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Matrices generated by semilattices

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Abstract

We give a characterization of 0–1 matrices M which are generated by semilattices in the way that $M_{ij} = 0$ if and only if $x_i \wedge x_j = \hat{0}$ where $x_i, x_j, \hat{0}$ are elements in a semilattice.

1. Introduction

Given a meet-semilattice L with least element $\hat{0}$ we can define a 0–1 matrix M corresponding to L by letting $M_{ij} = 0$ if $x_i \wedge x_j = \hat{0}$, where x_i, x_j are elements in L, and $M_{ij} = 1$ otherwise. In this paper we give a characterization of such matrices which leads to a polynomial-time algorithm for deciding whether a given matrix is of this type or not.

The motivation to study this problem comes mainly from the problem of communication complexity and the works of Lovász and Saks [2, 3]. Let us describe the problem in the following way.

Let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$ be two sets and $f: A \times B \to \{0, 1\}$ a function defined on all pairs (a, b) such that $a \in A, b \in B$. Suppose that we have two persons P_A and P_B. P_A has an element a_i from the set A and P_B has an element b_j from B. Neither knows the element of the other. Their objective is to find the value of $f(a_i, b_j)$ and at the same time bring the amount of information transmitted between them to a minimum. The naive approach is of course that P_A tells P_B the element he has and lets P_B calculate $f(a_i, b_j)$. This is sometimes, but not always, the best thing to do. A case in which they can do better is when P_A and P_B each have an integer and $f(a_i, b_j) = 1$ if and only if $a_i + b_j$ is odd. Obviously, P_A only has to tell P_B whether a_i is even or odd in order to enable P_B to calculate $f(a_i, b_j)$.

To every such communication problem we can associate the so-called communication matrix C defined by $C_{ij} = f(a_i, b_j)$. By $\kappa(C)$, the communication complexity, we mean the minimal number of bits of information, i.e., 0 or 1, which has to be exchanged between P_A and P_B in order to solve the problem in the worst case. It can be shown that $\log_2 r(C) \leqslant \kappa(C) \leqslant r(C)$ where $r(C)$ is the rank of C (see [2]).

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It is however desirable to get better bounds for $\kappa(C)$. The best investigated and perhaps most interesting special case is when $A = B$ is a semilattice and $f(a_i, a_j) = 0$ if and only if $a_i \wedge a_j = \hat{0}$. In the case of such matrices there are special Möbius function techniques for determining the rank (see [2]). It is therefore interesting to gain insight into the structure of such matrices and to find a way to decide if a given communication problem is of this semilattice type.

Two things are to be noticed in this paper. The first thing is that we have adopted the convention that $M_{ij} = 1$ if $x_i \wedge x_j \neq \hat{0}$ while Lovász and Saks say that $M_{ij} = 1$ if $x_i \wedge x_j = \hat{0}$. From the communication problem point of view there is not any real difference but our convention seems to be most practical for our purposes.

The second thing is that we will not assume that the element $\hat{0}$ of L has any corresponding row in the matrix M. If there was a corresponding row this row would consist entirely of 0's and would add nothing of interest to the problem.

In Section 2 we will see that it is natural both to specialize as well as to generalize the problem. The specialization is that we consider matrices generated by atomic semilattices. The generalization is that we consider matrices generated not merely by atomic semilattices but by atomic partially ordered sets. It will be shown that any matrix generated by a semilattice can be reduced to a matrix which is generated by an atomic semilattice. In Section 3 we then present two characterization theorems: one for matrices generated by atomic partially ordered sets; one for matrices generated by atomic semilattices.

In Section 4 we generalize to matrices generated by arbitrary partially ordered sets and semilattices. We will assume that the reader is familiar with the basic concepts of lattice theory. The standard reference is [1].

2. Atomic semilattices and partially ordered sets

Let P be a finite partially ordered set and A its set of minimal elements, i.e., elements a such that $a > x$ for no x in P. For every $x \in P$ we define $B(x) = \{a \in A : a \leqslant x\}$.

Definition. We will say that a matrix M is generated by P if there is an ordering $x_1, x_2, \ldots, x_n$ of the elements in P such that

$$M_{ij} = \begin{cases} 1 & \text{if } B(x_i) \cap B(x_j) \neq \emptyset, \\ 0 & \text{if } B(x_i) \cap B(x_j) = \emptyset. \end{cases}$$

If L is a semilattice (by which we always mean meet-semilattice) then $L - \{\hat{0}\}$ is a partially ordered set and the minimal elements of $L - \{\hat{0}\}$ are the atoms of L. Suppose that M is generated by $L - \{\hat{0}\}$. In that case it is easily seen that

$$M_{ij} = \begin{cases} 1 & \text{if } x_i \wedge x_j \neq 0, \\ 0 & \text{if } x_i \wedge x_j = 0. \end{cases}$$

We will use the following convention: When we say that a matrix M is generated by a semilattice L, we mean that M *is generated by* $L - \{\hat{0}\}$ according to the definition above.

Given any 0–1 matrix M of size $n \times n$ and a number i: $1 \leqslant i \leqslant n$ we define $S_i = \{j: M_{ij} = 1\}$, i.e., S_i is the support of row i.

If M is generated by a partially ordered set there is an obvious connection between the sets $B(x_i)$ and S_i.

Proposition 1. $S_i = \{j: x_j \geqslant x_k$ *for some* $x_k \in B(x_i)\}$.

Proof. Assume that $j \in S_i$. Then $M_{ij} = 1$, $B(x_i) \cap B(x_j) \neq \emptyset$ and therefore $x_j \geqslant x_k$ for some $x_k \in B(x_i)$. Conversely, if $x_j \geqslant x_k$ for some $x_k \in B(x_i)$ then $B(x_j) \cap B(x_i) \neq \emptyset$, $M_{ij} = 1$ and $j \in S_i$. □

Corollary. $S_i \subseteq S_j$ *if and only if* $B(x_i) \subseteq B(x_j)$.

A consequence is that $S_i = S_j$ if and only if $B(x_i) = B(x_j)$. An atomic semilattice is a semilattice in which every element x is the join of the atoms smaller than x. It is easily seen that a semilattice is atomic if and only if $B(x_i) = B(x_j)$ is equivalent with $x_i = x_j$. In the same way, we will say that a partially ordered set is atomic if and only if $B(x_i) = B(x_j)$ implies $x_i = x_j$. A matrix generated by such an atomic partially ordered set will have no identical rows or columns.

3. Characterization

We now present the characterization theorems. First we make the following definition.

If $M_n(0, 1)$ is the set of 0–1 matrices of size $n \times n$ then obviously $M_n(0, 1)$ is not closed under multiplication of its elements. There is however a natural way of defining a binary operation $\circ: M_n(0, 1) \times M_n(0, 1) \to M_n(0, 1)$ which corresponds to the normal multiplication.

Given any real matrix M we define a 0–1 matrix $H(M)$ by

$$[H(M)]_{ij} = \begin{cases} 1 & \text{if } M_{ij} > 0, \\ 0 & \text{otherwise}. \end{cases}$$

We now let $A \circ B = H(AB)$ for any pair A, B of 0–1 matrices such that the size of the rows of A equals the size of the columns of B. (The operation $\circ$ can easily be seen to be equivalent to the multiplication AB under the use of the rules of Boolean arithmetic, i.e., $1 + 1 = 1$, etc.).

Theorem 1. *A quadratic, symmetric* 0–1 *matrix* M *of size* n *is generated by an atomic partially ordered set if and only if* M, *after a suitable transformation* $M' = C^{\mathrm{T}}MC$

where C is a permutation matrix, can be written

$$M' = \begin{pmatrix} I & V^{\mathrm{T}} \\ V & W \end{pmatrix}$$

where I is the $k \times k$ unit matrix, $1 \leqslant k \leqslant n$, V is a $(n-k) \times k$-matrix with at least two 1's in every row and no rows identical and $W = V \circ V^{\mathrm{T}}$.

Proof. We first assume that M is generated by an atomic partially ordered set P. Let us order the elements in P so that $x_1, x_2, \ldots, x_k$ is the set of minimal elements and then, if necessary, reorder M so the first k rows and columns correspond to $x_1, x_2, \ldots, x_k$. This can be done by the transformation $M' = C^{\mathrm{T}}MC$ for some permutation matrix C. For every x_m with $m > k$ we can find at least two minimal elements x_i, x_j such that $\{x_i, x_j\} \subseteq B(x_m)$. If this was not the case x_m would be a minimal element. Therefore, every row of V contains at least two 1's. No rows in V can be identical since, if this were so, we would have $B(x_i) = B(x_j)$ for some nonminimal elements x_i, x_j, $x_i \neq x_j$, contrary to the requirement that P should be atomic.

We now prove that $W = V \circ V^{\mathrm{T}}$. We have to show that $W_{ij} = 1$ if and only if $\sum_{s=1}^{k} V_{is} V_{js} > 0$. Now, $W_{ij} = 1$ implies $x_q \in B(x_{i+k}) \cap B(x_{j+k})$ for some minimal element x_q. Then $V_{iq} = V_{jq} = 1$ so $\sum_{s=1}^{k} V_{is} V_{js} \geqslant 1 > 0$.

Conversely, if $\sum_{s=1}^{k} V_{is} V_{js} > 0$ then $V_{iq} V_{jq} = 1$ for some q such that $1 \leqslant q \leqslant k$. Therefore, $x_q \in B(x_{i+k}) \cap B(x_{j+k})$ and $M'_{i+k,j+k} = 1$, i.e., $W_{ij} = 1$.

Let us now assume that M is a quadratic, symmetric 0–1 matrix which meets the requirements in the theorem. We will construct an atomic partially ordered set P that generates M. From the definition in Section 2 we can see that a partially ordered set P generates a matrix M if and only if it generates any matrix $M' = C^{\mathrm{T}}MC$. Therefore, we just have to find a P that generates M', having the specific form stated in the theorem.

The obvious way to do this is to let P consist of n elements $y_1, y_2, \ldots, y_n$ and define $y_i \leqslant y_j$ if and only if $S_i \subseteq S_j$, i.e., the support of row i is included in the support of row j. It is trivially verified that P, defined in this way, is an atomic partially ordered set, whose minimal elements correspond to the first k rows. (The atomic structure of P follows from the structure of V.) It remains to be shown that P generates M'. This will be the case if $M'_{ij} = 1$ if and only if there exists a q such that $1 \leqslant q \leqslant k$ and $q \in S_i \cap S_j$.

Let us first assume that $M'_{ij} = 1$. If $i \leqslant k$ or $j \leqslant k$ we can take $\min(i,j)$ as q. The nontrivial case is $i, j > k$. $M'_{ij} = 1$ implies $W_{i-k,j-k} = 1$ and since $W = V \circ V^{\mathrm{T}}$ there exists a q such that $1 \leqslant q \leqslant k$ and $V_{i-k,q} V_{j-k,q} = 1$, i.e., $q \in S_i \cap S_j$.

On the other hand, if $1 \leqslant q \leqslant k$ and $q \in S_i \cap S_j$ then if $i \leqslant k$ we have $q = i$ and $M'_{ij} = M'_{qj} = 1$. In the same way, if $j \leqslant k$ we have $M'_{ij} = M'_{iq} = 1$. Finally, if $i, j > k$ we have $V_{i-k,q} V_{j-k,q} = 1$ and therefore $W_{i-k,j-k} = 1$ and $M'_{ij} = 1$ which proves the theorem. $\square$

Let us now consider an atomic semilattice L with $n + 1$ elements. If M is generated by $L - \{\hat{0}\}$ then obviously M fulfills the conditions of Theorem 1. One might expect that there are some extra restrictions on the matrix M which provide necessary and sufficient conditions for M to be generated by an atomic semilattice. Such a condition can be found. First let us write M in the form M' and for every i such that $1 \leqslant i \leqslant n - k$, define $D_i = \{j: V_{ij} = 1\}$. We see that $j \in D_i$ if and only if $x_j \in B(x_{k+i})$.

Theorem 2. *A quadratic, symmetric 0–1 matrix M of size n is generated by an atomic semilattice if and only if M satisfies the conditions in Theorem 1 and we for every pair i, j such that $|D_i \cap D_j| \geqslant 2$ can find an m such that $D_m = D_i \cap D_j$.*

Proof. Let us assume that M is generated by an atomic semilattice and $|D_i \cap D_j| \geqslant 2$. Then $x_{i+k} \wedge x_{j+k} \neq \hat{0}$ and we have $x_p = x_{i+k} \wedge x_{j+k}$ for some p. Since $|D_i \cap D_j| \geqslant 2$ we see that x_p cannot be a minimal element and $k < p \leqslant n$. It can now be shown that $D_{p-k} = D_i \cap D_j$. Indeed, if $r \in D_i \cap D_j$ then $x_r \leqslant x_{i+k}$, $x_r \leqslant x_{j+k}$ and $x_r \leqslant x_{i+k} \wedge x_{j+k} = x_p$, so $r \in D_{p-k}$. If $r \in D_{p-k}$ then $x_r \leqslant x_{i+k} \wedge x_{j+k}$ so $x_r \leqslant x_{i+k}$, $x_r \leqslant x_{j+k}$, i.e., $r \in D_i$, $r \in D_j$ and $r \in D_i \cap D_j$. Therefore $D_{p-k} = D_i \cap D_j$.

Conversely, if we for every pair i, j with $|D_i \cap D_j| \geqslant 2$ can find an m with $D_m = D_i \cap D_j$ then we can construct an atomic semilattice L which generates M' (and therefore generates M). Let L consist of the elements $\hat{0}, y_1, y_2, \ldots, y_n$ and define $y_i \leqslant y_j$ in the same way as in Theorem 1, i.e., $y_i \leqslant y_j$ if and only if $S_i \subseteq S_j$. We then define $\hat{0} < y_i$ for all y_i. From Theorem 1 we know that the partially ordered set $L - \{\hat{0}\}$ generates M. We have to show that L is in fact a semilattice.

To see this we set $B(y_i) = \{y_j: y_j \text{ is a minimal element in } L - \{\hat{0}\}, y_j \leqslant y_i\}$. From Proposition 1 we know that $y_i \leqslant y_j$ if and only if $B(y_i) \subseteq B(y_j)$.

It is now easily seen that $\hat{0} \wedge \hat{0} = \hat{0}$, $\hat{0} \wedge y_i = \hat{0}$ for all y_i and $y_i \wedge y_j = \hat{0}$ if $B(y_i) \cap B(y_j) = \emptyset$. If $B(y_i) \cap B(y_j) = \{y_p\}$ where y_p is a minimal element we have $y_i \wedge y_j = y_p$. What is left to be shown is that $y_i \wedge y_j$ exists for y_i, y_j with $|B(y_i) \cap B(y_j)| \geqslant 2$. Since $y_p \in B(y_i) \cap B(y_j)$ if and only if $p \in D_{i-k} \cap D_{j-k}$ we know that $|D_{i-k} \cap D_{j-k}| \geqslant 2$ and there exists an m with $D_m = D_{i-k} \cap D_{j-k}$ and $B(y_{m+k}) = B(y_i) \cap B(y_j)$. Obviously, $y_i \wedge y_j = y_{m+k}$ and the theorem is proved. □

4. The nonatomic case

The nonatomic case can be reduced to the atomic one. If M is generated by a nonatomic partially ordered set P then M will have some rows and columns equal to each other. If we remove all but one row and column in every multiple occurrence of rows and columns we get a matrix M^*. In the same manner, let us define an equivalence relation $\sim$ on P by: $x \sim y$ if and only if $B(x) = B(y)$. Let x_i^* be the equivalence class of x_i and $x_{i_1}^*, x_{i_2}^*, \ldots, x_{i_s}^*$ the partition of P into equivalence classes. We now define a partially ordered set P^* by letting $x_{i_p}^* \leqslant x_{i_q}^*$ if $B(x_{i_p}) \subseteq B(x_{i_q})$. It can be seen that P^* is an atomic partially ordered set with minimal elements

$x_{a_1}^*, x_{a_2}^*, \dots, x_{a_m}^*$ where $x_{a_1}, x_{a_2}, \dots, x_{a_m}$ are the minimal elements of P. With the help of Proposition 1 we see that P^* generates M^*.

In the same way, if M is generated by $L - \{\hat{0}\}$ where L is a semilattice then M^* is generated by $(L - \{\hat{0}\})^*$ and $(L - \{\hat{0}\})^*$ is an atomic semilattice if we add a zero element.

Conversely, if we have a matrix M such that M^* fulfills the conditions in Theorem 1 or 2 we can construct a partially ordered set P^* as in the proofs of the theorems. (The element y_i^* corresponds to row i in M^*.) We then replace each y_i^* by a chain $y_{i1}, y_{i2}, \dots, y_{ic_i}$ where c_i is the number of rows in M corresponding to row i in M^*. We then let $y_{ij} < y_{kl}$ if $i \neq k$ and $y_i^* < y_k^*$ and $y_{ij} \leqslant y_{il}$ if $j \leqslant l$.

The partially ordered set P obtained in this way generates M. Furthermore, if M^* is as in Theorem 2 then P becomes a semilattice if we add a zero element $\hat{0}$.

References

[1] G. Birkhoff, Lattice Theory (American Mathematical Society, Providence, RI, 3rd ed., 1967).

[2] L. Lovász, Communication complexity: a survey, in: B. Korte, L. Lovász, H.J. Prömel and A. Schrijver, eds., Paths, Flows, and VLSI-Layout (Springer, Berlin, 1990) 235–265.

[3] L. Lovász and M. Saks, Lattices, Möbius functions and communication complexity, in: Proceedings of the 29th IEEE FOCS (1988) 81–90.