

## RESEARCH

## Open Access



# Existence of nonoscillatory solutions for system of higher-order neutral differential equations with distributed coefficients and delays

Youjun Liu<sup>1\*</sup>, Huanhuan Zhao<sup>1</sup> and Jurang Yan<sup>2</sup>\*Correspondence: [lyj9791@126.com](mailto:lyj9791@126.com)<sup>1</sup>College of Mathematics and Computer Sciences, Shanxi Datong University, Datong, Shanxi 037009, P.R. China

Full list of author information is available at the end of the article

## Abstract

In this paper we consider the existence of nonoscillatory solutions for a system of higher-order neutral differential equations with distributed coefficients and delays. We use the *Banach* contraction principle to obtain new sufficient conditions for the existence of nonoscillatory solutions.

**Keywords:** system; higher-order; distributed coefficients and delays; nonoscillatory solutions; Banach contraction principle

## 1 Introduction and preliminary

In this paper, we consider the system of higher-order neutral differential equations with distributed coefficients and delays

$$\left[ r(t)\mathbf{x}(t) + \int_a^b p(t, \theta)\mathbf{x}(t - \theta) d\theta \right]^{(n)} + (-1)^{n+1} \left[ \int_c^d Q_1(t, \tau)\mathbf{x}(t - \tau) d\tau - \int_e^f Q_2(t, \sigma)\mathbf{x}(t - \sigma) d\sigma - \mathbf{h}(t) \right] = \mathbf{0}, \quad (1)$$

- (1) where  $n$  is a positive integer,  $n \geq 1$ ,  $0 < a < b$ ,  $0 < c < d$ ,  $0 < e < f$ ;
- (2)  $r \in C([t_0, \infty), \mathbf{R}^+)$ ,  $r(t) > 0$ ,  $p \in C([t_0, \infty) \times [a, b], \mathbf{R})$ ,  $\mathbf{h} \in C([t_0, \infty), \mathbf{R})$ ,
- (3)  $\mathbf{x} \in \mathbf{R}^n$ ,  $Q_i$  is continuous  $n \times n$  matrix on  $[t_0, \infty)$ ,  $i = 1, 2$ .

Recently there have been a lot of activities concerning the existence of nonoscillatory solutions for neutral differential equations with positive and negative coefficients. In 2005, the existence of nonoscillatory solutions of the first-order linear neutral delay differential equations

$$\frac{d}{dt} [x(t) + P(t)x(t - \tau)] + Q_1(t)x(t - \sigma_1) - Q_2(t)x(t - \sigma_2) = 0$$

was investigated by Zhang *et al.* [1]. In 2012, Candan [2] studied the higher-order nonlinear differential equation

$$[r(t)[x(t) + P(t)x(t - \tau)]^{(n-1)'} + (-1)^{n+1}[Q_1(t)g_1(x(t - \sigma_1)) - Q_2(t)g_2(x(t - \mu)) - f(t)] = 0.$$

In 2013, Candan [3] has investigated the existence of nonoscillatory solutions for the system of higher-order nonlinear neutral differential equations

$$[x(t) + P(t)x(t - \theta)]^{(n)} + (-1)^{n+1}[Q_1(t)x(t - \sigma_1) - Q_2(t)x(t - \sigma_2)] = \mathbf{0}.$$

In the same year, Liu *et al.* [4] has obtained the existence of nonoscillatory solutions for the system of higher-order neutral differential equations

$$[r(t)[x(t) + P(t)x(t - \theta)]^{(n-1)'} + (-1)^n \left( \int_c^d Q_1(t, \tau)x(t - \tau) d\tau - \int_e^f Q_2(t, \sigma)x(t - \sigma) d\sigma \right) = \mathbf{0}.$$

As can be seen from the development process of the above equations, the delay of neutral part in the discussed differential equations were all constant delays. However, the case for distributed deviating arguments is rather rare; see [5, 6]. In 2015, Candan and Gecgel [6] studied the systems of higher-order neutral differential equations with distributed delay

$$\left[ \left[ x(t) + \int_{a_3}^{b_3} \tilde{P}(t, \xi)x(t - \xi) d\xi \right] \right]' + (-1)^{n+1} \left[ \int_{a_1}^{b_1} Q_1(t, \xi)x(t - \xi) d\xi - \int_{a_2}^{b_2} Q_2(t, \xi)x(t - \xi) d\xi \right] = \mathbf{0}, \tag{2}$$

the discussion only covered the condition for the coefficient being  $0 < \int_{a_3}^{b_3} \tilde{P}(t, \xi)x(t - \xi) d\xi < \frac{1}{2}$  and  $-\frac{1}{2} < \int_{a_3}^{b_3} \tilde{P}(t, \xi)x(t - \xi) d\xi < 0$ . However, in this paper, the difficulty in establishing a feasible operator was settled by skillful use of  $r(t)$ , and the coefficients  $\int_a^b p_2(t, \xi) d\xi$  in the neutral part were all discussed in four cases, that is,  $(-\infty, -1), (-1, 0), (0, 1), (1, +\infty)$ . Thus, in view of the above, this paper may have theoretical value as well as practical application value. For related work, we refer the reader to [7–10].

A solution of the system of equations (1) is a continuous vector function  $x(t)$  defined on  $([t_1 - \mu, \infty), \mathbf{R}^n)$ , for some  $t_1 > t_0$ , such that  $r(t)x(t) - \int_a^b p(t, \theta)x(t - \theta) d\theta$  is  $n$  times continuously differentiable and the system of equations (1) holds for all  $n \geq 1$ . Here,  $\mu = \max\{b, \tau, \sigma\}$ .

### 2 The main results

**Theorem 1** Assume that  $0 \leq \int_a^b p(t, \theta) d\theta \leq p_1 < 1$  and

$$\int_{t_0}^\infty s^{n-1} \left\| \int_c^d Q_1(t, \tau) d\tau \right\| ds < \infty, \quad \int_{t_0}^\infty s^{n-1} \left\| \int_e^f Q_2(t, \sigma) d\sigma \right\| ds < \infty, \tag{3}$$

$$\int_{t_0}^\infty s^{n-1} \|h(s)\| ds < \infty.$$

Then equation (1) has a bounded nonoscillatory solution.

*Proof* Let  $\Lambda$  be the set of all continuous and bounded vector functions on  $[t_0, \infty)$  with the sup norm. Set  $A = \{x \in \Lambda, M_1 \leq \|x(t)\| \leq M_2, t \geq t_0\}$ , where  $M_1, M_2$  are two positive constants and  $c$  is a constant vector, such that  $p_1 M_2 + \frac{M_1}{p_1} < \|c\| < M_2, 1 \leq r(t) \leq \frac{1}{p_1}$ . From (3), one can choose a  $t_1 \geq t_0, t_1 \geq t_0 + \mu$ , sufficiently large,  $t \geq t_1$ , such that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_2 \left\| \int_c^d Q_1(t, \tau) d\tau \right\| + \|h(s)\| \right] ds \leq M_2 - \|c\|, \tag{4}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_2 \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| + \|h(s)\| \right] ds \leq \|c\| - p_1 M_2 + \frac{M_1}{p_1}, \tag{5}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(t, \tau) d\tau \right\| + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \right] ds \leq 1 - p_1, \tag{6}$$

and one defines an operator  $T$  on  $A$  as follows:

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{r(t)} \left\{ c - \int_a^b p(t, \theta) \mathbf{x}(t - \theta) d\theta + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}(s - \tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}(s - \sigma) d\sigma - h(s) \right] ds \right\} & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases}$$

It is easy to see that  $T\mathbf{x}$  is continuous, for  $t \geq t_1, \mathbf{x} \in A$ , by using (4), we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\leq \frac{1}{r(t)} \left\{ \|c\| + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}(s - \tau) d\tau - h(s) \right] ds \right\| \right\} \\ &\leq \|c\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_2 \left\| \int_c^d Q_1(t, \tau) d\tau \right\| + \|h(s)\| \right] ds \\ &\leq M_2, \end{aligned}$$

and taking (5) into account, we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\geq \frac{1}{r(t)} \left\{ \|c\| - \int_a^b p(t, \theta) \|\mathbf{x}(t - \theta)\| d\theta \right. \\ &\quad \left. - \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_e^f Q_2(s, \sigma) \mathbf{x}(s - \sigma) d\sigma - h(s) \right] ds \right\| \right\} \\ &\geq p_1 \left\{ \|c\| - p_1 M_2 - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left( M_1 \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| + \|h(s)\| \right) ds \right\} \\ &\geq M_1. \end{aligned}$$

These show that  $TA \subset A$ , since  $A$  is a bounded, closed, and convex subset of  $\Lambda$ , in order to apply the contraction principle we have to show that  $T$  is a contraction mapping on  $A$ . For  $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$ , and  $t \geq t_1$ ,

$$\begin{aligned} &\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\ &\leq \frac{1}{r(t)} \left\{ \int_a^b p(t, \theta) \|\mathbf{x}_1(t - \theta) - \mathbf{x}_2(t - \theta)\| d\theta \right. \\ &\quad \left. + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}_1(s - \tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}_1(s - \sigma) d\sigma - h(s) \right] ds \right\| \right\} \end{aligned}$$

$$\begin{aligned} & - \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}_2(s-\tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}_2(s-\sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \Big\} \\ & \leq \frac{1}{r(t)} \left\{ p_1 \|\mathbf{x}_1 - \mathbf{x}_2\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(t, \tau) d\tau \right\| \|\mathbf{x}_1(s-\tau) - \mathbf{x}_2(s-\tau)\| \right. \right. \\ & \quad \left. \left. + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \|\mathbf{x}_1(s-\sigma) - \mathbf{x}_2(s-\sigma)\| \right] ds \right\}. \end{aligned}$$

Using (6),

$$\begin{aligned} & \|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\ & \leq \|\mathbf{x}_1 - \mathbf{x}_2\| \left( p_1 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(t, \tau) d\tau \right\| + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \right] ds \right) \\ & < \|\mathbf{x}_1 - \mathbf{x}_2\|. \end{aligned}$$

which shows that  $T$  is a contraction mapping on  $A$  and therefore there exists a unique solution, obviously a bounded positive solution of (1)  $\mathbf{x} \in A$ , such that  $T\mathbf{x} = \mathbf{x}$ . The proof is complete.  $\square$

**Theorem 2** Assume that  $1 < p_3 \leq \int_a^b p(t, \theta) d\theta \leq p_2 < 2p_3 < +\infty$ , and that (3) holds. Then equation (1) has a bounded nonoscillatory solution.

*Proof* Let  $\Lambda$  be the set of all continuous and bounded vector functions on  $[t_0, \infty)$  with the sup norm. Set  $A = \{x \in \Lambda, M_3 \leq \|\mathbf{x}(t)\| \leq M_4, t \geq t_0\}$ , where  $M_3, M_4$  are two positive constants such that  $p_2M_4 + 2p_2M_3 < \|\mathbf{c}\| < 2p_3M_4, 2p_3 \leq r(t) \leq 2p_2$ . From (3), one can choose a  $t_1 \geq t_0 + b$ , sufficiently large  $t \geq t_1$ , such that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_4 \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \|\mathbf{h}(s)\| \right] ds \leq 2p_3M_4 - \|\mathbf{c}\|, \tag{7}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_4 \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| + \|\mathbf{h}(s)\| \right] ds \leq \|\mathbf{c}\| - p_2M_4 - 2p_2M_3, \tag{8}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \right] ds \leq 2p_3 - p_2, \tag{9}$$

and one defines an operator  $T$  on  $A$  as follows:

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{r(t)} \{ \mathbf{c} - \int_a^b p(t, \theta) \mathbf{x}(t-\theta) d\theta + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} [ \int_c^d Q_1(s, \tau) \mathbf{x}(s-\tau) d\tau \\ - \int_e^f Q_2(s, \sigma) \mathbf{x}(s-\sigma) d\sigma - \mathbf{h}(s) ] ds \} & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases}$$

It is easy to see that  $T$  is continuous, for  $t \geq t_1, \mathbf{x} \in A$ . By using (7), we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| & \leq \frac{1}{r(t)} \left\{ \|\mathbf{c}\| + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}(s-\tau) d\tau - \mathbf{h}(s) \right] ds \right\| \right\} \\ & \leq \frac{1}{2p_3} \left\{ \|\mathbf{c}\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_4 \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \|\mathbf{h}(s)\| \right] ds \right\} \leq M_4, \end{aligned}$$

and taking (8) into account, we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\geq \frac{1}{r(t)} \left\{ \|\mathbf{c}\| - \int_a^b p(t, \theta) \|\mathbf{x}(t - \theta)\| d\theta \right. \\ &\quad \left. + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_e^f Q_2(s, \sigma) \mathbf{x}(s - \sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \right\} \\ &\geq \frac{1}{2p_2} \left\{ \|\mathbf{c}\| - p_2 M_4 - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left( M_4 \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| + \|\mathbf{h}(s)\| \right) ds \right\} \\ &\geq M_3. \end{aligned}$$

These show that  $TA \subset A$ , since  $A$  is a bounded, closed, and convex subset of  $\Lambda$ , in order to apply the contraction principle, we have to show that  $T$  is a contraction mapping on  $A$ . For  $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$ , and  $t \geq t_1$ ,

$$\begin{aligned} &\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\ &\leq \frac{1}{r(t)} \left\{ \int_a^b p(t, \theta) \|\mathbf{x}_1(t - \theta) - \mathbf{x}_2(t - \theta)\| d\theta \right. \\ &\quad + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}_1(s - \tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}_1(s - \sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \\ &\quad - \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}_2(s - \tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}_2(s - \sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \right\} \\ &\leq \frac{1}{r(t)} \left\{ p_2 \|\mathbf{x}_1 - \mathbf{x}_2\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(s, \tau) d\tau \right\| \|\mathbf{x}_1(s - \tau) - \mathbf{x}_2(s - \tau)\| \right. \right. \\ &\quad \left. \left. + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \|\mathbf{x}_1(s - \sigma) - \mathbf{x}_2(s - \sigma)\| \right] ds \right\}, \end{aligned}$$

using (9),

$$\begin{aligned} &\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\ &\leq \frac{1}{2p_3} \|\mathbf{x}_1 - \mathbf{x}_2\| \left\{ p_2 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \right] ds \right\} \\ &< \|\mathbf{x}_1 - \mathbf{x}_2\|, \end{aligned}$$

which shows that  $T$  is a contraction mapping on  $A$  and therefore there exists a unique solution, obviously a bounded positive solution of (1)  $\mathbf{x} \in A$ , such that  $T\mathbf{x} = \mathbf{x}$ . The proof is complete.  $\square$

**Theorem 3** Assume that  $-1 < p_4 \leq \int_a^b p(t, \theta) d\theta \leq 0$  and that (2) holds. Then equation (1) has a bounded nonoscillatory solution.

*Proof* Let  $\Lambda$  be the set of all continuous and bounded functions on  $[t_0, \infty)$  with the sup norm. Set  $A = \{x \in \Lambda, M_5 \leq \|x(t)\| \leq M_6, t \geq t_0\}$ , where  $M_5, M_6$  are two positive constants such that  $\frac{M_5}{-p_4} < \|\mathbf{c}\| < (1 + p_4)M_6, 1 \leq r(t) \leq \frac{1}{-p_4}$ . From (3), one can choose a  $t_1 \geq t_0 + b$ ,

sufficiently large  $t \geq t_1$  such that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_6 \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \|\mathbf{h}(s)\| \right] ds \leq (1 + p_4)M_6 - \|\mathbf{c}\|, \tag{10}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_6 \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| + \|\mathbf{h}(s)\| \right] ds \leq \|\mathbf{c}\| + \frac{M_5}{p_4}, \tag{11}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \right] ds \leq 1 + p_4, \tag{12}$$

and one defines an operator  $T$  on  $A$  as follows:

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{r(t)} \left\{ \mathbf{c} - \int_a^b p(t, \theta) \mathbf{x}(t - \theta) d\theta + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}(s - \tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}(s - \sigma) d\sigma - \mathbf{h}(s) \right] ds \right\} & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases}$$

It is easy to see that  $T$  is continuous, for  $t \geq t_1, \mathbf{x} \in A$ . By using (10), we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\leq \frac{1}{r(t)} \left\{ \|\mathbf{c}\| - \int_a^b p(t, \theta) \|\mathbf{x}(t - \theta)\| d\theta \right. \\ &\quad \left. + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}(s - \tau) d\tau - \mathbf{h}(s) \right] ds \right\| \right\} \\ &\leq \|\mathbf{c}\| - p_4 M_6 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_6 \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \|\mathbf{h}(s)\| \right] ds \\ &\leq M_6, \end{aligned}$$

and taking (11) into account, we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\geq \frac{1}{r(t)} \left\{ \|\mathbf{c}\| - \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_e^f Q_2(s, \sigma) \mathbf{x}(s - \sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \right\} \\ &\geq -p_4 \left\{ \|\mathbf{c}\| - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_6 \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| + \|\mathbf{h}(s)\| \right] ds \right\} \\ &\geq M_5. \end{aligned}$$

These show that  $TA \subset A$ , since  $A$  is a bounded, closed, and convex subset of  $\Lambda$ , in order to apply the contraction principle, we have to show that  $T$  is a contraction mapping on  $A$ . For  $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$ , and  $t \geq t_1$ ,

$$\begin{aligned} &\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\ &\leq \frac{1}{r(t)} \left\{ \int_a^b p(t, \theta) \|\mathbf{x}_1(t - \theta) - \mathbf{x}_2(t - \theta)\| d\theta \right. \\ &\quad + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}_1(s - \tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}_1(s - \sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \\ &\quad - \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}_2(s - \tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}_2(s - \sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \Big\} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{r(t)} \left\{ -p_4 \|\mathbf{x}_1 - \mathbf{x}_2\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(s, \tau) d\tau \right\| \|\mathbf{x}_1(s-\tau) - \mathbf{x}_2(s-\tau)\| \right. \right. \\ &\quad \left. \left. + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \|\mathbf{x}_1(s-\sigma) - \mathbf{x}_2(s-\sigma)\| \right] ds \right\}. \end{aligned}$$

Or using (12),

$$\begin{aligned} &\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\ &\leq \|\mathbf{x}_1 - \mathbf{x}_2\| \left( -p_4 + KL \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \right] ds \right) \\ &< \|\mathbf{x}_1 - \mathbf{x}_2\|, \end{aligned}$$

which shows that  $T$  is a contraction mapping on  $A$  and therefore there exists a unique solution, obviously a bounded positive solution of (1)  $\mathbf{x} \in A$ , such that  $T\mathbf{x} = \mathbf{x}$ . The proof is complete.  $\square$

**Theorem 4** Assume that  $-\infty < 2p_5 < p_6 \leq \int_a^b p(t, \theta) d\theta \leq p_5 < -1$  and that (3) holds. Then equation (1) has a bounded nonoscillatory solution.

*Proof* Let  $\Lambda$  be the set of all continuous and bounded functions on  $[t_0, \infty)$  with the sup norm. Set  $A = \{\mathbf{x} \in \Lambda, M_7 \leq \|\mathbf{x}(t)\| \leq M_8, t \geq t_0\}$ , where  $M_7, M_8$  are two positive constants such that  $-2p_6M_7 < \|\mathbf{c}\| < (-2p_5 + p_6)M_8, -2p_5 < r(t) < -2p_6$ . From (3), one can choose a  $t_1 \geq t_0 + b$ , sufficiently large  $t \geq t_1$ , such that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_8 \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \|\mathbf{h}(s)\| \right] ds \leq -2p_5M_8 + p_6M_8 - \|\mathbf{c}\|, \tag{13}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_8 \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| + \|\mathbf{h}(s)\| \right] ds \leq \|\mathbf{c}\| + 2p_6M_7, \tag{14}$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \right] ds \leq p_6 - 2p_5, \tag{15}$$

and one defines an operator  $T$  on  $A$  as follows:

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{r(t)} \left\{ \mathbf{c} - \int_a^b p(t, \theta) \mathbf{x}(t-\theta) d\theta + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}(s-\tau) d\tau \right. \right. \\ \quad \left. \left. - \int_e^f Q_2(s, \sigma) \mathbf{x}(s-\sigma) d\sigma - \mathbf{h}(s) \right] ds \right\} & t \geq t_1, \\ (T\mathbf{x})(t_1) & t_0 \leq t \leq t_1. \end{cases}$$

It is easy to see that  $T$  is continuous, for  $t \geq t_1, \mathbf{x} \in A$ . By using (13), we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\leq \frac{1}{r(t)} \left\{ \|\mathbf{c}\| - \int_a^b p(t, \theta) \|\mathbf{x}(t-\theta)\| d\theta \right. \\ &\quad \left. + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}(s-\tau) d\tau - \mathbf{h}(s) \right] ds \right\| \right\} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{-2p_5} \left\{ \|\mathbf{c}\| - p_6 M_8 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_8 \left\| \int_c^d Q_1(s, \tau) d\tau \right\| + \|\mathbf{h}(s)\| \right] ds \right\} \\ &\leq M_8, \end{aligned}$$

and taking (14) into account, we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\geq \frac{1}{r(t)} \left\{ \|\mathbf{c}\| - \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_e^f Q_2(s, \sigma) \mathbf{x}(s-\sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \right\} \\ &\geq \frac{1}{-2p_6} \left\{ \|\mathbf{c}\| - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ M_8 \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| + \|\mathbf{h}(s)\| \right] ds \right\} \\ &\geq M_7. \end{aligned}$$

These show that  $TA \subset A$ , since  $A$  is a bounded, closed, and convex subset of  $\Lambda$ , in order to apply the contraction principle, we have to show that  $T$  is a contraction mapping on  $A$ . For  $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$ , and  $t \geq t_1$ ,

$$\begin{aligned} &\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\ &\leq \frac{1}{r(t)} \left\{ \int_a^b p(t, \theta) \|\mathbf{x}_1(t-\theta) - \mathbf{x}_2(t-\theta)\| d\theta \right. \\ &\quad + \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}_1(s-\tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}_1(s-\sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \\ &\quad - \left\| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \int_c^d Q_1(s, \tau) \mathbf{x}_2(s-\tau) d\tau - \int_e^f Q_2(s, \sigma) \mathbf{x}_2(s-\sigma) d\sigma - \mathbf{h}(s) \right] ds \right\| \Big\} \\ &\leq \frac{1}{r(t)} \left\{ -p_6 \|\mathbf{x}_1 - \mathbf{x}_2\| + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[ \left\| \int_c^d Q_1(s, \tau) d\tau \right\| \|\mathbf{x}_1(s-\tau) - \mathbf{x}_2(s-\tau)\| \right. \right. \\ &\quad \left. \left. + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \|\mathbf{x}_1(s-\sigma) - \mathbf{x}_2(s-\sigma)\| \right] ds \right\}. \end{aligned}$$

Or using (15),

$$\begin{aligned} &\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\ &\leq \frac{1}{-2p_5} \|\mathbf{x}_1 - \mathbf{x}_2\| \left\{ -p_6 + \frac{1}{(n-1)!} \int_t^\infty \frac{(s-t)^{n-1}}{r(s)} \left[ \left\| \int_c^d Q_1(s, \tau) d\tau \right\| \right. \right. \\ &\quad \left. \left. + \left\| \int_e^f Q_2(s, \sigma) d\sigma \right\| \right] ds \right\} \\ &< \|\mathbf{x}_1 - \mathbf{x}_2\|, \end{aligned}$$

which shows that  $T$  is a contraction mapping on  $A$  and therefore there exists a unique solution, obviously a bounded positive solution of (1)  $\mathbf{x} \in A$ , such that  $T\mathbf{x} = \mathbf{x}$ . The proof is complete. □



### 3 Example

Consider the higher-order neutral differential equation with distributed coefficients and delays

$$\begin{aligned} & \left( \frac{2}{2 + \sin t} \mathbf{x}(t) + \int_{\frac{\pi}{2}}^{\pi} e^{-t} \mathbf{x}(t - \theta) d\theta \right)^{(3)} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tau e^{-t} \begin{pmatrix} \frac{5}{2} & \frac{7}{2} \\ 5 & 1 \end{pmatrix} \mathbf{x}(t - \tau) d\tau \\ & - \int_{\pi}^{2\pi} e^{-t} \begin{pmatrix} 2 & 4 \\ \frac{7}{3} & \frac{11}{3} \end{pmatrix} \mathbf{x}(t - \sigma) d\sigma \\ & = e^{-t} (-4 \cos t - 12\pi \sin t - 7\pi + \pi^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \tag{16}$$

Here,  $n = 3$ ,

$$\begin{aligned} r(t) &= \frac{2}{2 + \sin t}, & a = c &= \frac{\pi}{2}, & b = e &= \pi, & d &= \frac{3\pi}{2}, \\ f &= 2\pi, & \mathbf{h}(t) &= e^{-t} (-4 \cos t - 12\pi \sin t - 7\pi + \pi^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ Q_1(t) &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tau e^{-t} \begin{pmatrix} \frac{5}{2} & \frac{7}{2} \\ 5 & 1 \end{pmatrix} d\tau, & Q_2(t) &= \int_{\pi}^{2\pi} e^{-t} \begin{pmatrix} 2 & 4 \\ \frac{7}{3} & \frac{11}{3} \end{pmatrix} d\sigma. \end{aligned}$$

It is easy to see that  $1 \leq r(t) \leq 2$ ,

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\pi} e^{-t} d\theta &= \frac{\pi}{2} e^{-t} > 1 (t > 0), & \int_t^{\infty} s^2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tau e^{-s} \left\| \begin{pmatrix} \frac{5}{2} & \frac{7}{2} \\ 5 & 1 \end{pmatrix} \right\| d\tau ds &< \infty, \\ \int_t^{\infty} s^2 \int_{\pi}^{2\pi} e^{-s} \left\| \begin{pmatrix} 2 & 4 \\ \frac{7}{3} & \frac{11}{3} \end{pmatrix} \right\| d\sigma ds &< \infty, \\ \int_t^{\infty} s^2 e^{-s} (-4 \cos s - 12\pi \sin s - 7\pi + \pi^2) \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| ds &< \infty, \end{aligned}$$

thus Theorem 2 holds.

In fact,  $\mathbf{x}(t) = \begin{pmatrix} 2 + \sin t \\ 2 + \sin t \end{pmatrix}$  is a nonoscillatory solution of equation (16).

### 4 Remark

When  $r(t) \equiv 1, \mathbf{h}(t) = \mathbf{0}$ , equation (1) becomes equation (2), thus this paper improves results of Candan and Gecgel [6].

#### Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

#### Author details

<sup>1</sup>College of Mathematics and Computer Sciences, Shanxi Datong University, Datong, Shanxi 037009, P.R. China. <sup>2</sup>School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, P.R. China.

**Acknowledgements**

This research is supported by Scientific Research Project Shanxi Datong University (No. 2011K3).

Received: 12 July 2016 Accepted: 7 December 2016 Published online: 07 February 2017

**References**

1. Zhang, W, Feng, W, Yan, J, Song, J: Existence of nonoscillatory solutions of first-order linear neutral delay differential equations. *Comput. Math. Appl.* **49**, 1021-1027 (2005)
2. Candan, T: The existence of nonoscillatory solutions of higher order nonlinear neutral differential equations. *Appl. Math. Lett.* **25**, 412-416 (2012)
3. Candan, T: Existence of nonoscillatory solutions for system of higher order neutral differential equations. *Math. Comput. Model.* **57**, 375-381 (2013)
4. Liu, Y, Zhang, J, Yan, J: Existence of nonoscillatory solutions for system of higher-order neutral differential equations with distributed deviating arguments. *Discrete Dyn. Nat. Soc.* **2013**, 1-8 (2013)
5. Candan, T: Existence of nonoscillatory solutions of first-order nonlinear neutral differential equations. *Appl. Math. Lett.* **26**, 1182-1186 (2013)
6. Candan, T, Gecgel, AM: Existence of nonoscillatory solutions for system of higher order neutral differential equations with distributed delay. *J. Comput. Anal. Appl.* **18**, 266-276 (2015)
7. Candan, T, Dahiya, RS: Existence of nonoscillatory solutions of first and second order neutral differential equations with distributed deviating arguments. *J. Franklin Inst.* **347**, 1309-1316 (2010)
8. Liu, Y, Zhang, J, Yan, J: Existence of nonoscillatory solutions of higher order neutral differential equations with distributed deviating arguments. *Acta Math. Appl. Sin.* **38**(2), 235-243 (Chinese) (2015)
9. Zhou, Y, Zhang, BG: Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients. *Appl. Math. Lett.* **15**, 867-874 (2002)
10. Györi, I, Ladas, G: *Oscillation Theory of Delay Differential Equations with Applications*. Clarendon Press, Oxford (1991)

Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

---

Submit your next manuscript at ► [springeropen.com](http://springeropen.com)

---