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Decomposition and Control

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1. INTRODUCTION

Various methods, both approximate and exact, have been used to replace the mathematical description of a differential system by a pair, or by a sequence, of descriptions of systems of lower order. The decompositions, expressed in some convenient "canonical" form, are then employed to deduce the behavior of solutions of the overall system from characteristics of its interconnected subsystems. This procedure, often applied to linear time-invariant systems, is used here to treat nonlinear, nonautonomous control systems describable by ordinary vector differential equations.

We give, in Section 2, immediately following some needed definitions and the requirements to be put on the differential equations, sufficient conditions for certain "perturbed" control systems, and for control-linear systems to be completely controllable if their subsystems are. We obtain similar results, also, for systems whose subsystems are locally controllable, locally path controllable, or globally path controllable.

The existence of a decomposed system is assumed; we do not address the

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problem of obtaining the decompositions used. Others have found conditions for the existence of some decompositions, and their work is cited, when appropriate, below.

2. Controllability of Systems Having a Decomposition

Our aim in this section is to provide sufficient conditions for several controllability properties of systems in decomposed form. The first definition specifies the regularity that we will require of the control systems under consideration.

Definition 2.1. A mapping $f\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is said to be *quasi-C^1* if the following conditions are satisfied:

(a) for every $t \in \mathbb{R}$ the mapping $(x, w) \mapsto f(x, w, t)$ is C^1;

(b) for every $(x, w) \in \mathbb{R}^n \times \mathbb{R}^m$ the mappings $t \mapsto f(x, w, t)$, $t \to D_1(x, w, t)$, and $t \mapsto D_2 f(x, w, t)$ are piecewise-continuous ($D_i f$ denotes the partial derivative of f with respect to its ith variable);

(c) the mappings f, $D_1 f$, and $D_2 f$ are locally bounded on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$. It is clear that a C^1 mapping $f\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is quasi-C^1. The control systems discussed here will be of the form

$$\dot{x} = f(x, u(t), t), \tag{1}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is quasi-C^1 and the controls u are elements of PC^m_∞, the set of all piecewise-continuous functions $u\colon \mathbb{R} \to \mathbb{R}^m$ for which $\sup\{\|u(t)\|\colon t \in \mathbb{R}\}$ is finite. We note that while each control is bounded, there is no a priori bound for the entire family of admissible controls. We will also have occasion to deal with autonomous system

$$\dot{x} = f(x, u(t)), \tag{2}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1.

We say that the nonautonomous system (1) admits a *serial decomposition* if the state $x \in \mathbb{R}^n$ and the control $u(t) \in \mathbb{R}^m$ can be split up into r components

$$x = \operatorname{col}(x_1, \ldots, x_r), \qquad u(t) = \operatorname{col}(u_1(t), \ldots, u_r(t)), \tag{3}$$

where $1 < r \leqslant \min\{n, m\}$, $x_i \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, $1 \leqslant i \leqslant r$, $\sum_{i=1}^r n_i = n$, $\sum_{i=1}^r m_i = m$, and with respect to these components (1) has the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u_1(t), t), \\ \dot{x}_2 &= f_2(x_1, x_2, u_1(t), u_2(t), t), \\ &\vdots \\ \dot{x}_r &= f_r(x_1,..., x_r,..., u_r(t), t). \end{aligned} \tag{4}$$

There is an obvious corresponding notion of a serial decomposition of the autonomous system (2). The component labeling (3) of the state and control will be used throughout and the meaning of the notation will not be repeated in the statements of our results.

System (1) is said to have a *parallel decomposition* if it can be written as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u_1(t), t), \\ \dot{x}_2 &= f_2(x_2, u_2(t), t), \\ &\vdots \\ \dot{x}_r &= f_r(x_r, u_r(t), t), \end{aligned} \tag{5}$$

where the component labeling given in (3) has been used. The corresponding formulation for autonomous systems is evident.

Respondek [9] has given necessary and sufficient conditions for autonomous systems (2) that are linear in the control to be transformable by a *local* change of coordinates, $y = \psi(x)$, into a system

$$\dot{y} = h(y, u(t)), \tag{6}$$

which admits either a serial decomposition

$$\begin{aligned} \dot{y}_1 &= h_1(y_1, u_1(t)), \\ \dot{y}_2 &= h_2(y_2, y_2, u_2(t)), \\ &\vdots \\ \dot{y}_r &= h_r(y_1,..., y_r, u_r(t)) \end{aligned} \tag{6a}$$

or (under more stringent requirements) a *parallel decomposition*

$$\begin{aligned} \dot{y}_1 &= h_1(y_1, u_1(t)), \\ \dot{y}_2 &= h_2(y_2, u_2(t)), \\ &\vdots \\ \dot{y}_r &= h_r(y_r, u_r(t)). \end{aligned} \tag{6b}$$

Other results on decompositions of control systems having symmetries can be found in [5].

The control systems treated here are assumed to have exact global decompositions. Our concern is not with how the decomposition is obtained, but rather with what one can say about certain controllability properties of the system given its decomposition.

Before proceeding with our results, we recall a few basic definitions. A point $z \in \mathbb{R}^n$ is said to be *attainable* from a point $y \in \mathbb{R}^n$ via the system (1) on the interval $[t_0, t_1]$ if there exists a control $u \in PC_\infty^m$ and an absolutely continuous function ϕ: $[t_0, t_1] \to \mathbb{R}^n$ such that $\dot{\phi}(t) = f(\phi(t), u(t), t)$ a.e. for $t \in [t_0, t_1]$ and $\phi(t_0) = y$, $\phi(t_1) = z$. The function ϕ is called a *response* of (1) corresponding to the control u. We denote by $\mathscr{A}(y;\ t_0, t_1)$ the set of all $z \in \mathbb{R}^n$ that are attainable from y via the system (1) on the interval $[t_0, t_1]$. The system (1) is *completely controllable* on the interval $[t_0, t_1]$ if $\mathscr{A}(y;\ t_0, t_1) = \mathbb{R}^n$ for every $y \in \mathbb{R}^n$. Equivalently (1) is completely controllable on $[t_0, t_1]$ if every pair of points y, $z \in \mathbb{R}^n$ can be joined by a response of (1) defined on $[t_0, t_1]$.

The first results of this paper deal with the complete controllability of systems (1) that have a particular decomposition. Later, we will treat other types of controllability. The following theorem is well known, but its proof is short and is included for completeness. More general versions of this theorem can be found, e.g., in [10].

THEOREM 2.2. *Consider the linear control system*

$$\dot{x} = A(t)x + B(t)\,u(t), \tag{7}$$

where $A(t)$, $B(t)$ are $n \times n$, $n \times m$ matrices, respectively, with entries that are piecewise-continuous functions of $t \in \mathbb{R}$. If (7) is completely controllable on an interval $[t_0, t_1]$, then for every piecewise-continuous function g: $[t_0, t_1] \to \mathbb{R}^n$ the system

$$\dot{x} = A(t)x + B(t)\,u(t) + g(t) \tag{8}$$

is also completely controllable on $[t_0, t_1]$.

Proof. Fix an initial point $\bar{x} \in \mathbb{R}^n$, let $\phi(t; u)$ denote the solution of (7) such that $\phi(t_0; u) = \bar{x}$, and let $\tilde{\phi}(t; u)$ denote the solution of (8) such that $\tilde{\phi}(t_0; u) = \bar{x}$. A simple application of the variation-of-parameters formula shows that

$$\tilde{\phi}(t; u) = \phi(t; u) + \int_{t_0}^{t} X(t, s)\, g(s)\, ds, \tag{9}$$

where $X(t, s)$ is the fundamental-matrix solution of the homogeneous linear system $\dot{x} = A(t)x$. The assumption that (7) is completely controllable on $[t_0, t_1]$ is equivalent to the assertion that the affine mapping $u \mapsto \phi(t_1; u)$ of PC_∞^m into $\mathbb{R}^n$ is surjective (for arbitrary $\bar{x}$). It follows from (9) that the mapping $u \mapsto \tilde{\phi}(t_1; u)$ is the translate of the surjective affine mapping $u \mapsto \phi(t_1; u)$ by the constant vector $\int_{t_0}^{t_1} X(t_1, s)\, g(s)\, ds$. Therefore, the affine mapping $u \mapsto \tilde{\phi}(t_1; u)$ is also surjective, which is equivalent to the complete controllability of (8) on the interval $[t_0, t_1]$. ∎

THEOREM 2.3. *Let system* (1) *have a ("hybrid") decomposition of the form*

$$\begin{aligned}
\dot{x}_1 &= A_1(t)x + B_1(t)\, u_1(t) + g_1(t),\\
\dot{x}_2 &= A_2(t)x + B_2(t)\, u_2(t) + g_2(x_2, u_2(t), t),\\
&\ \vdots\\
\dot{x}_r &= A_r(t)x_r + B_r(t)\, u_r(t) + g_r(x_1, ..., x_{r-1}, u_1(t), ..., u_{r-1}(t)),
\end{aligned} \tag{10}$$

where $A_i(t)$, $B_i(t)$ are $n_i \times n_i$, $n_i \times m_i$ matrices, respectively, with entries that are piecewise-continuous functions of $t \in \mathbb{R}$ and g_i is C^1 in all of its arguments, $1 \leqslant i \leqslant r$. If each linear subsystem

$$\dot{x}_i = A_i(t)x_i + B_i(t)\, u_i(t), \tag{11}$$

$1 \leqslant i \leqslant r$, is completely controllable on $[t_0, t_1]$, then (10) *is also completely controllable on $[t_0, t_1]$.*

Proof. Given arbitrary points y, z in $\mathbb{R}^n$, we must show that y and z can be joined by a response of (10) defined on the interval $[t_0, t_1]$. First, we write y and z in component form

$$y = \text{col}(y_1, ..., y_r), \qquad z = \text{col}(z_1, ..., z_r),$$

where $y_i, z_i \in \mathbb{R}^{n_i}$ for $1 \leqslant i \leqslant r$. The system (11) with $i = 1$ is completely controllable on $[t_0, t_1]$ by assumption. From Theorem 2.2 we infer that the perturbed system

$$\dot{x}_1 = A_1(t)x_1 + B_1(t)\, u_1(t) + g_1(t) \tag{12}$$

is also completely controllable on $[t_0, t_1]$. Thus we can choose a control $\bar{u} \in PC_\infty^{m_1}$ whose corresponding response $\bar{\phi}_1$ of (12) satisfies

$$\bar{\phi}_1(t_0) = y_1, \qquad \bar{\phi}_1(t_1) = z_1.$$

For $t_0 \leqslant t \leqslant t_1$ we set $\tilde{g}_2(t) = g_2(\bar{\phi}_1(t), \bar{u}_1(t), t)$ and we observe that $\tilde{g}_2$ is piecewise continuous. The system (11) with $i=2$ is completely controllable on $[t_0, t_1]$ by assumption. Again from Theorem 2.2 we infer that the perturbed system

$$\begin{aligned}\dot{x}_2 &= A_2(t)x_2 + B_2(t)\,u_2(t) + \tilde{g}_2(t)\\ &= A_2(t)x_2 + B_2(t)\,u_2(t) + g_2(\bar{\phi}_1(t), \bar{u}_1(t), t)\end{aligned}$$

is also completely controllable on $[t_0, t_1]$. Proceeding serially, we obtain controls $\bar{u}_2 \in PC_\infty^{m},..., \bar{u}_r \in PC_\infty^{m_r}$ whose corresponding responses

$$\dot{x}_i = A_i(t)x_i + B_i(t)\,u_i(t) + \tilde{g}_i(t),$$

where $\tilde{g}_i(t) = g_i(\bar{\phi}_1(t),..., \bar{\phi}_{i-1}(t), \bar{u}_1(t),..., \bar{u}_{i-1}(t), t)$, satisfy $\bar{\phi}_i(t_0) = y_i$, $\bar{\phi}_i(t_1) = z$ for $2 \leqslant i \leqslant r$. It follows that $\bar{\phi} = \text{col}(\bar{\phi}_1,..., \bar{\phi}_r)$ is the desired response of (10) corresponding to the control $\bar{u} = \text{col}(\bar{u}_1,..., \bar{u}_r)$ that joins y to z on the interval $[t_0, t_1]$. ∎

Note that the linear parts of the system (10) have been assumed to be decomposed in parallel, while the nonlinear parts, the "perturbations," are taken to be in series; we have called this a "hybrid" decomposition. We emphasize that no assumptions have been made about the size of the perturbations, which may be large or small compared to the linear terms.

Now, we consider another class of nonlinear systems. A nonlinear autonomous control system is said to be (homogeneous) *control linear* if it has the form

$$\dot{x} = F(x)\,u(t), \tag{13}$$

where $F(x)$ is an $n \times m$ matrix whose entries are at least C^1-functions of x; such systems have also been called *symmetric* by other authors. A useful feature of control-linear systems is that, in spite of their nonlinearity, one can give computable sufficient conditions for their complete controllability. Specifically, it is known that if $F(x)$ is C^∞ and if the Lie algebra generated by the column vectors of $F(x)$ has dimension n at each $x \in \mathbb{R}^n$, then (13) is completely controllable on every interval $[t_0, t_1]$ with $t_0 \geqslant t_1$ (see [6, 8] for details). This condition is also necessary for complete controllability if $F(x)$ is real analytic.

Our next result is similar in spirit to Theorem 2.2. We omit the proof but note that the techniques of proof are similar to those found in [3].

THEOREM 2.4. *Consider the control-linear system* (13) *where the entries* *of the matrix $F(x)$ are C^1-functions of x. If* (13) *is completely controllable on* $[t_0, t_1]$, *then for every quasi-C^1 mapping g: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ the system*

$$\dot{x} = g(x, t) + F(x)\, u(t)$$

is also completely controllable on $[t_0, t_1]$.

THEOREM 2.5. *Let system* (1) *have the hybrid decomposition of the form*

$$\begin{aligned}
\dot{x}_1 &= F_1(x_1)\, u_1(t) + g_1(x_1, t),\\
\dot{x}_2 &= F_2(x_2)\, u_2(t) + g_2(x_1, x_2, u_1(t), t),\\
&\ \vdots\\
\dot{x} &= F_r(x_r)\, u_r(t) + g_r(x_1, ..., x_r, u_1(t), ..., u_{r-1}(t), t),
\end{aligned} \tag{14}$$

where $F_i(x_i)$ is an $n_i \times m_i$ matrix with entries that are C^1-functions of $x_i \in \mathbb{R}^{n_i}$ and g_i is C^1 in all of its arguments, $1 \leqslant i \leqslant r$. If each control-linear subsystem

$$\dot{x}_i = F_i(x_i)\, u_i(t), \tag{15}$$

$1 \leqslant i \leqslant r$, *is completely controllable on* $[t_0, t_1]$, *then* (14) *is also completely controllable on* $[t_0, t_1]$.

Proof. The argument proceeds serially and follows from Theorem 2.4 in much the same way that Theorem 2.3 follows from Theorem 2.2. We omit the details. ∎

We turn our attention to some more restrictive types of controllability for systems having a decomposition. Given a control $\bar{u} \in PC^m_\infty$ and a response $\bar{\phi}$: $[t_0, t_u] \to \mathbb{R}^n$ of (1) corresponding to $\bar{u}$ with initial condition $\bar{\phi}(t_0) = \bar{x}$, we say that (1) is *locally controllable* along $\bar{\phi}$ at time $t_1 \in [t_0, t_u]$ if $\bar{\phi}(t) \in \operatorname{int} \mathscr{A}(\bar{x}; t_0, t_1)$ (int $\mathscr{A}$ denotes the interior of a subset $\mathscr{A}$ of $\mathbb{R}^n$). A well-known sufficient condition for the local controllability of (1) along the response $\bar{\phi}$ at time t_1 is that the *linear variational control system*

$$\dot{x} = D_1 f(\bar{\phi}(t), \bar{u}(t), t)x + D_2 f(\bar{\phi}(t), \bar{u}(t), t)\, u(t) \tag{16}$$

be completely controllable on $[t_0, t_1]$ (see [7]). In what follows we let $w = \operatorname{col}(w_1, ..., w_r)$ denote the component form of a vector $w \in \mathbb{R}^m$ with respect to the decomposition (3).

THEOREM 2.6. *Assume that the system* (1) *admits the serial decomposition* (4), *let* $\bar{\phi}$: $[t_0, t_{\bar{u}}] \to \mathbb{R}^n$ *be a response of* (1) *corresponding to a control* $\bar{u} \in PC^m_\infty$, *and assume that each linearized subsystem*

$$\dot{x}_i = \frac{\partial f_i}{\partial x_i}(\bar{\phi}_1(t),..., \bar{\phi}_i(t), \bar{u}_1(t),..., \bar{u}_i(t), t) x_i$$
$$+ \frac{\partial f_i}{\partial w_i}(\bar{\phi}_1(t),..., \bar{\phi}_i(t), \bar{u}_1(t),..., \bar{u}_i(t), t)\, u_i(t) \tag{17}$$

is completely controllable on $[t_0, t_1]$ *for* $t_1 \in [t_0, t_{\bar{u}}]$ *and* $1 \leqslant i \leqslant r$. *Then* (1) *is locally controllable along* $\bar{\phi}$ *at time* t_1.

Proof. Using the decomposition (4), we see that the linear variational control system (15) admits the decomposition

$$\dot{x}_1 = \frac{\partial f_i}{\partial x_i}(\bar{\phi}_1(t), \bar{u}_1(t), t), t) x_1 + \frac{\partial f}{\partial w}(\bar{\phi}_1(t), \bar{u}_1(t), t)\, u_1(t),$$
$$\vdots$$
$$\dot{x}_r = \frac{\partial f_r}{\partial x_r}(\bar{\phi}_1(t),..., \bar{\phi}_r(t), \bar{u}_1(t),..., \bar{u}_r(t), t) x_r$$
$$+ \frac{\partial f_r}{\partial w_r}(\bar{\phi}_1(t),..., \bar{\phi}_r(t), \bar{u}_1(t),..., \bar{u}_r(t), t)\, u_r(t) \tag{18}$$
$$+ \sum_{i=1}^{r-1} \left\{ \frac{\partial f_r}{\partial x_i}(\bar{\phi}_1(t),..., \bar{\phi}_r(t), \bar{u}_1(t),..., \bar{u}_r(t), t) x_i \right.$$
$$\left. + \frac{\partial f_r}{\partial w_i}(\bar{\phi}_1(t),..., \bar{\phi}_r(t), \bar{u}_1(t),..., \bar{u}_r(t), t)\, u_i(t) \right\}.$$

From the assumed complete controllability of (17) on $[t_0, t_1]$ for $i = 1,..., r$, and from Theorem 2.3, it follows that (18), and hence (16), is completely controllable on $[t_0, t_1]$, whence the result. ▮

In some situations it is of interest not only to reach a specified state, but to do so along a specified path (or response). This type of controllability is much stronger than the complete and local controllability properties discussed above. Consequently it is obtained only under rather stringent hypotheses. One illustration of such a local "path-controllability" result is given in

THEOREM 2.7. *Let the mapping* f *in the system* (1) *be* C^1 (*as opposed to just quasi-*C^1) *and let* $\bar{u}$: $\mathbb{R} \to \mathbb{R}^m$ *be a bounded continuous control that generates a response* $\bar{\phi}$: $[t_0, t_1] \to \mathbb{R}^n$. *Suppose that for every* $t \in [t_0, t_1]$ *we have*

$$\operatorname{rank} D_2 f(\bar{\phi}(t), \bar{u}(t), t) = n. \tag{19}$$

Then there exists a $\delta > 0$ *such that if* ϕ: $[t_0, t_1] \to \mathbb{R}^n$ *is any* C^1 *function for which*

$$\max\{\|\phi(t) - \bar{\phi}(t)\| + \|\dot{\phi}(t) - \dot{\bar{\phi}}(t)\| : t \in [t_0, t_1]\} < \delta,$$

then there exists a bounded continuous control u: $\mathbb{R} \to \mathbb{R}^m$ *that generates* ϕ *as a response of* (1) *on* $[t_0, t_1]$.

Proof. See [1, Theorem 4]; the output function g that appears in the referenced theorem will simply be the identity mapping $g(t, x) = x$ here. ▮

The controllability property in Theorem 2.7 can be referred to as the *local* C^1 *path controllability* of the system (1) near the response $\bar{\phi}$. We note that it is only claimed that paths which are close to $\bar{\phi}$ in the C^1 norm can be generated as responses of (1). It is a routine matter to reformulate Theorem 2.7 in the case where the system (1) admits the serial decomposition (4).

THEOREM 2.8. *Let the mapping* f *in the system* (1) *be* C^1 *and let* $\bar{u}$: $\mathbb{R} \to \mathbb{R}^m$ *be a bounded continuous control that generates a response* $\bar{\phi}$: $[t_0, t_1] \to \mathbb{R}^n$. *Assume that the system* (1) *admits the serial decomposition* (4) *and assume that for every* $t \in [t_0, t_1]$ *and* $i = 1, \ldots, r$ *we have*

$$\operatorname{rank} \frac{\partial f_i}{\partial w_i}(\bar{\phi}_1(t), \ldots, \bar{\phi}_i(t), \bar{u}_1(t), \ldots, \bar{u}_i(t), t) = n_i. \tag{20}$$

Then the system is locally C^1 *path controllable near the response* $\bar{\phi}$.

Proof. It is easy to see that the rank conditions (20) for $1 \leqslant i \leqslant r$ imply the rank condition (19), so the result follows from Theorem 2.7. ▮

Under stronger assumptions, we can obtain global results on C^1 path controllability. The sytem (1) is said to be *completely* C^1 *path controllable* on the interval $[t_0, t_1]$ if for every C^1 mapping ϕ: $[t_0, t_1] \to \mathbb{R}^n$ there exists a control $u \in PC^m_\infty$ that generates ϕ as a response of (1). The following theorem gives a sufficient condition for (1) to be completely C^1 path controllable (see [4, Theorem 3.1 and Remark 3.3]).

THEOREM 2.9. *Let the mapping* f *in the system* (1) *be* C^1 *and suppose that* f *satisfies the following two conditions*:

(a) *for every* $(x, t) \in \mathbb{R}^n \times [t_0, t_1]$ *the mapping* $w \mapsto f(x, w, t)$ *of* $\mathbb{R}^m$ *into* $\mathbb{R}^n$ *is surjective*;

(b) *for every* $(x, w, t) \in \mathbb{R}^n \times \mathbb{R}^m \times [t_0, t_1]$ *we have* $\operatorname{rank} D_2 f(x, w, t) = n$.

Then the system (1) *is completely* C^1 *path controllable on* $[t_0, t_1]$. ▮

Our next result gives an instance of complete C^1 path controllability for systems having a decomposition.

THEOREM 2.10. *Let the system* (1) *have a hybrid decomposition of the form*

$$\begin{aligned}\dot{x}_1 &= h_1(x_1, u_1(t), t) + g_1(x_1, t),\\ \dot{x}_2 &= h_2(x_2, u_2(t), t) + g_2(x_1, x_2, u_1(t), t),\\ &\vdots\\ \dot{x}_r &= h_r(x_r, u_r(t), t) + g_r(x_1,..., x_r, u_1(t),..., u_{r-1}(t), t),\end{aligned} \tag{21}$$

where the mappings h_i and g_i are C^1 for $1 \leqslant i \leqslant r$. Suppose, in addition, that for each $i = i,..., r$ the following two conditions are satisfied:

(a) *for every $(x_i, t) \in \mathbb{R}^{n_i} \times [t_0, t_1]$ the mapping $w_i \mapsto h_i(x_i, w_i, t)$ of $\mathbb{R}^{m_i}$ into $\mathbb{R}^{n_i}$ is surjective*;

(b) *for every $(x_i, w_i, t) \in \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \times [t_0, t_1]$ we have* rank $\partial h_i/\partial w_i(x_i, w_i, t) = n_i$. *Then the system* (1) *is completely C^1 path controllable on $[t_0, t_1]$.*

Proof. It is clear that conditions (a) and (b) for $i = 1,..., r$ imply that the right-hand side of the aggregrate system (21) (i.e., the decomposed form of (1)) satisfies conditions (a) and (b) of Theorem 2.9, whence the result. ∎

Observe that no special assumptions (other than continuous differentiability) are required for the mappings g_i in (21).

We close this paper by stating a result analogous to Theorem 2.10 for autonomous systems.

THEOREM 2.11. *Let the autonomous control system* (2) *admit the serial decomposition*

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, u_1(t)),\\ \dot{x}_2 &= f_2(x_1, x_2, u_1(t), u_2(t)),\\ &\vdots\\ \dot{x}_r &= f_r(x_1,..., x_r, u_1(t),..., u_r(t)),\end{aligned} \tag{22}$$

and suppose that for each $i = 1,..., r$ and for each

$$(x_1,..., x_i) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_i}, \qquad (w_1,..., w_{i-1}) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_{i-1}},$$

there exists $w_i \in \mathbb{R}^{m_i}$ *such that*

$$f_i(x_1, \ldots, x_i, w_1, \ldots, w_i) = 0,$$

$$\operatorname{rank} \frac{\partial f_i}{\partial w_i}(x_1, \ldots, x_i, w_1, \ldots, w_i) = n_i.$$

Then for every C^1 *curve* ϕ: $[0, T] \to \mathbb{R}^n$ *there exists a constant* $\lambda > 0$ *such that the reparametrized curve*

$$\psi(t) = \phi(\lambda t), \qquad 0 \leqslant t \leqslant T/\lambda,$$

can be realized as a response of (2).

Proof. The assumptions on the mappings f_i in the decomposition (22) imply that the mapping f: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ defining the right-hand side of (2) has the following property: for every $x \in \mathbb{R}^n$ there exists a $w \in \mathbb{R}^m$ such that $f(x, w) = 0$ and rank $D_2 f(x, w) = n$. The assertion of the theorem then follows by a minor variation of the argument used in [2, Theorem 2]. ∎

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