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On Morita's Localization

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INTRODUCTION

Any injective module $V \in R$-Mod defines a torsion radical (torsion theory) and an associated quotient category R-Mod$/V$, which is a full reflective Grothendieck subcategory of R-Mod with exact reflector (quotient functor). In this case, V can be chosen so that the ring of quotients it determines can be realized as the bicommutator (double centralizer) of V. Morita [13, 16] has generalized this construction by considering certain modules for which the reflector into the quotient category need not be exact, although he requires that the generalized ring of quotients must coincide with the bicommutator of the module.

It is possible to omit the latter condition, as is shown by Theorem 1.8 and this enlarges the class of modules that can be considered to include, in particular, any module that is injective modulo its annihilator. The new conditions obtained are not only more general, but appear to simplify the proofs as well.

Heinicke [7] has characterized localization functors as idempotent, left exact monads, and quotient categories as their categories of algebras. Given any monad T of R-Mod, there is an associated monad Q_T studied by Lambek and Rattray [10], which is idempotent and left exact if T is left exact and which gives the usual localization functor if $T = \mathrm{Hom}(\mathrm{Hom}(-, V), V)$ for ${}_RV$ injective. It is shown in Theorem 2.10 that if T is left exact when restricted to $Q_T(R)$-Mod, then the category of Q_T-algebras is a generalized quotient category in Morita's sense, thus extending Morita's theory of noncommutative localization. (This gives another proof of part of Theorem 1.8, in which, as in all of Section 1, the categorical language has been suppressed so as to provide easier access to the theory.)

The final section gives some applications. Morita's characterization of balanced modules [15] is modified in Theorem 3.1, while Theorem 3.2 extends

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a theorem of Lambek [8] giving a condition under which the generalized ring of quotients determined by V is a dense subring of the bicommutator of V. Theorem 3.5 generalizes results of [1], and Corollary 3.6 shows that if K is any ideal of R, then any subring of $Q_{\max}(R/K)$ determined by a radical of R-Mod is a generalized ring of quotients of R. Finally, Theorem 3.7 applies Theorem 2.10 to the monad $T = \mathrm{Hom}(P, P \otimes \text{-})$ and extends results of [5].

In the standard theory of noncommutative localization, for any ring of quotients Q of R, the kernel of the natural homomorphism $R \to Q$ must be the annihilator of an injective module. Morita's theory allows any ideal to be such a kernel and in addition, includes as a generalized ring of quotients any epimorphism in the category of rings. It is hoped that this theory will provide a language for certain more specialized constructions that are not standard rings of quotients.

1. On Morita's Localization

Let R be an associative ring with identity and let R-Mod denote the category of unital left R-modules. A subfunctor ρ of the identity on R-Mod is called a radical if $\rho(M/\rho(M)) = 0$ for all $M \in R$-Mod and a torsion radical if $\rho(M') = M' \cap \rho(M)$ for all submodules $M' \subseteq M$. Any class of modules defines a radical by assigning to each module the intersection of kernels of homomorphisms into the class and conversely, every radical is defined by the corresponding class of torsionfree modules. (Recall that M is ρ-torsion free if $\rho(M) = 0$ and ρ-torsion if $\rho(M) = M$.)

For $V \in R$-Mod, let rad_V be the radical of R-Mod defined for $M \in R$-Mod by

$$\mathrm{rad}_V(M) = \{m \in M \mid f(m) = 0 \text{ for all } f \in \mathrm{Hom}_R(M, V)\}.$$

If M' is a submodule of M, then the rad_V-closure of M' in M is

$$\{m \in M \mid f(m) = 0 \text{ for all } f \in \mathrm{Hom}_R(M, V) \text{ such that } f(M') = 0\}.$$

We say that M' is rad_V-closed in M if this closure is M' itself, which occurs if and only if M/M' is rad_V-torsionfree. If $f \in \mathrm{Hom}_R(M, N)$ and $f(M') \subseteq N'$ for submodules $M' \subseteq M$ and $N' \subseteq N$, then f maps the rad_V-closure of M' into the rad_V-closure of N'.

For $V \in R$-Mod, let $E = \mathrm{End}({}_RV)$ and $B = \mathrm{Bic}({}_RV) = \mathrm{End}(V_E)$. The composition of $\mathrm{Hom}_R(-, V)\colon R\text{-Mod} \to \text{Mod-}E$ and $\mathrm{Hom}_E(-, V)\colon \text{Mod-}E \to B\text{-Mod}$ is covariant (where Mod-E is the category of right E-modules). We denote this composition by $T_V\colon R\text{-Mod} \to B\text{-Mod}$ and note that T_V can be viewed as merely an endofunctor of R-Mod. If $\eta_M\colon M \to T_V(M)$ is defined by $\eta_M(m) = [f \to f(m)]$ for $m \in M$ and $f \in \mathrm{Hom}_R(M, V)$, then $\eta: I \to T_V$ defines a natural transformation, where I is the identity functor, and $\ker(\eta_M) = \mathrm{rad}_V(M)$. (When the meaning is obvious, we will write η instead of η_M.) If V is injective, then rad_V is a torsion radical and since $T_V(M)$ belongs to the quotient category determined by rad_V [8, Proposition 3.1], it follows that the rad_V-closure of $\eta(M)$ in $T_V(M)$ gives the module of quotients of M. This motivates the following definition given for any $V \in R$-Mod.

DEFINITION 1.1. For $V, M \in R$-Mod, let $Q_V(M)$ be the rad_V-closure of $\eta(M)$ in $T_V(M)$. In particular,

$$Q_V(R) = \{b \in \mathrm{Bic}({}_R V) \mid f(b) = 0 \text{ for all } f \in \mathrm{Hom}_R(\mathrm{Bic}({}_R V), V) \text{ such that } f\eta(R) = 0\}.$$

Note that Q_V: R-Mod $\to$ R-Mod is a functor, since, if $f \in \mathrm{Hom}_R(M, N)$, then $T_V(f)\,\eta_M = \eta_N f$ and so $T_V(f)$ maps $Q_V(M)$ into $Q_V(N)$. Thus, $Q_V(f)$ can be defined as the restriction of $T_V(f)$ to $Q_V(M)$.

$$\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[d, "f"'] & \eta(M) \subseteq Q_V(M) \subseteq T_V(M) \arrow[d, "T_V(f)"] \\
N \arrow[r, "\eta_N"] & \eta(N) \subseteq Q_V(N) \subseteq T_V(N)
\end{tikzcd}$$

The next proposition shows that actually, Q_V can be viewed as a functor from R-Mod to $Q_V(R)$-Mod.

PROPOSITION 1.2. *Let* $V \in R$-Mod.

(a) $Q_V(R)$ *is a subring of* $\mathrm{Bic}({}_R V)$.

(b) *Any* rad_V*-closed* R*-submodule of a* $\mathrm{Bic}({}_R V)$*-module is a* $Q_V(R)$*-submodule.*

(c) *If* $M, N \in \mathrm{Bic}({}_R V)$-Mod *and* $\mathrm{rad}_V(N) = 0$, *then any* R*-homomorphism* $f: M \to N$ *is a* $Q_V(R)$*-homomorphism.*

Proof. (c) If $f: M \to N$ satisfies the stated condition, then for $m \in M$, define $g \in \mathrm{Hom}_R(B, N)$ by $g(b) = bf(m) - f(bm)$, for all $b \in B = \mathrm{Bic}({}_R V)$. Since $g\eta(R) = 0$ and $\mathrm{rad}_V(N) = 0$, it follows that $g(Q_V(R)) = 0$ and thus, $qf(m) = f(qm)$ for all $q \in Q_V(R)$.

(b) If ${}_R M' \subseteq {}_B M$ is rad_V-closed, then it follows easily from (c) that $qm \in M'$ for all $q \in Q_V(R)$ and $m \in M'$.

(a) This follows from (b). ∎

PROPOSITION 1.3. *Let $V \in R$-Mod and let $0 \to M' \to^{i} M \to^{f} M'' \to 0$ be an exact sequence in R-Mod. Then, $Q_V(f)$ is an isomorphism if* $\mathrm{Hom}_R(i, V) = 0$.

Proof. If $\mathrm{Hom}_R(i, V) = 0$, then $\mathrm{Hom}_R(f, V)$ is an isomorphism and hence, $T_V(f)$ is an isomorphism. Since f is epic, $T_V(f)(\eta(M)) = \eta(M'')$ and this implies that $T_V(f)(Q_V(M)) = Q_V(M'')$. ∎

COROLLARY 1.4. *Let V, $M \in R$-Mod.*

(a) *$Q_V(M/M') \simeq Q_V(M)$ for any R-submodule $M' \subseteq \mathrm{rad}_V(M)$.*

(b) *$Q_V((R/K) \otimes_R M) \simeq Q_V(M)$ for any ideal $K \subseteq \mathrm{Ann}({}_R V)$.*

In Corollary 1.4, both isomorphisms are natural isomorphisms. Many questions are reduced by the second isomorphism to the case of faithful modules, since, if $K = \mathrm{Ann}({}_R V)$, then for any R/K-module M, the construction of $Q_V(M)$ is the same whether M and V are regarded as R/K-modules or as R-modules.

LEMMA 1.5. *The following conditions are equivalent for $V \in R$-Mod.*

(1) *For $M \in Q_V(R)$-Mod, any R-homomorphism $f: M \to V$ is a $Q_V(R)$-homomorphism.*

(2) *The natural map $\phi: V \to \mathrm{Hom}_R(Q_V(R), V)$, defined by $\phi(v)(q) = qv$ for $v \in V$, $q \in Q_V(R)$, is an isomorphism.*

(3) *Every R-homomorphism $f: Q_V(R) \to V$ can be extended to* $\mathrm{Bic}({}_R V)$.

Proof. (1) $\Rightarrow$ (2). This is immediate, since $V \in Q_V(R)$-Mod.

(2) $\Rightarrow$ (3). If $f: Q_V(R) \to V$, then by assumption, there exists $v \in V$ such that $f(q) = qv$ for all $q \in Q_V(R)$. This can be extended to g: $\mathrm{Bic}({}_R V) \to V$ by defining $g(b) = bv$, for all $b \in \mathrm{Bic}({}_R V)$.

(3) $\Rightarrow$ (1). As in the proof of Proposition 1.2 (c), given $f: M \to V$ and $m \in M$, define $g: Q_V(R) \to V$ and then extend g to $\mathrm{Bic}({}_R V)$ and apply Proposition 1.2. ∎

An additional condition equivalent to those of Lemma 1.5 is that the category of rad_V-torsionfree $Q_V(R)$-modules is a full subcategory of R-Mod. This can be shown by using condition (1). It follows that if the conditions are satisfied, then V determines the same radical rad_V of $Q_V(R)$-Mod whether viewed as a $Q_V(R)$-module or an R-module.

Following Morita, for $V \in R$-Mod, we let $D(V)$ denote the full subcategory of R-Mod determined by all modules ${}_R M$ for which there exists an exact sequence $0 \to M \to X_0 \to X_1$ in R-Mod, such that X_0 and X_1 are each isomorphic to a direct product of copies of V. Note that there exists such an exact sequence for M if and only if M is isomorphic to a rad_V-closed submodule of a direct product of copies of V. In this case, it follows from Proposition (1.2) that M is a $Q_V(R)$-module.

If M is any right module over $E = \mathrm{End}({}_RV)$, then a free resolution $\bigoplus_I E \to \bigoplus_J E \to M \to 0$ gives an exact sequence $0 \to \mathrm{Hom}_E(M, V) \to \mathrm{Hom}_E(\bigoplus_J E, V) \to \mathrm{Hom}_E(\bigoplus_I E, V)$, which is just an exact sequence $0 \to \mathrm{Hom}_E(M, V) \to \prod_J V \to \prod_I V$. This shows that $\mathrm{Hom}_E(-, V)$ is a functor from Mod-E into $\mathscr{D}(V)$ and in particular, T_V is a functor from R-Mod into $\mathscr{D}(V)$.

If ${}_RV$ is injective, then the quotient category R-Mod$/V$, determined by rad_V, coincides with $\mathscr{D}(V)$ and in fact, every quotient category can be obtained in this manner [13, Theorem 5.4]. In this case, $\mathscr{D}(V)$ is a reflective subcategory of R-Mod and the next proposition gives a more general condition under which this occurs. In fact, conditions (b) and (c) are equivalent without any assumptions [11, Proposition 1.2].

LEMMA 1.6. *Let ${}_RV$ be injective in $\mathscr{D}(V)$.*

(a) *If M is an R-submodule of $N \in \mathscr{D}(V)$, then $M \in \mathscr{D}(V) \Leftrightarrow M$ is rad_V-closed in N.*

(b) *For all $M \in R$-Mod, every R-homomorphism $f: Q_V(M) \to V$ can be extended to $T_V(M)$.*

(c) *Q_V: E-Mod $\to \mathscr{D}(V)$ is left adjoint to the inclusion $U_V: \mathscr{D}(V) \to$ R-Mod.*

Proof. (a) Assume that N is a rad_V-closed submodule of some direct product $\prod_I V$ of copies of V.

($\Leftarrow$) If $x \in N \backslash M$, then since M is rad_V-closed, there exists $f: N \to V$ with $f(x) \neq 0$ and $f(M) = 0$. Since V is injective in $\mathscr{D}(V)$, f can be extended to $\prod_I V$. On the other hand, if $x \in \prod_I V \backslash N$, then since N is rad_V-closed in $\prod_I V$, there exists $f: \prod_I V \to V$ with $f(M) \subseteq f(N)$ and $f(x) \neq 0$. Thus, M is rad_V-closed in $\prod_I V$.

($\Rightarrow$) Assume that M is a rad_V-closed submodule of $\prod_J V$ and let M' be the rad_V-closure of M in N. Then, in the diagram

$$\begin{tikzcd} M \arrow[d, "1"'] & \subseteq & M' & \subseteq & N \arrow[dlll, "f"] \\ M & \subseteq & \Pi_J V \end{tikzcd}$$

the identity $1: M \to M$ can be extended to $f: N \to \prod_J V$, since V is injective in $\mathscr{D}(V)$ and so M must be a direct summand of M', since $f(M') \subseteq M$. Thus, if $M \neq M'$, there exists $0 \neq g: M' \to V$ with $g(M) = 0$ and this can be extended to N, since, by the converse (proved above), $M' \in \mathscr{D}(V)$ and this contradicts the definition of M'.

(b) If $f: Q_V(M) \to V$, then f can be extended to $T_V(M)$ since by (a), $Q_V(M) \in \mathscr{D}(V)$.

(c) If $M \in \mathscr{D}(V)$, then by (a) M is rad_V-closed in $T_V(M)$ and so $Q_V(M) = M$. If $f: N \to M$ in R-Mod, then $Q_V(f): Q_V(N) \to Q_V(M) = M$ is an extension of f and the extension is unique, since (b) implies that $\text{Hom}_R(Q_V(M)/\eta(M), V) = 0$. This shows that Q_V is a left adjoint for U_V. ▌

PROPOSITION 1.7. *The following conditions are equivalent for* $V \in R$-Mod.

(1) *V is injective in both* $Q_V(R)$-Mod *and* $\mathscr{D}(V)$.

(2) *V is injective in* $Q_V(R)$-Mod *and the natural map* $V \to \text{Hom}_R(Q_V(R), V)$ *is an isomorphism.*

(3) *Every R-homomorphism from a $Q_V(R)$-submodule of* $\text{Bic}({}_RV)$ *into V can be extended to* $\text{Bic}({}_RV)$.

Proof. (1) ⇒ (2). This follows from Lemma 1.6(b) and Lemma 1.5.

(2) ⇒ (3). By Lemma 1.5, every R-homomorphism from a $Q_V(R)$-submodule of $\text{Bic}({}_RV)$ into V is a $Q_V(R)$-homomorphism, so it can be extended by the injectivity of V to $\text{Bic}({}_RV)$.

(3) ⇒ (1). Baer's criterion for injectivity shows that V is injective in $Q_V(R)$-Mod. If $M \in \mathscr{D}(V)$, then M is a $Q_V(R)$-module and by Lemma 1.5, every R-homomorphism from M into V is a $Q_V(R)$-homomorphism, since by assumption condition (3) of Lemma 1.5 is satisfied. ▌

THEOREM 1.8. *If* $V \in R$-Mod *satisfies the conditions of Proposition* 1.7, *then* $\mathscr{D}(V)$ *is a full reflective Grothendieck subcategory of* R-Mod.

Conversely, if $\mathscr{B}$ *is a full reflective Grothendieck subcategory of* R-Mod, *with reflector* Q: R-Mod $\to \mathscr{B}$, *then any injective cogenerator V of $\mathscr{B}$ satisfies the conditions of Proposition* 1.7 *and* $\mathscr{B} = \mathscr{D}(V)$ *with* $Q \simeq Q_V$.

Proof. If V satisfies the conditions of Proposition 1.7, then by Lemma 1.5, the category of rad_V-torsionfree $Q_V(R)$-modules is a full subcategory of R-Mod and so the $Q_V(R)$-module structure of any module ${}_RM \in \mathscr{D}(V)$ uniquely extends its R-module structure. Furthermore, any exact sequence $0 \to M \to \prod_I V \to \prod_J V$ in R-Mod is also in $Q_V(R)$-Mod. In $Q_V(R)$-Mod, rad_V is a torsion radical since V is injective and so $\mathscr{D}(V)$ is equivalent to $Q_V(R)$-Mod$/V$, which is a Grothendieck category. By Lemma 1.6, $\mathscr{D}(V)$ is a reflective subcategory with reflector Q_V.

Conversely, assume that $\mathscr{B}$ is a full Grothendieck subcategory with reflector Q: R-Mod $\to \mathscr{B}$. Let V be an injective cogenerator of $\mathscr{B}$. If $B \in \mathscr{B}$, then in $\mathscr{B}$, there exists an exact sequence $0 \to B \to \prod_I V \to \prod_J V$, and this is exact in R-Mod since the inclusion functor must be left exact, so $\mathscr{B} \subseteq \mathscr{D}(V)$. On the other hand, the inclusion is full and preserves kernels and direct products, so if there exists an exact sequence $0 \to M \to \prod_I V \to \prod_J V$ in R-Mod, then $M \in \mathscr{B}$ and so $\mathscr{D}(V) \subseteq \mathscr{B}$. Now, since V is injective in $\mathscr{B}$, Lemma 1.6 implies that Q_V is a left adjoint for the inclusion and then $Q_V \simeq Q$ by uniqueness of adjoints. Thus, $Q_V(R)$ is a generator for $\mathscr{B}$ and since $\mathscr{B}$ is a full subcategory, $Q_V(R) \simeq \mathrm{End}({}_RQ_V(R))$. By the Gabriel–Popescu theorem [17, Theorem 10.3], $\mathrm{Hom}_R(Q_V(R), -)\colon \mathscr{B} \to Q_V(R)$-Mod has an exact left adjoint and so $\mathrm{Hom}_R(Q_V(R), -)$ must preserve injectives. Thus, V is injective as a $Q_V(R)$-module and so it satisfies condition (1) of Proposition 1.7. ∎

Note that in Theorem 1.8, the first implication can be shown without using Lemma 1.6, since once it has been established that $\mathscr{D}(V)$ is equivalent to $Q_V(R)$-Mod$/V$, then the latter has a reflector $Q_V{}^*$: $Q_V(R)$-Mod $\to$ $Q_V(R)$-Mod$/V$, so $Q_V(R) \otimes_R -$ followed by $Q_V{}^*$ gives the required adjoint. The ring of quotients of $Q_V(R)$ constructed with respect to rad_V is just $Q_V(R)$, since the $Q_V(R)$-bicommutator of V is just $\mathrm{Bic}({}_RV)$. This makes it possible to apply many of the standard results on rings of quotients.

In [13], a module ${}_RV$ is said by Morita to be of type *FI* if: (i) V is injective in $\mathrm{Bic}({}_RV)$-Mod; (ii) the natural map $V \to \mathrm{Hom}_R(\mathrm{Bic}({}_RV), V)$ is an isomorphism; and (iii) V is finitely generated over $\mathrm{End}({}_RV)$. (Condition (iii) is equivalent to Morita's condition by [16, Lemma 9.3].) If V satisfies the conditions of Proposition 1.7 and is finitely generated over $\mathrm{End}({}_RV)$, then applying [13, Theorem 5.6] to V in $Q_V(R)$-Mod, it follows that $Q_V(R) = \mathrm{Bic}({}_RV)$ and so V is of type *FI*. Conversely, if V is of type *FI*, then it follows from condition (ii) that $\mathrm{Hom}_R(\mathrm{Bic}({}_RV)/\eta(R), V) = 0$ and so $Q_V(R) = \mathrm{Bic}({}_RV)$. Thus, ${}_RV$ is of type *FI* if and only if ${}_RV$ satisfies the conditions of Proposition 1.7 and V is finitely generated over $\mathrm{End}({}_RV)$.

Theorem 1.8 considerably enlarges the class of modules that may be used in constructing a localization in Morita's sense, in which the quotient functor need not be exact, since any module ${}_RV$ that is injective in $R/\mathrm{Ann}(V)$-Mod satisfies condition (3) of Proposition 1.7, while it need not be finitely generated over $\mathrm{End}({}_RV)$. Of course, by [14, Theorem 1.1], the localization can be constructed with respect to some module ${}_RW$ of type *FI*. Since Q_V is a left adjoint, it is exact if and only if U_V preserves injectives, which occurs if and only if ${}_RV$ is injective, in which case we have just the standard localization.

The next proposition characterizes rad_V-torsionfree $Q_V(R)$-modules.

PROPOSITION 1.9. *Let $V \in R$-Mod satisfy the conditions of Lemma 1.5. If $M \in R$-Mod with $\mathrm{rad}_V(M) = 0$, then M has a $Q_V(R)$-module structure extending its R-module structure if and only if M is generated by the R-module $Q_V(R)$.*

Proof. We prove only the "if" part and for this, it is sufficient to assume that M is a submodule of a direct product $\prod_I V$ of copies of V. Then, by assumption, M is the image of an R-homomorphism from a direct sum $\bigoplus_J Q_V(R)$ of copies of $Q_V(R)$ into $\prod_I V$. If V satisfies Lemma 1.5, then this R-homomorphism is a $Q_V(R)$-homomorphism, so M must be a $Q_V(R)$-submodule of $\prod_I V$. ∎

If $R \to S$ is an epimorphism in the category of rings, then S-Mod is a full reflective Grothendieck subcategory of R-Mod [17, Proposition 13.7] and so $S\text{-Mod} = \mathscr{D}(V)$ for any injective cogenerator $V \in S$-Mod. Then $Q_V \simeq S \otimes_R -$ and it follows from the construction of $Q_V(R)$ and the fact that V is a cogenerator in S-Mod that $Q_V(R) = S$. The next theorem shows that conversely, if V satisfies the conditions of Proposition 1.7 and $Q_V(R)\text{-Mod} = \mathscr{D}(V)$, then V is a cogenerator in $Q_V(R)$-Mod, $Q_V \simeq Q_V(R) \otimes_R -$ and $R \to Q_V(R)$ is an epimorphism in the category of rings.

If ${}_RV$ satisfies the conditions of Proposition 1.7, then we have the following diagram.

$$\begin{tikzcd}
R\text{-Mod} \arrow[d, shift right, "Q_V(R)\otimes_R -"'] \arrow[dd, bend right=60, "Q_V"'] \\
Q_V(R)\text{-Mod} \arrow[u, shift right, "U"'] \arrow[d, shift right, "Q_V^*"'] \\
\mathscr{D}(V) \arrow[u, shift right, "U_V^*"'] \arrow[uu, bend right=60, "U_V"']
\end{tikzcd}$$

It has been shown in Theorem 1.8 that $\mathscr{D}(V)$ is a quotient category of $Q_V(R)$-Mod and so $Q_V{}^*$ is the quotient functor defined by considering V as a $Q_V(R)$-module. The functor U is the forgetful functor and has a left adjoint $Q_V(R) \otimes_R -$. The next theorem generalizes the standard theorem on perfect quotient functors [17, Theorem 13.1], with the exception of the conditions stated in terms of the filter of left ideals of R. However, U_V is exact and commutes with direct sums if and only if $U_V{}^*$ does, so the conditions could be stated in terms of the filter of left ideals of $Q_V(R)$ determined by rad_V.

THEOREM 1.10. *If ${}_RV$ satisfies the conditions of Proposition* 1.7, *then with reference to the above diagram, the following conditions are equivalent*:

(1) *$U_V{}^*$ is an equivalence.*

(2) *V is a cogenerator in $Q_V(R)$*-Mod.

(3) *Every $Q_V(R)$-module is* rad_V-*torsionfree.*

(4) *U_V is exact and commutes with direct sums.*

(5) $U_V{}^* Q_V \simeq Q_V(R) \otimes_R -$.

(6) $R \to Q_V(R)$ *is an epimorphism in the category of rings and for all R-homomorphisms f, $Q_V(f) = 0$ implies that $Q_V(R) \otimes_R f = 0$.*

(7) $Q_V(f) = 0$ *implies that $Q_V(R) \otimes_R f = 0$, for all R-homomorphisms f.*

Proof. Conditions (1)–(5) can be shown to be equivalent either directly (the proofs are straightforward), or by reducing to the case when V is of type *FI* and applying [16, Theorem 4.2].

(1), (5) $\Rightarrow$ (6). If $U_V{}^*$ is an equivalence, then U must be a full functor, so $R \to Q_V(R)$ is a ring epimorphism [17, Proposition 13.7].

(6) $\Rightarrow$ (7). Immediate.

(7) $\Rightarrow$ (3). Let $M \in Q_V(R)$-Mod. If $\operatorname{rad}_V(M) \neq 0$, then there exists $0 \neq f: Q_V(R) \to \operatorname{rad}_V(M) \subseteq M$ in $Q_V(R)$-Mod with $Q_V(R) \otimes_R f \neq 0$. This is a contradiction, since applying Q_V to the exact sequence $Q_V(R) \xrightarrow{f} M \xrightarrow{p} \operatorname{coker}(f) \to 0$ shows that $Q_V(f) = 0$, since Q_V is right exact and $Q_V(p)$ is an isomorphism by Corollary 1.4. ∎

PROPOSITION 1.11. *Let $\alpha: R \to S$ be an epimorphism in the category of rings and let $V \in S$-Mod. Then, $Q_V(R) = Q_V(S)$ and V satisfies the conditions of Proposition* 1.7 *in R-Mod if and only if it satisfies them in S-Mod.*

Proof. Since $\alpha: R \to S$ is epic, S-Mod is a full subcategory of R-Mod, so $\operatorname{End}({}_RV) = \operatorname{End}({}_SV)$. Therefore, $\operatorname{Bic}({}_RV) = \operatorname{Bic}({}_SV)$. If $q \in Q_V(R)$ and $f \in \operatorname{Hom}_S(\operatorname{Bic}({}_RV), V)$ with $f\eta(S) = 0$, then $f\eta\alpha(R) = 0$ and hence, $f(q) = 0$. Thus, $Q_V(R) \subseteq Q_V(S)$. Conversely, if $q \in Q_V(S)$ and $f \in \operatorname{Hom}_R(\operatorname{Bic}({}_RV), V)$, then $f\eta\alpha(R) = 0$ implies that $f\eta(S) = 0$, since f must be an S-homomorphism and hence, $f(q) = 0$. Thus, $Q_V(S) \subseteq Q_V(R)$. The second part is then obvious. ∎

We remark that a similar result holds if S is a ring of quotients R_σ of R with respect to some torsion radical σ and ${}_RV$ belongs to the quotient category R-Mod$/\sigma$.

PROPOSITION 1.12. *Let K be an ideal of R and let V be the injective envelope of R/K in R/K-Mod. Then, $Q_V(R) \simeq Q_{\max}(R/K)$ and so $Q_V(R)$ is semisimple Artinian if and only if R/K has finite dimension and zero singular ideal in R/K-Mod.*

Proof. By Corollary 1.4, $Q_V(R) \simeq Q_V(R/K \otimes_R R) \simeq Q_V(R/K)$ and since V is the injective envelope of R/K, $Q_V(R/K)$ is just the complete ring of quotients $Q_{\max}(R/K)$ of R/K. The remainder is a well-known result. ∎

2. Monads and Localization

Recall that a monad of R-Mod is an endofunctor T: R-Mod $\to$ R-Mod with natural transformations $\eta: I \to T$ and μ: $T^2 \to T$, such that the following diagrams commute. (I is the identity functor.)

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} IT \arrow[r, "\eta T"] \arrow[d, equal] & T^2 \arrow[d, "\mu"] & TI \arrow[l, "T\eta"'] \arrow[d, equal] \\ T \arrow[r, equal] & T \arrow[r, equal] & T \end{tikzcd}$$

If F: R-Mod $\to \mathscr{A}$ and $G: \mathscr{A} \to R$-Mod are covariant functors I with F a left adjoint of G, then GF is a monad, with $\eta: I \to GF$ and $\mu = G\,\delta F$: $GFGF \to GF$, where η and $\delta: FG \to I$ are the natural transformations associated with the adjoint situation. Similarly, for $V \in R$-Mod, the functor T_V defined in Section 1 is a monad in a natural way, since it is the composite of two (contravariant) adjoint functors.

Throughout this section, T will be a fixed additive monad of R-Mod. A T-algebra is a pair $\langle M, \alpha\rangle$ with $M \in R$-Mod and $\alpha \in \operatorname{Hom}_R(T(M), M)$, such that the following diagrams commute.

$$\begin{tikzcd} T^2(M) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_M"'] & T(M) \arrow[d, "\alpha"] \\ T(M) \arrow[r, "\alpha"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, "1"'] & T(M) \arrow[d, "\alpha"] \\ & M \end{tikzcd}$$

A T-morphism (of T-algebras) $f: \langle M, \alpha\rangle \to \langle N, \beta\rangle$ is an R-homomorphism $f: M \to N$, such that the following diagram commutes.

$$\begin{tikzcd} T(M) \arrow[r, "T(f)"] \arrow[d, "\alpha"'] & T(N) \arrow[d, "\beta"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Assigning to each module ${}_RM$ and R-homomorphism f the T-algebra $\langle T(M), \mu_M\rangle$ and T-morphism $T(f)$ defines a functor from R-Mod into the category of T-algebras that is left adjoint to the forgetful functor and in fact, this pair of adjoints defines the monad T.

Heinicke [7], calls a left exact idempotent monad a localization functor. If σ is a torsion radical of R-Mod with quotient category R-Mod$/\sigma$, then the quotient functor Q_σ: R-Mod $\to$ R-Mod is a localization functor, with R-Mod$/\sigma$ isomorphic to the category of Q_σ-algebras. Conversely, if T is any localization functor, then $T \simeq Q_\sigma$ for some torsion radical σ [7, Theorem 2.4]. The following definition is motivated by Lambek's result [8, Corollary 3.1a], that if ${}_RV$ is injective and $T = T_V$, then Q_T is the corresponding localization functor.

Definition 2.1. For the monad $\langle T, \eta, \mu \rangle$, let rad_T denote the largest radical ρ of R-Mod, such that $T(M)$ is ρ-torsion free for all $M \in R$-Mod. For $M \in R$-Mod, let $Q_T(M)$ denote the rad_T-closure of $\eta(M)$ in $T(M)$.

If ${}_RM$ is a direct summand of $T(M)$, then M is rad_T-closed in $T(M)$ and so $Q_T(M) = M$. This occurs, for instance, if M is injective and rad_T-torsionfree. The observation proves the following proposition and shows that $Q_T T$ is naturally isomorphic to T, since $T(M)$ is a T-algebra for all $M \in R$-Mod. (Note that since η_M must be a monomorphism, we can identify M and $\eta(M)$.)

Proposition 2.2. *If $M \in R$-Mod is a T-algebra, then $Q_T(M) = M$.*

Proposition 2.3. *Let $M \in R$-Mod.*

(a) $\mathrm{rad}_T(M) = \ker(\eta_M)$

(b) $Q_T(M) = \ker(\eta_{T(M)} - T(\eta_M))$.

(c) $Q_T(M) = T(M) \Leftrightarrow \eta_{T(M)}$ *is an isomorphism.*

Proof. (a) Since $T(M)$ is rad_T-torsionfree, $\eta_M(\mathrm{rad}_T(M)) = 0$ and it follows that $\mathrm{rad}_T(M) \subseteq \ker(\eta_M)$. On the other hand, if $f: M \to T(N)$ for some $N \in R$-Mod, then $T(f)\,\eta_M i = 0$ for i: $\ker(\eta_M) \to M$ and so $\eta_{T(N)} f i = 0$, which implies $fi = 0$, since $\eta_{T(N)}$ is monic. Thus, $\ker(\eta_M) \subseteq \mathrm{rad}_T(M)$.

(b) Let $K = \ker(\eta_{T(M)} - T(\eta_M))$. Since $\eta_{T(M)}$ and $T(\eta_M)$ agree on $\eta(M)$, it follows from the definition of $Q_T(M)$ that they agree on $Q_T(M)$ and hence, $Q_T(M) \subseteq K$. On the other hand, if f: $T(M) \to T(N)$ with $f\eta_M = 0$, then $T(f)\,T(\eta_M) = 0$ implies that $T(f)\,\eta_{T(M)}(K) = T(f)\,T(\eta_M)(K) = 0$. Thus, $\eta_{T(N)} f(K) = 0$, which implies that $f(K) = 0$, since $\eta_{T(N)}$ is monic and so $K \subseteq Q_T(M)$.

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "\eta_M"'] & K \subseteq T(M) \arrow[r, "f"] \arrow[d, "\eta_{T(M)}"] & T(N) \arrow[d, "\eta_{T(N)}"] \\ T(M) \arrow[r, "T(\eta_M)"'] & T^2(M) \arrow[r, "T(f)"'] & T^2(N) \end{tikzcd}$$

(c) If $\eta_{T(M)}$ is an isomorphism, then $\mu_M \eta_{T(M)} = 1_{T(M)} = \mu_M T(\eta_M)$ shows that $T(\eta_M) = \mu_M^{-1} = \eta_{T(M)}$ and so $Q_T(M) = T(M)$ by (b). Conversely, if $Q_T(M) = T(M)$, then by (b), $\eta_{T(M)} = T(\eta_M)$. Using this and the fact that μ

is a natural transformation, $\eta_{T(M)}\mu_M = T(\eta_M)\,\mu_M = \mu_{T(M)}T(T(\eta_M)) = \mu_{T(M)}\,T(\eta_{T(M)}) = 1_{T(M)}$, and $\eta_{T(M)}$ is an isomorphism. ▮

When $T = GF$ for an adjoint pair (with $\eta: I \to GF$, $\delta: FG \to I$), then rad_T is also the largest radical ρ such that every module of the form $G(A)$ is ρ-torsionfree. In this setting, Proposition 2.2(b) is just [11, Proposition 1.5]. By [4, Lemma 4.7], the following conditions are equivalent for $M \in R$-Mod: (1) $Q_T(M) = M$; (2) $\delta_{F(M)}$ is an epimorphism; (3) $F(\eta_M)$ is a monomorphism; and (4) $\eta_{T(M)}$ is an isomorphism.

Proposition 2.2b shows that Q_T is the construction used by Fakir [6], so that Q_T is a monad, with $Q_T = T$ if and only if T is idempotent. To show that Q_T is a monad, factor $\eta: I \to T$ as $\eta = \epsilon\eta'$, where $\eta': I \to Q_T$ and $\epsilon: Q_T \to T$. Since $Q_T(T(M)) = T(M)$, $Q_T(\epsilon) = \mu\eta_T Q_T(\epsilon) = \mu T(\epsilon)\,\epsilon Q_T$. Using this and the fact that $\eta_T\mu\epsilon_T Q_T(\epsilon) = T(\eta)\,\mu\epsilon_T Q_T(\epsilon)$, it can be shown that $Q_T(\epsilon)$ factors through ϵ, say, $\epsilon\mu' = Q_T(\epsilon)$. It can be verified that, with these natural transformations, $\langle Q_T, \eta', \mu'\rangle$ is a monad. Note that if $T(\epsilon)$ is monic, then so is $Q_T(\epsilon)$ and hence, μ', so that μ' must be an isomorphism. Thus, Q_T is idempotent if $T(\epsilon_M)$ is monic for all $M \in R$-Mod.

PROPOSITION 2.4. *If T is left exact, then Q_T is a localization functor.*

Proof. By the preceding remarks, Q_T must be idempotent. Let $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ be an exact sequence in R-Mod. Then, $Q_T(ji) = 0$ and $Q_T(i)$ is monic, since Q_T is just the restriction of T. If $x \in \ker(Q_T(j))$, then $x \in \ker(T(j)) = \mathrm{Im}(T(i))$, so that $x = T(i)(x')$ for some $x' \in T(M')$. Thus, $\eta_{T(M)}T(i)(x') = \eta_{T(M)}(x) = T(\eta_M)(x) = T(\eta_M)\,T(i)(x')$, since $x \in Q_T(M)$ and so $T^2(i)\eta_{T(M')}(x') = T^2(i)T(\eta_{M'})(x')$. This shows that $x' \in \ker(\eta_{T(M')} - T(\eta_{M'})) = Q_T(M')$, since T is exact and therefore, $T^2(i)$ is monic. Thus, $Q_T(M)$ is left exact. ▮

PROPOSITION 2.5. *Let $N = M \oplus M' \in R$-Mod, with $i: M \to N$ and $p: N \to M$ the inclusion and projection, respectively. If $\langle N, \beta\rangle$ is a T-algebra, then for $\alpha = p\beta T(i)$, the following conditions are equivalent.*

(1) *$\langle M, \alpha\rangle$ is a T-algebra and p is a T-morphism.*

(2) *$p\beta T(ip) = p\beta$.*

(3) *$\beta(T(M')) \subseteq M'$.*

Proof. (1) $\Rightarrow$ (2). If p is a T-morphism, then $p\beta = \alpha T(p) = p\beta T(i)\,T(p)$.

(2) $\Rightarrow$ (1). By assumption, $\alpha T(\alpha) = p\beta T(i)\,T(p\beta T(i)) = p\beta T(\beta)T^2(i)$ and so $\alpha T(\alpha) = p\beta T(i)\,\mu_M = \alpha\mu_M$, which shows that $\langle M, \alpha\rangle$ is a T-algebra. It is immediate from the assumption, that p is a T-morphism.

$$
\begin{tikzcd}
T^2(M) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_M"'] & T(M) \arrow[r, shift left, "T(i)"] \arrow[d, "\alpha"] & T(N) \arrow[l, shift left, "T(p)"] \arrow[d, "\beta"] \\
T(M) \arrow[r, "\alpha"'] & M \arrow[r, shift left, "i"] & N \arrow[l, shift left, "p"]
\end{tikzcd}
$$

(2) $\Leftrightarrow$ (3). $p\beta T(ip) = p\beta \Leftrightarrow \beta(T(ip) - 1) \subseteq \ker p$. The result then follows, since $\ker p = M'$ and the image of $T(ip) - 1_{T(N)}$ is $T(M')$. ▮

PROPOSITION 2.6. *A* rad_T*-closed submodule of a* T*-algebra is a* $\underset{\sim}{Q}_T$*-algebra and in this case, any* T*-morphism restricts to a* $\underset{\sim}{Q}_T$*-morphism.*

Proof. Let ${}_RM$ be a rad_T-closed submodule of a T-algebra $\langle N, \beta\rangle$. Since $\underset{\sim}{Q}_T T = T$ and $\underset{\sim}{Q}_T(N) = N$, any R-homomorphism $f: N \to T(X)$, $X \in R$-Mod, is a $\underset{\sim}{Q}_T$-morphism, so $\ker(f)$ is a $\underset{\sim}{Q}_T$-algebra, since the category of $\underset{\sim}{Q}_T$-algebras has kernels. Since the category also has intersections, M must be a $\underset{\sim}{Q}_T$-algebra. In fact, if $i: M \to N$ is the inclusion, then $\beta T(i)\, \epsilon_M$ factors through i, say, $i\alpha = \beta T(i)\, \epsilon_M$ and $\langle M, \alpha\rangle$ gives the required $\underset{\sim}{Q}_T$-algebra structure. If $\langle N', \gamma\rangle$ is a T-algebra and $f: N \to N'$ is a T-morphism, then $\underset{\sim}{Q}_T(N') = N'$ and $fi\alpha = f\beta T(i)\, \epsilon_M = \gamma T(f)\; T(i)\, \epsilon_M = \gamma\eta_{N'}\underset{\sim}{Q}_T(fi) = \underset{\sim}{Q}_T(fi)$, so fi is a $\underset{\sim}{Q}_T$-morphism. ▮

DEFINITION 2.7. The category of $\underset{\sim}{Q}_T$-algebras will be denoted by R-Mod$/T$.

Propositions 2.5 and 2.6 show that modules in R-Mod$/T$ can be characterized as certain direct summands of rad_T-closed submodules of T-algebras.

THEOREM 2.8. *For* $M \in R$-Mod, *the following conditions are equivalent.*

(1) $\eta_M: M \to T(M)$ *is an isomorphism.*

(2) $\eta_{T(M)}$ *is an isomorphism and* M *is isomorphic to a* rad_T*-closed submodule of* $T(N)$, *for some* $N \in R$-Mod.

(3) $\underset{\sim}{Q}_T(M) = T(M)$ *and* M *is a* $\underset{\sim}{Q}_T$*-algebra.*

Proof. That (1) $\Rightarrow$ (2) is obvious and (2) $\Rightarrow$ (3) follows from Propositions 2.3 and 2.6. If (3) holds, then M is a direct summand of $\underset{\sim}{Q}_T(M) = T(M)$, since it is a $\underset{\sim}{Q}_T$-algebra and so $M = \underset{\sim}{Q}_T(M)$ by Proposition 2.2, and thus, (1) holds. ▮

If $\langle M, \alpha\rangle$ is a T-algebra, then for $m \in M$, let $\rho_m{}^* = \alpha T(\rho_m)$: $T(R) \to M$, where $\rho_m = [r \to rm]$: $R \to M$. For $q \in T(R)$, define $qm = \rho_m{}^*(q)$. Then, $\rho_m{}^*\rho_q{}^* = \rho_{qm}^*$, since $\rho_m{}^*\rho_q{}^*$ is a T-morphism and extends ρ_{qm} and furthermore, $\rho_m{}^* + \rho_n{}^* = \rho_{m+n}^*$ for any $n \in M$, since T is assumed to be additive. These remarks can be used to show that $T(R)$ is a ring and that any T-algebra is a $T(R)$-module. In addition, any T-morphism is a $T(R)$-homomorphism.

Of course, since Q_T is a monad, $Q_T(R)$ is also a ring. As in Proposition 1.2, if $M, N \in T(R)$-Mod and $\mathrm{rad}_T(N) = 0$, then any R-homomorphism $f: M \to N$ is a $Q_T(R)$-homomorphism.

In particular, if T is idempotent, then $Q_T = T$ and any R-homomorphism from a $T(R)$-module into a T-algebra is a $T(R)$-homomorphism. More generally, the following result holds.

LEMMA 2.9. *The following conditions are equivalent for any T-algebra $\langle M, \alpha\rangle$.*

(1) *μ_M is an isomorphism.*

(2) *η_M is an isomorphism.*

(3) *For any T-algebra $\langle N, \beta\rangle$, every R-homomorphism $\gamma: M \to N$ is a T-morphism.*

Proof. (1) $\Leftrightarrow$ (2). If μ_M is an isomorphism, then $T(\eta_M) = \eta_{T(M)}$ is an isomorphism, so $\eta_M\alpha = T(\alpha)\,\eta_{T(M)} = T(\alpha)\,T(\eta_M) = T(\alpha\eta_M) = 1$ and η_M is an isomorphism. Conversely, if η_M is an isomorphism, then so is $T(\eta_M)$ and hence, also μ_M.

(2) $\Leftrightarrow$ (3). If η_M is an isomorphism and $f: M \to N$, then $f\alpha = \beta\eta_N f\alpha = \beta T(f)\,\eta_M\alpha = \beta T(f)$, since $\eta_M\alpha = 1$ and thus, f is a T-morphism.

Assuming the converse shows that $\eta_M: M \to T(M)$ is a T-morphism, so $\eta_M\alpha = \mu_M T(\eta_M) = 1$ and η_M is an isomorphism. ∎

THEOREM 2.10. *Let $U: Q_T(R)$-Mod $\to R$-Mod be the forgetful functor. If TU is left exact, then the following conditions hold.*

(a) *Q_T is idempotent.*

(b) *$Q_T U$ is a localization functor of $Q_T(R)$-Mod.*

(c) *R-Mod$/T$ is a full Grothendieck subcategory of R-Mod.*

Proof. (a) The monomorphism $\epsilon_M: Q_T(M) \to T(M)$ is the kernel of the Q_T-morphism $\eta_{T(M)} - T(\eta_M)$ and so it is a $Q_T(R)$-homomorphism. By assumption, $TU(\epsilon_M)$ is monic and so $\epsilon_M\mu_M' = Q_T(\epsilon_M)$ is monic and thus, μ_M' is an isomorphism.

(b) Since Q_T is idempotent, by Lemma 2.9 the R-homomorphism $\eta_M: M \to T(M)$ is a $Q_T(R)$-homomorphism for all $M \in Q_T(R)$-Mod. Using the fact that $T(M)$ has a unique $Q_T(R)$-module structure extending the R-module structure, it follows that $(TU)^2 = T^2U$ and it can be shown that $\langle TU, \eta, \mu\rangle$ is a monad of $Q_T(R)$-Mod. By assumption, TU is left exact and since $Q_T U(M) = \ker(TU(\eta_M) - \eta_{TU(M)})$, Proposition 2.4 implies that $Q_T U$ is left exact and idempotent.

(c) Since Q_T is idempotent, Lemma 2.9 shows that R-Mod$/T$ is a full subcategory and so any Q_T-algebra has a unique $Q_T(R)$-module structure extending the R-module structure. Thus, U is an isomorphism of categories when restricted to a functor from Q_TU-algebras to Q_T-algebras and this shows that R-Mod$/T$ is a Grothendieck category, since Q_TU is a localization functor. ▮

3. Applications

If $V \in R$-Mod and $T = T_V$ is the monad defined by $\operatorname{Hom}_R(-, V)$, then $Q_T = Q_V$ and we will denote the category of Q_V-algebras by R-Mod$/V$. By Proposition 2.6, each module $M \in D(V)$ belongs to R-Mod$/V$. If V satisfies the equivalent conditions of Proposition 1.7, then the two categories coincide and it follows from Lemma 1.5, that $Q_VU: Q_V(R)\text{-Mod} \to R\text{-Mod}$ is left exact since $\operatorname{Hom}_R(-, V)$ is exact on $Q_V(R)$-Mod. Thus, the first part of Theorem 1.8 is a special case of Theorem 2.10, which shows in addition that Q_V is idempotent, while the second part provides a converse to Theorem 2.10 in that any full reflective Grothendieck subcategory is of the form R-Mod$/T$ for some monad T such that TU is left exact.

Recall that a module ${}_RV$ is called balanced if $\eta_R: R \to \operatorname{Bic}({}_RV)$ is an isomorphism. Applying Theorem 2.8 to the monad T_V, gives the following generalization of [15, Theorem 5.1].

Theorem 3.1. *Let V be a faithful left R-module. Then, V is balanced $\Leftrightarrow R \in R$-Mod$/V$ and $Q_V(R) = \operatorname{Bic}({}_RV)$.*

The next theorem is a generalization of part (1) of [8, Theorem 4.2], where a proof is given in case ${}_RV$ is injective. Recall that $Q_V(R)$ is said to be a dense subring of $\operatorname{Bic}({}_RV)$ if for each $E = \operatorname{End}({}_RV)$-finitely generated submodule $F \subseteq V_E$ and $b \in \operatorname{Bic}({}_RV)$, there exists $q \in Q_V(R)$, such that $(b - q)F = 0$. Note that the hypothesis of the theorem is satisfied if $U_V: R\text{-Mod}/V \to R\text{-Mod}$ is exact and the conditions of Proposition 1.7 hold for V.

Theorem 3.2. *Q_V is dense in* $\operatorname{Bic}({}_RV)$ *if for any direct sum V^n, $n > 0$, each cyclic $Q_V(R)$-submodule is* rad_V*-closed.*

Proof. If $v_1, \ldots, v_n \in V$, let $v = (v_1, \ldots, v_n) \in V^n$. If $bv \notin Q_V(R)v$ for some $b \in \operatorname{Bic}({}_RV)$, then, since $Q_V(R)v$ is rad_V-closed in V^n, there exists $g \in \operatorname{Hom}_R(V^n, V)$ with $g(Q_V(R)v) = 0$, but $g(bv) \neq 0$. This is a contradiction, since $g(v) \in g(Q_V(R)v) = 0$ implies that $g(bv) = \sum_{i=1}^n g_i(bv_i) = b \sum_{i=1}^n g_i(v_i) = bg(v) = 0$, since b commutes with the components $g_i \in \operatorname{End}({}_RV)$ of g. Thus,

$\mathrm{Bic}({}_RV)v = Q_V(R)v$ and for each $b \in \mathrm{Bic}({}_RV)$, there exists $q \in Q_V(R)$, such that $qv_i = bv_i$ for $1 \leqslant i \leqslant n$. ∎

If ${}_RV$ is injective, then $R/\mathrm{Ann}(V)$ can be embedded in some direct product W of copies of V and so $Q_W(R) = \mathrm{Bic}({}_RW)$ [16, Theorem 3.2]. But two injective modules that cogenerate each other determine isomorphic rings of quotients and so $\mathrm{Bic}({}_RW) \simeq Q_V(R)$. These results can be recovered in a much more general setting.

The bicommutator of $V_1 \oplus V_2 \in R$-Mod is the set of matrices of the form $\left(\begin{smallmatrix} b_1 & 0 \\ 0 & b_2 \end{smallmatrix}\right)$ such that $b_1 \in \mathrm{Bic}(V_1)$, $b_2 \in \mathrm{Bic}(V_2)$ and $b_2 f_{21} = f_{21} b_1$, $b_1 f_{12} = f_{12} b_2$ for all $f_{21} \in \mathrm{Hom}_R(V_1, V_2)$, $f_{12} \in \mathrm{Hom}_R(V_2, V_1)$ (see [1]). There is a natural ring homomorphism from $\mathrm{Bic}(V_1 \oplus V_2)$ into $\mathrm{Bic}(V_1)$ that is a monomorphism if V_1 cogenerates V_2 and an isomorphism if V_1 both generates and cogenerates V_2.

Similarly, if $W = \prod_I V$ is a direct product of copies of ${}_RV$, then $\mathrm{Bic}(V \oplus W)$ is the set of matrices $\left(\begin{smallmatrix} b & 0 \\ 0 & \bar{b} \end{smallmatrix}\right)$ such that $b \in \mathrm{Bic}(V)$, $\bar{b}$ acts on W by componentwise multiplication by b and $f\bar{b} = bf$ for all $f \in \mathrm{Hom}_R(W, V)$. (The latter condition guarantees that $\bar{b} \in \mathrm{Bic}(W)$ and the condition that $fb = \bar{b}f$ for $f \in \mathrm{Hom}_R(V, W)$ is automatically satisfied.) Note that the natural homomorphism ϕ: $\mathrm{Bic}(V \oplus W) \to \mathrm{Bic}(V)$ is an isomorphism if V is finitely generated over $\mathrm{End}({}_RV)$, since in this case, V both generates and cogenerates $\prod_I V$ [4, Proposition 2.7].

Proposition 3.3. *If* $V \in R$-Mod *and* $\mathrm{Bic}(V)$ *is embedded in a direct product* $W = \prod_{\alpha \in I} V_\alpha$ *of copies of* V, *then the following conditions hold for the natural ring homomorphism* ϕ: $\mathrm{Bic}(V \oplus W) \to \mathrm{Bic}(V)$.

(a) $\mathrm{Im}(\phi)$ *is* rad_V*-closed in* $\mathrm{Bic}(V)$.

(b) $\mathrm{Im}(\phi) = Q_V(R)$ *if every* R*-homomorphism* $f: \mathrm{Bic}(V) \to V$ *can be extended to* W.

Proof. (a) If $a \in \mathrm{Bic}(V) \setminus \mathrm{Im}(\phi)$, then by the preceding remarks characterizing $\mathrm{Bic}(V \oplus W)$, there exists $f \in \mathrm{Hom}_R(W, V)$, with $f(aw) \neq af(w)$ for some $w \in W$. Define $g: \mathrm{Bic}(V) \to V$ by $g(b) = f(bw) - bf(w)$, for all $b \in \mathrm{Bic}(V)$. Since g is an R-homomorphism, $g\phi = 0$ and $g(a) \neq 0$, this shows that $\mathrm{Im}(\phi)$ is rad_V-closed in $\mathrm{Bic}(V)$.

(b) Because $\mathrm{Im}(\phi) \supseteq \eta(R)$, by (a) we must have $\mathrm{Im}(\phi) \supseteq Q_V(R)$. If $f \in \mathrm{Hom}_R(\mathrm{Bic}(V), V)$ with $f\eta(R) = 0$, then by assumption, f can be extended to $g: W \to V$. If $b \in \mathrm{Im}(\phi)$, then $f(b) = g(b) = bg(1) = bf(1) = 0$, since b must commute with elements of $\mathrm{Hom}_R(W, V)$. Thus, $\mathrm{Im}(\phi) \subseteq Q_V(R)$. ∎

Proposition 3.4. *If* V, $W \in R$-Mod *and* V *and* W *both generate and cogenerate each other, then* $Q_V(R) \simeq Q_W(R)$.

Proof. By assumption, the homomorphisms $\mathrm{Bic}(V \oplus W) \to \mathrm{Bic}(V)$ and $\mathrm{Bic}(V \oplus W) \to \mathrm{Bic}(W)$ are isomorphisms. Since V and W cogenerate each other, $\mathrm{rad}_V = \mathrm{rad}_W$ and it follows that $Q_V(R) \simeq Q_W(R)$. ▌

The module ${}_R V$ is called fully divisible if it is generated by the injective envelope $E(R)$ of the module ${}_R R$. In [1], it has been shown that if V is faithful, fully divisible, and finitely generated over $\mathrm{End}({}_R V)$, then $\mathrm{Bic}({}_R V)$ is isomorphic to the rad_V-closure of R in $E(R)$, a subring of $Q_{\max}(R)$. The next theorem uses techniques of this paper to generalize the result to the case in which V is merely faithful and fully divisible. In this situation, if ${}_R M$ is fully divisible and rad_V-torsion free, then M can be embedded as a $Q_V(R)$-submodule in a direct product W of copies of V. That is, M inherits the $Q_V(R)$-module structure of W (which is unique), but there may be other $Q_V(R)$-module structures extending the R-module structure (see [2]).

THEOREM 3.5. *If ${}_R V$ is faithful and fully divisible, then $Q_V(R)$ is isomorphic to the subring of $Q_{\max}(R)$ defined by the* rad_V*-closure of R in $E(R)$.*

Proof. Let S be the rad_V-closure of R in $E(R)$, which is a subring of $Q_{\max}(R)$ by [1, Theorem 2.3]. Since V is fully divisible, every R-homomorphism from R to V extends to $E(R)$ [1, Proposition 1.5] and so $\mathrm{Hom}_R(i, V)$ is epic for the inclusion $i: R \to E(R)$, which implies that $T_V(i)$ and hence, $Q_V(i)$ is a monomorphism. Because $\mathrm{rad}_V \leqslant \mathrm{rad}_{E(R)}$, $\eta_{E(R)}: E(R) \to T_V(R)$ is monic and since $E(R)$ is injective, $\eta_{E(R)}$ has a splitting map π with $\pi\eta_{E(R)} = 1$. Thus, $Q_V(E(R)) = E(R)$ and so $Q_V(i)(Q_V(R)) = \pi\eta_{E(R)}Q_V(i)(Q_V(R)) = \pi T_V(i)\,\epsilon_R(Q_V(R)) \subseteq S$ since $\pi T_V(i)$ maps R into R and therefore, maps the rad_V-closure of R in $\mathrm{Bic}({}_R V)$ into the rad_V-closure of R in $E(R)$. Since $T_V(i)\,\epsilon_R = \eta_R Q_V(i)$, then $s = Q_V(i)(q)$ has the property (for all $q \in Q_V(R)$) that

$$[f \to qf(1)] = T_V(i)\,\epsilon_R(q) = \eta_{E(R)}Q_V(i)(q) = \eta_{E(R)}(S) = [f \to f(s)],$$

i.e., $qf(1) = f(s)$, for all $f \in \mathrm{Hom}_R(E(R), V)$.

Now, V is an S-module by extending $\rho_v = [r \to rv]: R \to V$ to $\rho_v': E(R) \to V$ (the extension is unique on S) and defining $sv = \rho_v'(s)$ for all $s \in S$ and $v \in V$. Define $h: S \to \mathrm{Bic}({}_R V)$ by $h(s) = [v \to sv]$, for $s \in S$, $v \in V$. Then, for $q \in Q_V(R)$,

$$\begin{aligned} hQ_V(i)(q) &= h(s) = [v \to sv] = [v \to \rho_v'(s)] \\ &= [v \to q\rho_v'(1)] = [v \to qv] = q, \end{aligned}$$

where $s = Q_V(i)(q)$ and so $hQ_V(i) = 1$.

We have shown that the monomorphism $Q_V(i): Q_V(R) \to Q_V(E(R)) = E(R)$

factors through S and in fact, $Q_V(R)$ is a direct summand of S. This forces $\mathrm{Im}(Q_V(i)) = S$, since $E(R)$ is an essential extension of $R \subseteq \mathrm{Im}(Q_V(i))$. ▮

COROLLARY 3.6. *Let ρ be a radical of R-*Mod, *with $\rho(R) = K$ and let S be the ρ-closure of R/K in $E({}_{R/K}R/K)$. If $S \subseteq Q_{\max}(R/K)$, then there exists $V \in R$-*Mod *with $S \simeq Q_V(R)$.*

Proof. If $S \subseteq Q_{\max}(R/K)$, then it is a subring by [1, Lemma 2.1]. Furthermore S is the rad_V-closure of R/K in $E({}_{R/K}R/K)$ for $V = E({}_{R/K}R/K) \oplus E({}_{R/K}R/K)/S$ and so, by Theorem 3.5, $S \simeq Q_V(R/K)$. (In fact, by [1, Proposition 3.1], $S \simeq \mathrm{Bic}({}_RV)$). By Proposition 1.11, we have that $S \simeq Q_V(R)$. ▮

The final theorem concerns the monad of R-Mod determined by $P \otimes_R -$. By [4, Lemma 4.7] any module of type *FP* [13] satisfies the hypothesis of the theorem, so this includes the case when P_R is finitely generated projective [13, Corollary 1.2]. The theorem generalizes [5, Theorem 2.1], which shows that the ring of quotients determined by a projective module coincides with its bicommutator.

THEOREM 3.7. *Let $P \in$* Mod-R, *$E = \mathrm{End}(P_R)$ and $T(M) = \mathrm{Hom}_E(P, P \otimes_R M)$ for all $M \in R$-*Mod. *If P is flat in* Mod-$Q_T(R)$ *and the natural map ϕ: $P \otimes_R Q_T(R) \to P$ given by $\phi(p \otimes q) = pq$ for all $p \in P$, $q \in Q_T(R)$ is an isomorphism, then the following conditions hold.*

(a) *R-*Mod*$/T$ is a full Grothendieck subcategory.*

(b) *If P is projective in* Mod-$Q_T(R)$, *then* $Q_T(R) = \mathrm{Bic}(P_R)$.

(c) *If P is projective and finitely generated in* Mod-$Q_T(R)$, *then T is idempotent and consequently, $Q_T = T$.*

Proof. (a) Let $Q = Q_T(R)$ and let $U: Q$-Mod $\to R$-Mod. The result follows from Theorem 2.10, since, if P_R satisfies the assumption, then for $M \in Q$-Mod, $P \otimes_R M \simeq P \otimes_R (Q \otimes_Q M) \simeq (P \otimes_R Q) \otimes_Q M \simeq P \otimes_Q M$ and so TU is left exact since P_Q is flat.

(b) Since $\mathrm{Bic}(P_Q) = \mathrm{Bic}(P_R)$ and Q is its own ring of quotients with respect to $\mathrm{Hom}_E(P, P \otimes_Q -) \simeq TU$, we can apply [5, Theorem 2.1] to show that $Q = \mathrm{Bic}(P_R)$, since Q is just the ring of quotients constructed with respect to $\ker(TU)$.

(c) If P_Q is finitely generated and projective, then it is well known that the natural homomorphism $P \otimes_R \mathrm{Hom}_E(P, M) \to M$ is an isomorphism for all $M \in E$-Mod. Since this homomorphism is used to define μ: $T^2 \to T$ in the monad, μ must be an isomorphism and T is idempotent. ▮

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