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A MATRIX FOR COMPUTING THE JONES POLYNOMIAL OF A KNOT

LOUIS ZULLI

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0. INTRODUCTION

IN the mid-80s, the field of knot theory was revolutionized by the discovery (creation?) of a slew of new polynomial invariants of knots and links. The first—and, now, perhaps the most widely known—of these new knot polynomials was the Jones polynomial, discovered by Vaughan Jones while he was investigating representations of the braid group into operator algebras arising in physics.

Soon thereafter, Louis Kauffman introduced his bracket polynomial for knots and links, which provided a simple, diagrammatic model for the Jones polynomial.

In this article we describe a scheme which associates to each unoriented knot diagram a symmetric matrix over the field $\mathbf{Z}_2$, and we show how the bracket polynomial (and, thus, the Jones polynomial) of the knot can be calculated from this matrix using elementary linear algebra.

This paper is organized as follows: Section 1 gives a brief introduction to Kauffman's bracket polynomial, and also provides the diagram-theoretic conventions and notations used throughout the article. Section 2 contains a statement of the main result, which is then proved in Section 3. In Section 4 we employ our new approach to the Jones polynomial to offer a linear-algebraic proof of a fundamental inequality relating the span of that polynomial to the number of crossings in a knot diagram.

1. PRELIMINARIES

In this section we give an abbreviated introduction to Louis Kauffman's bracket polynomial. Here and throughout, we restrict our attention to unoriented *knot* diagrams. For a fuller account of the material in this section, the reader is directed to [4].

For an unoriented knot diagram K, let $\mathscr{C}(K)$ denote the set of crossings in K. A *state* S for a diagram K is a function $S:\mathscr{C}(K) \to \{A, B\}$, that is, a choice, at each crossing in K, of a label, A or B. The set of all states for a diagram K will be denoted $\mathscr{S}(K)$.

At a crossing in a knot diagram, the local regions immediately counter-clockwise from the overcrossing strand are called the *A regions*. The local regions immediately counter-clockwise from the undercrossing strand are called the *B regions*; see Fig. 1.

If a diagram K' is obtained from a diagram K by the local alteration suggested in Fig. 2, we say that K' is obtained from K by *opening the A channel* at crossing $i \in \mathscr{C}(K)$. If a diagram K' is obtained from a diagram K by the local alteration suggested in Fig. 3, we say that K' is obtained from K by *opening the B channel* at crossing $i \in \mathscr{C}(K)$.

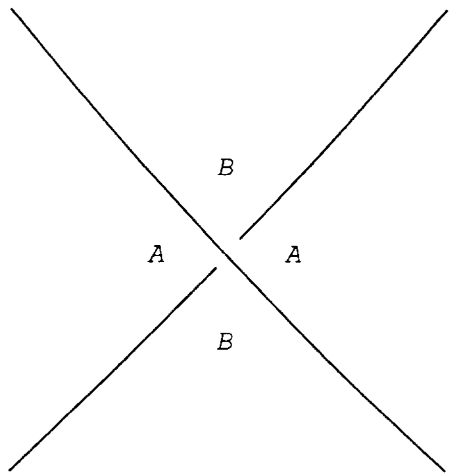

Fig. 1. *A* and *B* regions.

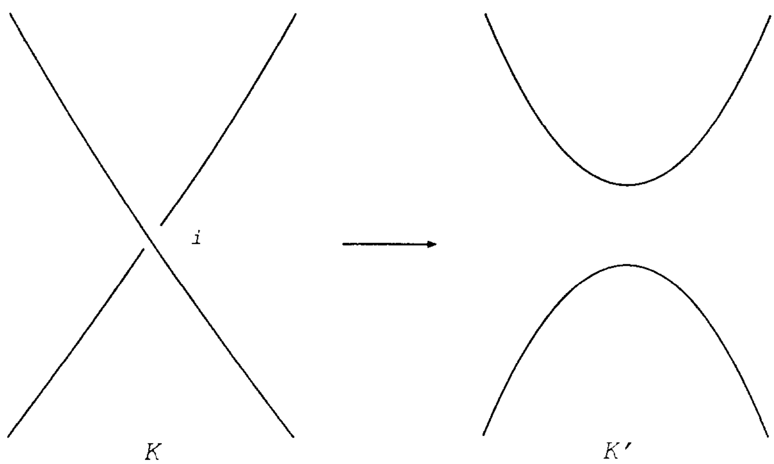

Fig. 2. Opening the *A* channel at crossing *i*.

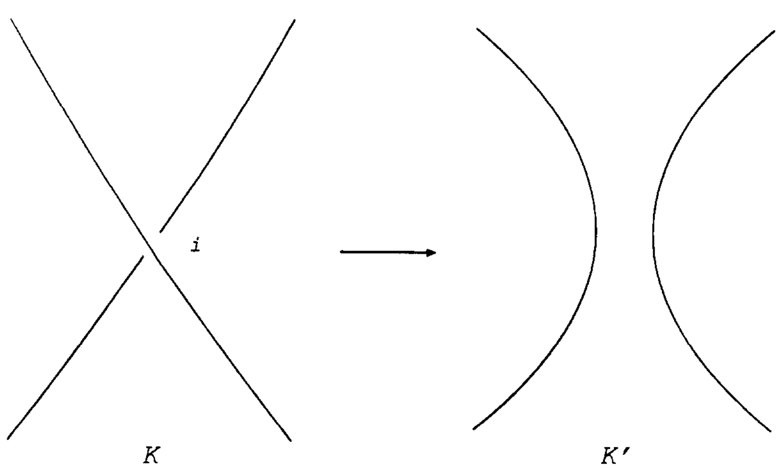

Fig. 3. Opening the *B* channel at crossing *i*.

For a diagram K and a state $S \in \mathscr{S}(K)$, let $K \mid S$ denote the diagram obtained from K by opening the $S(i)$ channel at crossing i for each $i \in \mathscr{C}(K)$. Each connected component of $K \mid S$ will be called a *circle*.

Example. For the trefoil diagram K in Fig. 4, and the state $S = ABB$, $K \mid S$ consists of two circles.

Let $\#(K \mid S)$ denote the number of circles in the split-open diagram $K \mid S$. Let $A(S)$ denote the number of A labels in the state S, and let $B(S)$ denote the number of B labels in the state S.

For an unoriented knot diagram K, define $[K] \in \mathbf{Z}[A, B, d]$ by

$$[K] = \begin{cases} 1 & \text{if } \mathscr{C}(K) = \emptyset \\ \sum_{S \in \mathscr{S}(K)} A^{A(S)} B^{B(S)} d^{\#(K \mid S) - 1} & \text{if } \mathscr{C}(K) \neq \emptyset \end{cases}$$

Let $\langle K \rangle \in \mathbf{Z}[A, A^{-1}]$ denote the Laurent polynomial obtained from $[K]$ by setting $B = A^{-1}$ and $d = -A^2 - A^{-2}$. The polynomial $\langle K \rangle$ is Kauffman's ***bracket polynomial*** for K. (With these special values for B and d, $[\ ]$ becomes invariant under a Type 2 Reidemeister move. Invariance of $[\ ]$ under Type 2 moves then implies invariance under Type 3 moves. Thus, $\langle\ \rangle$ is a regular isotopy invariant.)

At an *oriented* crossing i in a knot diagram K, the *crossing sign* $\varepsilon(i) \in \{+1, -1\}$ is defined as follows. Imagine placing the palm of the right hand upon the overcrossing strand at crossing i, with the fingers pointing in the direction of the arrow. If the thumb of that hand points in the direction of the arrow on the undercrossing strand, then $\varepsilon(i) = +1$. Otherwise $\varepsilon(i) = -1$; see Fig. 5.

For an unoriented knot diagram K, the *writhe* of K, denoted $w(K)$, is defined as follows: Place an orientation on K—which of the two orientations of K you choose is immaterial. Then

$$w(K) = \sum_{i \in \mathscr{C}(K)} \varepsilon(i).$$

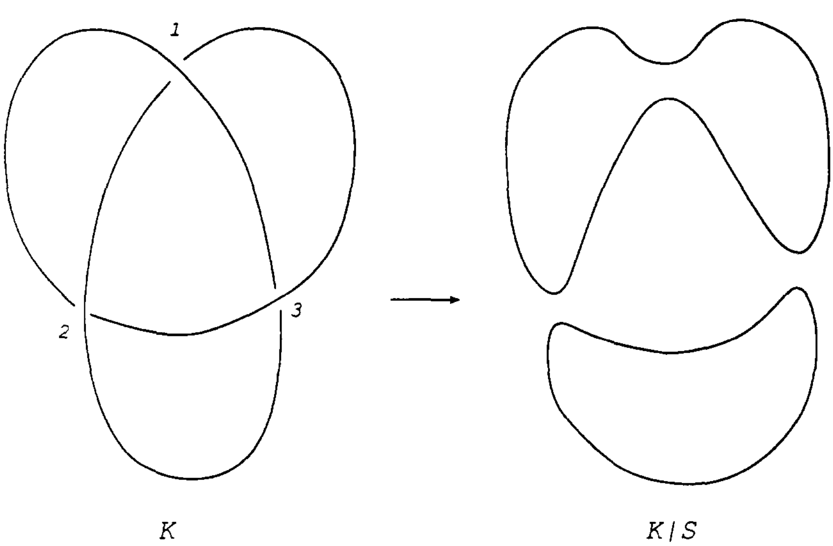

Fig. 4. Splitting a diagram according to a state.

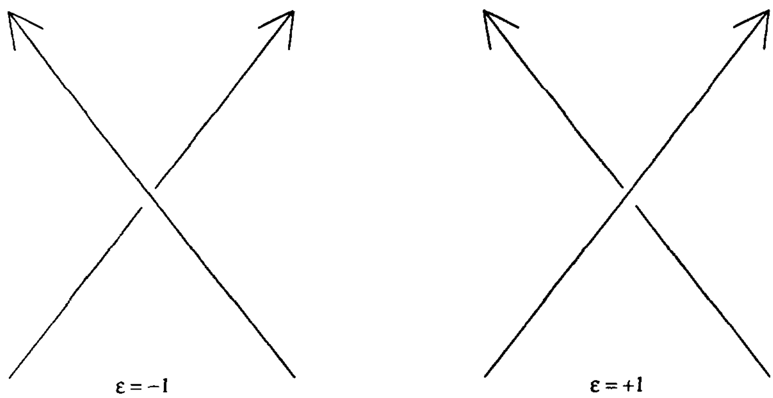

Fig. 5. Crossing signs.

For an unoriented knot diagram K, define $f_K \in \mathbf{Z}[A, A^{-1}]$ by

$$f_K = (-A^{-3})^{w(K)} \langle K \rangle.$$

THEOREM 1 (Kauffman). *The Laurent polynomial f_K is invariant under all three Reidemeister moves, hence f_K is a knot invariant. Furthermore, setting $A = t^{-1/4}$ in f_K yields the Jones polynomial V_K.*

2. RESULTS

In this section we present our results. In particular, we describe a scheme which associates to each unoriented knot diagram a symmetric matrix over the field $\mathbf{Z}_2$, and we show how the bracket polynomial (and hence the Jones polynomial) of the knot may be calculated directly from this matrix using elementary linear algebra. A key point to bear in mind is that, in order to compute $[K]$, it suffices to be able to compute $\#(K \mid S)$, the number of circles obtained when the diagram K is split open according to an arbitrary state S.

We begin by discussing how to *adorn* an unoriented knot diagram K. Given such a diagram, enumerate the crossings $1, 2, \ldots, n$ in any manner whatsoever. At each crossing $i \in \mathscr{C}(K)$, place an arrow (i.e. a local orientation) on the overcrossing strand at crossing i. There are two choices at each crossing for the direction of this arrow; you may choose either. In particular, no global consistency of these local orientations is required. Let i^+ denote the overcrossing arrow at crossing i. Let i^+_+ denote the vertex of i^+ represented by the arrow-head, and let i^+_- denote the vertex of i^+ marking the tail of the arrow. At each crossing $i \in \mathscr{C}(K)$, place an arrow on the undercrossing strand so that the crossing sign $\varepsilon(i)$ is $+1$. Let i^- denote the undercrossing arrow at crossing i. Let i^-_+ denote the head of i^-, and let i^-_- denote the tail of i^-; see Figs. 6 and 7.

From this adorned diagram, define an $n \times n$ matrix T over $\mathbf{Z}_2$ as follows. For $i \neq j$, T_{ij} is defined to be the number of times (mod 2) that a traveler passes through crossing i while making the following trip—the traveler begins on the overcrossing arrow j^+, and proceeds along the knot in the direction of that arrow until he returns to crossing j (i.e. until he reaches the undercrossing arrow j^-). For $i = j$, T_{ij} is defined as follows—if, upon completing the trip defined above, the traveler finds the arrow j^- "urging him onward" (i.e. if j^-_- is the vertex of j^- first encountered upon returning to crossing j) then $T_{ij} = 0$. If the traveler finds the arrow j^- commanding him to "go back" (i.e. if j^-_+ is the vertex of j^- first encountered

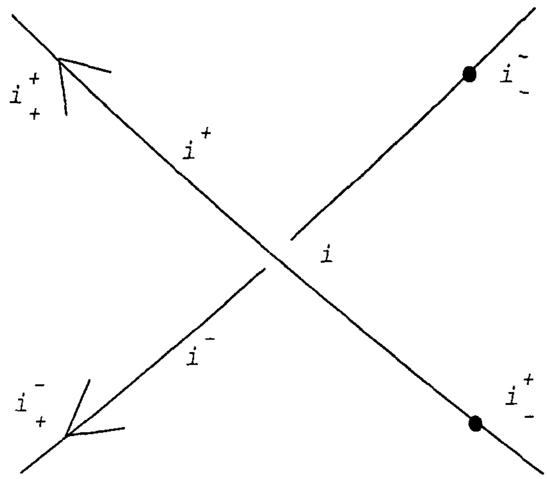

Fig. 6. An adorned crossing.

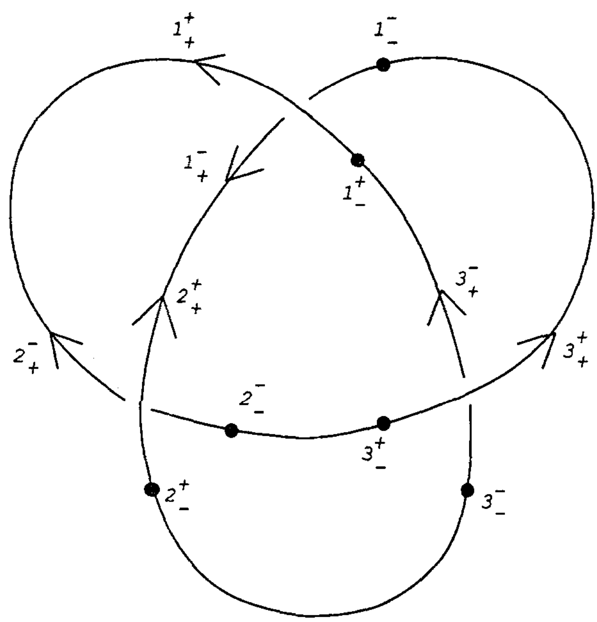

Fig. 7. An adorned diagram.

upon returning to crossing j) then $T_{ij} = 1$. The matrix T so constructed will be called the (mod 2) *trip matrix* of the adorned diagram K. One may check that this matrix is symmetric, and that it does not depend on which directions are chosen for the overcrossing arrows in the process of adorning the diagram K. (Both of these facts follow immediately from elementary observations about arrangements of pairs of points on a circle.) The symmetry of the matrix T will not be used in any of what follows.

Example. For the adorned trefoil diagram of Fig. 7, we have

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The matrix T contains sufficient information to compute the Jones polynomial of the knot K. Specifically, we have

THEOREM 2. (a) *The writhe of K is the number of zeros on the diagonal of T less the number of ones on the diagonal of T.*

(b) *Suppose the state S is obtained from the state $AA \ldots A$ by toggling the labels in positions $i_1, i_2, \ldots, i_m$. Let T_S be the matrix obtained from the matrix T by toggling the entries in the corresponding positions along the diagonal of T. Then:*

(i) *nullity*$(T_S) = \#(K \mid S) - 1$.

(ii) *The ith standard basis vector for $(\mathbf{Z}_2)^n$ lies in the column space of T_S if and only if, in the split-open diagram $K \mid S$, there is a self-approaching circle at site i.*

Notes. (1) By "toggle" we mean "change from A to B (or *vice versa*)" or "change from 0 to 1 (or *vice versa*)".

(2) Knowing at which sites self-approaching circles occur is essential in determining which sites in the split-open diagram $K \mid S$ are so-called *active sites.* The bracket polynomial can be computed entirely from those split-open diagrams $K \mid S$ consisting of precisely one circle, if the activities of all the sites in such diagrams can be determined: see [7, 4] for more details.

3. THE PROOF

In this section we prove the theorem. Our proof combines a geometric construction and an elementary calculation using homology with $\mathbf{Z}_2$-coefficients. In particular, to each adorned knot diagram K and each state $S \in \mathscr{S}(K)$, we associate a compact surface Σ_S having precisely $\#(K \mid S)$ boundary components, and we then compute this number of boundary components homologically, using the long exact homology sequence of the pair $(\Sigma_S, \partial\Sigma_S)$. (We use $\mathbf{Z}_2$-coefficients because the surface Σ_S may be non-orientable.)

Before detailing the construction of Σ_S, we first give a more precise description of how to open channels in an adorned diagram K.

Note that opening the A channel at an adorned crossing $i \in \mathscr{C}(K)$ can be accomplished as follows. Remove the interior of the arrow i^+ and the interior of the arrow i^-. Introduce a new arc α_i^+ joining i^-_- to i^+_+, and a second new arc α_i^- joining i^+_- to i^-_+. The arcs α_i^+ and α_i^- should be disjoint. Similarly, opening the B channel at an adorned crossing $i \in \mathscr{C}(K)$ can be accomplished as follows. Remove the interior of the arrow i^+ and the interior of the arrow i^-. Introduce a new arc α_i^+ joining i^-_+ to i^+_+, and a second new arc α_i^- joining i^-_- to i^+_-. The arcs α_i^+ and α_i^- should be disjoint. Note that in both channel openings we are following the convention: the arc α_i containing i^+_+ is given the superscript $+$, and the other arc α_i is given the superscript $-$; see Figs. 8 and 9.

At this point, our notation is sufficient to describe the construction of the surface Σ_S. Given an adorned knot diagram K, we may view K as representing an embedding $\mathbf{S}^1 \hookrightarrow \mathbf{R}^3$, and thus we may "pull back" K to give an adorned, standardly drawn $\mathbf{S}^1$, which we will denote K^*; see Fig. 10.

For each adorned knot diagram K and state $S \in \mathscr{S}(K)$, we may construct a diagram $K^* \mid S$ by "pulling back" the channel openings in K dictated by S to corresponding operations on the adorned circle K^*. That is, we remove the interior of each arrow in K^*, and we introduce arcs α_i^+ and α_i^-, $i = 1, 2, \ldots, n$, in the same fashion that the corresponding arcs are being introduced in K. *We draw these pairs of arcs α_i^+ and α_i^- as parallel, possibly*

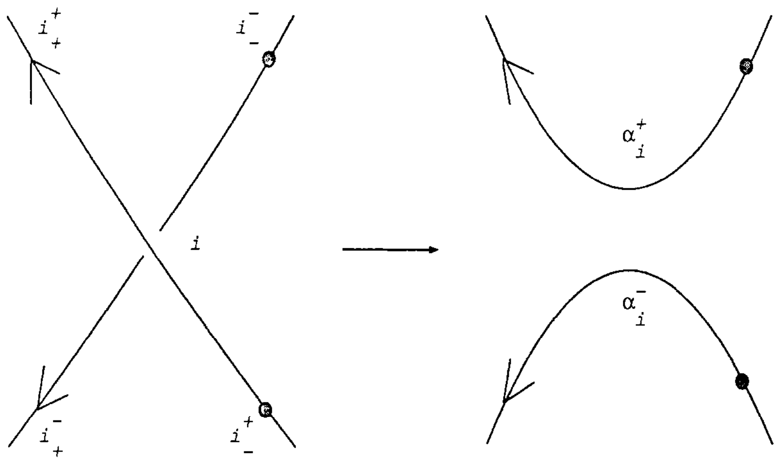

Fig. 8. Opening the A channel.

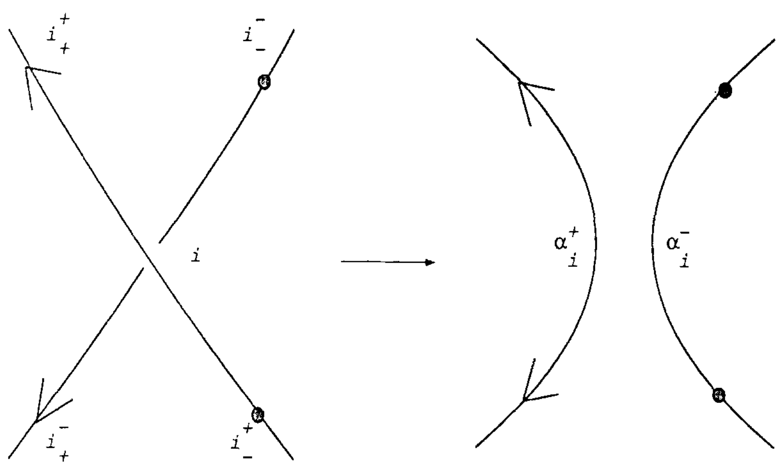

Fig. 9. Opening the B channel.

twisted, strands in the exterior of K^*. Thus, the diagram $K^* \mid S$ represents a collection of circles in $\mathbf{R}^3$, whereas the diagram $K \mid S$ represents a collection of circles in $\mathbf{R}^2$. Of course, by construction, the number of circles in the diagram $K \mid S$ is identical to the number of circles in the diagram $K^* \mid S$.

Example. For an adorned trefoil diagram and the state $S = ABB$, the diagram $K^* \mid S$ is illustrated in Fig. 11.

The surface Σ_S is obtained by "shading in" the diagram $K^* \mid S$. That is, Σ_S is obtained from the standard disk bounded by the circle K^* by attaching (possibly twisted) bands, one band for each crossing in the diagram K. By construction, the number of boundary components of Σ_S is precisely $\#(K \mid S)$.

Since each component of $\partial\Sigma_S$ is a circle, it is not difficult to count the number of components of $\partial\Sigma_S$ homologically, Thus, $\#(K \mid S)$ may be determined homologically. Consider the long exact homology sequence of the pair $(\Sigma_S, \partial\Sigma_S)$, where all homology

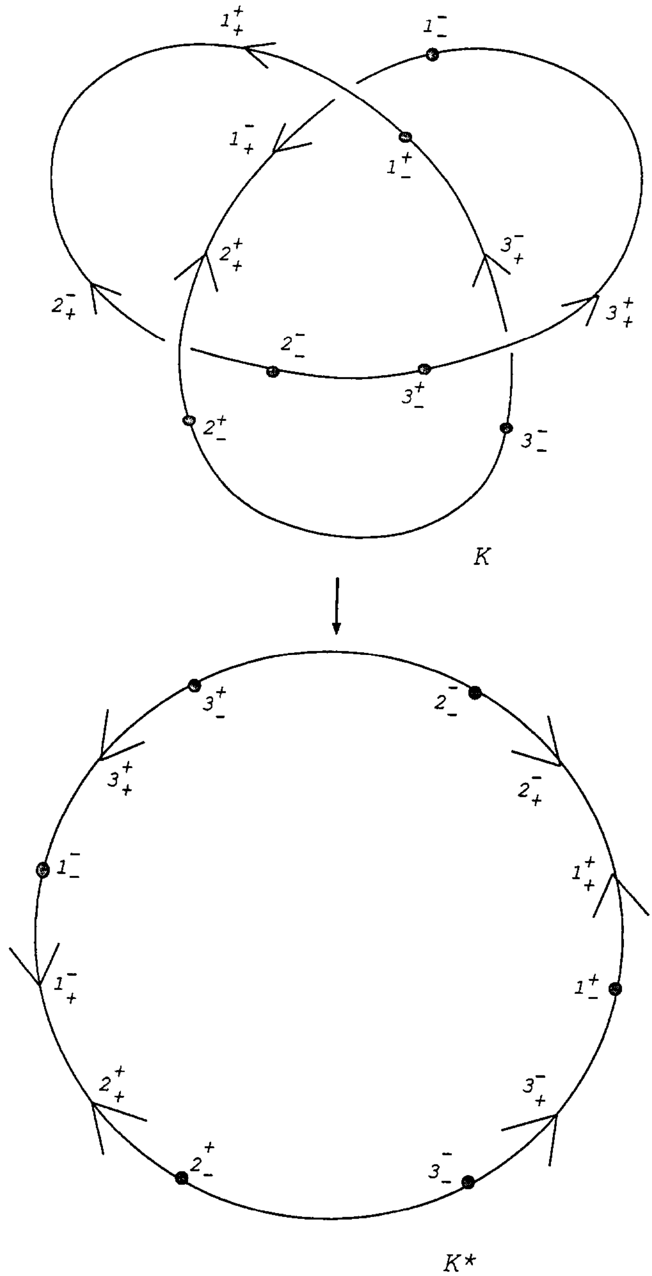

Fig. 10. Pulling back an adorned diagram.

groups are understood to have coefficients in $\mathbf{Z}_2$:

$$0 \to H_2(\Sigma_S, \partial\Sigma_S) \to H_1(\partial\Sigma_S) \to H_1(\Sigma_S) \xrightarrow{j_*} H_1(\Sigma_S, \partial\Sigma_S) \to \cdots .$$

This yields an exact sequence

$$0 \to H_2(\Sigma_S, \partial\Sigma_S) \to H_1(\partial\Sigma_S) \to H_1(\Sigma_S) \xrightarrow{j_*} \mathrm{im}(j_*) \to 0$$

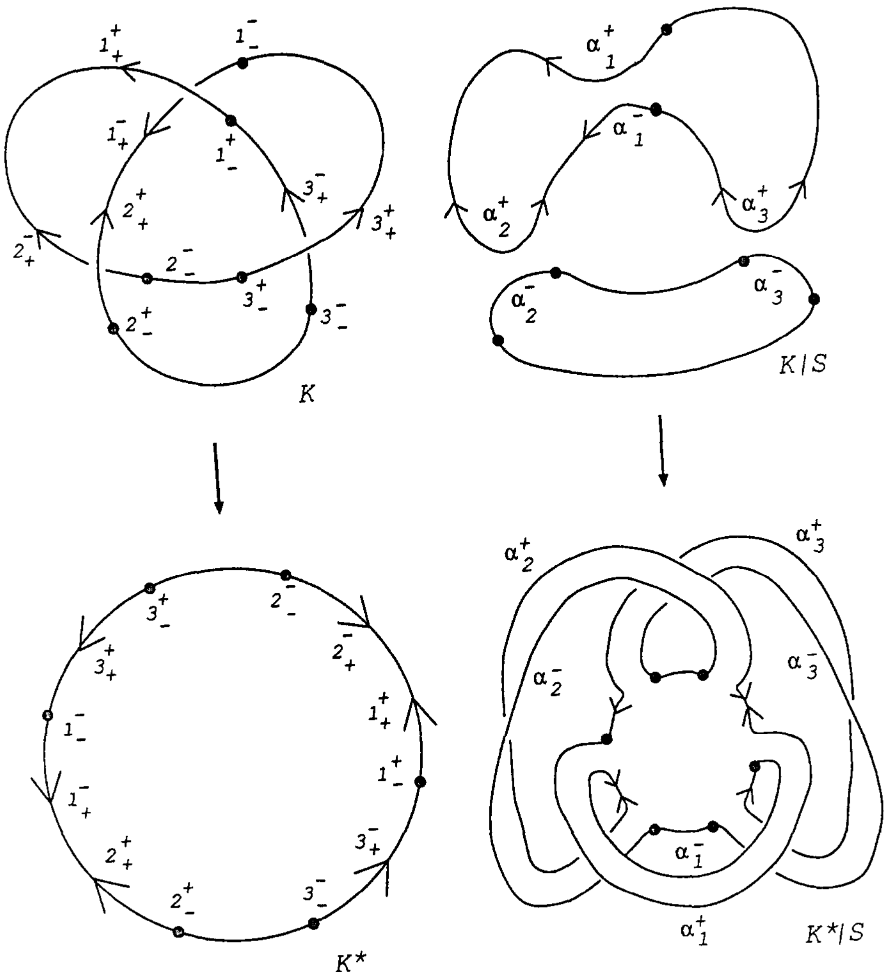

Fig. 11. Constructing $K^* \mid S$.

and so

$$\dim H_2(\Sigma_S, \partial\Sigma_S) - \dim H_1(\partial\Sigma_S) + \dim H_1(\Sigma_S) - \dim \operatorname{im}(j_*) = 0$$

Thus,

$$\dim H_1(\partial\Sigma_S) = \dim H_2(\Sigma_S, \partial\Sigma_S) + \dim \ker(j_*) = 1 + \operatorname{nullity}(j_*)$$

that is,

$$\#(K \mid S) = 1 + \operatorname{nullity}(j_*).$$

At this point, and until further notice, let us consider the surface Σ_S *for the special state* $S = AA \ldots A$. We will describe "canonical" bases for $H_1(\Sigma_S)$ and $H_1(\Sigma_S, \partial\Sigma_S)$, and we will show that the matrix of j_* relative to these bases is precisely the trip matrix T of the adorned knot diagram K.

By Lefschetz duality, $H_1(\Sigma_S, \partial\Sigma_S) \approx H^1(\Sigma_S) \approx H_1(\Sigma_S)^*$. A canonical basis $\{1, 2, \ldots, n\}$ for $H_1(\Sigma_S)$ and the dual basis $\{1^*, 2^*, \ldots, n^*\}$ for $H_1(\Sigma_S, \partial\Sigma_S)$ are shown in Fig. 12 (for an adorned figure-eight knot diagram).

The matrix of j_* relative to these canonical bases is now readily determined. Given a cycle i in the basis for $H_1(\Sigma_S)$, replace it by the homologous cycle i' defined as follows: i'

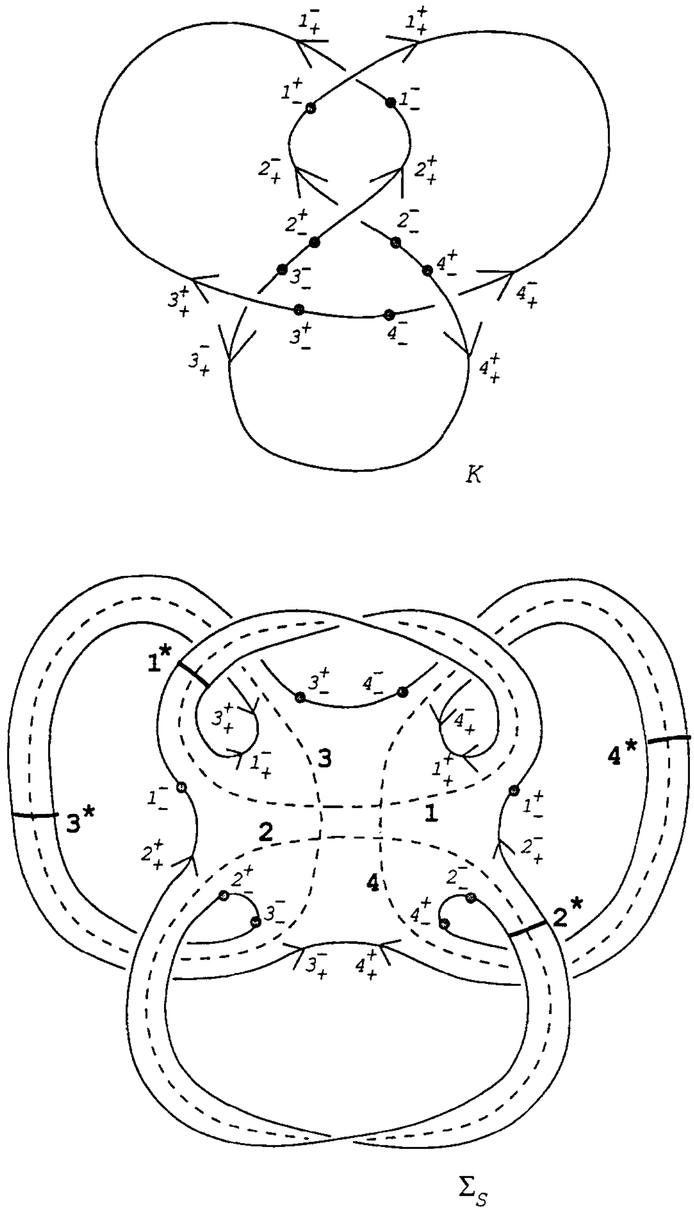

Fig. 12. Canonical bases for $H_1(\Sigma_S)$ and $H_1(\Sigma_S, \partial\Sigma_S)$.

consists of the arc α_i^+, together with a circular arc, along what was formerly K^*, from i_+^+ to i_-^-. There are, of course, two circular arcs so described—choose the one leading away from i_+^+ in the direction of that arrowhead; see Fig. 13.

To compute $j_*(i) = j_*(i')$, we simply discard those portions of the cycle i' which lie in $\partial\Sigma_S$, as indicated in Fig. 14.

It is now easy to verify that the matrix of j_* relative to the canonical bases $\{1, 2, \ldots, n\}$ and $\{1^*, 2^*, \ldots, n^*\}$ is precisely the trip matrix T described in Section 2. Indeed, the relative cycles which comprise $j_*(i')$ correspond precisely to the crossings encountered by the traveler in his trip from the overcrossing arrow i^+ to the undercrossing arrow i^-. Thus, the trip matrix T for an adorned knot diagram K is the matrix of j_* (relative to the canonical bases) *for the special state* $S = AA \ldots A$.

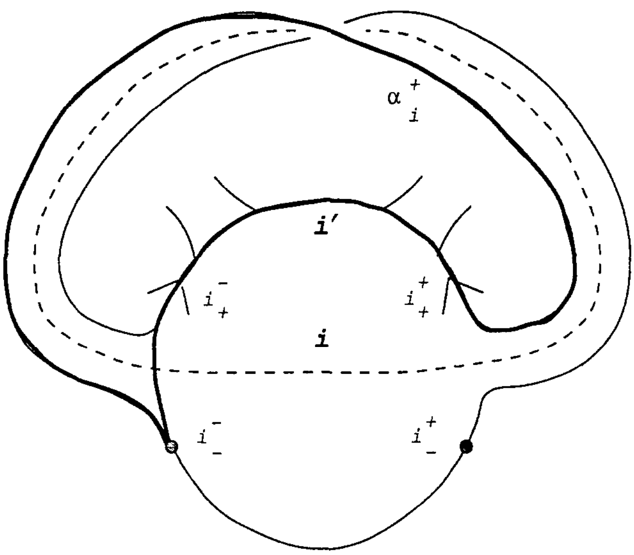

Fig. 13. Replacing i by i'.

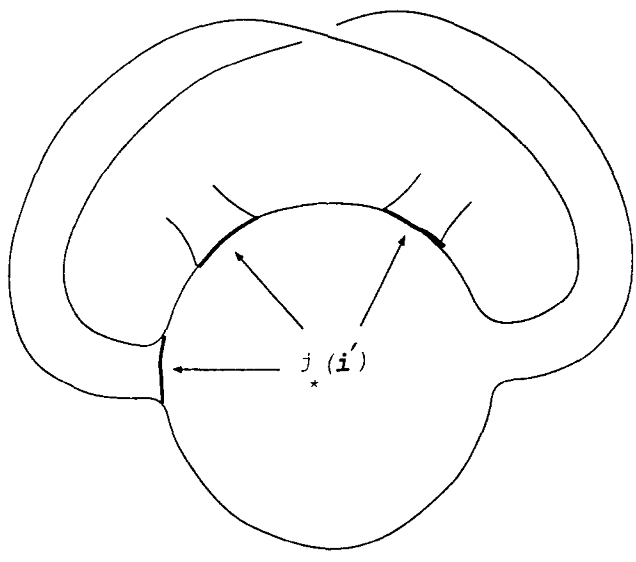

Fig. 14. Computing $j_*(i')$.

To complete the proof of (i) of Theorem 2(b), we must now consider how the matrix of j_* for an arbitrary state S is related to the matrix of j_* for the special state $AA \ldots A$.

It suffices to consider how the surface $\Sigma_{S'}$ is related to the surface Σ_S, where the state S' is obtained from the state S by toggling the label at crossing $i \in \mathscr{C}(K)$ from A to B. It is easy to verify that $\Sigma_{S'}$ may be obtained from Σ_S by introducing (or removing—up to homeomorphism there is no distinction) a half-twist to the band corresponding to crossing i. Thus, using the "same" bases for $H_1(\Sigma_{S'})$ and $H_1(\Sigma_{S'}, \partial\Sigma_{S'})$ that were used for $H_1(\Sigma_S)$ and $H_1(\Sigma_S, \partial\Sigma_S)$, it is not difficult to see that the matrix of j_* for the state S' differs from the matrix of j_* for the state S precisely at the ith diagonal element. Thus, the matrix of j_* for any state $S \in \mathscr{S}(K)$ can be obtained from the trip matrix T by simply toggling the appropriate entries on the diagonal of T.

To prove (ii) of Theorem 2(b), we simply need to note that, for an arbitrary state $S \in \mathcal{S}(K)$, there will be a self-approaching circle at site i in $K \mid S$ if and only if the arcs α_i^+ and α_i^- lie in the same component of $\partial\Sigma_S$. But this last condition can be detected homologically—α_i^+ and α_i^- lie in the same component of $\partial\Sigma_S$ if and only if the relative cycle i^* lies in the kernel of $\partial: H_1(\Sigma_S, \partial\Sigma_S) \to H_0(\partial\Sigma_S)$. Since $\ker(\partial) = \mathrm{im}(j_*)$, we have proved that there is a self-approaching circle at site i in $K \mid S$ if and only if i^* lies in the image of j_*.

Finally, Theorem 2(a) follows immediately from the definition of the writhe and our orientation convention for crossings in an adorned diagram.

4. AN APPLICATION

In this section we use our theorem to give a quick proof of the following basic inequality.

PROPOSITION. *For an unoriented knot diagram K with n crossings,*

$$\mathrm{span}\langle K\rangle \leq 4n.$$

Notes. (1) For any Laurent polynomial f, $\mathrm{span}(f)$ is defined to be $\max\deg(f) - \min\deg(f)$. For example, $\mathrm{span}(8A^3 - 7A + 4A^{-2}) = 3 - (-2) = 5$.

(2) $\mathrm{span}\langle K\rangle = \mathrm{span}(f_K)$ is a knot invariant. Thus, this proposition gives a lower bound (namely, $\frac{1}{4}\mathrm{span}\langle K\rangle$) for the number of crossings in *any* diagram for a knot having the polynomial $\langle K\rangle$.

(3) This inequality was originally proved, independently, by Murasugi and Thistlethwaite. A simpler proof was later given by Kauffman; see [5, 7, 3].

Proof of proposition. The term in $\langle K\rangle$ contributed by the state $AA \ldots A$ is $A^n d^{\mathrm{nullity}(T)}$, where T is the trip matrix for K. Since $d = (-A^2 - A^{-2})$, the max degree *of this summand* is $n + 2\,\mathrm{nullity}(T)$. An elementary argument then shows that no other state can contribute a term with a greater max degree. (See the proof of 3.1 in [4] for the details.) Thus,

$$\max\deg\langle K\rangle \leq n + 2\,\mathrm{nullity}(T).$$

An analogous argument shows that

$$\min\deg\langle K\rangle \geq -n - 2\,\mathrm{nullity}(T')$$

where T' is the matrix corresponding to the state $BB \ldots B$. Note that $T + T' = I_n$, the $n \times n$ identity matrix over $\mathbf{Z}_2$. Thus,

$$\begin{aligned} \mathrm{span}\langle K\rangle &\leq (n + 2\,\mathrm{nullity}(T)) - (-n - 2\,\mathrm{nullity}(T')) \\ &= 2n + 2(\mathrm{nullity}(T) + \mathrm{nullity}(T')) \\ &\leq 2n + 2(n + \mathrm{nullity}(T + T')) \\ &= 2n + 2(n + \mathrm{nullity}(I_n)) \\ &= 4n. \end{aligned}$$

□

Note. If K is an alternating knot diagram, then the trip matrix T is a projection, i.e. $T^2 = T$ [6]. This implies that T' is a projection as well. Using this fact, along with the additional assumption that the diagram K is reduced, it is not difficult to show that both of the inequalities above becomes equalities. The first inequality becomes an equality because there can be no self-approaching circles in either $K \mid AA \ldots A$ or $K \mid BB \ldots B$. (If there were a self-approaching circle, say at site i in $K \mid AA \ldots A$, then, by our main result, the ith standard basis vector for $(\mathbf{Z}_2)^n$ would lie in the column space of T. But if T is a projection this vector would then be fixed by T, implying that the ith column of T has a single non-zero entry in position i. This cannot occur if the diagram K is reduced.) The second inequality becomes an equality because, if T is a projection, $\ker(T') = \mathrm{im}(T)$. Thus, for a reduced, alternating knot diagram K, we recover the equality span $\langle K \rangle = 4n$.

Acknowledgements—I would especially like to thank Richard Stong for pointing out that the trip matrix of an alternating knot diagram is a projection. Also, a thank you to Louis Kauffman for a helpful conversation, and to Allen Hatcher, for his guidance as my thesis advisor.

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Department of Mathematics
Rice University
P.O. Box 1892
Houston
TX 77251, U.S.A.