamma(F)$. We have $u_{s+t}(\mathrm{id} \odot u_{st}) = u_t(u_s \odot \mathrm{id})$. In the sequel, we no longer write u_{st} nor u_t and just use the identifications $\mathbb{I}\Gamma_s(F) \odot \mathbb{I}\Gamma_t(F) = \mathbb{I}\Gamma_{s+t}(F)$ and $\mathbb{I}\Gamma(F) \odot \mathbb{I}\Gamma_t(F) = \mathbb{I}\Gamma(F)$. Notice that in the second identification $\mathscr{S}_t(a) = a \odot \mathrm{id}_{\mathbb{I}\Gamma_t(F)} \in \mathscr{B}^a(\mathbb{I}\Gamma(F) \odot \mathbb{I}\Gamma_t(F)) = \mathscr{B}^a(\mathbb{I}\Gamma(F))$. We explain this more detailed in a more general context in Section 4.4.

In the symmetric Fock space we may define an *exponential vector* to any element in the one-particle sector. In the time ordered Fock module we must be more careful.

2.3.8. Definition. For a step function $x \in \mathfrak{S}(\mathbb{R}_+, F)$ we define the *exponential vector* $\psi(x) \in \mathrm{I\Gamma}(F)$ as

$$\psi(x) = \sum_{n=0}^{\infty} \Delta_n x^{\odot n}$$

with $x^{\odot 0} = \omega$. (Observe that if x has support $[0, t]$ and $\|x(s)\| \leqslant c \in \mathbb{R}_+$, then $\|\Delta_n x^{\odot n}\|^2 \leqslant \frac{t^n c^{2n}}{n!}$ where $\frac{t^n}{n!}$ is the volume of the set $\{(t_n, \ldots, t_1) : t \geqslant t_n \geqslant \cdots \geqslant t_1 \geqslant 0\}$ so that $\|\psi(x)\|^2 \leqslant e^{tc^2} < \infty$.)

Let $\mathrm{t} = (t_n, \ldots, t_1) \in \mathbb{I}_t$, put $t_0 = 0$, and let $x = \sum_{i=1}^n \zeta_i \mathbb{I}_{[t_{i-1}, t_i)}$. Then we easily check

$$\psi(x) = \psi(\zeta_n \mathbb{I}_{[0, t_n - t_{n-1})}) \odot \cdots \odot \psi(\zeta_1 \mathbb{I}_{[0, t_1 - t_0)}). \tag{2.3.2}$$

2.3.9. Theorem. *For all $t \in [0, \infty]$ the exponential vectors to elements $x \in \mathfrak{S}([0, t], F)$ form a total subset of $\mathrm{I\Gamma}_t(F)$.*

The proof goes very much along the lines for the symmetric Fock space. A detailed version can be found in [Ske01a].

2.3.10. Remark. Obviously, the definition of the exponential vectors extends to elements $x \in L^\infty(\mathbb{R}_+, F) \cap L^2(\mathbb{R}_+, F)$. It is also not difficult to see that it makes sense for Bochner square integrable functions $x \in L^2_B(\mathbb{R}_+, F) \subset L^2(\mathbb{R}_+, F)$. ($\psi(x)$ depends continuously on x in L^2_B-norm.) It is, however, unclear, whether it is possible to define $\psi(x)$ for arbitrary $x \in L^2(\mathbb{R}_+, F)$. We can only say that if $x \in E_{[0,s]}$, $y \in E_{[0,t]}$ are such that $\psi(x)$, $\psi(y)$ exist, then $\psi(\mathscr{S}_t x \oplus y) = \psi(x) \odot \psi(y)$ exists, too. Observe that, in general, $\psi(x) \odot \psi(y)$ and $\psi(y) \odot \psi(x)$ are very much different elements of $\mathrm{I\Gamma}_{s+t}(F)$.

The exponential vectors $\xi_t = \psi(\zeta \mathbb{I}_{[0,t)})$ ($\zeta \in F$) play a distinguished role. They fulfill the factorization

$$\xi_s \odot \xi_t = \xi_{s+t} \tag{2.3.3}$$

and $\xi_0 = \omega$. In accordance with Definition 4.2.1 we call such a family $\xi^\odot = (\xi_t)_{t \in \mathbb{R}_+}$ a *unit*. Notice that $T_t(b) = \langle \xi_t, b\xi_t \rangle$ defines a CP-semigroup on $\mathcal{B}$ (see Proposition 4.2.5). Additionally, $\psi(\zeta \mathbb{I}_{[0,t)})$ depends continuously on t so that the corresponding semigroup is uniformly continuous (cf. Theorem 4.4.12). We ask, whether there are other continuous units $\xi^\odot$ than these *exponential units*. The answer is given by the following theorem from [LS01].

2.3.11. Theorem. *Let $\beta \in \mathcal{B}$, $\zeta \in F$, and let $\xi^0 = (\xi^0_t)_{t\in\mathbb{R}_+}$ with $\xi^0_t = e^{t\beta}$ be the uniformly continuous semigroup in $\mathcal{B}$ with generator β. Then $\xi^\odot(\beta,\zeta) = (\xi_t(\beta,\zeta))_{t\in\mathbb{R}_+}$ with the component ξ^n_t of $\xi_t(\beta,\zeta) \in \mathbb{I}\Gamma_t$ in the n-particle sector defined as*

$$\xi^n_t(t_n, \dots, t_1) = \xi^0_{t-t_n}\zeta \odot \xi^0_{t_n-t_{n-1}}\zeta \odot \cdots \odot \xi^0_{t_2-t_1}\zeta\xi^0_{t_1} \tag{2.3.4}$$

(*and, of course,* ξ^0_t *for* $n = 0$), *is a unit. Moreover, both functions* $t \mapsto \xi_t \in \mathbb{I}\Gamma(F)$ *and the CP-semigroup* $T^{(\beta,\xi)}$ *with* $T^{(\beta,\xi)}_t = \langle \xi_t(\beta,\xi), \bullet\xi_t(\beta,\xi)\rangle$ *are uniformly continuous and the generator of* $T^{(\beta,\xi)}$ *is*

$$b \mapsto \langle \zeta, b\zeta \rangle + b\beta + \beta^* b. \tag{2.3.5}$$

Conversely, let $\xi^\odot$ *be a unit such that* $t \mapsto \xi_t \in \mathbb{I}\Gamma(F)$ *is a continuous function. Then there exist unique* $\beta \in \mathcal{B}$ *and* $\zeta \in F$ *such that* $\xi_t = \xi_t(\beta,\zeta)$ *as defined by* (2.3.4).

2.3.12. Remark. We see that $T^{(\beta,\zeta)}$ has a generator of Christensen–Evans type; see Appendix C.

2.3.13. Remark. The exponential units $\psi(\zeta \mathbb{I}_{[0,t)})$ correspond to $\xi_t(0,\zeta)$. We may consider $\xi_t(\beta,\zeta)$ as $\xi(0,\zeta)$ *renormalized* by the semigroup $e^{t\beta}$. This is motivated by the observation that for $\mathcal{B} = \mathbb{C}$ all factors $e^{(t_i-t_{i-1})\beta}$ in (2.3.4) come together and give $e^{t\beta}$. The other way round, in the noncommutative context we have to distribute the normalizing factor $e^{t\beta}$ over the time intervals $[t_{i-1}, t_i)$.

2.3.14. Observation. In the case of a von Neumann module F, the characterization of continuous units in Theorem 2.3.11 remains true also, if we allow ξ_t to be in the bigger space $\mathbb{I}\Gamma^s_t(F)$. This follows, because the proof in [LS01] that continuous units must have the form $\xi_t(\beta,\zeta)$ works as before.

2.3.15. Remark. Fixing a semigroup ξ^0 and an element ζ in F, Eq. (2.3.4) gives more general units. For that it is sufficient to observe that ξ^0 is bounded by Ce^{ct} for suitable constants C, c (so that ξ^n_t are summable). An example from [LS01] shows that we may not hope to generalize Theorem 2.3.11 to units which are continuous in a weaker topology only. On the other hand, this example also shows that there are interesting non-continuous units (giving rise to strongly continuous CP-semigroups), although the time ordered Fock module is spanned by its continuous units.

3. Kernels

Positive definite kernels on some set S with values in $\mathbb{C}$ (i.e. functions $k : S \times S \to \mathbb{C}$ such that $\sum_{i,j} \bar{c}_i k^{\sigma_i,\sigma_j} c_j \geqslant 0$ for all choices of finitely many $c_i \in \mathbb{C}, \sigma_i \in S$) are well-established objects. There are basically two important results on such kernels.

One is the *Kolmogorov decomposition* which provides us with a Hilbert space H and an embedding $i: S \to H$ (unique, if the set $i(S)$ is total) such that $k^{\sigma,\sigma'} = \langle i(\sigma), i(\sigma') \rangle$.

The other main result is that the *Schur product* of two positive definite kernels (i.e. the pointwise product) is again positive definite. Semigroups of such kernels were studied, for instance, in [Gui72] or [PS72]. The kernel obtained by (pointwise) derivative at $t = 0$ of such a semigroup is *conditionally positive definite*, and any such kernel defines a positive definite semigroup via (pointwise) exponential.

The goal of this section is to find suitable generalizations of the preceding notions to the $\mathcal{B}$-valued case. Suitable means, of course, that we will have plenty of occasion to see these notions at work. Positive definite $\mathcal{B}$-valued kernels together with the Kolmogorov decomposition generalize easily (Section 3.1). They are, however, not sufficient, mainly, because for noncommutative $\mathcal{B}$ the pointwise product of two kernels does not preserve positive definiteness. For this reason we have to pass to *completely positive definite kernels* (Section 3.2). These kernels take values in the bounded mappings on the C^*-algebra $\mathcal{B}$, fulfilling a condition closely related to complete positivity. Instead of the pointwise product of elements in $\mathcal{B}$ we consider the composition (pointwise on $S \times S$) of mappings on $\mathcal{B}$. Also here we have a Kolmogorov decomposition for a completely positive definite kernel, we may consider *Schur semigroups* of such (CPD-semigroups) and their generators (Section 3.4).

Both completely positive mappings and completely positive definite kernels have realizations as *matrix elements* with vectors of a suitably constructed two-sided Hilbert module. In both cases we can understand the composition of two such objects in terms of the tensor product of the underlying Hilbert modules (GNS-modules or Kolmogorov modules). In fact, we find the results for completely positive definite kernels by reducing the problems to completely positive mappings (between $n \times n$-matrix algebras) with the help of Lemmata 3.2.1 and 3.4.6, and then applying the crucial constructions in Section 2.2. In both cases the tensor product plays a distinguished role. An attempt to realize a whole semigroup, be it of mappings or of kernels, on the same Hilbert module, leads us directly to the notion of tensor product systems of Hilbert modules, namely, the GNS-system in Section 4.3.

It is a feature of CPD-semigroups on S that they restrict to CPD-kernels, when $S = \{s\}$ consists of a single element. Sometimes, the proofs of statements on CPD-semigroups are straightforward analogues of those for CPD-semigroups. However, often they are not. In this chapter we put emphasis on the first type of statements which, therefore, will help us in the remaining chapters to analyze product systems. To prove the other type of statements like Theorem 3.5.2 we have to wait for Section 5.4.

Although slightly different, our notion of completely positive definite kernels is inspired very much by the corresponding notion in [AK99]. The idea to consider CP-semigroups on $M_n(\mathcal{B})$ (of which the CPD-semigroups are a direct generalization) is entirely due to [AK99].

3.1. Positive definite kernels

3.1.1. Definition. Let S be a set and let $\mathcal{B}$ be a pre-C^*-algebra. A *$\mathcal{B}$-valued kernel* or short *kernel* on S is a mapping $\mathfrak{k} : S \times S \to \mathcal{B}$. We say a kernel $\mathfrak{k}$ is *positive definite*, if

$$\sum_{\sigma,\sigma' \in S} b_\sigma^* \mathfrak{k}^{\sigma,\sigma'} b_{\sigma'} \geqslant 0 \tag{3.1.1}$$

for all choices of $b_\sigma \in \mathcal{B}$ $(\sigma \in S)$ where only finitely many b_σ are different from 0.

3.1.2. Observation. Condition (3.1.1) is equivalent to

$$\sum_{i,j} b_i^* \mathfrak{k}^{\sigma_i,\sigma_j} b_j \geqslant 0 \tag{3.1.2}$$

for all choices of finitely many $\sigma_i \in S, b_i \in \mathcal{B}$. To see this, define b_σ $(\sigma \in S)$ to be the sum over all b_i for which $\sigma_i = \sigma$. Then (3.1.2) transforms into (3.1.1). The converse direction is trivial.

3.1.3. Proposition. *Let $\mathcal{B}$ be a unital pre-C^*-algebra and let $\mathfrak{k}$ be a positive definite $\mathcal{B}$-valued kernel on S. Then there exists a pre-Hilbert $\mathcal{B}$-module E and a mapping $i : S \to E$ such that*

$$\mathfrak{k}^{\sigma,\sigma'} = \langle i(\sigma), i(\sigma') \rangle$$

and $E = \mathsf{span}(i(S)\mathcal{B})$. Moreover, if (E', i') is another pair with these properties, then $i(\sigma) \mapsto i'(\sigma)$ establishes an isomorphism $E \to E'$.

Proof. Let $S_\mathcal{B}$ denote the free right $\mathcal{B}$-module generated by S (i.e. $\bigoplus_{\sigma \in S} \mathcal{B} = \{(b_\sigma)_{\sigma \in S} : b_\sigma \in \mathcal{B}, \#\{\sigma \in S : b_\sigma \neq 0\} < \infty\}$ or, in other words, $S_\mathbb{C} \otimes \mathcal{B}$ where $S_\mathbb{C}$ is a vector space with basis S). Then by (3.1.1)

$$\langle (b_\sigma), (b'_\sigma) \rangle = \sum_{\sigma,\sigma' \in S} b_\sigma^* \mathfrak{k}^{\sigma\sigma'} b'_{\sigma'}$$

defines a semiinner product on $S_\mathcal{B}$. We set $E = S_\mathcal{B} / \mathcal{N}_{S_\mathcal{B}}$ and $i(\sigma) = (\delta_{\sigma\sigma'} \mathbf{1})_{\sigma' \in S} + \mathcal{N}_{S_\mathcal{B}}$. Then the pair (E, i) has all desired properties. Uniqueness is clear. □

3.1.4. Remark. If $\mathcal{B}$ is non-unital, then we still may construct E as before as a quotient of $S_\mathbb{C} \otimes \mathcal{B}$, but we do not have the mapping i. We have, however, a mapping $\hat{i} : S \times \mathcal{B} \to E$, sending (σ, b) to $(\delta_{\sigma\sigma'} b)_{\sigma' \in S} + \mathcal{N}_{S_\mathcal{B}}$, such that $b^* \mathfrak{k}^{\sigma,\sigma'} b' = \langle \hat{i}(\sigma, b), \hat{i}(\sigma', b') \rangle$ with similar cyclicity and uniqueness properties.

The easiest way to have a mapping like i also in the non-unital case, is by observing that $\mathfrak{k}$ is positive definite also as kernel with values in $\tilde{\mathcal{B}}$. (To see this approximate $\tilde{\mathbf{1}} \in \tilde{\mathcal{B}}$ strictly by an approximate unit for $\mathcal{B}$.) If $(\tilde{E}, \tilde{i})$ is the corresponding pair, then $\tilde{E}$ contains E as a dense submodule. After completion the difference disappears.

3.1.5. Definition. We refer to the pair (E, i) as the *Kolmogorov decomposition* for $\mathfrak{k}$ and to E as its *Kolmogorov module*.

3.1.6. Example. For $\mathbb{C}$-valued positive definite kernels we recover the usual Kolmogorov decomposition. For instance, usual proofs of the Stinespring construction for a completely positive mapping $T : \mathcal{A} \to \mathscr{B}^a(G)$ start with a Kolmogorov decomposition for the kernel $((a, g), (a', g')) \mapsto \langle g, T(a^*a')g' \rangle$ on $\mathcal{A} \times G$ and obtain in this way the pre-Hilbert space $H = E \odot G$ where E is the GNS-module of T; cf. Remark 2.1.5.

For $\mathcal{B} = \mathscr{B}^a(F)$ for some pre-Hilbert $\mathcal{C}$-module F we recover the Kolmogorov decomposition in the sense of Murphy [Mur97]. He recovers the module $E \odot F$ of the KSGNS-construction for a completely positive mapping $T : \mathcal{A} \to \mathscr{B}^a(F)$ (cf. [Lan95]) as Kolmogorov decomposition for the kernel $((a, y), (a', y')) \mapsto \langle y, T(a^*a')y' \rangle$ on $\mathcal{A} \times F$.

3.2. Completely positive definite kernels

For $\mathbb{C}$-valued kernels there is a positivity preserving product, namely, the *Schur product* which consists in multiplying two kernels pointwise. For non-commutative $\mathcal{B}$ this operation is also possible, but will, in general, not preserve positive definiteness. It turns out that we have to consider kernels which take as values mappings between algebras rather than kernels with values in algebras. Then the pointwise multiplication in the Schur product is replaced by pointwise composition of mappings. Of course, this includes the usual Schur product of $\mathbb{C}$-valued kernels, if we interpret $z \in \mathbb{C}$ as mapping $w \mapsto zw$ on $\mathbb{C}$.

3.2.1. Lemma. *Let S be a set and let $\mathfrak{K} : S \times S \to \mathscr{B}(\mathcal{A}, \mathcal{B})$ be a kernel with values in the bounded mappings between pre-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Then the following conditions are equivalent*:

1. *We have*
$$\sum_{i,j} b_i^* \mathfrak{K}^{\sigma_i, \sigma_j}(a_i^* a_j) b_j \geqslant 0$$
for all choices of finitely many $\sigma_i \in S$, $a_i \in \mathcal{A}$, $b_i \in \mathcal{B}$.
2. *The kernel $\mathfrak{k} : (\mathcal{A} \times S) \times (\mathcal{A} \times S) \to \mathcal{B}$ with $\mathfrak{k}^{(a,\sigma),(a',\sigma')} = \mathfrak{K}^{\sigma,\sigma'}(a^*a')$ is positive definite.*
3. *The mapping*
$$a \mapsto \sum_{i,j} b_i^* \mathfrak{K}^{\sigma_i, \sigma_j}(a_i^* a a_j) b_j$$
is completely positive for all choices of finitely many $\sigma_i \in S$, $a_i \in \mathcal{A}$, $b_i \in \mathcal{B}$.

4. *For all choices* $\sigma_1, \ldots, \sigma_n \in S$ $(n \in \mathbb{N})$ *the mapping*

$$\mathfrak{K}^{(n)} : (a_{ij}) \mapsto (\mathfrak{K}^{\sigma_i,\sigma_j}(a_{ij}))$$

from $M_n(\mathcal{A})$ *to* $M_n(\mathcal{B})$ *is completely positive.*

5. *For all choices* $\sigma_1, \ldots, \sigma_n \in S$ $(n \in \mathbb{N})$ *the mapping* $\mathfrak{K}^{(n)}$ *is positive.*

Moreover, each of these conditions implies the following conditions.

6. *The mapping*

$$a \mapsto \sum_{\sigma,\sigma' \in S} b_\sigma^* \mathfrak{K}^{\sigma,\sigma'}(a) b_{\sigma'}$$

is completely positive for all choices of $b_\sigma \in \mathcal{B}$ $(\sigma \in S)$ *where only finitely many* b_σ *are different from* 0.

7. *The mapping*

$$a \mapsto \sum_{\sigma,\sigma' \in S} \mathfrak{K}^{\sigma,\sigma'}(a_\sigma^* a a_{\sigma'})$$

is completely positive for all choices of $a_\sigma \in \mathcal{A}$ $(\sigma \in S)$ *where only finitely many* a_σ *are different from* 0.

Proof. Conditions 1 and 2 are equivalent by Observation 3.1.2.

Condition 3 means

$$\sum_{k,\ell \in K} \sum_{i,j \in I} \beta_k^* b_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(a_i^* \alpha_k^* \alpha_\ell a_j) b_j \beta_\ell \geqslant 0 \tag{3.2.1}$$

for all finite sets I, K and $a_i, \alpha_k \in \mathcal{A}$ and $b_i, \beta_k \in \mathcal{B}$. To see $3 \Rightarrow 1$ we choose K consisting of only one element and we replace α_k and β_k by an approximate unit for $\mathcal{A}$ and an approximate unit for $\mathcal{B}$, respectively. By a similar procedure we see $3 \Rightarrow 6$ and $3 \Rightarrow 7$.

To see $1 \Rightarrow 3$, we choose $P = I \times K$, $\sigma_{(i,k)} = \sigma_i$, $a_{(i,k)} = \alpha_k a_i$, and $b_{(i,k)} = b_i \beta_k$. Then (3.2.1) transforms into

$$\sum_{p,q \in P} b_p^* \mathfrak{K}^{\sigma_p,\sigma_q}(a_p^* a_q) b_q \geqslant 0,$$

which is true by 1.

To see $2 \Rightarrow 4$, we do the Kolmogorov decomposition $(E, \hat{\imath})$ for the kernel $\mathfrak{k}$ in the sense of Remark 3.1.4. If $\mathcal{A}$ and $\mathcal{B}$ are unital, then we set $x_j = \hat{\imath}(\mathbf{1}, \sigma_j, \mathbf{1}) \in E$ $(j =$

$1, \ldots, n$). Then the mapping in 4 is completely positive as explained in Section 2.2. If $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital, then we set $x_j = \hat{i}(u_\lambda, \sigma_j, v_\mu)$ for some approximate units (u_λ) and (v_μ) for $\mathcal{A}$ and $\mathcal{B}$, respectively, and we obtain the mapping in 4 as limit (pointwise in norm of $M_n(\mathcal{B})$) of completely positive mappings.

Conditions 4 and 5 are equivalent by simple index manipulations.

To see $5 \Rightarrow 1$ we apply 5 to the positive element $A = (a_i^* a_j) \in M_n(\mathcal{A})$ which means that $\langle B, \mathfrak{K}^{(n)}(A)B \rangle$ is positive for all $B = (b_1, \ldots, b_n) \in \mathcal{B}^n$ and, therefore, implies 1. □

3.2.2. Definition. We call a kernel $\mathfrak{K} : S \times S \to \mathscr{B}(\mathcal{A}, \mathcal{B})$ *completely positive definite*, if it fulfills one of conditions 1–5 in Lemma 3.2.1. By $\mathcal{K}_S(\mathcal{A}, \mathcal{B})$ we denote the set of completely positive definite kernel on S from $\mathcal{A}$ to $\mathcal{B}$. A kernel fulfilling conditions 6 and 7 in Lemma 3.2.1 is called *completely positive definite for* $\mathcal{B}$ and *completely positive definite for* $\mathcal{A}$, respectively.

3.2.3. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital, and let $\mathfrak{K}$ be in $\mathcal{K}_S(\mathcal{A}, \mathcal{B})$. Then there exists a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module E (i.e. the canonical representation of $\mathcal{A}$ is a contraction) and a mapping $i : S \to E$ such that*

$$\mathfrak{K}^{\sigma,\sigma'}(a) = \langle i(\sigma), a i(\sigma') \rangle,$$

and $E = \mathsf{span}(\mathcal{A} i(S) \mathcal{B})$. Moreover, if (E', i') is another pair with these properties, then $i(\sigma) \mapsto i'(\sigma)$ establishes an isomorphism $E \to E'$.

Conversely, if E is a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$-module and S a collection of elements of E, then $\mathfrak{K}$ defined by setting $\mathfrak{K}^{\sigma,\sigma'}(a) = \langle \sigma, a\sigma' \rangle$ is completely positive definite.

3.2.4. Corollary. *A kernel $\mathfrak{K} \in \mathcal{K}_S(\mathcal{A}, \mathcal{B})$ is hermitian, i.e. $\mathfrak{K}^{\sigma,\sigma'}(a^*) = \mathfrak{K}^{\sigma',\sigma}(a)^*$. (This remains true, also if $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital.)*

Proof of Theorem 3.2.3. By Proposition 3.2.3 we may do the Kolmogorov decomposition for the kernel $\mathfrak{k}$ and obtain a pre-Hilbert $\mathcal{B}$-module E with an embedding $i_\mathfrak{k}$. We have

$$\mathfrak{k}^{\sigma',\sigma''}(a'^* a a'') = \langle i_\mathfrak{k}(a', \sigma'), i_\mathfrak{k}(a a'', \sigma'') \rangle = \langle i_\mathfrak{k}(a^* a', \sigma'), i_\mathfrak{k}(a'', \sigma'') \rangle.$$

Therefore, by Observation 2.1.1 setting $a i_\mathfrak{k}(a', \sigma') = i_\mathfrak{k}(a a', \sigma')$ we define a left action of $\mathcal{A}$ on E. This action is non-degenerate, because $\mathcal{A}$ is unital, and the unit acts as unit on E. It is contractive, because all mappings $\mathfrak{K}^{\sigma,\sigma'}$ are bounded, so that in the whole construction we may assume that $\mathcal{A}$ is complete. Setting $i(\sigma) = i_\mathfrak{k}(\mathbf{1}, \sigma)$, the pair (E, i) has the desired properties.

The converse direction is clear from Section 2.2. □

3.2.5. Definition. We refer to the pair (E, i) as the *Kolmogorov decomposition* for $\mathfrak{K}$ and to E as its *Kolmogorov module*.

3.2.6. Observation. If $\mathcal{B}$ is a von Neumann algebra, then we may pass to the strong closure $\bar{E}^s$. It is not necessary that also $\mathcal{A}$ is a von Neumann algebra, and also if $\mathcal{A}$ is a von Neumann algebra, then $\bar{E}^s$ need not be a two-sided von Neumann module. However, for *normal* kernels (i.e. all mappings $\mathfrak{K}^{\sigma,\sigma'}$ are σ-weak) $\bar{E}^s$ is a von Neumann $\mathcal{A}$–$\mathcal{B}$-module.

Our notion of completely positive definite kernels differs from that given by Accardi and Kozyrev [AK99]. Their completely positive definite kernels fulfill only our requirement for kernels completely positive definite for $\mathcal{B}$. The weaker requirement in [AK99] is compensated by an additional property of their concrete kernel (essentially coming due to the simpler structure in the case $\mathcal{B} = \mathscr{B}(G)$); see [Ske01a] for details.

3.3. Partial order of kernels

We say, a completely positive mapping T *dominates* another S, if the difference $T - S$ is also completely positive. In this case, we write $T \geqslant S$. Obviously, $\geqslant$ defines a partial order. As shown by Arveson [Arv69] in the case of $\mathscr{B}(G)$ and extended by Paschke [Pas73] to arbitrary von Neumann algebras, there is an order isomorphism from the set of all completely positive mappings dominated by a fixed completely positive mapping T and certain mappings on the GNS-module of T (or the representation space of the Stinespring representation in the case of $\mathscr{B}(G)$).

In this section we extend these notions and the result to kernels and their Kolmogorov decomposition. Theorem 3.3.3 is the basis for Theorem A.7 which provides us with a powerful tool to establish whether a dilation of a completely positive semigroup is its *GNS-dilation*. In Lemma 3.3.2 we need self-duality. So we stay with von Neumann modules.

3.3.1. Definition. We say, a kernel $\mathfrak{K}$ on S from $\mathcal{A}$ to $\mathcal{B}$ *dominates* another kernel $\mathfrak{L}$, if the difference $\mathfrak{K} - \mathfrak{L}$ is in $\mathcal{K}_S(\mathcal{A}, \mathcal{B})$. For $\mathfrak{K} \in \mathcal{K}_S(\mathcal{A}, \mathcal{B})$ we denote by $\mathcal{D}_{\mathfrak{K}} = \{\mathfrak{L} \in \mathcal{K}_S(\mathcal{A}, \mathcal{B}) : \mathfrak{K} \geqslant \mathfrak{L}\}$ the set of all completely positive definite kernels *dominated* by $\mathfrak{K}$.

3.3.2. Lemma. *Let $\mathcal{A}$ be a unital C^*-algebra, let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space G, and let $\mathfrak{K} \geqslant \mathfrak{L}$ be kernels in $\mathcal{K}_S(\mathcal{A}, \mathcal{B})$. Let (E, i) denote the Kolmogorov decomposition for $\mathfrak{K}$. Then there exists a unique positive contraction $w \in \mathscr{B}^{\mathrm{a,bil}}(\bar{E}^s)$ such that $\mathfrak{L}^{\sigma,\sigma'}(a) = \langle i(\sigma), wai(\sigma') \rangle$.*

Proof. Let (F, j) denote the Kolmogorov decomposition for $\mathfrak{L}$. As $\mathfrak{K} - \mathfrak{L}$ is completely positive, the mapping $v : i(\sigma) \mapsto j(\sigma)$ extends to an $\mathcal{A}$–$\mathcal{B}$-linear contraction

$E \to F$. Indeed, for $x = \sum_k a_k i(\sigma_k) b_k$ we find

$$\langle x, x \rangle - \langle vx, vx \rangle = \sum_{k,\ell} b_k^* (\mathfrak{K}^{\sigma_k,\sigma_\ell} - \mathfrak{L}^{\sigma_k,\sigma_\ell})(a_k^* a_\ell) b_\ell \geqslant 0,$$

such that $\|x\| \geqslant \|vx\|$. Of course, v extends further to a contraction $\bar{E}^{\mathrm{s}} \to \bar{F}^{\mathrm{s}}$. Since von Neumann modules are self-dual, v has an adjoint $v^* \in \mathscr{B}^{\mathrm{a}}(\bar{F}^{\mathrm{s}}, \bar{E}^{\mathrm{s}})$. Since adjoints of bilinear mappings and compositions among them are bilinear, too, it follows that also $w = v^* v$ is bilinear. Of course, $\langle i(\sigma), wai(\sigma') \rangle = \langle i(\sigma), v^* vai(\sigma') \rangle = \langle j(\sigma), aj(\sigma') \rangle = \mathfrak{L}^{\sigma,\sigma'}(a)$. □

3.3.3. Theorem. *Let S be a set, let $\mathcal{A}$ be a unital C^*-algebra, let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space G, and let $\mathfrak{K}$ be a kernel in $\mathcal{K}_S(\mathcal{A}, \mathcal{B})$. Denote by (E, i) the Kolmogorov decomposition of $\mathfrak{K}$. Then the mapping $\mathfrak{O} : w \mapsto \mathfrak{L}_w$ with*

$$\mathfrak{L}_w^{\sigma,\sigma'}(a) = \langle i(\sigma), wai(\sigma') \rangle$$

establishes an order isomorphism from the positive part of the unit ball in $\mathscr{B}^{\mathrm{a,bil}}(\bar{E}^{\mathrm{s}})$ onto $\mathcal{D}_{\mathfrak{K}}$.

Moreover, if (F, j) is another pair such that $\mathfrak{K}^{\sigma,\sigma'}(a) = \langle j(\sigma), aj(\sigma') \rangle$, then $\mathfrak{O}$ is still a surjective order homomorphism. It is injective, if and only if (F, j) is (unitarily equivalent to) the Kolmogorov decomposition of $\mathfrak{K}$.

Proof. Let us start with the more general (F, j). Clearly, $\mathfrak{O}$ is order preserving. As $E \subset F$ and $\mathscr{B}^{\mathrm{a}}(\bar{E}^{\mathrm{s}}) = p\mathscr{B}^{\mathrm{a}}(\bar{F}^{\mathrm{s}})p \subset \mathscr{B}^{\mathrm{a}}(\bar{F}^{\mathrm{s}})$ where p is the projection onto $\bar{E}^{\mathrm{s}}$, Lemma 3.3.2 tells us that $\mathfrak{O}$ is surjective. If p is non-trivial, then $\mathfrak{O}$ is certainly not injective, because $\mathfrak{L}_p = \mathfrak{L}_{\mathbf{1}}$. Otherwise, it is injective, because the elements $j(\sigma)$ are strongly total, hence, separate the elements of $\mathscr{B}^{\mathrm{a}}(\bar{F}^{\mathrm{s}})$. It remains to show that in the latter case also the inverse $\mathfrak{O}^{-1}$ is order preserving. But this follows from Lemma 2.1.9. □

3.3.4. Remark. By restriction to completely positive mappings (i.e. $\#S = 1$) we obtain Paschke's result [Pas73]. Passing to $\mathcal{B} = \mathscr{B}(G)$ and doing the Stinespring construction, we find Arveson's result [Arv69].

3.4. Schur product and semigroups of kernels

Now we come to products, or better, compositions of kernels. The following definition generalizes the Schur product of a matrix of mappings and a matrix as discussed in Section 2.2.

3.4.1. Definition. Let $\mathfrak{K} \in \mathcal{K}_S(\mathcal{A}, \mathcal{B})$ and let $\mathfrak{L} \in \mathcal{K}_S(\mathcal{B}, \mathcal{C})$. Then the *Schur product* of $\mathfrak{L}$ and $\mathfrak{K}$ is the kernel $\mathfrak{L} \circ \mathfrak{K} \in \mathcal{K}_S(\mathcal{A}, \mathcal{C})$, defined by setting $(\mathfrak{L} \circ \mathfrak{K})^{\sigma,\sigma'}(a) = \mathfrak{L}^{\sigma,\sigma'} \circ \mathfrak{K}^{\sigma,\sigma'}(a)$.

3.4.2. Theorem. $\mathfrak{L}\circ\mathfrak{K}$ is completely positive definite, too.

Proof. If all algebras are unital, then this follows directly from Theorem 3.2.3. Indeed, by the forward direction of Theorem 3.2.3 we have the Kolmogorov decompositions (E, i) and (F, j) for $\mathfrak{K}$ and $\mathfrak{L}$, respectively. Like in Section 2.2 we find $\mathfrak{L}^{\sigma,\sigma'}\circ\mathfrak{K}^{\sigma,\sigma'}(a) = \langle i(\sigma)\odot j(\sigma), ai(\sigma')\odot j(\sigma')\rangle$ from which $(\mathfrak{L}\circ\mathfrak{K})^{\sigma,\sigma'}$ is completely positive definite by the backward direction of Theorem 3.2.3. If the algebras are not necessarily unital, then (as in the proof of 2 ⇒ 4 in Lemma 3.2.1) we may apply the same argument, replacing $i(\sigma)$ by $\hat{i}(u_\lambda, \sigma, v_\mu)$ (and similarly for j) and approximating in this way $\mathfrak{L}\circ\mathfrak{K}$ by completely positive definite kernels. □

3.4.3. Observation. The proof shows that, like the GNS-construction of completely positive mappings, the Kolmogorov decomposition of the composition $\mathfrak{L}\circ\mathfrak{K}$ can be obtained from those for $\mathfrak{K}$ and $\mathfrak{L}$. More precisely, we obtain it as the two-sided submodule of $E\odot F$ generated by $\{i(\sigma)\odot j(\sigma) : \sigma\in S\}$ and the embedding $i\odot j$: $\sigma\mapsto i(\sigma)\odot j(\sigma)$.

3.4.4. Definition. A family $(\mathfrak{T}_t)_{t\in\mathbb{R}_+}$ of kernels on S from $\mathcal{B}$ to $\mathcal{B}$ is called a (*uniformly continuous*) *Schur semigroup* of kernels, if for all $\sigma, \sigma'\in S$ the mappings $\mathfrak{T}_t^{\sigma,\sigma'}$ form a (uniformly continuous) semigroup on $\mathcal{B}$; see Definition C.1. A (*uniformly continuous*) *CPD-semigroup* of kernels, is a (uniformly continuous) Schur semigroup of completely positive definite kernels.

Like for CP-semigroups, the generators of (uniformly continuous) CPD-semigroups can be characterized by a *conditional* positivity condition.

3.4.5. Definition. A kernel $\mathfrak{L}$ on S from $\mathcal{B}$ to $\mathcal{B}$ is called *conditionally completely positive definite*, if

$$\sum_{i,j} b_i^*\mathfrak{L}^{\sigma_i,\sigma_j}(a_i^*a_j)b_j \geqslant 0 \tag{3.4.1}$$

for all choices of finitely many $\sigma_i\in S$, $a_i, b_i\in\mathcal{B}$ such that $\sum_i a_i b_i = 0$.

3.4.6. Lemma. *For a kernel $\mathfrak{L}$ on S from $\mathcal{B}$ to $\mathcal{B}$ the following conditions are equivalent*:

1. *$\mathfrak{L}$ is conditionally completely positive definite.*
2. *For all choices $\sigma_1, \ldots, \sigma_n\in S$ $(n\in\mathbb{N})$ the mapping*

$$\mathfrak{L}^{(n)} : (a_{ij})\mapsto(\mathfrak{L}^{\sigma_i,\sigma_j}(a_{ij}))$$

on $M_n(\mathcal{B})$ is conditionally completely positive, i.e. for all $A^k, B^k\in M_n(\mathcal{B})$ such that $\sum_k A^k B^k = 0$ we have $\sum_{k,\ell} B^{k*}\mathfrak{L}^{(n)}(A^{k*}A^\ell)B^\ell \geqslant 0$.

Proof. By Lemma 2.1.9 an element $(b_{ij}) \in M_n(\mathcal{B})$ is positive, if and only if $\sum_{i,j} b_i^* b_{ij} b_j \geqslant 0$ for all $b_1, \ldots, b_n \in \mathcal{B}$. Therefore, Condition 2 is equivalent to

$$\sum_{i,j,p,q,k,\ell,r} b_i^* b_{pi}^{k*} \mathfrak{L}^{\sigma_p,\sigma_q}(a_{rp}^{k*} a_{rq}^{\ell}) b_{qj}^{\ell} b_j \geqslant 0$$

for all $\sigma_1, \ldots, \sigma_n \in S$, $b_1, \ldots, b_n \in \mathcal{B}$ $(n \in \mathbb{N})$, and finitely many $(a_{ij}^k) \in M_n(\mathcal{A})$, $(b_{ij}^k) \in M_n(\mathcal{B})$ such that $\sum_{p,k} a_{ip}^k b_{pj}^k = 0$ for all i,j. Assume that 1 is true, choose $b_i \in \mathcal{B}$, and choose $a_{rp}^k, b_{pi}^k \in \mathcal{B}$ such that $\sum_{p,k} a_{rp}^k b_{pi}^k = 0$ for all r, i. Then $\sum_{p,k} a_{rp}^k (\sum_i b_{pi}^k b_i) = 0$ for all r and 1 implies that $\sum_{i,j,p,q,k,\ell} b_i^* b_{pi}^{k*} \mathfrak{L}^{\sigma_p,\sigma_q}(a_{rp}^{k*} a_{rq}^{\ell}) b_{qj}^{\ell} b_j \geqslant 0$ for each r separately. (Formally, we pass to indices (p,k) and set $\sigma_{(p,k)} = \sigma_p$ as in the proof of Lemma 3.2.1.) Summing over r we find 2.

Conversely, assume that 2 is true and choose $a_i, b_i \in \mathcal{B}$ such that $\sum_i a_i b_i = 0$. Set $a_{rp} = \delta_{1r} a_p$ and $b_{pi} = b_p$. Then $\sum_p a_{rp} b_{pi} = \delta_{1r} \sum_p a_p b_p = 0$ for all r, i and 2 implies that the matrix $(\sum_{p,q,r} b_{pi}^* \mathfrak{L}^{\sigma_p,\sigma_q}(a_{rp}^* a_{rq}) b_{qj})_{i,j} = (\sum_{p,q} b_p^* \mathfrak{L}^{\sigma_p,\sigma_q}(a_p^* a_q) b_q)_{i,j}$ is positive. As any of the (equal) diagonal entries $\sum_{p,q} b_p^* \mathfrak{L}^{\sigma_p,\sigma_q}(a_p^* a_q) b_q$ must be positive in $\mathcal{B}$, we find 1. □

3.4.7. Theorem. *Let $\mathcal{B}$ be a unital C^*-algebra and let S be a set. Then the formula*

$$\mathfrak{T}_t = e^{t\mathfrak{L}} \tag{3.4.2}$$

(*where the exponential is that for the Schur product of kernels*) *establishes a one-to-one correspondence between uniformly continuous CPD-semigroups $(\mathfrak{T}_t)_{t \in \mathbb{R}_+}$ of positive definite kernels $\mathfrak{L}$ on S from $\mathcal{B}$ to $\mathcal{B}$ and hermitian* (*see Corollary* 3.2.4) *conditionally completely positive definite kernels on S from $\mathcal{B}$ to $\mathcal{B}$.*

Proof. First of all, let us remark that (3.4.2) establishes a one-to-one correspondence between uniformly continuous Schur semigroups and kernels $\mathfrak{L} : S \times S \to \mathscr{B}(\mathcal{B})$. This follows simply by the same statement for the uniformly continuous semigroups $\mathfrak{T}_t^{\sigma,\sigma'}$ and their generators $\mathfrak{L}^{\sigma,\sigma'}$. So the only problem we have to deal with is positivity.

Let $\mathfrak{T}$ by a CPD-semigroup. By Lemma 3.2.1(4) this is equivalent to complete positivity of the semigroup $\mathfrak{T}_t^{(n)}$ on $M_n(\mathcal{B})$ for each choice of $\sigma_1, \ldots, \sigma_n \in S$ $(n \in \mathbb{N})$. So let us choose $A^k, B^k \in M_n(\mathcal{B})$ such that $\sum_k A^k B^k = 0$. Then

$$\sum_{k,\ell} B^{k*} \mathfrak{L}^{(n)}(A^{k*} A^{\ell}) B^{\ell} = \lim_{t \to 0} \frac{1}{t} \sum_{k,\ell} B^{k*} \mathfrak{T}_t^{(n)}(A^{k*} A^{\ell}) B^{\ell} \geqslant 0.$$

In other words, $\mathfrak{L}^{(n)}$ is conditionally completely positive and by Lemma 3.4.6(2) $\mathfrak{L}$ is conditionally completely positive definite. As limit of hermitian kernels, also $\mathfrak{L}$ must be hermitian.

Conversely, let $\mathfrak{L}$ be hermitian and conditionally completely positive definite, so that $\mathfrak{L}^{(n)}$ is hermitian conditionally completely positive for each choice of $\sigma_1, \ldots, \sigma_n \in S$ $(n \in \mathbb{N})$. We follow Evans and Lewis [EL77, Theorem 14.2 (3 ⇒ 1)] to show that $\mathfrak{T}_t^{(n)}$ is positive, which by Lemma 3.2.1(5) implies that $\mathfrak{T}_t$ is completely positive definite.

Let $A \geqslant 0$ and B in $M_n(\mathcal{B})$ such that $AB = 0$. Then also $\sqrt{A}B = 0$, hence $B^* \mathfrak{L}^{(n)}(A)B \geqslant 0$, because $\mathfrak{L}^{(n)}$ is conditionally completely positive. Let $0 \leqslant \varepsilon < \|\mathfrak{L}^{(n)}\|^{-1}$, hence $\mathsf{id} - \varepsilon\mathfrak{L}^{(n)}$ is invertible. Now let $A = A^*$ be an arbitrary self-adjoint element in $M_n(\mathcal{B})$. We show that $A \geqslant 0$ whenever $(\mathsf{id} - \varepsilon\mathfrak{L}^{(n)})(A) \geqslant 0$, which establishes the hermitian mapping $(\mathsf{id} - \varepsilon\mathfrak{L}^{(n)})^{-1}$ as positive. We write $A = A_+ - A_-$ where A_+, A_- are unique positive elements fulfilling $A_+A_- = 0$. Therefore, $A_-\mathfrak{L}^{(n)}(A_+)A_- \geqslant 0$. Indeed,

$$0 \leqslant A_-(\mathsf{id} - \varepsilon\mathfrak{L}^{(n)})(A)A_- = A_-(\mathsf{id} - \varepsilon\mathfrak{L}^{(n)})(A_+)A_- - A_-(\mathsf{id} - \varepsilon\mathfrak{L}^{(n)})(A_-)A_-$$

$$= -\varepsilon A_-\mathfrak{L}^{(n)}(A_+)A_- - A_-^3 + \varepsilon A_-\mathfrak{L}^{(n)}(A_-)A_-,$$

hence

$$A_-^3 \leqslant A_-^3 + \varepsilon A_-\mathfrak{L}^{(n)}(A_+)A_- \leqslant \varepsilon A_-\mathfrak{L}^{(n)}(A_-)A_-.$$

If $A_- \neq 0$, then $\|A_-\|^3 = \|A_-^3\| \leqslant \|\varepsilon A_-\mathfrak{L}^{(n)}(A_-)A_-\| \leqslant \varepsilon\|\mathfrak{L}^{(n)}\| \, \|A_-\|^3 < \|A_-\|^3$, a contradiction, hence $A_- = 0$. We have $\mathfrak{T}_t^{(n)} = \lim_{m\to\infty}(\mathbf{1} - \frac{t}{m}\mathfrak{L}^{(n)})^{-m}$ which is positive as limit of compositions of positive mappings. □

By appropriate applications of Lemmata 3.2.1 and 3.4.6 to a kernel on a one-element set S, we find the following well-known result.

3.4.8. Corollary. *The formula $T_t = e^{t\mathcal{L}}$ establishes a one-to-one correspondence between uniformly continuous CP-semigroups on $\mathcal{B}$ (i.e. semigroups of completely positive mappings on $\mathcal{B}$) and hermitian conditionally completely positive mappings $\mathcal{L} \in \mathscr{B}(\mathcal{B})$.*

3.4.9. Observation. A CP-semigroup on a von Neumann algebra is normal, if and only if its generator is σ-weak. (This follows from the observation that norm limits of σ-weak mappings are σ-weak.)

We find a simple consequence, by applying this argument to the CP-semigroups $\mathfrak{T}_t^{(n)}$.

3.4.10. Corollary. *A CPD-semigroup $\mathfrak{T}$ on a von Neumann algebra is normal (i.e. each mapping $\mathfrak{T}_t^{\sigma,\sigma'}$ is σ-weak), if and only if its generator $\mathfrak{L}$ is σ-weak.*

3.4.11. Remark. It is easily possible to show first Corollary 3.4.8 as in [EL77], and then apply it to $\mathfrak{T}_t^{(n)} = e^{t\mathfrak{L}^{(n)}}$ to show the statement for CPD-semigroups. Notice, however, that also in [EL77] in order to show Corollary 3.4.8, it is necessary to know at least parts of Lemma 3.2.1 in a special case.

We say a CPD-semigroup $\mathfrak{T}$ *dominates* another $\mathfrak{T}'$ (denoted by $\mathfrak{T} \geqslant \mathfrak{T}'$), if $\mathfrak{T}_t \geqslant \mathfrak{T}'_t$ for all $t \in \mathbb{T}$. The following lemma reduces the analysis of the order structure of uniformly continuous CPD-semigroups to that of the order structure of their generators.

3.4.12. Lemma. *Let $\mathfrak{T}$ and $\mathfrak{T}'$ be uniformly continuous CPD-semigroups on S in $\mathcal{K}_S(\mathcal{B})$ with generators $\mathfrak{L}$ and $\mathfrak{L}'$, respectively. Then $\mathfrak{T} \geqslant \mathfrak{T}'$, if and only if $\mathfrak{L} \geqslant \mathfrak{L}'$.*

Proof. Since $\mathfrak{T}_0 = \mathfrak{T}'_0$, we have $\frac{\mathfrak{T}_t - \mathfrak{T}'_t}{t} = \frac{\mathfrak{T}_t - \mathfrak{T}_0}{t} - \frac{\mathfrak{T}'_t - \mathfrak{T}'_0}{t} \to \mathfrak{L} - \mathfrak{L}'$ for $t \to 0$ so that $\mathfrak{T} \geqslant \mathfrak{T}'$ certainly implies $\mathfrak{L} \geqslant \mathfrak{L}'$. Conversely, assume that $\mathfrak{L} \geqslant \mathfrak{L}'$. Choose $n \in \mathbb{N}$ and $\sigma_i \in S$ $(i = 1, \ldots, n)$. From the proof of Theorem 3.4.7 we know that $(\mathbf{1} - \varepsilon\mathfrak{L}^{(n)})^{-1} \geqslant 0$ and $(\mathbf{1} - \varepsilon\mathfrak{L}'^{(n)})^{-1} \geqslant 0$ for all sufficiently small $\varepsilon > 0$. Moreover, by Theorem 3.4.2

$$(\mathbf{1} - \varepsilon\mathfrak{L}^{(n)})^{-1} - (\mathbf{1} - \varepsilon\mathfrak{L}'^{(n)})^{-1} = \varepsilon(\mathbf{1} - \varepsilon\mathfrak{L}^{(n)})^{-1}(\mathfrak{L}^{(n)} - \mathfrak{L}'^{(n)})(\mathbf{1} - \varepsilon\mathfrak{L}'^{(n)})^{-1} \geqslant 0,$$

because all three factors are $\geqslant 0$. This implies $(\mathbf{1} - \frac{t}{m}\mathfrak{L}^{(n)})^{-m} - (\mathbf{1} - \frac{t}{m}\mathfrak{L}'^{(n)})^{-m} \geqslant 0$ for m sufficiently big. Letting $m \to \infty$, we find $\mathfrak{T}_t^{(n)} \geqslant \mathfrak{T}_t'^{(n)}$ and further $\mathfrak{T} \geqslant \mathfrak{T}'$ by Lemma 3.2.1(4). $\square$

3.5. *The CPD-semigroup of the time ordered Fock module and its generator*

Let $\mathcal{B}$ be a unital C^*-algebra, let ζ be an element in a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module F, and let $\beta \in \mathcal{B}$. Then

$$\mathcal{L}(b) = \langle \zeta, b\zeta \rangle + b\beta + \beta^* b \tag{3.5.1}$$

is obviously conditionally completely positive and hermitian so that $T_t = e^{t\mathcal{L}}$ is a uniformly continuous CP-semigroup. We say the generator of T has *Christensen–Evans form* (or is a *CE-generator*). Theorem C.4 by Christensen and Evans [CE79] asserts that generators $\mathcal{L}$ of normal CP-semigroups T on a von Neumann algebra $\mathcal{B}$ always have the form (3.5.1) where F is some von Neumann $\mathcal{B}$–$\mathcal{B}$-module.

In this section we study the CPD-semigroup associated with the time ordered Fock module. From the form of its generator we conjecture the correct generalization of the CE-form of a generator from CP-semigroups to CPD-semigroups, and we state as Theorem 3.5.2 that the generators of normal uniformly continuous CPD-semigroups always have that form. It is one of the main goals in the remainder of these notes to proof Theorem 3.5.2, but we will not achieve this before Section 5.4.

For us it will be extremely important that F can be chosen in a minimal way, as it follows from Lemma C.2 (and its Corollary C.3 which asserts that bounded derivations with values in von Neumann modules are inner). Therefore, we consider Lemma C.2 rather than Theorem C.4 (which is a corollary of Lemma C.2) as the main result of [CE79]. The results in [CE79] are stated for (even non-unital) C^*-algebras $\mathcal{B}$. However, the proof runs (more or less) by embedding $\mathcal{B}$ into the bidual von Neumann algebra $\mathcal{B}^{**}$. Hence, the inner product on F takes values in $\mathcal{B}^{**}$ and also $\beta \in \mathcal{B}^{**}$. Only the combinations in (3.5.1) remain in $\mathcal{B}$. As this causes unpleasant complications in formulations of statements, usually, we restrict to the case of von Neumann algebras.

Now we use the set $\mathcal{U}_c(F)$ of continuous units for the time ordered Fock module $\mathbb{I}\Gamma^{\odot}(F)$ over a Hilbert $\mathcal{B}$–$\mathcal{B}$-module F to define its associated CPD-semigroup. Theorem 2.3.11 tells us that $\mathcal{U}_c(F)$ can be parametrized by the set $\mathcal{B} \times F$. (In Section 5.2 we will also sometimes use the natural vector space structure of $\mathcal{B} \times F$.)

Let

$$\mathbb{I}\Gamma_t^{\mathcal{U}_c}(F) = \operatorname{span}\{b_n \xi_{t_n}(\beta_n, \zeta_n) \odot \cdots \odot b_1 \xi_{t_1}(\beta_1, \zeta_1) b_0 | \\ \mathfrak{t} \in \mathbb{J}_t; b_0, \ldots, b_n, \beta_1, \ldots, \beta_n \in \mathcal{B}; \zeta_1, \ldots, \zeta_n \in F\}.$$

Then $\mathbb{I}\Gamma_s^{\mathcal{U}_c}(F) \odot \mathbb{I}\Gamma_t^{\mathcal{U}_c}(F) = \mathbb{I}\Gamma_{s+t}^{\mathcal{U}_c}(F)$ by restriction of u_{st} in Theorem 2.3.6. (Cf. also Proposition 4.2.6.)

Let $\xi^{\odot}, \xi'^{\odot}$ be two units. Obviously, also the mappings $b \mapsto \langle \xi_t, b\xi'_t \rangle$ form a semigroup on $\mathcal{B}$ (of course, in general not CP; cf. again Proposition 4.2.5). If ξ_t, ξ'_t are continuous, then so is the semigroup. Another way to say this is that the kernels

$$\mathfrak{T}_t : \mathcal{U}_c(F) \times \mathcal{U}_c(F) \to \mathfrak{T}_t^{(\beta,\zeta),(\beta',\zeta')} = \langle \xi_t(\beta, \zeta), \bullet \xi_t(\beta', \zeta') \rangle$$

form a uniformly continuous CPD-semigroup $\mathfrak{T}$ of kernels on $\mathcal{U}_c(F)$ from $\mathcal{B}$ to $\mathcal{B}$. Similar to the proof of (2.3.5) (see [LS01]) one may show that the generator $\mathfrak{L}$ of $\mathfrak{T}$ is given by

$$\mathfrak{L}^{(\beta,\zeta),(\beta',\zeta')}(b) = \langle \zeta, b\zeta' \rangle + b\beta' + \beta^* b. \tag{3.5.2}$$

By Theorem 3.4.7 $\mathfrak{L}$ is a conditionally completely positive definite kernel. Of course, it is an easy exercise to check this directly.

Now it is clear how to define the analogue of the CE-generator for CPD-semigroups on some set S. Let $\mathcal{B}$ be a unital C^*-algebra, let ζ_σ $(\sigma \in S)$ be elements in a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module F, and let $\beta_\sigma \in \mathcal{B}$ $(\sigma \in S)$. Then the kernel $\mathfrak{L}$ on S defined, by setting

$$\mathfrak{L}^{\sigma,\sigma'}(b) = \langle \zeta_\sigma, b\zeta_{\sigma'} \rangle + b\beta_{\sigma'} + \beta_\sigma^* b \tag{3.5.3}$$

is conditionally completely positive definite and hermitian. (The first summand is completely positive definite. Each of the remaining summands is conditionally completely positive definite, but the sum cannot be arbitrary, because $\mathfrak{L}$ should be hermitian.)

3.5.1. Definition. A generator $\mathfrak{L}$ of a uniformly continuous CPD-semigroup has *Christensen–Evans form* (or is a *CE-generator*), if it can be written in the form (3.5.3).

3.5.2. Theorem. *Let $\mathfrak{T}$ be a normal uniformly continuous CPD-semigroup on S on a von Neumann algebra $\mathcal{B}$ with generator $\mathfrak{L}$. Then there exist a von Neumann $\mathcal{B}$–$\mathcal{B}$-module F with elements $\zeta_\sigma \in F$ $(\sigma \in S)$, and elements $\beta_\sigma \in \mathcal{B}$ $(\sigma \in S)$ such that $\mathfrak{L}$ has the Christensen–Evans form in* (3.5.3). *Moreover, the strongly closed submodule of F generated by the elements $b\zeta_\sigma - \zeta_{\sigma'}b$ $(b \in \mathcal{B}; \sigma, \sigma' \in S)$ is determined by $\mathfrak{L}$ up to (two-sided) isomorphism.*

We prove this theorem (and semigroup versions of other theorems like Theorem 3.3.3) in Section 5 (after Theorem 5.4.1) with the help of product systems. A direct generalization of the methods of [CE79] as explained in Appendix C fails, however. This is mainly due to the following fact.

3.5.3. Observation. Although the von Neumann module F is determined uniquely by the cyclicity condition in Theorem 3.5.2, the concrete choice neither of ζ_σ nor of β_σ is unique. This makes it impossible to extend what the results from [CE79] assert for each $\mathfrak{T}^{(n)}$ $(\sigma_1, \dots, \sigma_n \in S)$ by an inductive limit over finite subsets of S to $\mathfrak{T}$.

We close with some totality results about the units in $\mathcal{U}_c(F)$. Theorem 2.3.9 tells us that the tensor products

$$\xi_{t_n}(0, \zeta_n) \odot \cdots \odot \xi_{t_1}(0, \zeta_1) \tag{3.5.4}$$

$(t_1 + \cdots + t_n = t)$ form a total subset of $\mathrm{I\!\Gamma}_t(F)$. Therefore, the closed linear span of such vectors contains also the units $\xi^\odot(\beta, \zeta)$. But, we can specify the approximation much better.

3.5.4. Lemma. *Let $\xi^\odot(\beta, \xi), \xi^\odot(\beta', \xi')$ be two continuous units.*

1. *For all $\varkappa, \varkappa' \in [0, 1]$, $\varkappa + \varkappa' = 1$ we have*

$$\lim_{n \to \infty} \left(\xi_{\frac{\varkappa t}{n}}(\beta, \zeta) \odot \xi_{\frac{\varkappa' t}{n}}(\beta', \zeta')\right)^{\odot n} = \xi_t(\varkappa\beta + \varkappa'\beta', \varkappa\zeta + \varkappa'\zeta')$$

   in the $\mathcal{B}$-weak topology.

2. *For all $b \in \mathcal{B}$ we have*

$$\lim_{n \to \infty} \left(e^{b\frac{t}{n}} \xi_{\frac{t}{n}}(\beta, \zeta)\right)^{\odot n} = \lim_{n \to \infty} \left(\xi_{\frac{t}{n}}(\beta, \zeta) e^{b\frac{t}{n}}\right)^{\odot n} = \xi_t(\beta + b, \zeta)$$

   in norm.

3. *For all* $\varkappa, \varkappa' \in \mathbb{C}$, $\varkappa + \varkappa = 1$ *we have*

$$\lim_{n \to \infty} (\varkappa \xi_{\frac{t}{n}}(\beta, \zeta) + \varkappa' \xi_{\frac{t}{n}}(\beta, \zeta))^{\odot n} = \xi_t(\varkappa\beta + \varkappa'\beta', \varkappa\zeta + \varkappa'\zeta')$$

in norm.

Part 1 is a generalization from an observation in [Arv89]. Part 2 is trivial in the case $\mathcal{B} = \mathbb{C}$. We used it first together with Part 1 in Skeide [Ske01b] for $\mathcal{B} = \mathbb{C}^2$. Both may be considered as a direct consequence of the Trotter product formula; see [Ske01a] for a detailed argument. Part 3 is the straightforward generalization of an observation by Liebscher [Lie03].

3.5.5. Theorem. *Let S be a total subset of F containing* 0. *Then exponential vectors to S-valued step functions are total in* $\mathbb{I}\Gamma(F)$.

Proof. It is sufficient to show the statement for $\mathbb{I}\Gamma_t(F)$ for some fixed t. By Lemma 3.5.4(3) the closure of the span of exponentials to S-valued step functions contains the exponentials to step functions with values in the affine hull of S (i.e. all linear combinations $\sum_i \varkappa_i \zeta_i$ from S with $\sum_i \varkappa_i = 1$). Since $0 \in S$ the affine hull coincides with the span of S which is dense in F. Now the statement follows, because the units depend continuously on their parameters and from totality of (3.5.4). □

We find the following result on the exponential vectors of $\Gamma(L^2(\mathbb{R}_+))$ $(= \mathbb{I}\Gamma(\mathbb{C}))$. It was obtained first by Parthasarathy and Sunder [PS98] and later by [Bha01]. The proof in [Ske00b] arises by restricting the methods in this section to the bare essentials of the special case $\mathcal{B} = \mathbb{C}$ and fits into half a page.

3.5.6. Corollary. *Exponential vectors to indicator functions of finite unions of intervals are total in* $\mathbb{I}\Gamma(\mathbb{C}) = \Gamma(L^2(\mathbb{R}_+))$.

Proof. The set $S = \{0, 1\}$ is total in $\mathbb{C}$ and contains 0. □

In accordance with Definition 4.2.7 we may say that the set $\xi^{\odot}(0, S)$ of units is generating. Recall, however, that generating is a weaker property. Lemma 3.5.4(2) asserts, for instance, that what a single unit $\xi^{\odot}(\beta, \zeta)$ generates via expressions as in (4.2.3), contains the units $\xi^{\odot}(\beta + b, \zeta)$ for all $b \in \mathcal{B}$, in particular, the unit $\xi^{\odot}(0, \zeta)$.

3.5.7. Corollary. *Let S be a total subset of F containing* 0 *and for each $\zeta \in S$ choose $\beta_\zeta \in \mathcal{B}$. Then the set $\{\xi^{\odot}(\beta_\zeta, \zeta) : \zeta \in S\}$ is generating for* $\mathbb{I}\Gamma(F)$.

4. Tensor product systems of Hilbert modules

4.1. Definition and basic examples

4.1.1. Definition. Let $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{T} = \mathbb{N}_0$, and let $\mathcal{B}$ be a unital C^*-algebra. A *tensor product system of pre-Hilbert modules*, or for short a *product system*, is a family

$E^{\odot} = (E_t)_{t \in \mathbb{T}}$ of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-modules E_t with a family of two-sided unitaries $u_{st} : E_s \odot E_t \to E_{s+t}$ $(s, t \in \mathbb{T})$, fulfilling the associativity condition

$$u_{r(s+t)}(\mathsf{id} \odot u_{st}) = u_{(r+s)t}(u_{rs} \odot \mathsf{id}), \tag{4.1.1}$$

where $E_0 = \mathcal{B}$ and u_{s0}, u_{0t} are the identifications as in Definition 2.1.2. Once, the choice of u_{st} is fixed, we always use the identification

$$E_s \odot E_t = E_{s+t}. \tag{4.1.2}$$

We speak of tensor *product systems* of *Hilbert modules* $E^{\bar{\odot}}$ and of *von Neumann modules* $E^{\bar{\odot}^s}$, if $E_s \bar{\odot} E_t = E_{s+t}$ and $E_s \bar{\odot}^s E_t = E_{s+t}$, respectively.

A *morphism* of product systems $E^{\odot}$ and $F^{\odot}$ is a family $w^{\odot} = (w_t)_{t \in \mathbb{T}}$ of mappings $w_t \in \mathscr{B}^{\mathrm{a,bil}}(E_t, F_t)$, fulfilling

$$w_{s+t} = w_s \odot w_t \tag{4.1.3}$$

and $w_0 = \mathsf{id}_{\mathcal{B}}$. A morphism is *unitary*, *contractive*, etc., if w_t is for every $t \in \mathbb{T}$. An *isomorphism* of product systems is a unitary morphism.

A *product subsystem* is a family $E'^{\odot} = (E'_t)_{t \in \mathbb{T}}$ of $\mathcal{B}$–$\mathcal{B}$-submodules E'_t of E_t such that $E'_s \odot E'_t = E'_{s+t}$ by restriction of identification (4.1.2).

By the *trivial* product system we mean $(\mathcal{B})_{t \in \mathbb{T}}$ where $\mathcal{B}$ is equipped with its trivial $\mathcal{B}$–$\mathcal{B}$-module structure.

4.1.2. Observation. Notice that, in general, there need not exist a projection endomorphism of $E^{\odot}$ onto a subsystem $E'^{\odot}$ of $E^{\odot}$. If, however, each projection $p_t \in \mathscr{B}^{\mathrm{a}}(E_t)$ onto E'_t exists (hence, the p_t are two-sided), then the p_t form an endomorphism. Conversely, any projection endomorphism $p^{\odot}$ determines a product subsystem $E'_t = p_t E_t$. Therefore, in product systems of von Neumann modules there is a one-to-one correspondence between subsystems and projection endomorphisms.

4.1.3. Example. Let F be a (pre-)Hilbert $\mathcal{B}$–$\mathcal{B}$-module. By Theorem 2.3.6 the time ordered Fock modules $\underline{\mathbb{I}\Gamma}_t(F)$ form a product system of pre-Hilbert modules. We call $\underline{\mathbb{I}\Gamma}^{\odot}(F) = (\underline{\mathbb{I}\Gamma}_t(F))_{t \in \mathbb{T}}$ the product system (of pre-Hilbert modules) *associated* with the time ordered Fock module $\underline{\mathbb{I}\Gamma}(F)$. We use similar notations for $\mathbb{I}\Gamma(F)$ and $\mathbb{I}\Gamma^{s}(F)$. More generally, we speak of a *time ordered product system* $E^{\odot}$ (of Hilbert modules $E^{\bar{\odot}}$, of von Neumann modules $E^{\bar{\odot}^s}$), if $E^{\odot}$, $(E^{\bar{\odot}}, E^{\bar{\odot}^s})$ is isomorphic to $\underline{\mathbb{I}\Gamma}^{\odot}(F)$ (to $\mathbb{I}\Gamma^{\odot}(F)$, to $\mathbb{I}\Gamma^{s\odot}(F)$).

Let $\lambda > 0$. Then $[\mathcal{T}^{\lambda}_t f](s) = \sqrt{\lambda} f(\lambda s)$ $(s \in [0, \frac{t}{\lambda}])$ defines a two-sided isomorphism $L^2([0,t]) \to L^2([0, \frac{t}{\lambda}])$. Clearly, the family of second quantizations $\mathcal{F}(\mathcal{T}^{\lambda}_t) \restriction \underline{\mathbb{I}\Gamma}_t(F)$ defines an isomorphism from $\underline{\mathbb{I}\Gamma}^{\odot}(F)$ to the time rescaled product system $(\underline{\mathbb{I}\Gamma}_{\frac{t}{\lambda}}(F))_{t \in \mathbb{T}}$.

4.1.4. Example. Usually, our semigroup is $\mathbb{T} = \mathbb{R}_+$. However, also the case $\mathbb{T} = \mathbb{N}_0$ has interesting applications in the theory of quantum Markov chains. We describe this briefly. With each pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module E we can associate a *discrete* product system $(E^{\odot n})_{n \in \mathbb{N}_0}$. Conversely, any discrete product system $(E_n)_{n \in \mathbb{N}_0}$ can be obtained in that way from E_1.

4.2. Units and CPD-semigroups

4.2.1. Definition. A *unit* for a product system $E^{\odot} = (E_t)_{t \in \mathbb{T}}$ is a family $\xi^{\odot} = (\xi_t)_{t \in \mathbb{T}}$ of elements $\xi_t \in E_t$ such that

$$\xi_s \odot \xi_t = \xi_{s+t} \tag{4.2.1}$$

in identification (4.1.2) and $\xi_0 = \mathbf{1} \in \mathcal{B} = E_0$. By $\mathcal{U}(E^{\odot})$ we denote the set of all units for $E^{\odot}$. A unit $\xi^{\odot}$ is *unital* and *contractive*, if $\langle \xi_t, \xi_t \rangle = \mathbf{1}$ and $\langle \xi_t, \xi_t \rangle \leqslant \mathbf{1}$, respectively. A unit is *central*, if $\xi_t \in C_{\mathcal{B}}(E_t)$ for all $t \in \mathbb{T}$.

4.2.2. Remark. A unit can be *trivial*, i.e. $\xi_t = 0$ for $t > 0$. Of course, this will not occur, as soon as we pose continuity conditions on the unit.

4.2.3. Observation. Obviously, a morphism $w^{\odot}$ sends units to units. For this the requirement $w_0 = \mathsf{id}_{\mathcal{B}}$ is necessary. For a subset $S \subset \mathcal{U}(E^{\odot})$ of units for $E^{\odot}$ we denote by $w^{\odot} S \subset \mathcal{U}(F^{\odot})$ the subset of units for $F^{\odot}$, consisting of the units $w\xi^{\odot} = (w_t \xi_t)_{t \in \mathbb{T}}$ $(\xi^{\odot} \in S)$.

4.2.4. Example. Time ordered product systems have a central unital unit, namely, the vacuum unit. However, there are even simple product systems without any central unital unit.

Let $\mathcal{B} = \mathcal{K}(G) + \mathbb{C}\mathbf{1} \subset \mathcal{B}(G)$ be the unitization of the compact operators on some infinite-dimensional Hilbert space. Let $h \in \mathcal{B}(G)$ be a self-adjoint operator and define the uniformly continuous unital automorphism group $\alpha_t = e^{ith} \bullet e^{-ith}$ on $\mathcal{B}$. It is easy to see that the Hilbert $\mathcal{B}$–$\mathcal{B}$-modules $\mathcal{B}_t$ defined to coincide with $\mathcal{B}$ as right Hilbert modules and with left multiplication $b.x_t = \alpha_t(b)x_t$ form a product system $\mathcal{B}^{\odot}$ via the identification $x_s \odot y_t = \alpha_t(x_s)y_t$. A central element $\xi_t \in \mathcal{B}_t$ should fulfill

$$b.\xi_t = e^{ith} b e^{-ith} \xi_t = \xi_t b \quad \text{or} \quad b e^{-ith} \xi_t = e^{-ith} \xi_t b$$

for all $b \in \mathcal{B}$. In other words, since the center of $\mathcal{B}$ is trivial, $e^{-ith}\xi_t$ is a multiple of the identity so that ξ_t is a multiple of e^{ith}. If the ξ_t are different from 0, then we may normalize such that $\xi_t = e^{ith}$. It follows that $h = -i \frac{d\xi_t}{dt}|_{t=0}$ is an element of $\mathcal{B}$. Conversely, if $h \notin \mathcal{B}$, then $\mathcal{B}^{\odot}$ does not admit a central unital unit. Of course, $\mathcal{B}^{\odot}$ has a unital unit, namely, $\xi_t = \mathbf{1}$.

4.2.5. Proposition. *The family* $\mathfrak{U} = (\mathfrak{U}_t)_{t\in\mathbb{T}}$ *of kernels* $\mathfrak{U}_t$ *on* $\mathscr{U}(E^{\odot})$ *from* $\mathcal{B}$ *to* $\mathcal{B}$, *defined by setting*

$$\mathfrak{U}_t^{\xi,\xi'}(b) = \langle \xi_t, b\xi'_t \rangle$$

is a CPD-semigroup. More generally, the restriction $\mathfrak{U}\upharpoonright S$ *to any subset* $S\subset\mathscr{U}(E^{\odot})$ *is a CPD-semigroup.*

Proof. Completely positive definiteness follows from the second half of Theorem 3.2.3. The semigroup property follows from

$$\mathfrak{U}_{s+t}^{\xi,\xi'}(b) = \langle \xi_{s+t}, b\xi'_{s+t} \rangle = \langle \xi_s \odot \xi_t, b\xi'_s \odot \xi'_t \rangle = \langle \xi_t, \langle \xi_s, b\xi'_s \rangle \xi'_t \rangle = \mathfrak{U}_t^{\xi,\xi'} \circ \mathfrak{U}_s^{\xi,\xi'}(b)$$

and $\langle \xi_0, b\xi'_0 \rangle = b$. $\square$

Observe that here and on similar occasions, where it is clear that the superscripts refer to units, we prefer to write the shorter $\mathfrak{U}^{\xi,\xi'}$ instead of the more correct $\mathfrak{U}^{\xi^{\odot},\xi'^{\odot}}$.

In Section 4.3 we will see that any CPD-semigroup, i.e. in particular, any CP-semigroup, can be recovered in this way from its *GNS-system*. In other words, any CPD-semigroup is obtained from units of a product system. However, the converse need not be true as there are even Arveson systems which are not generated by their units (see [Tsi00]). Nevertheless, the units of a product system *generate* a product subsystem, determined uniquely by $\mathfrak{U}$. In the following proposition we explain this even for subsets $S\subset\mathscr{U}(E^{\odot})$. Although both statements are fairly obvious, we give a detailed proof of the first one, because it gives us immediately the idea of how to construct the product system of a CPD-semigroup.

4.2.6. Proposition. *Let* $E^{\odot}$ *be a product system and let* $S\subset\mathscr{U}(E^{\odot})$. *Then the spaces*

$$E_t^S = \mathsf{span}\{b_n\xi_{t_n}^n \odot \cdots \odot b_1\xi_{t_1}^1 b_0 \mid n\in\mathbb{N},\ b_i\in\mathcal{B}, \xi^i\odot\in S, (t_n, \ldots, t_1)\in\mathbb{J}_t\} \tag{4.2.2}$$

form a product subsystem $E^{S\odot}$ *of* $E^{\odot}$, *the* (*unique*) *subsystem generated by* S.

Moreover, if $E'^{\odot}$ *is another product system with a subset of units set-isomorphic to* S (*and, therefore, identified with* S) *such that* $\mathfrak{U}\upharpoonright S = \mathfrak{U}'\upharpoonright S$, *then* $E'^{S\odot}$ *is isomorphic to* $E^{S\odot}$ (*where the identification of the subset* $S\subset\mathscr{U}(E^{\odot})$ *and* $S\subset\mathscr{U}(E'^{\odot})$ *and extension via* (4.2.2) *gives the isomorphism*).

Proof. The restriction of u_{st} to $E_s^S \odot E_t^S$ in the identification (4.1.2) gives

$$\begin{aligned}&(b_{n+m}\xi_{r_{n+m}}^{n+m} \odot \cdots \odot b_{n+1}\xi_{r_{n+1}}^{n+1} b'_n) \odot (b_n\xi_{r_n}^n \odot \cdots \odot b_1\xi_{r_1}^1 b_0)\\ &\quad= b_{n+m}\xi_{r_{n+m}}^{n+m} \odot \cdots \odot b_{n+1}\xi_{r_{n+1}}^{n+1} \odot b'_n b_n\xi_{r_n}^n \odot \cdots \odot b_1\xi_{r_1}^1 b_0,\end{aligned}$$

where $(r_{n+m}, \ldots, r_{n+1})\in\mathbb{J}_s$ and $(r_n, \ldots, r_1)\in\mathbb{J}_t$. Therefore, $u_{st}(E_s^S \odot E_t^S \subset E_{s+t}^S$. To see surjectivity let $\mathfrak{r} = (r_k, \ldots, r_1)\in\mathbb{J}_{s+t}$ and $b_i\in\mathcal{B}$ $(i = 0, \ldots, k)$, $\xi^i\in S$ $(i = 1, \ldots, k)$. If $\mathfrak{r}$

hits t, i.e. $\mathfrak{r} = \mathfrak{s} \smile \mathfrak{t}$ for some $\mathfrak{s} \in \mathbb{J}_s, \mathfrak{t} \in \mathbb{J}_t$, then clearly

$$b_k \xi^k_{r_k} \odot \cdots \odot b_1 \xi^1_{r_1} b_0 \tag{4.2.3}$$

is in $u_{st}(E^S_s \odot E^S_t)$. If $\mathfrak{r}$ does not hit t, then we may easily achieve this by splitting that $\xi^\ell_{r_\ell}$ with $\sum_{i=1}^{\ell-1} r_i < t < \sum_{i=1}^{\ell} r_i$ into a tensor product of two; cf. Example 2.1.6. More precisely, we write $\xi^\ell_{r_\ell}$ as $\xi^\ell_{r'_2} \odot \xi^\ell_{r'_1}$ such that $r'_1 + r'_2 = r_\ell$ and $r'_1 + \sum_{i=1}^{\ell-1} r_i = t$. Also here we find that (4.2.3) is in $u_{st}(E^S_s \odot E^S_t)$. □

Like for Arveson systems, the question, whether a product system is generated by its units or even some subset of units in the stated way, is crucial for the classification of product systems. However, for Hilbert spaces the property of certain subset to be total, does not depend on the topology, whereas for Hilbert modules we must distinguish clearly between the several possibilities. Furthermore, we can opt to consider only subsets of units distinguished by additional properties like continuity (which, unlike for Arveson systems, again must be split into different topologies).

In our frame work it turns out that it is most convenient—convenient in the sense that the obtained classification results parallel best those for Arveson systems—to look at continuous sets of units. Here we call a single unit $\xi^\odot$ *continuous*, if the CP-semigroup $T^\xi_t = \langle \xi_t, \bullet \xi_t \rangle$ is uniformly continuous. More generally, a set S of units is *continuous*, if the CPD-semigroup $\mathfrak{U} \upharpoonright S$ is uniformly continuous.

4.2.7. Definition. A product system $E^\odot = (E_t)_{t \in \mathbb{T}}$ of pre-Hilbert modules is of *type* $\underline{\mathrm{I}}$, if it is *generated* by some continuous set $S \subset \mathscr{U}(E^\odot)$ of units, i.e. if $E^\odot = E^{S\odot}$. It is of *type* I and of *type* I^{s}, if $E^\odot$ is the closure of $E^{S\odot}$ in norm and in strong topology, respectively. We say the set S is *generating* (in the respective topology).

We add subscripts s and n, if S can be chosen such that $\mathfrak{U} \upharpoonright S$ is strongly continuous and normal, respectively. If we can find an arbitrary generating sets of units (without continuity conditions), then we add the subscript a (for algebraic).

Obviously, type I implies type I_s and each of them implies I_a (and similarly for types $\underline{\mathrm{I}}$ and I^{s}), whereas n is a *local* property of the CPD-semigroup which may or may not hold independently (and which is automatic for von Neumann modules). For each subscript type $\underline{\mathrm{I}}$ implies type I implies type I^{s}.

The GNS-system of a CP-semigroup constructed in [BS00] is generated by a single unit. Whereas a product system of pre-Hilbert spaces generated by a single unit is the trivial one. In Example 4.2.4 we have seen that the supply of central units depends on the closure. The product system $\mathcal{B}^\odot$ considered there is clearly type I, but it does not contain a central unit. Therefore, it is not a time ordered system. Passing to strong closure, the central unit $(e^{ith})_{t \in \mathbb{R}_+}$ is now contained in $\mathcal{B}^{\bar{\odot}^s}$.

Similarly, the following example shows that the required continuity properties for the generating set of units may affect the type.

4.2.8. Example. We look again at a product system constructed like $\mathcal{B}^{\odot}$ in Example 4.2.4 from an automorphism group on a C^*-algebra $\mathcal{B}$. Now for $\mathcal{B}$ we choose $L^\infty(\mathbb{R})$ with the time shift endomorphism $\mathscr{S}_t$. Clearly, the members $\mathcal{B}_t$ $(t>0)$ of that product system do not contain non-zero centered elements. But, even worse, the time shift is only strongly continuous. Therefore, a non-zero CP-semigroup composed of mappings $f \mapsto \langle \xi_t, f\xi_t \rangle = \langle \xi_t, \xi_t \rangle \mathscr{S}_t f$ cannot be continuous either. Consequently, there is not a single continuous unit in $\mathcal{B}^{\odot}$. Nevertheless, the product system is generated by the single strongly continuous unit $(\mathbf{1})_{t\in\mathbb{R}_+}$ and, therefore it is type I_s.

Restriction to $L^\infty(\mathbb{R}_-)$ gives us a similar example starting from an E_0-semigroup. We find our experience from [Ske01b] reconfirmed that, in particular, commutative C^*-algebras provide us with simple counter examples for what we know from the extreme non-commutative case $\mathscr{B}(G)$.

4.2.9. Example. Let F be a Hilbert $\mathcal{B}$–$\mathcal{B}$-module and consider the time ordered product system $\mathrm{I\!\Gamma}^{\odot}(F)$ of Hilbert modules with the set $\mathscr{U}_c(F) = \{\xi^{\odot}(\beta,\zeta) : \beta\in\mathcal{B}, \zeta\in F\}$ of units. As argued in Section 3.5 $\mathfrak{U}\restriction\mathscr{U}_c(F)$ is a uniformly continuous CPD-semigroup. By Theorem 2.3.9 the exponential units $\xi^{\odot}(0,\zeta)$ $(\zeta\in F)$ alone generate $\mathrm{I\!\Gamma}^{\odot}(F)$. Therefore, $\mathrm{I\!\Gamma}^{\odot}(F)$ is type I. Similarly, if $\mathcal{B}$ is a von Neumann algebra and F is also a von Neumann $\mathcal{B}$-module, then the product system $\mathrm{I\!\Gamma}^{s\odot}(F)$ is type I^{s}. So far, it need not be type I^{s}_{n}. Only if F is a two-sided von Neumann module, then $\mathrm{I\!\Gamma}^{s\odot}(F)$ is a time ordered product system of von Neumann modules and, therefore, type I^{s}_{n}. We will use these notions interchangeably. If F is centered (i.e., F is generated by its center in some topology) then the exponential units to elements in the center of F are already generating for that topology. Theorem 2.3.11 and Observation 2.3.14 tell us that for both $\mathrm{I\!\Gamma}^{\odot}(F)$ and $\mathrm{I\!\Gamma}^{s\odot}(F)$ the set $S = \mathscr{U}_c(F) = \{\xi^{\odot}(\beta,\zeta)\colon \beta\in\mathcal{B}\}$ has no proper extension such that the CPD-semigroup associated with this extension is still uniformly continuous. ($\mathfrak{U}\restriction\mathscr{U}_c(F)$ is maximal continuous.)

4.3. CPD-semigroups and product systems

In this section we construct for each CPD-semigroup $\mathfrak{T}$ on S a product system $E^{\odot}$ with a generating set of units such that $\mathfrak{T}$ is recovered as in Proposition 4.2.5 by matrix elements with these units. The construction is a direct generalization from CP-semigroups to CPD-semigroups of the construction in [BS00], and it contains the case of CP-semigroups as the special case where S consists of one element.

The idea can be looked up from the proof of Proposition 4.2.6 together with Example 2.1.6 and its generalization to completely positive definite kernels by the methods in Section 2.2 and Observation 3.4.3. Indeed, the two-sided submodule of E_t^S in Proposition 4.2.6 generated by $\{\xi_t(\xi^{\odot}\in S)\}$ is just the Kolmogorov module $\breve{E}_t$ of the kernel $\mathfrak{U}_t\restriction S\in\mathcal{K}_S(\mathcal{B})$. Splitting ξ_t into $\xi_{t-s}\odot\xi_s$ (for all $\xi^{\odot}\in S$), as done in that proof, means to embed $\breve{E}_t$ into the bigger space $\breve{E}_{t-s}\odot\breve{E}_s$. By definition we obtain all of E_t^S, if we continue this procedure by splitting the interval $[0, t)$ into more and more disjoint subintervals. In other words, E_t^S is the inductive limit over tensor products of an increasing number of Kolmogorov modules $\breve{E}_{t_i}$ (t_i summing up to t) of $\mathfrak{U}_{t_i} \upharpoonright S$.

For a general CPD-semigroup $\mathfrak{T}$ on some set S we proceed precisely in the same way, with the only exception that now the spaces E_t^S do not yet exist. We must construct them. So let $(\breve{E}_t, \breve{\xi}_t)$ denote the Kolmogorov decomposition for $\mathfrak{T}_t$, where $\breve{\xi}_t : \sigma \mapsto \breve{\xi}_t^\sigma$ is the canonical embedding. (Observe that $\breve{E}_0 = \mathcal{B}$ and $\breve{\xi}_0^\sigma = \mathbf{1}$ for all $\sigma \in S$.) Let $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_t$. We define

$$\breve{E}_\mathfrak{t} = \breve{E}_{t_n} \odot \cdots \odot \breve{E}_{t_1} \quad \text{and} \quad \breve{E}_{()} = \breve{E}_0.$$

In particular, we have $\breve{E}_{(t)} = \breve{E}_t$. By obvious generalization of Example 2.1.6

$$\breve{\xi}_t^\sigma \mapsto \breve{\xi}_\mathfrak{t}^\sigma := \breve{\xi}_{t_n}^\sigma \odot \cdots \odot \breve{\xi}_{t_1}^\sigma$$

defines an isometric two-sided homomorphism $\beta_{\mathfrak{t}(t)} : \breve{E}_t \to \breve{E}_\mathfrak{t}$.

Now suppose that $\mathfrak{t} = (t_n, \ldots, t_1) = \mathfrak{s}_m \smile \cdots \smile \mathfrak{s}_1 \geqslant \mathfrak{s} = (s_m, \ldots, s_1)$ with $|\mathfrak{s}_j| = s_j$. By

$$\beta_{\mathfrak{t}\mathfrak{s}} = \beta_{\mathfrak{s}_m(s_m)} \odot \cdots \odot \beta_{\mathfrak{s}_1(s_1)}$$

we define an isometric two-sided homomorphism $\beta_{\mathfrak{t}\mathfrak{s}} : \breve{E}_\mathfrak{s} \to \breve{E}_\mathfrak{t}$. Obviously, $\beta_{\mathfrak{t}\mathfrak{r}}\beta_{\mathfrak{r}\mathfrak{s}} = \beta_{\mathfrak{t}\mathfrak{s}}$ for all $\mathfrak{t} \geqslant \mathfrak{r} \geqslant \mathfrak{s}$. See the appendix of [BS00] for details about inductive limits. We obtain the following result.

4.3.1. Proposition. *The family $(\breve{E}_\mathfrak{t})_{\mathfrak{t} \in \mathbb{J}_t}$ together with $(\beta_{\mathfrak{t}\mathfrak{s}})_{\mathfrak{s} \leqslant \mathfrak{t}}$ forms an inductive system of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-modules. Hence, also the inductive limit $E_t = \lim \operatorname{ind}_{\mathfrak{t} \in \mathbb{J}_t} \breve{E}_\mathfrak{t}$ is a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module and the canonical mappings $i_\mathfrak{t} : \breve{E}_\mathfrak{t} \to E_t$ are isometric two-sided homomorphisms.*

In order to distinguish this inductive limit, where the involved isometries preserve left multiplication, from a different one in Section 4.4, where this is not the case, we refer to it as the *two-sided inductive limit*. This is a change of nomenclature compared with [BS00], where this limit was referred to as the *first inductive limit*.

Before we show that the E_t form a product system, we observe that the elements $\breve{\xi}_t^\sigma$ survive the inductive limit.

4.3.2. Proposition. *Let $\xi_t^\sigma = i_{(t)}\breve{\xi}_t^\sigma$ for all $\sigma \in S$. Then $i_\mathfrak{t}\breve{\xi}_\mathfrak{t}^\sigma = \xi_t^\sigma$ for all $\mathfrak{t} \in \mathbb{J}_t$. Moreover,*

$$\langle \xi_t^\sigma, b\xi_t^{\sigma'} \rangle = \mathfrak{T}_t^{\sigma,\sigma'}(b). \tag{4.3.1}$$

Proof. Let $\mathfrak{s}, \mathfrak{t} \in \mathbb{J}_t$ and choose $\mathfrak{r}$, such that $\mathfrak{r} \geqslant \mathfrak{s}$ and $\mathfrak{r} \geqslant \mathfrak{t}$. Then $i_\mathfrak{s}\breve{\xi}_\mathfrak{s}^\sigma = i_\mathfrak{r}\beta_{\mathfrak{r}\mathfrak{s}}\breve{\xi}_\mathfrak{s}^\sigma = i_\mathfrak{r}\breve{\xi}_\mathfrak{r}^\sigma = i_\mathfrak{r}\beta_{\mathfrak{r}\mathfrak{t}}\breve{\xi}_\mathfrak{t}^\sigma = i_\mathfrak{t}\breve{\xi}_\mathfrak{t}^\sigma$.

Moreover,
$\langle \xi_t^\sigma, b\xi_t^{\sigma'} \rangle = \langle i_{(t)}\breve{\xi}_t^\sigma, bi_{(t)}\breve{\xi}_t^{\sigma'} \rangle = \langle i_{(t)}\breve{\xi}_t^\sigma, i_{(t)}b\breve{\xi}_t^{\sigma'} \rangle = \langle \breve{\xi}_t^\sigma, b\breve{\xi}_t^{\sigma'} \rangle = \mathfrak{T}_t^{\sigma,\sigma'}(b).$ □

4.3.3. Corollary. $(\xi_t^\sigma)^* i_{\mathfrak{t}} = \breve{\xi}_{\mathfrak{t}}^{\sigma *}$ *for all* $\mathfrak{t} \in \mathbb{J}_t$. *Therefore,* $\breve{\xi}_{\mathfrak{t}}^{\sigma *}\beta_{\mathfrak{t}\mathfrak{s}} = \breve{\xi}_{\mathfrak{s}}^{\sigma *}$ *for all* $\mathfrak{s} \leqslant \mathfrak{t}$.

4.3.4. Remark. Clearly, $E_0 = \breve{E}_0 = \mathcal{B}$ and $\xi_0^\sigma = \breve{\xi}_0^\sigma = \mathbf{1}$ such that $E_t = E_0 \odot E_t = \xi_0 \odot E_t$ in the identification according to Definition 2.1.2.

4.3.5. Theorem. *The family* $E^\odot = (E_t)_{t\in\mathbb{T}}$ *(with* E_t *as in Proposition* 4.3.1*) forms a product system. Each of the families* $\xi^{\sigma\odot} = (\xi_t^\sigma)_{t\in\mathbb{T}}$ *(with* ξ_t^σ *as in Proposition* 4.3.2*) forms a unit and the set* $\mathcal{U}(S) = \{\xi^{\sigma\odot}\,(\sigma\in S)\}$ *of units is generating for* $E^\odot$.

Proof. Let $s, t \in \mathbb{T}$ and choose $\mathfrak{s}\in\mathbb{J}_s$ and $\mathfrak{t}\in\mathbb{J}_t$. Then the proof that the E_t form a product system is almost done by observing that

$$\breve{E}_{\mathfrak{s}} \odot \breve{E}_{\mathfrak{t}} = \breve{E}_{\mathfrak{s}\smile\mathfrak{t}}. \tag{4.3.2}$$

From this, intuitively, the mapping $u_{st} : i_{\mathfrak{s}}x_{\mathfrak{s}} \odot i_{\mathfrak{t}}y_{\mathfrak{t}} \mapsto i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}})$ should define a surjective isometry. Surjectivity is clear, because (as in the proof of Proposition 4.2.6) elements of the form $i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}})$ are total in $\breve{E}_{s+t}$. To see isometry we observe that $i_{\mathfrak{s}}x_{\mathfrak{s}} = i_{\hat{\mathfrak{s}}}\beta_{\hat{\mathfrak{s}}\mathfrak{s}}x_{\mathfrak{s}}$ and $i_{\mathfrak{t}}y_{\mathfrak{t}} = i_{\hat{\mathfrak{t}}}\beta_{\hat{\mathfrak{t}}\mathfrak{t}}y_{\mathfrak{t}}$ for $\hat{\mathfrak{t}} \geqslant \mathfrak{t}$ and $\hat{\mathfrak{s}} \geqslant \mathfrak{s}$. Similarly, $i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}}) = i_{\hat{\mathfrak{s}}\smile\hat{\mathfrak{t}}}(\beta_{\hat{\mathfrak{s}}\mathfrak{s}}x_{\mathfrak{s}} \odot \beta_{\hat{\mathfrak{t}}\mathfrak{t}}y_{\mathfrak{t}})$. Therefore, for checking the equation

$$\langle i_{\mathfrak{s}}x_{\mathfrak{s}} \odot i_{\mathfrak{t}}y_{\mathfrak{t}}, i_{\mathfrak{s}'}x'_{\mathfrak{s}'} \odot i_{\mathfrak{t}'}y'_{\mathfrak{t}'} \rangle = \langle i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}}), i_{\mathfrak{s}'\smile\mathfrak{t}'}(x'_{\mathfrak{s}'} \odot y'_{\mathfrak{t}'}) \rangle,$$

we may assume that $\mathfrak{t}' = \mathfrak{t}$ and $\mathfrak{s}' = \mathfrak{s}$. Now isometry is clear, because both $i_{\mathfrak{s}} \odot i_{\mathfrak{t}} : \breve{E}_{\mathfrak{s}} \odot \breve{E}_{\mathfrak{t}} \to E_s \odot E_t$ and $i_{\mathfrak{s}\smile\mathfrak{t}} : \breve{E}_{\mathfrak{s}\smile\mathfrak{t}} = \breve{E}_{\mathfrak{s}} \odot \breve{E}_{\mathfrak{t}} \to \breve{E}_{s+t}$ are (two-sided) isometries. The associativity condition follows directly from associativity of (4.3.2).

The fact that the ξ_t^σ form a unit is obvious from Proposition 4.3.2 and Observation 3.4.3. The set $\mathcal{U}(S)$ of units is generating, because E_t is generated by vectors of the form $i_{\mathfrak{t}}(b_n\breve{\xi}_{t_n}^n \odot \cdots \odot b_1\breve{\xi}_{t_1}^1 b_0)$ $(b_i \in \mathcal{B}, \xi^i \odot \in \mathcal{U}(S))$. □

4.3.6. Remark. We, actually, have shown, using identifications (4.1.2) and (4.3.2), that $i_{\mathfrak{s}} \odot i_{\mathfrak{t}} = i_{\mathfrak{s}\smile\mathfrak{t}}$.

4.3.7. Definition. We refer to $E^\odot$ as the *GNS-system* of $\mathfrak{T}$. Proposition 4.2.6 tells us that the pair $(E^\odot, \mathcal{U}(S))$ is determined up to isomorphism by the requirement that $\mathcal{U}(S)$ be a generating set of units fulfilling (4.3.1). We refer to $E^{\bar{\odot}}$ as the GNS-system of *Hilbert* modules. If $\mathcal{B}$ is a von Neumann algebra and $\mathfrak{T}$ a normal CPD-semigroup, then all $\bar{E}_t^s$ are von Neumann modules. We refer to $E^{\bar{\odot}^s}$ as the GNS-system of *von Neumann* modules.

4.4. Unital units, E_0-semigroups and local cocycles

In this section we provide the necessary results to replace the continuity of the units in Theorem 2.3.11 (which is a property relative to $\mathbb{I}\Gamma(F)$) by an intrinsic property of $\mathbb{I}\Gamma^{\odot}(F)$. Without these results we cannot show Lemma 5.3.1.

A unit vector $\xi \in E$ gives rise to an isometric embedding $\xi \odot \mathsf{id} : F \to E \odot F, y \mapsto \xi \odot y$ with adjoint $\xi^* \odot \mathsf{id} : x \odot y \mapsto \langle \xi, x \rangle y$. Hence, we may utilize a *unital unit* $\xi^{\odot}$ for a product system $E^{\odot}$ to embed E_s into E_t for $t \geqslant s$ and, finally, end up with a *second inductive limit* (in the nomenclature of [BS00]). However, since the embeddings no longer preserve left multiplication, we do not have a unique left multiplication on the inductive limit $E = \lim \mathrm{ind}_{t \to \infty} E_t$. We, therefore, refer to it as the *one-sided inductive limit*. The identification by (4.1.2) has a counter part obtained by sending, formally, s to ∞. The embedding of $\mathscr{B}^{\mathrm{a}}(E_s)$ into $\mathscr{B}^{\mathrm{a}}(E_{s+t})$, formally, becomes an embedding $\mathscr{B}^{\mathrm{a}}(E_{``\infty"})$ into $\mathscr{B}^{\mathrm{a}}(E_{``\infty+t"})$, i.e. an endomorphism of $\mathscr{B}^{\mathrm{a}}(E)$. This endomorphism depends, however, on t. The family formed by all these endomorphisms turns out to be an E_0-semigroup.

Let $t, s \in \mathbb{T}$ with $t \geqslant s$. We define the isometry

$$\gamma_{ts} = \xi_{t-s} \odot \mathsf{id} : E_s \to E_{t-s} \odot E_s = E_t.$$

Let $t \geqslant r \geqslant s$. Since $\xi^{\odot}$ is a unit, we have

$$\gamma_{ts} = \xi_{t-s} \odot \mathsf{id} = \xi_{t-r} \odot \xi_{r-s} \odot \mathsf{id} = \gamma_{tr}\gamma_{rs}.$$

That leads to the following result.

4.4.1. Proposition. *The family $(E_t)_{t \in \mathbb{T}}$ together with $(\gamma_{ts})_{s \leqslant t}$ forms an inductive system of right pre-Hilbert $\mathcal{B}$-modules. Hence, also the inductive limit $E = \lim \mathrm{ind}_{t \to \infty} E_t$ is a right pre-Hilbert $\mathcal{B}$-module. Moreover, the canonical mappings $k_t : E_t \to E$ are isometries.*

E contains a distinguished unit vector.

4.4.2. Proposition. *Let $\xi = k_0 \xi_0$. Then $k_t \xi_t = \xi$ for all $t \in \mathbb{T}$. Moreover, $\langle \xi, \xi \rangle = \mathbf{1}$.*

Proof. Precisely, as in Proposition 4.3.2. $\square$

4.4.3. Theorem. *For all $t \in \mathbb{T}$ we have*

$$E \odot E_t = E, \tag{4.4.1}$$

extending (4.1.2) *in the natural way. Moreover,*

$$E \odot (E_s \odot E_t) = (E \odot E_s) \odot E_t. \tag{4.4.2}$$

Proof. The mapping $u_t : k_s x_s \odot y_t \mapsto k_{s+t}(x_s \odot y_t)$ defines a surjective isometry. We see that this is an isometry precisely as in the proof of Theorem 4.3.5. To see surjectivity recall that any element in E can be written as $k_r x_r$ for suitable $r \in \mathbb{T}$ and $x_r \in E_r$. If $r \geqslant t$ then consider x_r as an element of $E_{r-t} \odot E_t$ and apply the prescription to see that $k_r x_r$ is in the range of u_t. If $r < t$, then apply the prescription to $\xi_0 \odot \gamma_{tr} x_r \in E_0 \odot E_t$. Of course,

$$u_{s+t}(\mathrm{id} \odot u_{st}) = u_t(u_s \odot \mathrm{id}) \tag{4.4.3}$$

which, after identifications (4.4.1) and (4.1.2), implies (4.4.2). □

4.4.4. Corollary. *The family* $\vartheta = (\vartheta_t)_{t \in \mathbb{T}}$ *of endomorphisms* $\vartheta_t : \mathscr{B}^{\mathrm{a}}(E) \to \mathscr{B}^{\mathrm{a}}(E \odot E_t) = \mathscr{B}^{\mathrm{a}}\ (E)$ *defined by setting*

$$\vartheta_t(a) = a \odot \mathrm{id}_{E_t} \tag{4.4.4}$$

is a strict E_0*-semigroup.*

Proof. The semigroup property follows directly from $E \odot E_{s+t} = E \odot (E_s \odot E_t) = (E \odot E_s) \odot E_t$. Strictness of each ϑ_t trivially follows from the observation that vectors of the form $x \odot x_t$ $(x \in E, x_t \in E_t)$ span E. □

4.4.5. Remark. Making use of identification (4.4.1), the proof of Theorem 4.4.3, actually, shows that, $k_s \odot \mathrm{id} = k_{s+t}$. Putting $s = 0$ and making use of Remark 4.3.4, we find

$$k_t = (k_0 \odot \mathrm{id})(\xi_0 \odot \mathrm{id}) = \xi \odot \mathrm{id}.$$

In particular, $\xi = \xi \odot \xi_t$.

4.4.6. Corollary. k_t *is an element of* $\mathscr{B}^{\mathrm{a}}(E_t, E)$. *The adjoint mapping is*

$$k_t^* = \xi^* \odot \mathrm{id}\ : E = E \odot E_t \to E_t.$$

Therefore, $k_t^* k_t = \mathrm{id}_{E_t}$ *and* $k_t k_t^*$ *is a projection onto the range of* k_t.

4.4.7. Example. The one-sided inductive limit over the product system $\mathrm{I\!\Gamma}^{\odot}(F)$ of time ordered Fock modules for the vacuum unit $\omega^{\odot}$ is just $\mathrm{I\!\Gamma}(F)$ and ϑ is the restriction of the time shift group $\mathscr{S}$ on $\mathscr{B}^{\mathrm{a}}(\breve{\mathrm{I\!\Gamma}}(F))$ to an E_0-semigroup on $\mathscr{B}^{\mathrm{a}}(\mathrm{I\!\Gamma}(F))$.

Let $w^{\odot} = (w_t)_{t \in \mathbb{T}}$ be an endomorphism of $E^{\odot}$. Then, clearly, setting $\mathfrak{w}_t = \mathrm{id} \odot w_t$ we define a local cocycle $\mathfrak{w} = (\mathfrak{w}_t)_{t \in \mathbb{T}}$ for ϑ (*local* means that $\mathfrak{w}_t$ commutes with $\vartheta_t(\mathscr{B}^{\mathrm{a}}(E))$, what is clear because $\vartheta_t(\mathscr{B}^{\mathrm{a}}(E))$ commutes with $\mathscr{B}^{\mathrm{a,bil}}(E_t) = \mathrm{id}_E \odot \mathscr{B}^{\mathrm{a,bil}}(E_t) \subset \mathscr{B}^{\mathrm{a}}(E)$ and *cocycle* means that $\mathfrak{w}_{s+t} = \vartheta_t(\mathfrak{w}_s)\mathfrak{w}_t = \mathfrak{w}_t \vartheta_t(\mathfrak{w}_s)$ and $\mathfrak{w}_0 = \mathbf{1}$). By Bhat and Skeide [BS00, Lemma 7.5] also the converse is true.

4.4.8. Theorem. *The formula* $\mathfrak{w}_t = \mathsf{id} \odot w_t$ *establishes a one-to-one correspondence between local cocycles* $\mathfrak{w}$ *for* ϑ *and endomorphisms* $w^\odot$ *of* $E^\odot$.

4.4.9. Observation. The E_0-semigroup ϑ, or better the space $\mathscr{B}^a(E)$ where it acts, depends highly on the choice of a (unital) unit. (However, if two inductive limits coincide for two unital units $\xi^\odot, \xi'^\odot$, then the corresponding E_0-semigroups are outer conjugate; see [Ske02].) On the contrary, the set of endomorphisms is an intrinsic property of $E^\odot$ not depending on the choice of a unit. Therefore, we prefer very much to study product systems by properties of their endomorphisms, instead of cocycles with respect to a fixed E_0-semigroup.

4.4.10. Remark. We mention a small error in [BS00] where we did not specify the value of a cocycle at $t = 0$, which is, of course, indispensable, if we want that cocycles map units to units (cf. Observation 4.2.3).

Cocycles may be continuous or not. In Theorem 2.3.11 we have computed all units for $\mathbb{I}\Gamma^\odot(F)$ which are continuous in $\mathbb{I}\Gamma(F)$. In Example 4.4.7 we explained that $\mathbb{I}\Gamma(F)$ is the one-sided inductive limit over $\mathbb{I}\Gamma^\odot(F)$ for the vacuum unit. Now we investigate how such continuity properties can be expressed intrinsically, without reference to the inductive limit.

We say a unit $\xi^\odot$ is *continuous*, if the associated CP-semigroup $T_t^\xi(b) = \langle \xi_t, b\xi_t \rangle$ is uniformly continuous. More generally, a set S of units is *continuous*, if $\mathfrak{U}\restriction S$ is uniformly continuous.

4.4.11. Lemma. *Let* $\xi^\odot$ *be a unital continuous unit for* $E^\odot$, *and denote by* E *the one-sided inductive limit for* $\xi^\odot$. *Let* $\zeta^\odot$ *be another unit. Then the following conditions are equivalent.*

1. *The function* $t \mapsto \xi \odot \zeta_t \in E$ *is continuous.*
2. *The semigroups* $\mathfrak{U}^{\zeta,\xi}$ *and* T^ζ *are uniformly continuous.*
3. *The functions* $t \mapsto \langle \zeta_t, \xi_t \rangle$ *and* $t \mapsto \langle \zeta_t, \zeta_t \rangle$ *are continuous.*

Moreover, if $\zeta^\odot, \zeta'^\odot$ *are two units both fulfilling one of the three conditions above, then also the function* $t \mapsto \langle \zeta_t, \zeta'_t \rangle$ *is continuous, hence, also the semigroup* $\mathfrak{U}^{\zeta,\zeta'}$ *is uniformly continuous.*

Proof. The crucial step in the proof is the observation that the norm of mappings on $\mathcal{B}$ of the form $b \mapsto \langle x, by \rangle$ (for x, y in some pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module) can be estimated by $\|x\|\,\|y\|$.

$1 \Rightarrow 2$: We have

$$\xi \odot \zeta_{t+\varepsilon} - \xi \odot \zeta_t = \xi \odot \zeta_\varepsilon \odot \zeta_t - \xi \odot \xi_\varepsilon \odot \zeta_t = \xi \odot (\zeta_\varepsilon - \xi_\varepsilon) \odot \zeta_t, \tag{4.4.5}$$

so that $t \mapsto \xi \odot \zeta_t$ is continuous, if and only if $\|\zeta_t - \xi_t\| \to 0$ for $t \to 0$. Thus, 1 implies

$$\|\mathfrak{U}_t^{\zeta,\xi} - \mathrm{id}\| \leqslant \|\mathfrak{U}_t^{\zeta,\xi} - T_t^{\xi}\| + \|T_t^{\xi} - \mathrm{id}\| \to 0,$$

because the norm of $\mathfrak{U}_t^{\zeta,\xi} - T_t^{\xi} : b \mapsto \langle \zeta_t - \xi_t, b\xi_t \rangle$ is smaller than $\|\zeta_t - \xi_t\| \, \|\xi_t\| \to 0$, and

$$\|T_t^{\zeta} - \mathrm{id}\| \leqslant \|T_t^{\zeta} - \mathfrak{U}_t^{\zeta,\xi}\| + \|\mathfrak{U}_t^{\zeta,\xi} - \mathrm{id}\| \to 0,$$

because the norm of $T_t^{\zeta} - \mathfrak{U}_t^{\zeta,\xi} : b \mapsto \langle \zeta_t, b(\zeta_t - \xi_t) \rangle$ is smaller than $\|\zeta_t\| \, \|\zeta_t - \xi_t\| \to 0$ and by the preceding estimate.

$2 \Rightarrow 3$ is trivial, so let us come to $3 \Rightarrow 1$. We have

$$\|\zeta_t - \xi_t\|^2 \leqslant \|\langle \zeta_t, \zeta_t \rangle - \mathbf{1}\| + \|\langle \zeta_t, \xi_t \rangle - \mathbf{1}\| + \|\langle \xi_t, \zeta_t \rangle - \mathbf{1}\| + \|\langle \xi_t, \xi_t \rangle - \mathbf{1}\|$$

which tends to 0 for $t \to 0$, if 3 holds. Then (4.4.5) implies continuity of $\xi \odot \zeta_t$.

Now let $\zeta^{\odot}, \zeta'^{\odot}$ be two units fulfilling 3. Then

$$\|\langle \zeta_t, \zeta_t' \rangle - \mathbf{1}\| \leqslant \|\langle \zeta_t, \zeta_t' - \xi_t \rangle\| + \|\langle \zeta_t - \xi_t, \xi_t \rangle\| + \|\langle \xi_t, \xi_t \rangle - \mathbf{1}\| \to 0$$

for $t \to 0$ so that $t \mapsto \langle \zeta_t, \zeta_t' \rangle$ is continuous. As before, this implies that $\mathfrak{U}^{\zeta,\zeta'}$ is uniformly continuous. □

The following theorem is simple corollary of Theorem 4.3.5 and Lemma 4.4.11. Taking into account also the extensions following Corollary 5.4.3 which assert that a continuous unit is contained in a time ordered product systems of von Neumann $\mathcal{B}^{**}$–$\mathcal{B}^{**}$-modules, and the fact that by Lemma 3.5.4(2) units in such product systems may be normalized within that system, one may show that we can drop the assumption in brackets.

4.4.12. Theorem. *For a CPD-semigroup $\mathfrak{T}$ on a set S containing an element σ such that $\mathfrak{T}^{\sigma,\sigma}$ is uniformly continuous (and that $\mathfrak{T}_t^{\sigma,\sigma}(\mathbf{1}) = \mathbf{1}$ for all $t \in \mathbb{R}_+$) the following statements are equivalent*:

1. *$\mathfrak{T}$ is uniformly continuous.*
2. *The functions $t \mapsto \mathfrak{T}_t^{\sigma'',\sigma'}(\mathbf{1})$ are continuous for all $\sigma'', \sigma' \in S$.*
3. *The functions $t \mapsto \mathfrak{T}_t^{\sigma,\sigma'}(\mathbf{1})$ and $t \mapsto \mathfrak{T}_t^{\sigma',\sigma'}(\mathbf{1})$ are continuous for all $\sigma' \in S$.*

The main idea in the proof of Lemma 4.4.11 is that a certain (completely bounded) mapping can be written as $b \mapsto \langle x, by \rangle$ for some vectors in some *GNS-space*. Theorem 4.4.12 is an intrinsic result about CPD-semigroups obtained, roughly speaking, by rephrasing all statements from Lemma 4.4.11 involving units in terms of the associated CPD-semigroup. It seems difficult to show Theorem 4.4.12 directly without reference to the GNS-system of the CPD-semigroup.

Another consequence of Lemma 4.4.11 concerns continuity properties of local cocycles.

4.4.13. Corollary. *Let $E^\odot$ be generated by a subset $S \subset \mathscr{U}(E^\odot)$ of units such that $\mathfrak{U}\restriction S$ is a uniformly continuous CPD-semigroup. Let $\xi^\odot \in S$ be a unital unit, and denote by E the one-sided inductive limit for $\xi^\odot$. Then for a morphism $w^\odot$ and the associated local cocycle $\mathfrak{w} = (\mathsf{id} \odot w_t)_{t \in \mathbb{T}}$ the following equivalent conditions*

1. *The CPD-semigroup $\mathfrak{U}\restriction(S \cup w^\odot S)$ (see Observation 4.2.3) is uniformly continuous. (In particular, if S is maximal continuous, then $w^\odot$ leaves S invariant.)*
2. *For some $\xi'^\odot \in S$ all functions $t \mapsto \langle \xi'_t, \zeta_t \rangle$, $t \mapsto \langle \zeta_t, \zeta_t \rangle$ $(\zeta^\odot \in w^\odot S)$ are continuous.*

both imply that $\mathfrak{w}$ is strongly continuous.

Proof. By simple applications of Lemma 4.4.11(1) and (2) are equivalent, and for the remaining implication it is sufficient to choose $\xi'^\odot = \xi^\odot$. So assume that all functions $t \mapsto \langle \zeta_t, \zeta_t \rangle$, $t \mapsto \langle \xi_t, \zeta_t \rangle$ $(\zeta^\odot \in S \cup w^\odot S)$ are continuous. Then

$$||w_t\zeta_t - \zeta_t|| = ||\xi \odot w_t\zeta_t - \xi \odot \zeta_t|| \leqslant ||\xi \odot w_t\zeta_t - \xi|| + ||\xi \odot \zeta_t - \xi|| \to 0 \qquad (4.4.6)$$

for $t \to 0$. Applying $\mathfrak{w}_{s+\varepsilon} - \mathfrak{w}_s = \mathsf{id} \odot (w_\varepsilon - \mathsf{id}_{E_\varepsilon}) \odot w_s$ to a vector of the form $\xi \odot x_t$ where $x_t \in E_t$ is as in (4.2.3), we conclude from (4.4.6) (choosing $\varepsilon > 0$ so small that $w_\varepsilon - \mathsf{id}_{E_\varepsilon}$ comes to act on a single unit only) that the function $s \mapsto \mathfrak{w}_s(\xi \odot x_t)$ is continuous. Since the vectors $\xi \odot x_t$ span E, $\mathfrak{w}$ is strongly continuous. □

4.4.14. Observation. If $\mathfrak{w}$ is bounded locally uniformly (for instance, if $w^\odot$ is contractive) or, equivalently, if the extension of $\mathfrak{w}$ to $\bar{E}$ is also strongly continuous, then also the reverse implication holds. (We see by the same routine arguments that the inner product $\langle \xi_t, w_t\zeta_t \rangle = \langle \xi \odot \xi_t, \xi \odot w_t\zeta_t \rangle = \langle \xi, \mathfrak{w}_t(\xi \odot \zeta_t) \rangle$ depends continuously on t and, similarly, also $\langle w_t\zeta_t, w_t\zeta_t \rangle$.)

4.4.15. Definition. A morphism $w^\odot$ is *continuous*, if $S \cup w^\odot S$ is continuous for some generating continuous subset S of units.

5. Type I product systems

In this chapter we show that type I^{s} product systems of von Neumann modules are time ordered Fock modules. This is the analogue of Arveson's result that type I Arveson systems are symmetric Fock spaces [Arv89].

In Section 5.1 we show that a product system is contained in a time ordered product system, if it contains at least one (continuous) central unit. In Section 5.2 we study the continuous endomorphisms of the time ordered Fock module. We find its projection morphisms. In Section 5.3 and provide a necessary and sufficient criterion for that a given set of (continuous) units is (strongly) generating. The basic idea (used by Bhat [Bha01] for a comparable purpose) is that a product system of von Neumann modules is generated by a set of units, if and only if there is precisely one projection endomorphism (namely, the identity morphism), leaving the units of this set invariant. In Section 5.4 we utilize the Christensen–Evans Lemma C.2 to show that the GNS-system of a uniformly continuous CPD-semigroup has a central unit and, therefore, is contained in a time ordered Fock module by Section 5.1. By Section 5.3 these units generate a whole time ordered subsystem. We point out that the result by Christensen and Evans is equivalent to show existence of a central unit in any type $\mathrm{I^s}$ system.

5.1. *Central units in type* I *product systems*

In this section we show that type I product systems are contained in time ordered Fock modules, if at least one of the continuous units is central. So let $\omega^\odot$ be a central unit in an arbitrary product system and let $\xi^\odot$ be any other unit. Then

$$\mathfrak{U}_t^{\xi,\omega}(b) = \langle \xi_t, b\omega_t \rangle = \langle \xi_t, \omega_t \rangle b = \mathfrak{U}_t^{\xi,\omega}(\mathbf{1})b \tag{5.1.1}$$

and

$$\mathfrak{U}_{s+t}^{\xi,\omega}(\mathbf{1}) = \mathfrak{U}_t^{\xi,\omega}(\mathfrak{U}_s^{\xi,\omega}(\mathbf{1})) = \mathfrak{U}_t^{\xi,\omega}(\mathbf{1})\mathfrak{U}_s^{\xi,\omega}(\mathbf{1}).$$

In other words, $\mathfrak{U}^{\xi,\omega}(\mathbf{1})$ is a semigroup *in* $\mathcal{B}$ and determines $\mathfrak{U}^{\xi,\omega}$ by (5.1.1). In particular, $\mathfrak{U}^{\omega,\omega}(\mathbf{1})$ is a semigroup in $C_{\mathcal{B}}(\mathcal{B})$. If $\omega^\odot$ is continuous, then all $\mathfrak{U}_t^{\omega,\omega}(\mathbf{1})$ are invertible. Henceforth, we may assume without loss of generality that $\omega^\odot$ is unital, i.e. $T^\omega = \mathsf{id}$ is the trivial semigroup.

5.1.1. Lemma. *Let $\omega^\odot$ be a central unital unit and let $\xi^\odot$ be another unit for a product system $E^\odot$ such that the CPD-semigroup $\mathfrak{U}\restriction\{\omega^\odot, \xi^\odot\}$ is uniformly continuous. Let $\beta \in \mathcal{B}$ denote the generator of the semigroup $\mathfrak{U}^{\omega,\xi}(\mathbf{1})$ in $\mathcal{B}$, i.e. $\mathfrak{U}_t^{\omega,\xi}(\mathbf{1}) = e^{t\beta}$, and let $\mathcal{L}^\xi$ denote the generator of the CP-semigroup T^ξ on $\mathcal{B}$. Then the mapping*

$$b \mapsto \mathcal{L}^\xi(b) - b\beta - \beta^* b \tag{5.1.2}$$

is completely positive, i.e. $\mathcal{L}^\xi$ is a CE-generator.

Proof. We consider the CP-semigroup $\mathfrak{U}^{(2)} = (\mathfrak{U}_t^{(2)})_{t\in\mathbb{R}_+}$ on $M_2(\mathcal{B})$ with $\mathfrak{U}_t^{(2)} = \begin{pmatrix} \mathfrak{U}_t^{\omega,\omega} & \mathfrak{U}_t^{\omega,\xi} \\ \mathfrak{U}_t^{\xi,\omega} & \mathfrak{U}_t^{\xi,\xi} \end{pmatrix}$ whose generator is

$$\mathfrak{L}^{(2)}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \frac{d}{dt}\bigg|_{t=0}\begin{pmatrix} \mathfrak{U}_t^{\omega,\omega}(b_{11}) & \mathfrak{U}_t^{\omega,\xi}(b_{12}) \\ \mathfrak{U}_t^{\xi,\omega}(b_{21}) & \mathfrak{U}_t^{\xi,\xi}(b_{22}) \end{pmatrix} = \begin{pmatrix} 0 & b_{12}\beta \\ \beta^* b_{21} & \mathcal{L}^\xi(b_{22}) \end{pmatrix}.$$

By Theorem 3.4.7 and Lemma 3.4.6 $\mathfrak{L}^{(2)}$ is conditionally completely positive. Let $A_i = \begin{pmatrix} 0 & 0 \\ a_i & a_i \end{pmatrix}$ and $B_i = \begin{pmatrix} 0 & -b_i \\ 0 & b_i \end{pmatrix}$. Then $A_i B_i = 0$, i.e. $\sum_i A_i B_t = 0$, so that

$$0 \leqslant \sum_{i,j} B_i^* \mathfrak{L}^{(2)}(A_i^* A_j) B_j = \sum_{i,j} B_i^* \begin{pmatrix} 0 & a_i^* a_j \beta \\ \beta^* a_i^* a_j & \mathcal{L}^{\xi}(a_i^* a_j) \end{pmatrix} B_j$$
$$= \sum_{i,j} \begin{pmatrix} 0 & 0 \\ 0 & b_i^*(\mathcal{L}^{\xi}(a_i^* a_j) - a_i^* a_j \beta - \beta^* a_i^* a_j) b_j \end{pmatrix}.$$

This means that (5.1.2) is completely positive. □

Now we show how the generator of CPD-semigroups (i.e. many units) in product systems with a central unit boils down to the generator $\mathcal{L}^{\xi}$ of a CP-semigroup (i.e. a single unit) as in Lemma 5.1.1. Once again in these notes, we exploit the ideas of Section 2.2.

5.1.2. Theorem. *Let $E^{\odot}$ be a product system with a subset $S \subset \mathscr{U}(E^{\odot})$ of units and a central (unital) unit $\omega^{\odot}$ such that $\mathfrak{U} \upharpoonright S \cup \{\omega^{\odot}\}$ is a uniformly continuous CPD-semigroup. Then the generator $\mathfrak{L}$ of the (uniformly continuous) CPD-semigroup $\mathfrak{T} = \mathfrak{U} \upharpoonright S$ is a CE-generator.*

Proof. For $\xi^{\odot} \in S$ denote by $\beta_{\xi} \in \mathcal{B}$ the generator of the semigroup $\mathfrak{U}^{\omega,\xi}(\mathbf{1})$ in $\mathcal{B}$. We claim as in Lemma 5.1.1 that the kernel $\mathfrak{L}_0$ on S defined by setting

$$\mathfrak{L}_0^{\xi,\xi'}(b) = \mathfrak{L}^{\xi,\xi'}(b) - b\beta_{\xi'} - \beta_{\xi}^* b$$

(for $(\xi^{\odot}, \xi'^{\odot}) \in S \times S$) is completely positive definite, what shows the theorem. By Lemma 3.2.1(4) it is equivalent to show that the mapping $\mathfrak{L}_0^{(n)}$ on $M_n(\mathcal{B})$ defined by setting

$$(\mathfrak{L}_0^{(n)}(B))_{ij} = \mathfrak{L}^{\xi^i,\xi^j}(b_{ij}) - b_{ij}\beta_{\xi^j} - \beta_{\xi^i}^* b_{ij}$$

is completely positive for all choices of $n \in \mathbb{N}$ and $\xi^i \odot \in S$ $(i = 1, \ldots, n)$.

First, observe that by Section 2.2 $M_n(E^{\odot}) = (M_n(E_t))_{t \in \mathbb{T}}$ is a product system of $M_n(\mathcal{B})$–$M_n(\mathcal{B})$-modules. Clearly, the diagonal matrices $\Xi_t \in M_n(E_t)$ with entries $\xi_t^i \delta_{ij}$ form a unit $\Xi^{\odot}$ for $M_n(E^{\odot})$. Moreover, the unit $\Omega^{\odot}$ with entries $\delta_{ij}\omega^{\odot}$ is central and unital. For the units $\Omega^{\odot}$ and $\Xi^{\odot}$ the assumptions of Lemma 5.1.1 are fulfilled. The generator $\hat{\beta}$ of the semigroup $\mathfrak{U}^{\Omega,\Xi}(\mathbf{1})$ is the matrix with entries $\delta_{ij}\beta_{\xi^i}$. Now (5.1.2) gives us back $\mathfrak{L}_0^{(n)}$ which, therefore, is completely positive. □

5.1.3. Corollary. *The GNS-system $E^{\odot}$ of $\mathfrak{T}$ is embedable into a time ordered product system. More precisely, let (F, ζ) be the (completed) Kolmogorov decomposition for the* *kernel* $\mathfrak{L}_0$ *with the canonical mapping* $\zeta : \xi^{\odot} \mapsto \zeta_{\xi}$. *Then*

$$\xi^{\odot} \mapsto \xi^{\odot}(\beta_{\xi}, \zeta_{\xi})$$

extends as an isometric morphism $E^{\odot} \to \mathrm{I\!\Gamma}^{\odot}(F)$.

Notice that (in the notations of Theorem 5.1.2) the preceding morphism may be extended to $E^{S_{\omega}\odot}$ where $S_{\omega} = S \cup \{\omega^{\odot}\}$, by sending $\omega^{\odot} \in \mathcal{U}(E^{\odot})$ to $\omega^{\odot} \in \mathcal{U}_c(F)$.

5.2. Morphisms of the time ordered Fock module

In the preceding section we found that, roughly speaking, type I product systems with a central unit may be embedded into a time ordered Fock module. In the following section we want to find criteria to decide, whether this Fock module is generated by such a subsystem. To that goal, in this section we study the endomorphisms of $\mathrm{I\!\Gamma}^{\odot}(F)$.

After establishing the general form of (possibly unbounded, but adjointable) continuous morphisms, we find very easily characterizations of isometric, coisometric, unitary, positive, and projection morphisms. The generalizations of ideas from Bhat's "cocycle computations" in [Bha01] are straightforward. Contractivity requires slightly more work and, because we do not need it for our main goal, we postpone it to Appendix A.

Besides (4.1.3), the crucial property of a morphism is to consist of adjointable mappings. Adjointability, checked on some total subset of vectors, assures well-definedness by Observation 2.1.1. If $w^{\odot}$ is a morphism (on an algebraic product system) except that the w_t are allowed to be unbounded, then we speak of a *possibly unbounded* morphism. As product systems we consider the algebraic subsystems $\mathrm{I\!\Gamma}^{\mathcal{U}_c\odot}(F) = (\mathrm{I\!\Gamma}_t^{\mathcal{U}_c}(F))_{t\in\mathbb{R}_+}$ of the time ordered systems $\mathrm{I\!\Gamma}^{\odot}(F)$ which are generated by the sets $\mathcal{U}_c(F)$ of continuous units.

Recall that a continuous morphism $w^{\odot}$ of time ordered Fock modules corresponds to a transformation

$$\xi^{\odot}(\beta, \zeta) \mapsto \xi^{\odot}(\gamma_w(\beta, \zeta), \eta_w(\beta, \zeta)) \tag{5.2.1}$$

among sets of continuous units. We want to know which transformations of the parameter space $\mathcal{B} \times F$ of the continuous units define operators w_t by extending (5.2.1) to vectors of the form (4.2.3).

5.2.1. Theorem. *Let* F *and* F' *be Hilbert* $\mathcal{B}$–$\mathcal{B}$-*modules. Then setting*

$$w_t\xi_t(\beta, \zeta) = \xi_t(\gamma + \beta + \langle \eta, \zeta \rangle, \eta' + a\zeta), \tag{5.2.2}$$

we establish a one-to-one correspondence between possibly unbounded continuous morphisms $w^{\odot} = (w_t)_{t \in \mathbb{R}_+}$ *from* $\mathbb{I}\Gamma^{\mathscr{U}_c \odot}(F)$ *to* $\mathbb{I}\Gamma^{\mathscr{U}_c \odot}(F')$ *and matrices*

$$\Gamma = \begin{pmatrix} \gamma & \eta^* \\ \eta' & a \end{pmatrix} \in \mathscr{B}^{\mathrm{a,bil}}(\mathcal{B} \oplus F, \mathcal{B} \oplus F') = \begin{pmatrix} C_{\mathcal{B}}(\mathcal{B}) & C_{\mathcal{B}}(F)^* \\ \mathbb{C}_{\mathcal{B}}(F') & \mathscr{B}^{\mathrm{a,bil}}(F, F') \end{pmatrix}.$$

Moreover, the adjoint of $w^{\odot}$ *is given by the adjoint matrix* $\Gamma^* = \begin{pmatrix} \gamma^* & \eta'^* \\ \eta & a^* \end{pmatrix}$.

Proof. From bilinearity and adjointability of w_t we have

$$\langle \xi_t(\beta, \zeta), b\xi_t(\gamma_{w^*}(\beta', \zeta'), \eta_{w^*}(\beta', \zeta')) \rangle = \langle \xi_t(\gamma_w(\beta, \zeta), \eta_w(\beta, \zeta)), b\xi_t(\beta', \zeta') \rangle \quad (5.2.3)$$

for all $t \in \mathbb{R}_+$, $\beta, \beta' \in \mathcal{B}$, $\zeta \in F$, $\zeta' \in F'$ or, equivalently, by differentiating at $t = 0$ and (3.5.2)

$$\langle \zeta, b\eta_{w^*}(\beta', \zeta') \rangle + b\gamma_{w^*}(\beta', \zeta') + \beta^* b = \langle \eta_w(\beta, \zeta), b\zeta' \rangle + b\beta' + \gamma_w(\beta, \zeta)^* b. \quad (5.2.4)$$

It is easy to check that validity of (5.2.2) implies (5.2.4) and, henceforth, (5.2.3). Therefore, (5.2.2) defines a unique adjointable bilinear operator $\hat{w}_t$ from the bimodule generated by all $\xi_t(\beta, \zeta)$ ($\beta \in \mathcal{B}, \zeta \in F$) (i.e. the Kolmogorov decomposition of $\mathfrak{U}_t \upharpoonright \mathscr{U}_c(F)$) into $\mathbb{I}\Gamma_t^{\mathscr{U}_c}(F')$. It is clear that (as in the proof of Proposition A.6) the $\hat{w}_t$ define an operator on $\mathbb{I}\Gamma_t^{\mathscr{U}_c}(F)$, that this operator is the extension of (5.2.1) to vectors of the form (4.2.3), and that the operators fulfill (4.1.3). We put $w_0 = \mathsf{id}_{\mathcal{B}}$, and the w_t form a morphism.

It remains to show that (5.2.2) is also a necessary condition on the form of the functions $\gamma_w : \mathcal{B} \times F \to \mathcal{B}$ and $\eta_w : \mathcal{B} \times F \to F'$. Putting $\zeta = 0, \zeta' = 0$ in (3.5.2), we find

$$b\gamma_{w^*}(\beta', 0) + \beta^* b = b\beta' + \gamma_w(\beta, 0)^* b. \quad (5.2.5)$$

Putting also $\beta = \beta' = 0$ and $b = \mathbf{1}$, we find $\gamma_{w^*}(0, 0)^* = \gamma_w(0, 0)$. We denote this element of $\mathcal{B}$ by γ. Reinserting arbitrary $b \in \mathcal{B}$, we find that $\gamma \in C_{\mathcal{B}}(\mathcal{B})$. Reinserting arbitrary $\beta \in \mathcal{B}$, we find $\gamma_w(\beta, 0) = \gamma + \beta$ and, similarly, $\gamma_{w^*}(\beta', 0) = \gamma^* + \beta'$.

Putting in 5.2.4 $\zeta = 0$, inserting $\gamma_w(\beta, 0)^*$ and subtracting $\beta^* b$, we obtain

$$b\gamma_{w^*}(\beta', \zeta') = \langle \eta_w(\beta, 0), b\zeta' \rangle + b\beta' + \gamma^* b = \langle \eta_w(\beta, 0), b\zeta' \rangle + b\gamma_{w^*}(\beta', 0)$$

(recall that γ commutes with b), or

$$b\gamma_{w^*}(\beta', \zeta') - b\gamma_{w^*}(\beta', 0) = \langle \eta_w(\beta, 0), b\zeta' \rangle. \quad (5.2.6)$$

We obtain a lot of information. Firstly, the left-hand side and the right-hand side cannot depend on β' and β, respectively. Therefore, $\eta_w(\beta, 0) = \eta_w(0, 0)$ which we denote by $\eta' \in F'$. Secondly, we put $b = \mathbf{1}$ and multiply again with an arbitrary $b \in \mathcal{B}$ from the left. Together with the original version of (5.2.6) we obtain that $\eta' \in C_{\mathcal{B}}(F')$. Finally, with $b = \mathbf{1}$ we obtain $\gamma_{w^*}(\beta', \zeta') = \gamma^* + \beta' + \langle \eta', \zeta' \rangle$. A similar computation starting from $\zeta' = 0$, yields $\eta_{w^*}(\beta', 0) = \eta_{w^*}(0,0) = \eta$ for some $\eta \in C_{\mathcal{B}}(F)$ and $\gamma_w(\beta, \zeta) = \gamma + \beta + \langle \eta, \zeta \rangle$.

Inserting the concrete form of $\gamma_{w^{(*)}}$ into (5.2.4) and subtracting $\gamma^* b + b\beta' + \beta^* b = b\gamma^* + b\beta' + \beta^* b$, we obtain

$$\langle \zeta, b\eta_{w^*}(\beta', \zeta') \rangle + b\langle \eta', \zeta' \rangle = \langle \eta_w(\beta, \zeta), b\zeta' \rangle + \langle \zeta, \eta \rangle b. \tag{5.2.7}$$

Again, we conclude that $\eta_{w^*}(\beta', \zeta') = \eta_{w^*}(0, \zeta')$ and $\eta_w(\beta, \zeta) = \eta_w(0, \zeta)$ cannot depend on β' and β, respectively. Putting $b = \mathbf{1}$, we find $\langle \zeta, \eta_{w^*}(0, \zeta') - \eta \rangle = \langle \eta_w(0, \zeta) - \eta', \zeta' \rangle$. It follows that the mapping $a : \zeta \mapsto \eta_w(0, \zeta) - \eta'$ has an adjoint, namely, $a^* : \zeta' \mapsto \eta_{w^*}(0, \zeta') - \eta$. Since F and F' are complete, a is an element of $\mathscr{B}^{\mathrm{a}}(F, F')$. Inserting a and a^* in (5.2.7), and taking into account that η and η' are central, we find that $a \in \mathscr{B}^{\mathrm{a,bil}}(F, F')$, and $\eta_w(\beta, \zeta) = \eta' + a\zeta$ and $\eta_{w^*}(\beta', \zeta') = \eta + a^*\zeta'$ as desired. □

5.2.2. Corollary. *A (possibly unbounded) continuous endomorphism $w^{\odot}$ of $\mathbb{\Gamma}^{\mathscr{U}_c \odot}(F)$ is self-adjoint, if and only if Γ is self-adjoint.*

Of course, the correspondence is not functorial in the sense that $ww'^{\odot} = (w_t w'_t)_{t \in \mathbb{R}_+}$ is not given by $\Gamma\Gamma'$. However, we easily check the following.

5.2.3. Corollary. *Let $w^{\odot}$ be a morphism with matrix Γ. Then*

$$\begin{pmatrix} \mathbf{1} & 0 & 0 \\ \gamma & \mathbf{1} & \eta^* \\ \eta' & 0 & a \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \beta \\ \zeta \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \gamma_w(\beta, \zeta) \\ \zeta_w(\beta, \zeta) \end{pmatrix} \quad \text{and the mapping } w^{\odot} \mapsto \hat{\Gamma} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ \gamma & \mathbf{1} & \eta^* \\ \eta' & 0 & a \end{pmatrix}$$

is functorial in the sense that $\hat{\Gamma}'' = \hat{\Gamma}\hat{\Gamma}'$ for $w''^{\odot} = ww'^{\odot}$.

5.2.4. Corollary. *The continuous morphism $w^{\odot}$ with the matrix $\Gamma = \begin{pmatrix} \gamma & \eta^* \\ \eta' & a \end{pmatrix}$ is isometric, if and only if a is isometric, $\eta' \in C_{\mathcal{B}}(F')$ arbitrary, $\eta = -a^*\eta'$, and $\gamma = ih - \frac{\langle \eta', \eta' \rangle}{2}$ for some $h = h^* \in C_{\mathcal{B}}(\mathcal{B})$. It is coisometric, if and only if a is coisometric, $\eta \in C_{\mathcal{B}}(F)$ arbitrary, $\eta' = -a\eta$, and $\gamma = ih - \frac{\langle \eta, \eta \rangle}{2}$ for some $h = h^* \in C_{\mathcal{B}}(\mathcal{B})$. It is unitary (i.e. an isomorphism), if and only if a is unitary, $\eta \in C_{\mathcal{B}}(F)$ arbitrary, $\eta' = -a\eta$, and $\gamma = ih - \frac{\langle \eta, \eta \rangle}{2}$ for some $h = h^* \in C_{\mathcal{B}}(\mathcal{B})$ or, equivalently, if a is unitary, $\eta' \in C_{\mathcal{B}}(F')$ arbitrary, $\eta = -a^*\eta'$, and $\gamma = ih - \frac{\langle \eta', \eta' \rangle}{2}$ for some $h = h^* \in C_{\mathcal{B}}(\mathcal{B})$.*

The form of these conditions reminds us very much of the form of the corresponding conditions for solutions of quantum stochastic differential equations; see e.g. [Ske00c].

After the characterizations of isomorphisms we come to projections. Of course, a projection endomorphism must be self-adjoint and so must be its matrix.

5.2.5. Corollary. *A continuous endomorphism* $w^{\odot}$ *of* $\mathbb{I}\Gamma^{\mathcal{U}_c\odot}(F)$ *is a projection morphism, if and only if its matrix* Γ *has the form*

$$\Gamma = \begin{pmatrix} -\langle \eta, \eta \rangle & \eta^* \\ \eta & p \end{pmatrix},$$

where p is a projection in $\mathscr{B}^{\mathrm{a,bil}}(F)$, *and* $\eta \in (\mathbf{1} - p)C_{\mathcal{B}}(F)$.

Since a continuous morphism of a product system $\mathbb{I}\Gamma^{\odot}(F)$ or $\mathbb{I}\Gamma^{\mathrm{s}\odot}(F)$ (or between such) sends continuous units to continuous units, it restricts to a morphism of $\mathbb{I}\Gamma^{\mathcal{U}_c\odot}(F)$ (or between such). Therefore, all characterizations extend to the case of Hilbert modules and the case of von Neumann modules.

5.3. Strongly generating sets of units

Now we characterize strongly generating sets of continuous units for time ordered product systems of von Neumann modules. The idea is that, if a set of units is not strongly generating, then by Observation 4.1.2 there exists a non-trivial projection morphism onto the subsystem generated by these units. In order to apply our methods we need to know that this morphism is continuous.

5.3.1. Lemma. *Let* $p^{\odot}$ *be a projection morphism leaving invariant (i.e.* $p\xi^{\odot} = \xi^{\odot}$ *for all* $\xi^{\odot} \in S$*) a non-empty subset* $S \subset \mathcal{U}_c(F)$ *of continuous units for* $\mathbb{I}\Gamma^{\mathrm{s}\odot}(F)$. *Then* $p^{\odot}$ *is continuous.*

Proof. By Lemma 3.5.4(2), the completion (therefore, *a fortiori* the strong closure) of what a single continuous unit $\xi^{\odot}(\beta, \zeta) \in S$ generates in a time ordered system contains the unital unit $\xi^{\odot}(-\frac{\langle \zeta, \zeta \rangle}{2}, \zeta)$. Therefore, we may assume that S contains a unital unit $\xi^{\odot}$. Now let $\xi'^{\odot}$ be an arbitrary continuous unit. Then the function $t \mapsto \langle \xi_t, p_t \xi'_t \rangle = \langle p_t \xi_t, \xi'_t \rangle = \langle \xi_t, \xi'_t \rangle$ is continuous. Moreover, we have

$$\langle p_t \xi'_t, p_t \xi'_t \rangle - \langle \xi_t, \xi_t \rangle = \langle \xi'_t - \xi_t, p_t \xi'_t \rangle + \langle \xi_t, p_t(\xi'_t - \xi_t) \rangle \to 0$$

for $t \to 0$. From this it follows as, for instance, in (4.4.5) that also the function $t \mapsto \langle p_t \xi'_t, p_t \xi'_t \rangle$ is continuous. By Lemma 4.4.11 also the unit $p\xi'^{\odot}$ is continuous. As $\xi'^{\odot}$ was arbitrary, $p^{\odot}$ is continuous. $\square$

5.3.2. Theorem. *Let F be a von Neumann* $\mathcal{B}$–$\mathcal{B}$*-module and let* $S \subset \mathcal{U}_c(F)$ *be a continuous subset of units for* $\mathbb{I}\Gamma^{\mathrm{s}\odot}(F)$. *Then S is strongly generating, if and only if the*

$\mathcal{B}$–$\mathcal{B}$-submodule

$$F_0 = \left\{ \sum_{i=1}^{n} a_i \zeta_i b_i \mid n \in \mathbb{N}; \zeta_i \in S_F; a_i, b_i \in \mathcal{B} : \sum_{i=1}^{n} a_i b_i = 0 \right\} \tag{5.3.1}$$

of F is strongly dense in F, where $S_F = \{\zeta \in F \mid \exists \beta \in \mathcal{B} : \xi^{\odot}(\beta, \zeta) \in S\}$.

Proof. Denote by $\mathbb{I}\Gamma^{S\odot}$ the strong closure of the product subsystem of $\mathbb{I}\Gamma^{s\odot}(F)$ generated by the units in S. We define another $\mathcal{B}$–$\mathcal{B}$-submodule

$$F^0 = \left\{ \sum_{i=1}^{n} a_i \zeta_i b_i \mid n \in \mathbb{N}; \zeta_i \in S_F; a_i, b_i \in \mathcal{B} \right\}$$

of F. We have $F \supset \overline{F^0}^{\mathrm{s}} \supset \overline{F_0}^{\mathrm{s}}$. Denote by p_0 and p^0 in $\mathscr{B}^{\mathrm{a,bil}}(F)$ the projections onto $\overline{F_0}^{\mathrm{s}}$ and $\overline{F^0}^{\mathrm{s}}$, respectively. (Since $\overline{F_0}^{\mathrm{s}}$ and $\overline{F^0}^{\mathrm{s}}$ are von Neumann modules, the projections exist, and since $\overline{F_0}^{\mathrm{s}}$ and $\overline{F^0}^{\mathrm{s}}$ are $\mathcal{B}$–$\mathcal{B}$-submodules, the projections are bilinear.) We have to distinguish three cases.

(i) $F \neq \overline{F^0}^{\mathrm{s}}$. In this case $p^0 \neq \mathbf{1}$ and the matrix $\begin{pmatrix} 0 & 0 \\ 0 & p^0 \end{pmatrix}$ defines a non-trivial projection morphism leaving $\mathbb{I}\Gamma^{S\odot}$ invariant.

(ii) $F = \overline{F^0}^{\mathrm{s}} \neq \overline{F_0}^{\mathrm{s}}$. Set $q = \mathbf{1} - p_0$. We may rewrite an arbitrary element of F^0 as

$$\sum_{i=1}^{n} a_i \zeta_i b_i = \sum_{i=1}^{n} (a_i \zeta_i - \zeta_i a_i) b_i + \sum_{i=1}^{n} (\zeta_i a_i - \zeta a_i) b_i + \zeta \sum_{i=1}^{n} a_i b_i,$$

where $\zeta \in S_F$ is arbitrary. We find $q \sum_{i=1}^{n} a_i \zeta_i b_i = q\zeta \sum_{i=1}^{n} a_i b_i$. Putting $a_i = b_i = \mathbf{1}\delta_{ik}$, we see that the element $\eta = q\zeta$ cannot depend on ζ. Varying $a_k = b$ for $\zeta_k = \zeta$, we see that $b\eta = \eta b$, i.e. $\eta \in C_{\mathcal{B}}(F)$. Finally, $p_0 \neq \mathbf{1}$ and $\eta \neq 0$. Hence, the matrix $\begin{pmatrix} -\langle \eta, \eta \rangle & \eta^* \\ \eta & p_0 \end{pmatrix}$ defines a non-trivial projection morphism leaving $\mathbb{I}\Gamma^{S\odot}$ invariant.

(iii) $F = \overline{F^0}^{\mathrm{s}} = \overline{F_0}^{\mathrm{s}}$. Consider the projection morphism with matrix $\begin{pmatrix} -\langle \eta, \eta \rangle & \eta^* \\ \eta & p \end{pmatrix}$ and suppose that it leaves $\mathbb{I}\Gamma^{S\odot}$ invariant. Then $\zeta = \eta + p\zeta$ for all $\zeta \in S_F$. Since η is in the center, an element in F_0 written as in (5.3.1) does not change, if we replace ζ_i with $p\zeta_i$. It follows $pF = p\overline{F_0}^{\mathrm{s}} = \overline{F_0}^{\mathrm{s}} = F$, whence $p = \mathbf{1}$ and $\eta = (\mathbf{1} - p)\eta = 0$. Therefore, the only (continuous) projection morphism leaving $\mathbb{I}\Gamma^{S\odot}$ invariant is the identity morphism. $\square$

5.3.3. Corollary. *A single unit $\xi^{\odot}(\beta, \zeta)$ is generating for $\mathbb{I}\Gamma^{s\odot}(F)$, if and only if*

$$F = \overline{\mathsf{span}}^{\mathrm{s}}\{(b\zeta - \zeta b)b' : b, b' \in \mathcal{B}\}.$$

5.3.4. Remark. In the case where $\mathcal{B} = \mathscr{B}(G)$ for some separable Hilbert space G we have $F = \mathscr{B}(G, G \bar{\otimes} \mathfrak{H})$ where $\mathfrak{H} \cong \mathsf{id} \otimes \mathfrak{H} = C_{\mathcal{B}}(F)$ is the center of F and

$\zeta = \sum_n b_n \otimes e_n$ for some ONB $(e_n)_{n \in N}$ (N a subset of $\mathbb{N}$) and $b_i \in \mathcal{B}$ such that $\sum_n b_n^* b_n < \infty$. The condition stated in [Bha01], which, therefore, should be equivalent to our cyclicity condition in Corollary 5.3.3, asserts that the set $\{\mathbf{1}, b_1, b_2, \dots\}$ should be linearly independent in a certain sense (stronger than usual linear independence).

5.3.5. Observation. We see explicitly that the property of the set S to be generating or not is totally independent of the parameters β_i of the units $\xi^{\odot}(\beta_i, \zeta_i)$ in S. Of course, we new this before from the proof of Lemma 5.3.1.

5.3.6. Remark. We may rephrase Step (ii) as $\overline{F^0}^{\mathrm{s}} = \overline{F_0}^{\mathrm{s}} \oplus q\mathcal{B}$ for some central projection in $q \in \mathcal{B}$ such that $q\mathcal{B}$ is the strongly closed ideal in $\mathcal{B}$ generated by $\langle \eta, \eta \rangle$. By the same argument as in Step (iii) we obtain the most important consequence.

5.3.7. Corollary. *The mapping*

$$\xi^{\odot}(\beta, \zeta) \mapsto \xi^{\odot}\left(\beta + \frac{\langle \eta, \eta \rangle}{2}, \zeta - \eta\right)$$

(*which is isometric by* (2.3.5)) *extends as an isomorphism from the subsystem of* $\mathrm{I\!\Gamma}^{\mathrm{s}\odot}(F)$ *generated by* S *onto* $\mathrm{I\!\Gamma}^{\mathrm{s}\odot}(\overline{F_0}^{\mathrm{s}})$. *In other words, each strongly closed product subsystem of the time ordered product system* $\mathrm{I\!\Gamma}^{\mathrm{s}\odot}(F)$ *of von Neumann modules generated by a subset* $S \subset \mathcal{U}_c(F)$ *of continuous units, is isomorphic to a time ordered product system of von Neumann modules over a von Neumann submodule of* F.

5.3.8. Remark. If $\overline{F_0}^{\mathrm{s}} \neq \overline{F^0}^{\mathrm{s}}$, then, clearly, the subsystem isomorphic to $\mathrm{I\!\Gamma}^{\mathrm{s}\odot}(\overline{F_0}^{\mathrm{s}})$ does not coincide with the subsystem $\mathrm{I\!\Gamma}^{\mathrm{s}\odot}(\overline{F_0}^{\mathrm{s}})$. It does not even contain the vacuum unit of $\mathrm{I\!\Gamma}^{\mathrm{s}\odot}(F)$.

5.3.9. Remark. If S contains a unit $\xi^{\odot}(\beta_0, \zeta_0)$ with $\zeta_0 = 0$ (in other words, as for the condition in Theorem 5.3.2 we may forget about β_0, if S contains the vacuum unit $\omega^{\odot} = \xi^{\odot}(0, 0)$), then $F_0 = F^0$. (Any value of $\sum_{i=1}^n a_i b_i$ may be compensated in $\sum_{i=0}^n a_i b_i$ by a suitable choice of a_0, b_0, because $a_0 \zeta_0 b_0$ does not contribute to the sum $\sum_{i=0}^n a_i \zeta_i b_i$.) We obtain a strong version of Theorem 3.5.5.

5.4. Type $\mathrm{I}_n^{\mathrm{s}}$ product systems

5.4.1. Theorem. *Let $T = (T_t)_{t \in \mathbb{R}_+}$ be a normal uniformly continuous CP-semigroup on a von Neumann algebra $\mathcal{B}$. Let F, $\zeta \in F$, and $\beta \in \mathcal{B}$ be as in Theorem* C.4 (*by* [CE79]), *i.e. F is a von Neumann $\mathcal{B}$–$\mathcal{B}$-module such that $F = \overline{\mathrm{span}}^{\mathrm{s}}\{(b\zeta - \zeta b)b' : b, b' \in \mathcal{B}\}$ and $T^{(\beta,\zeta)} = T$. Then the strong closure of the GNS-system of T is* (*up to isomorphism*) $\mathrm{I\!\Gamma}^{\mathrm{s}\odot}(F)$ *and the generating unit is $\xi^{\odot}(\beta, \zeta)$. Here F and $\xi^{\odot}(\beta, \zeta)$ are determined up to unitary isomorphism.*

Proof. This is a direct consequence of Theorem C.4 and Corollary 5.3.3 of Theorem 5.3.2. □

Proof of Theorem 3.5.2. By Theorem 5.4.1 the subsystem of the GNS-system generated by a single unit in S has a central (continuous) unit. By Theorem 5.1.2 the generator of $\mathfrak{T}$ is a CE-generator. The uniqueness statement follows as in Corollary 5.3.7 from the construction of the module $\overline{F_0}^{\mathrm{s}}$. □

5.4.2. Theorem. *Type* $\mathrm{I}_n^{\mathrm{s}}$ *product systems are time ordered product systems of von Neumann modules.*

Proof. By Theorem 3.5.2 (and Corollary 5.1.3) a type $\mathrm{I}_n^{\mathrm{s}}$ product system is contained in a time ordered product system. By Corollary 5.3.7 it is all of a time ordered product system. □

5.4.3. Corollary. *The* (*strong closure of the*) *GNS-system of a uniformly continuous normal CPD-semigroup is a time ordered product system of von Neumann modules.*

Extensions. Section 5.1 works for Hilbert modules F (even for pre-Hilbert modules, but honestly speaking, it is not reasonable to do so, because the construction of sufficiently many units in a time ordered Fock modules involves norm limits). Also the analysis of continuous morphisms in Section 5.2 works for Hilbert modules. In the proof of Theorem 5.3.2 we need projections onto submodules in two different places. Firstly, we need the projections onto the submodules $\overline{F_0}^{\mathrm{s}}$ and $\overline{F^0}^{\mathrm{s}}$ of F. Secondly, if S is not strongly generating, then there should exists projections onto the members of the subsystem strongly generated by S.

For both it is sufficient that F is a right von Neumann module (the left action of $\mathcal{B}$ need not be normal). Then the projections onto $\overline{F_0}^{\mathrm{s}}$ and $\overline{F^0}^{\mathrm{s}}$, clearly, exist. But, also for the second condition we simply may pass to the strong closure of the members of the product systems. (For this it is sufficient that $\mathcal{B}$ is a von Neumann algebra. Left multiplication by $b \in \mathcal{B}$ is strongly continuous as operation on the module. It just may happen that left multiplication is not strongly continuous as mapping $b \mapsto bx$.) This even shows that $\mathbb{I}\Gamma^{\odot}(F)$ and $\mathbb{I}\Gamma^{\mathrm{s}\odot}(F)$ have the same continuous morphisms (in particular, projection morphisms), as soon as F is a right von Neumann module (of course, still a Hilbert $\mathcal{B}$–$\mathcal{B}$-module), because any continuous morphism leaves invariant the continuous units and whatever is generated by them in whatever topology.

As Lemma C.2 does not need normality, Theorem 5.4.1 remains true for uniformly continuous CP-semigroups (still on a von Neumann algebra). We find Theorem 3.5.2 for uniformly continuous CPD-semigroups. Consequently, Theorem 5.4.2 remains true for type I^{s} product systems of (right) von Neumann modules and Corollary 5.4.3 remains true for uniformly continuous CPD-semigroups on von Neumann algebras.

Finally, all results can be extended in the usual way to the case when $\mathcal{B}$ is a (unital) C^*-algebra, by passing to the bidual $\mathcal{B}^{**}$. We obtain then the weaker statements that the type I product systems and GNS-systems of uniformly continuous CPD-semigroups are strongly dense subsystems of product systems of von Neumann modules associated with time ordered Fock modules. Like in the case of the CE-generator, we can no longer guarantee that the inner products of the canonical units $\xi^{\odot}$ and the β_{ξ} are in $\mathcal{B}$. Example 4.2.4 shows clearly (maybe, more clearly than existing examples) that we cannot discuss this away: There are product systems of uniformly continuous CP-semigroups (even automorphism groups) on a unital C^*-algebra whose generator cannot be written in CE-form.

Resumé. Notice that Theorem 5.4.1 is the first and the only time where we use the results by Christensen and Evans [CE79] quoted in Appendix C (in particular, Lemma C.2). In Sections 5.1 and 5.2 we reduced the proof of Theorem 5.4.2 to the problem to show existence of a central unit among the (continuous) units of a type $\mathrm{I}_n^{\mathrm{s}}$ product system. In fact, Lemma 5.1.1 together with Corollary 5.3.7 shows that existence of a central unit is equivalent to Lemma C.2. With our methods we are also able to conclude back from the form (3.5.1) of a generator to Lemma C.2, a result which seems not to be accessible by the methods in [CE79]. We summarize:

5.4.4. Theorem. *The following statements are equivalent*:

1. *Bounded derivations with values in a von Neumann module are inner.*
2. *The generator of a normal uniformly continuous CP-semigroup on a von Neumann algebra has CE-form.*
3. *The GNS-system of a normal uniformly continuous CP-semigroup on a von Neumann algebra has a central unital unit.*

If we are able to show existence of a central unit directly, then we will provide a new proof of the results by Christensen and Evans [CE79]. We do not yet have concrete results into that direction. But, we expect that a proof, if possible, should reduce the problem to the application of one deep theorem (like the Krein–Milman theorem or an existence theorem for solutions of quantum stochastic differential equations) and rather algebraic computations in product systems. Also the order structure of CPD-semigroups, which we discuss in Appendix A, could play an essential role.

We remark that the methods from Section 5.1 should work to some extent also for unbounded generators. More precisely, if $E^{\odot}$ is a product system with a central unital unit $\omega^{\odot}$ such that the semigroups $\mathfrak{U}^{\xi,\omega}$ in $\mathcal{B}$ have a reasonable generator (not in $\mathcal{B}$, but for instance, a closed operator on G, when $\mathcal{B} \subset \mathscr{B}(G)$), then this should be sufficient to split of a (possibly unbounded) completely positive part from the generator. It is far from being clear what a "GNS-construction" for such unbounded completely positive mappings could look like (see, for instance, the example from [LS01] mentioned in Remark 2.3.15). Nevertheless, the splitting of the generator alone, so far a postulated property in literature, would constitute a considerable improvement.

6. Outlook

In these notes we defined type I product systems and we clarified the structure of type I systems of von Neumann modules as being (up to isomorphism) time ordered systems. For type I systems of Hilbert modules we know at least that they are (strongly dense) subsystems of time ordered systems of von Neumann modules. Example 4.2.4 tells us that this may not be improved without additional assumptions.

In [Ske01c] the category of *spatial* product systems of Hilbert modules is defined as those which admit a central unital unit. It is shown that a spatial type I system of Hilbert modules (a so-called *completely spatial* system) is isomorphic to a time ordered system $\mathbb{I}\Gamma^{\odot}(F)$ for a two-sided Hilbert module F (unique up to isomorphism). Moreover, a spatial product system contains a unique maximal completely spatial subsystem. The *index* of a spatial system is defined as the two-sided module F of its maximal completely spatial subsystem and a product of spatial product systems is provided, under which the index is additive (direct sum).

So far, we have a theory of type $\mathrm{I}_n^{\mathrm{s}}$ systems and of spatial type I systems which parallels completely that of Arveson. A *uniform* definition of type I was possible, because the properties of a type I system do not depend on our choice of the generating set of continuous units. (A simple multiplication by a non-measurable phase function shows that incompatible choices are possible.) For more general product systems, those not of type I, it is no longer possible to express continuity requirements just in terms of units. Presently, we are working on a definition of continuous types II and III systems; see [Ske03]. For type II systems, where we fix a unital reference unit, our definition will be compatible with that notion of a continuous section which comes from the embedding of all E_t into the same inductive limit E; see Section 4.4. Example 4.2.8 provides us with a type III system.

We see in the case of spatial systems that we have to distinguish between two different types of units, such which are just continuous and central ones. Only in the case of von Neumann modules the difference between spatial and non-type III disappears. We mention also a construction from Liebscher [Lie03] who constructs from every Arveson system a type II Arveson system (with index $\{0\}$). This construction promises to work also for Hilbert modules and von Neumann modules. Presently, we apply it starting from both time ordered systems and our type III example.

With any Arveson system there is an associated *spectral* C^*-algebra. Zacharias [Zac00a,Zac00b] computed their K-theory and showed their pure infiniteness in the non-type III case. Also here it is likely that the same methods work for spatial product systems of Hilbert modules.

Appendix A. Morphisms and order

The goal of this appendix is to establish the analogue of Theorem 3.3.3 for the (strong closure of the) GNS-system of a (normal) CPD-semigroup $\mathfrak{T}$ in $\mathcal{K}_S(\mathcal{B})$ for some von Neumann algebra $\mathcal{B}$. It is a straightforward generalization of the result for CP-semigroups obtained in [BS00] and asserts that the set of CPD-semigroups *dominated* by $\mathfrak{T}$ is order isomorphic to the set of positive contractive morphisms of its GNS-system. Then we investigate this order structure for the time ordered Fock module with the methods from Section 5.2.

A.1. Definition. Let $\mathfrak{T}$ be a CPD-semigroup in $\mathcal{K}_S(\mathcal{B})$. By $\mathcal{D}_{\mathfrak{T}}$ we denote the set of CPD-semigroups $\mathfrak{S}$ in $\mathcal{K}_S(\mathcal{B})$ *dominated* by $\mathfrak{T}$, i.e. $\mathfrak{S}_t \in \mathcal{D}_{\mathfrak{T}_t}$ for all $t \in \mathbb{T}$, which we indicate by $\mathfrak{T} \geqslant \mathfrak{S}$. If we restrict to normal CPD-semigroups, then we write $\mathcal{K}_S^n(\mathcal{B})$ and $\mathcal{D}_{\mathfrak{T}}^n$, respectively.

Obviously, $\geqslant$ defines partial order among the CPD-semigroups.

A.2. Proposition. *Let $\mathfrak{T} \geqslant \mathfrak{S}$ be two CPD-semigroups in $\mathcal{K}_S(\mathcal{B})$. Then there exists a unique contractive morphism $v^{\odot} = (v_t)_{t\in\mathbb{T}}$ from the GNS-system $E^{\odot}$ of $\mathfrak{T}$ to the GNS-system $F^{\odot}$ of $\mathfrak{S}$, fulfilling $v_t \xi_t^{\sigma} = \zeta_t^{\sigma}$ for all $\sigma \in S$.*

Moreover, if all v_t have an adjoint, then $w^{\odot} = (v_t^* v_t)_{t\in\mathbb{T}}$ is the unique positive, contractive endomorphism of $E^{\odot}$ fulfilling $\mathfrak{S}_t^{\sigma,\sigma'}(b) = \langle \xi_t^{\sigma}, w_t b \xi_t^{\sigma'} \rangle$ for all $\sigma, \sigma' \in S$, $t \in \mathbb{T}$ and $b \in \mathcal{B}$.

Proof. This is a combination of the construction in the proof of Lemma 3.3.2 (which asserts that there is a family of contractions $\breve{v}_t$ from the Kolmogorov decomposition $\breve{E}_t$ of $\mathfrak{T}_t$ to the Kolmogorov decomposition $\breve{F}_t$ of $\mathfrak{S}_t$) and arguments like in Section 4.3. More precisely, denoting by $\beta_{\mathfrak{ts}}^{\mathfrak{T}}, i_{\mathfrak{t}}^{\mathfrak{T}}$ and $\beta_{\mathfrak{ts}}^{\mathfrak{S}}, i_{\mathfrak{t}}^{\mathfrak{S}}$ the mediating mappings and the canonical embeddings for the two-sided inductive limits for the CPD-semigroups $\mathfrak{T}$ and $\mathfrak{S}$, respectively, we have to show that the mappings $i_{\mathfrak{t}}^{\mathfrak{S}} \breve{v}_{\mathfrak{t}} \breve{E}_{\mathfrak{t}} \to F_t$, where $v_{\mathfrak{t}} = \breve{v}_{t_n} \odot \cdots \odot \breve{v}_{t_1}$ $(\mathfrak{t} \in \mathbb{J}_t)$, define a mapping $v_t : E_t \to F_t$ (obviously, contractive and bilinear). From

$$\breve{v}_{\mathfrak{s}} \odot \breve{v}_{\mathfrak{t}} = \breve{v}_{\mathfrak{s} \smile \mathfrak{t}} \tag{A.1}$$

we conclude $\beta_{\mathfrak{ts}}^{\mathfrak{S}} \breve{v}_{\mathfrak{s}} = \breve{v}_{\mathfrak{t}} \beta_{\mathfrak{ts}}^{\mathfrak{T}}$. Applying $i_{\mathfrak{t}}^{\mathfrak{S}}$ to both sides the statement follows. Again from (A.1) (and Remark 4.3.6) we find that $v_s \odot v_t = v_{s+t}$. Clearly, $v^{\odot}$ is unique, because we know the values on a generating set of units. The statements about $w^{\odot}$ are now obvious. $\square$

A.3. Theorem. *Let $E^{\bar{\odot}^s} = (E_t)_{t\in\mathbb{T}}$ be a product system of von Neumann $\mathcal{B}$–$\mathcal{B}$-modules E_t, and let $S \subset \mathscr{U}(E^{\bar{\odot}^s})$ be a subset of units for $E^{\bar{\odot}^s}$. Then the mapping $\mathfrak{O} : w^{\odot} \mapsto \mathfrak{S}_w$ defined by setting*

$$(\mathfrak{S}_w^{\xi,\xi'})_t(b) = \langle \xi_t, w_t b \xi_t' \rangle$$

for all $t \in \mathbb{T}$, $\xi, \xi' \in S$, $b \in \mathcal{B}$, establishes an order morphism from the set of contractive, positive morphisms of $E^{\bar{\odot}^s}$ (equipped with pointwise order) onto the set $\mathcal{D}_{\mathfrak{T}}^n$ of normal

CPD-semigroups $\mathfrak{S}$ *dominated by* $\mathfrak{T} = \mathfrak{U}\upharpoonright S$. *It is an order isomorphism, if and only if* $E^{S\bar{\odot}^s} = E^{\bar{\odot}^s}$.

Proof. If $E^{S\bar{\odot}^s} \neq E^{\bar{\odot}^s}$, then $\mathfrak{O}$ is not one-to-one, because the identity morphism $w_t = \mathsf{id}_{E_t}$ and the morphism $p^{\odot} = (p_t)_{t\in\mathbb{T}}$ of projections p_t onto $\overline{E_t^S}^s$ are different morphisms giving the same CPD-semigroup $\mathfrak{S}_w = \mathfrak{S}_p$. On the other hand, any morphism $w^{\odot}$ for $E^{S\bar{\odot}^s}$ extends to a morphism composed of mappings $w_t p_t$ of $E^{\bar{\odot}^s}$ giving the same Schur semigroup $\mathfrak{S}_w$. Therefore, we are done, if we show the statement for $E^{S\bar{\odot}^s} = E^{\bar{\odot}^s}$.

So let us assume that S is generating. Then $\mathfrak{O}$ is one-to-one. It is also order preserving, because $w^{\odot} \geqslant w'^{\odot}$ implies

$$\begin{aligned}(\mathfrak{S}_w^{\xi,\xi'})_t(b) - (\mathfrak{S}_{w'}^{\xi,\xi'})_t(b) &= \langle \xi_t, (w_t - w'_t)b\xi'_t \rangle \\ &= \langle \sqrt{w_t - w'_t}\xi_t, b\sqrt{w_t - w'_t}\xi'_t \rangle \end{aligned} \tag{A.2}$$

so that $(\mathfrak{S}_w)_t \geqslant (\mathfrak{S}_{w'})_t$ in $\mathcal{K}_S(\mathcal{B})$. By obvious extension of Proposition A.6 to von Neumann modules, which guarantees existence of v_t^*, we see that $\mathfrak{O}$ is onto. Now let $\mathfrak{T} \geqslant \mathfrak{S} \geqslant \mathfrak{S}'$ with morphisms $w^{\odot} = \mathfrak{O}^{-1}(\mathfrak{S})$ and $w'^{\odot} = \mathfrak{O}^{-1}(\mathfrak{S}')$ and construct $v_t \in \mathscr{B}^{\mathrm{a,bil}}(\bar{E}_t^{\mathrm{s}}, \bar{F}_t^{\mathrm{s}})$, $v'_t \in \mathscr{B}^{\mathrm{a,bil}}(\bar{E}_t^{\mathrm{s}}, \overline{F'}_t^{\mathrm{s}})$, and $u_t \in \mathscr{B}^{\mathrm{a,bil}}(\bar{F}_t, \overline{F'}_t^{\mathrm{s}})$, for the pairs $\mathfrak{T} \geqslant \mathfrak{S}$, $\mathfrak{T} \geqslant \mathfrak{S}'$, and $\mathfrak{S} \geqslant \mathfrak{S}'$, respectively, as in Proposition A.6 and extension to the strong closures. Then by uniqueness we have $v'_t = u_t v_t$. It follows $w_t - w'_t = v_t^*(\mathbf{1} - u_t^* u_t)v_t \geqslant 0$. This shows that also $\mathfrak{O}^{-1}$ respects the order and, therefore, is an order isomorphism. (Observe that for the last conclusion (A.2) is not sufficient, because the vectors $b\xi_t b'$ ($\xi^{\odot} \in S; b, b' \in \mathcal{B}$) do not span E_t.) $\square$

Observe that this result remains true, if we require that the morphisms respect some subset of units like, for instance, the continuous units in the time ordered Fock module. We investigate now the order structure of the set of (possibly unbounded) positive continuous morphisms on $\mathrm{I\!\Gamma}^{\mathcal{U}_c\odot}(F)$. We will see that it is mirrored by the positivity structure of the corresponding matrices $\Gamma \in \mathscr{B}^{\mathrm{a,bil}}(\mathcal{B} \oplus F)$ where F is an arbitrary Hilbert $\mathcal{B}$–$\mathcal{B}$-module. Recalling that by Lemma 2.1.9 positive contractions are dominated by $\mathbf{1}$, we find a simple criterion for contractive positive morphisms as those whose matrix Γ is dominated (in $\mathscr{B}^{\mathrm{a,bil}}(F)$) by the matrix $\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ of the identity morphism.

A.4. Lemma. *A (possibly unbounded) continuous endomorphism* $w^{\odot}$ *of* $\mathrm{I\!\Gamma}^{\mathcal{U}_c\odot}(F)$ *with the matrix* $\Gamma = \begin{pmatrix} \gamma & \eta^* \\ \eta & a \end{pmatrix}$ *is positive, if and only if it is self-adjoint and* a *is positive.*

Proof. $w^{\odot}$ is certainly positive, if it is possible to write it as a square of a self-adjoint morphism with matrix $\hat{\varDelta} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ \delta & \mathbf{1} & \chi^* \\ \chi & 0 & c \end{pmatrix}$ say (δ and c self-adjoint). In other words, we must have

$$\begin{pmatrix} \mathbf{1} & 0 & 0 \\ \gamma & \mathbf{1} & \eta^* \\ \eta & 0 & a \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ \delta & \mathbf{1} & \chi^* \\ \chi & 0 & c \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 & 0 \\ \delta & \mathbf{1} & \chi^* \\ \chi & 0 & c \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 2\delta + \langle \chi, \chi \rangle & \mathbf{1} & \chi^* + (c\chi)^* \\ \chi + c\chi & 0 & c^2 \end{pmatrix}.$$

This equation can easily be resolved, if $a \geqslant 0$. We put $c = \sqrt{a}$. Since $c \geqslant 0$ we have $\mathbf{1} + c \geqslant \mathbf{1}$ so that $\mathbf{1} + c$ is invertible. We put $\chi = (\mathbf{1} + c)^{-1}\eta$. Finally, we set $\delta = \frac{\gamma - \langle \chi, \chi \rangle}{2}$ $(= \delta^*)$. Then $\hat{\varDelta}$ determines a self-adjoint endomorphism whose square is $w^{\odot}$.

On the other hand, if $w^{\odot}$ is positive, then Γ is self-adjoint and the generator $\mathfrak{L}_w$ of the CPD-semigroup $\mathfrak{S}_w$ is conditionally completely positive definite. For $\mathfrak{L}_w$ we find (rewritten conveniently)

$$\mathfrak{L}_w^{(\beta,\zeta),(\beta',\zeta')}(b) = \langle \zeta, ba\zeta' \rangle + b\Big(\langle \eta, \zeta' \rangle + \beta' + \frac{\gamma}{2}\Big) + \Big(\langle \zeta, \eta \rangle + \beta^* + \frac{\gamma}{2}\Big)b.$$

For each $\zeta \in F$ we choose $\beta \in \mathcal{B}$ such that $\langle \zeta, \eta \rangle + \beta^* + \frac{\gamma}{2} = 0$. Then it follows as in Remark 5.3.9 ($\zeta = 0 \in F$) that the kernel $b \mapsto \langle \zeta, ba\zeta' \rangle$ on F is not only conditionally completely positive definite, but completely positive definite. This implies that $a \geqslant 0$. $\square$

A.5. Remark. By applying the lemma to the endomorphism with matrix $\hat{\varDelta}$, we see that it is positive, too.

A.6. Lemma. *For two self-adjoint possibly unbounded morphisms $w^{\odot}$ and $v^{\odot}$ with matrices $\Gamma = \begin{pmatrix} \gamma & \eta^* \\ \eta & a \end{pmatrix}$ and $\varDelta = \begin{pmatrix} \delta & \chi^* \\ \chi & c \end{pmatrix}$, respectively, we have $w^{\odot} \geqslant v^{\odot}$, if and only if $\Gamma \geqslant \varDelta$ in $\mathscr{B}^{\mathrm{a,bil}}(\mathcal{B} \oplus F)$.*

Proof. By Theorem A.3 and Lemma 3.4.12 we have $w^{\odot} \geqslant v^{\odot}$, if and only if $\mathfrak{S}_w \geqslant \mathfrak{S}_v$, if and only if $\mathfrak{L}_w \geqslant \mathfrak{L}_v$. By Eqs. (5.2.2) and (5.2.4) we see that in the last infinitesimal form $\mathfrak{L}_w - \mathfrak{L}_v$, only the difference $\Gamma - \varDelta$ enters. Furthermore, evaluating the difference of these kernels at concrete elements $\xi^{\odot}(\beta, \zeta), \xi^{\odot}(\beta', \zeta')$, the β, β' do not contribute. Therefore, it is sufficient to show the statement in the case when $\varDelta = 0$, i.e. $w^{\odot}$ dominates (or not) the morphism $v^{\odot}$ which just projects onto the vacuum, and to check completely positive definiteness only against exponential units. We find

$$\begin{aligned}\sum_{i,j} b_i^*(\mathfrak{L}_w - \mathfrak{L}_v)^{(0,\zeta_i),(0,\zeta_j)}(a_i^* a_j) b_j \\ &= \sum_{i,j} b_i^*(\langle \zeta_i, a_i^* a_j a \zeta_j \rangle + \langle \zeta_i, a_i^* a_j \eta \rangle + a_i^* a_j \langle \eta, \zeta_j \rangle + a_i^* a_j \gamma) b_j \\ &= \sum_{i,j} \langle a_i \zeta_i b_i, a a_j \zeta_j b_j \rangle + \langle a_i \zeta_i b_i, \eta \rangle a_j b_j + (a_i b_i)^* \langle \eta, a_j \zeta_j b_j \rangle + (a_i b_i)^* \gamma a_j b_j \\ &= \langle Z, \Gamma Z \rangle,\end{aligned}$$

where $Z = \sum_i (a_i b_i, a_i \zeta_i b_i) \in \mathcal{B} \oplus F$. Elements of the form Z do, in general, not range over all of $\mathcal{B} \oplus F$. However, to check positivity of Γ with $(\zeta, \beta) \in \mathcal{B} \oplus F$ we choose $\zeta_1 = \lambda\zeta$, $\zeta_2 = 0$, $a_1 = a_2 = \mathbf{1}$, and $b_1 = \frac{\mathbf{1}}{\lambda}$, $b_2 = \beta$. Then $Z \to (\beta, \zeta)$ for $\lambda \to \infty$. This means that $\mathfrak{L}_w - \mathfrak{L}_v \geqslant 0$, if and only if $\Gamma (= \Gamma - \Delta) \geqslant 0$. □

A.7. Corollary. *The set of contractive positive continuous morphisms of* $\mathbb{I}\Gamma^{\odot}(F)$ *is order isomorphic to the set of those self-adjoint matrices* $\Gamma \in \mathscr{B}^{\mathrm{a,bil}}(\mathcal{B} \oplus F)$ *with* $a \geqslant 0$ *and* $\Gamma \leqslant \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$.

It is possible to characterize these matrices further. We do not need this characterization.

Appendix B. CPD-semigroups in $\mathcal{K}_S(\mathcal{B})$ versus CP-semigroups on $\mathscr{B}(H_S)\bar{\otimes}^s\mathcal{B}$

In the proof of Theorem 5.1.2 we utilized the possibility to pass from a product system $E^{\odot}$ of $\mathcal{B}$–$\mathcal{B}$-modules to a product system $M_n(E^{\odot})$ of $M_n(\mathcal{B})$–$M_n(\mathcal{B})$-modules. Given a family $\xi^{i\odot}$ $(i = 1, \ldots, n)$ of units for $E^{\odot}$ we defined the diagonal unit $\Xi^{\odot}$ for $M_n(E^{\odot})$ with diagonal entries $\xi^{i\odot}$.

We remark that $\Xi^{\odot}$ is generating for $M_n(E^{\odot})$, if and only if the set $S = \{\xi^{1\odot}, \ldots, \xi^{n\odot}\}$ is generating for $E^{\odot}$. In this case $T^{\Xi}(B) = \langle \Xi_t, B\Xi_t \rangle$ is a CP-semigroup on $M_n(\mathcal{B})$ whose GNS-system is $M_n(E^{\odot})$. Moreover, T^{Ξ} is uniformly continuous, if and only if the CPD-semigroup $\mathfrak{U}(E^{\odot}) \upharpoonright S$ is (and the same holds for normality, if $\mathcal{B}$ is a von Neumann algebra). We may apply Theorem 5.4.1 to T^{Ξ} and obtain that the GNS-system of $M_n(\mathcal{B})$–$M_n(\mathcal{B})$-modules is isomorphic to a time ordered product system. Taking into account that as explained in Section 2.2 a product system of $M_n(\mathcal{B})$–$M_n(\mathcal{B})$-modules is always of the form $M_n(E_t)$ where the E_t form a product system, we obtain that the two descriptions are interchangeable. Specifying that, on the one hand, we look at product systems generated by not more than n units and, on the other hand, that we look only at CP-semigroups on $M_n(\mathcal{B})$ and units for $M_n(E^{\odot})$ which are diagonal, we obtain that the analogy is complete.

This way to encode the information of a CPD-semigroup into a single CP-semigroup is taken from Accardi and Kozyrev [AK99] which was also our motivation to study completely positive definite kernels and Schur semigroups of such. In [AK99] the authors considered only the case of the product system of *symmetric* (i.e. time ordered) Fock modules (see [Ske98]) $\Gamma^{s\odot}(L^2(\mathbb{R}_+,\mathscr{B}(G)))$ $\cong \mathbb{I}\Gamma^{s\odot}(\mathscr{B}(G))$, where two central exponential units, namely, the vacuum plus any other, are generating. They were lead to look at CP-semigroups on $M_2(\mathscr{B}(G))$. (Notice that in our case we have even interesting results with a single generating unit.) What we explained so far is the generalization to n generating units (in the case of $\mathcal{B} = \mathscr{B}(G)$ already known to the authors of [AK99]).

Now we want to extend the idea to generating sets S containing an arbitrary number of units. It is good to keep the intuitive idea of matrices, now of infinite, even possibly uncountable, dimension. Technically, it is better to change the picture from matrices $M_n(E)$ to exterior tensor products $M_n \otimes E$ as explained in Section 2.2. Now the diagonal unit $\Xi^{\odot}$ should have infinitely many entries. For that we must be able to control the norm of each entry. Some sort of continuity should be sufficient, but as we want to control also the norm of the generator, we restrict to the uniformly continuous case.

Let S be a set of continuous units for $\mathbb{I}\Gamma^{s\odot}(F)$ and denote by H_S the Hilbert space with ONB $(e_\xi)_{\xi^\odot \in S}$. We have

$$\begin{aligned} L^2(\mathbb{R}_+,\mathscr{B}(H_S)\bar{\otimes}^s F) &= L^2(\mathbb{R}_+)\bar{\otimes}^s(\mathscr{B}(H_S)\bar{\otimes}^s F) \\ &= \mathscr{B}(H_S)\bar{\otimes}^s(L^2(\mathbb{R}_+)\bar{\otimes}^s F) = \mathscr{B}(H_S)\bar{\otimes}^s L^2(\mathbb{R}_+,F), \end{aligned}$$

where $\mathscr{B}(H_S)\bar{\otimes}^s F$ and, henceforth, $L^2(\mathbb{R}_+,\mathscr{B}(H_S)\bar{\otimes}^s F)$ is a von Neumann $\mathscr{B}(H_S)\bar{\otimes}^s\mathcal{B}$–$\mathscr{B}(H_S)\bar{\otimes}^s\mathcal{B}$-module see Section 2.2. Consequently, we find

$$\mathscr{B}(H_S)\bar{\otimes}^s\mathbb{I}\Gamma^{s\odot}(F) = \mathbb{I}\Gamma^{s\odot}(\mathscr{B}(H_S)\bar{\otimes}^s F).$$

A continuous unit $\xi^\odot(B,Z)$ $(B \in \mathscr{B}(H_S)\bar{\otimes}^s\mathcal{B}, Z \in \mathscr{B}(H_S)\bar{\otimes}^s F)$ is *diagonal* (in the matrix picture), if and only if B and Z are diagonal. A diagonal unit $\xi^\odot(B,Z)$ is strongly generating for $\mathbb{I}\Gamma^{s\odot}(\mathscr{B}(H_S)\bar{\otimes}^s F)$, if and only if the set $\{\xi^\odot(\beta,\zeta)\}$ running over the diagonal entries of $\xi^\odot(B,Z)$ is strongly generating for $\mathbb{I}\Gamma^{s\odot}(F)$.

Can we put together the units from S to a single diagonal unit? In order that a family $(a_\xi)_{\xi \in S}$ of elements in $\mathcal{B}$ (in F) defines (as strong limit) an element in $\mathscr{B}(H_S)\bar{\otimes}^s\mathcal{B}$ (in $\mathscr{B}(H_S)\bar{\otimes}^s F$) with entries a_ξ in the diagonal, is it necessary and sufficient that it is uniformly bounded. This will, in general, not be the case. However, as long as we are only interested in whether S is generating or not, we may modify S without changing this property. By Observation 5.3.5 we may forget completely about the parameters β_ξ. Moreover, for the condition in Theorem 5.3.2 the length of the ζ_ξ is irrelevant (as long as it is not 0, of course). We summarize.

B.1. Theorem. *Let $\mathfrak{T}$ be a normal uniformly continuous CPD-semigroup on S in $\mathcal{K}_S(\mathcal{B})$. Then there exists a normal uniformly continuous CP-semigroup T on*

$\mathscr{B}(H_S)\bar{\otimes}^s\mathcal{B}$ such that the GNS-system (of von Neumann modules) of T is $\mathscr{B}(H_S)\bar{\otimes}^s E^{\bar{\odot}^s}$ where $E^{\bar{\odot}^s}$ is the GNS-system (of von Neumann modules) of $\mathfrak{T}$.

So far, we considered diagonal units for the time ordered Fock module $\mathbb{I}\Gamma^{s\odot}(\mathscr{B}(H_S)\bar{\otimes}^s F)$. Of course, $\xi^{\odot}(B,Z)$ is a unit for any choice of $B\in\mathscr{B}(H_S)\bar{\otimes}^s\mathcal{B}$ and $Z\in\mathscr{B}(H_S)\bar{\otimes}^s F$. The off-diagonal entries of such a unit fulfill a lot of recursive relations. In the case of Hilbert spaces ($\mathcal{B}=\mathbb{C}$) and finite sets S ($\mathscr{B}(H_S)=M_n$) we may hope to compute $\xi^{\odot}(B,Z)$ explicitly. This should have many applications in the theory of special functions, particularly those involving iterated integrals of exponential functions.

Appendix C. Generators of CP-semigroups

C.1. Definition. Let $\mathcal{A}$ be a unital Banach algebra and $\mathbb{T}=\mathbb{R}_+$ or $\mathbb{T}=\mathbb{N}_0$. A *semigroup in* $\mathcal{A}$ is a family $T=(T_t)_{t\in\mathbb{T}}$ of elements $T_t\in\mathcal{A}$ such that $T_0=\mathbf{1}$ and $T_sT_t=T_{s+t}$. If $\mathcal{A}=\mathscr{B}(B)$ is the algebra of bounded operators on a Banach space B (with composition $\circ$ as product), then we say T is a *semigroup on* B.

A semigroup $T=(T_t)_{t\in\mathbb{R}_+}$ in $\mathcal{A}$ is *uniformly continuous*, if

$$\lim_{t\to 0}||T_t-\mathbf{1}||=0.$$

If B is itself a Banach space of operators on another Banach space (for instance, if B is a von Neumann algebra), then T is *strongly continuous*, if $t\mapsto T_t(b)$ is strongly continuous in B for all $b\in B$.

The form of generators of uniformly continuous CP-semigroups was found by Christensen–Evans [CE79] for arbitrary, even non-unital, C^*-algebras $\mathcal{B}$. We quote the basic result [CE79, Theorem 2.1] rephrased in the language of *derivations* with values in a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module F, i.e. a mapping $d:\mathscr{B}\to F$ fulfilling

$$d(bb')=bd(b')+d(b)b'.$$

Then we repeat the cohomological discussion of [CE79] which allows to find the form of the generator in the case of von Neumann algebras.

C.2. Lemma. *Let d be a bounded derivation from a pre-C^*-algebra $\mathcal{B}$ $(\subset\mathscr{B}^a(G))$ to a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module F $(\subset\mathscr{B}^a(G,F\odot G))$. Then there exists $\zeta\in\overline{\mathsf{span}}^s d(\mathcal{B})\mathcal{B}$ $(\subset\bar{F}^s\subset\mathscr{B}(\bar{G},\overline{F\odot G}))$ such that*

$$d(b)=b\zeta-\zeta b. \tag{C.1}$$

Observe that $\overline{\mathsf{span}}^s d(\mathcal{B})\mathcal{B}$ is a two-sided submodule of $\bar{F}^s$. Indeed, we have $bd(b')=d(bb')-d(b)b'$ so that we have invariance under left multiplication.

Recall that a derivation of the form as in (C.1) is called *inner*, if $\zeta \in F$. Specializing to a von Neumann algebra $\mathcal{B}$ we reformulate as follows.

C.3. Corollary. *Bounded derivations from a von Neumann algebra $\mathcal{B}$ to a von Neumann $\mathcal{B}$–$\mathcal{B}$-module are inner* (*and*, *therefore*, *normal*).

Specializing further to the von Neumann module $\mathcal{B}$, we find the older result that bounded derivations on von Neumann algebras are inner; see e.g. [Sak71].

In the sequel, we restrict to normal CP-semigroups on von Neumann algebras. As an advantage (which is closely related to self-duality of von Neumann modules) we end up with simple statements as in Corollary C.3 instead of the involved ones in Lemma C.2. The more general setting does not give more insight (in fact, the only insight is that satisfactory results about the generator are only possible in the context of von Neumann algebras), but just causes unpleasant formulations.

C.4. Theorem (Christensen and Evans [CE79]). *Let T be a normal uniformly continuous CP-semigroup on a von Neumann algebra $\mathcal{B}$ with generator $\mathcal{L}$. Then there exist a von Neumann $\mathcal{B}$–$\mathcal{B}$-module F, an element $\zeta \in F$, and an element $\beta \in \mathcal{B}$ such that $\mathcal{L}$ has the Christensen–Evans form* (3.5.1). *Moreover*, *the strongly closed submodule of F generated by the derivation $d(b) = b\zeta - \zeta b$ is determined by $\mathcal{L}$ up to* (*two-sided*) *isomorphism.*

Proof. We proceed similarly as for the GNS-construction, and try to define an inner product on the $\mathcal{B}$–$\mathcal{B}$-module $\mathcal{B} \otimes \mathcal{B}$ with the help of $\mathcal{L}$. However, since $\mathcal{L}$ is only conditionally completely positive, we can define this inner product not for all elements in this module, but only for those elements in the two-sided submodule generated by elements of the form $b \otimes \mathbf{1} - \mathbf{1} \otimes b$. This is precisely the subspace of all $\sum_i a_i \otimes b_i$ for which $\sum_i a_i b_i = 0$ with inner product

$$\left\langle \sum_i a_i \otimes b_i, \sum_j a_j \otimes b_j \right\rangle = \sum_{i,j} b_i^* \mathcal{L}(a_i^* a_j) b_j. \tag{C.2}$$

We divide out the length-zero elements and denote by F the strong closure.

By construction, F is a von Neumann $\mathcal{B}$–$\mathcal{B}$-module and it is generated as a von Neumann module by the bounded derivation $d(b) = (b \otimes \mathbf{1} - \mathbf{1} \otimes b) + \mathcal{N}_F$. By Corollary C.3 there exists $\zeta \in F$ such that $d(b) = b\zeta - \zeta b$. Moreover, we have

$$\begin{aligned} &\mathcal{L}(bb') - b\mathcal{L}(b') - \mathcal{L}(b)b' + b\mathcal{L}(\mathbf{1})b' \\ &\quad = \langle \zeta, bb'\zeta \rangle - b\langle \zeta, b'\zeta \rangle - \langle \zeta, b\zeta \rangle b' + b\langle \zeta, \zeta \rangle b' \end{aligned}$$

from which it follows that the mapping $D : b \mapsto \mathcal{L}(b) - \langle \zeta, b\zeta \rangle - \frac{b(\mathcal{L}(\mathbf{1}) - \langle \zeta, \zeta \rangle) + (\mathcal{L}(\mathbf{1}) - \langle \zeta, \zeta \rangle)b}{2}$ is a bounded hermitian derivation on $\mathcal{B}$. Therefore, there exists $h = h^* \in \mathcal{B}$ such that $D(b) = ibh - ihb$. Setting $\beta = \frac{\mathcal{L}(\mathbf{1}) - \langle \zeta, \zeta \rangle}{2} + ih$ we find $\mathcal{L}(b) = \langle \zeta, b\zeta \rangle + b\beta + \beta^* b$.

Let F' be another von Neumann module with an element ζ' such that the derivation $d'(b) = b\zeta' - \zeta' b$ generates F' and such that $\mathcal{L}(b) = \langle \zeta', b\zeta' \rangle + b\beta' + \beta'^* b$ for some $\beta' \in \mathcal{B}$. Then the mapping $d(b) \mapsto d'(b)$ extends as a two-sided unitary $F \to F'$, because the inner product (C.2) does not depend on β. □

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