WHITEHEAD PRODUCTS IN STIEFEL MANIFOLDS AND K-THEORY

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§1.

Let \( i_{n+k,k}: S^{d(n+1)-1} = O_{n+1,1} \to O_{n+k,k} \) be the standard embedding of the sphere in the Stiefel manifold. Here \( d \) is 1, 2 or 4 and \( O_{n+k,k} \) is \( V_{n+k,k} = O(n+k)/O(n) \), \( W_{n+k,k} = U(n+k)/U(n) \) or \( X_{n+k,k} = Sp(n+k)/Sp(n) \) depending upon which of the real, complex or quaternionic cases is considered. It was shown by the second author in [10] that if \( n+1 \) is not a power of 2 and if in addition in the real case \( n \) is odd, then the Whitehead square \( [i_{n+k,k}, i_{n+k,k}] \) is of order 2 for all \( k \geq 1 \). When combined with earlier results of Mahowald [7] this implies that the order of \( [i_{n+k,k}, i_{n+k,k}] \) in \( 
\pi_{2d(n+1)-1}(U_{n+k,k}) \) is determined whenever \( n+1 \) is not a power of 2. The primary purpose of this note is to begin an investigation in the complex and quaternionic cases into what happens when \( n+1 \) is a power of 2. The real Stiefel manifolds \( V_{n+k,k} \) will not be considered for the remainder of this paper.

THEOREM 1.1. Let \( n+1 = 2^t \) and \( k \geq 1 \).

(a) In the complex case \( [i_{n+k,k}, i_{n+k,k}] \) has order 2 whenever \( k \leq n-t-1 \) and also when \( k = n-t \) if \( t \neq 2 \) mod 4.

(b) In the quaternionic case \( [i_{n+k,k}, i_{n+k,k}] \) has order 2 whenever \( k \leq n - \lfloor (t+1)/2 \rfloor \) and also when \( k = n - (t-1)/2 \) if \( t \equiv 3 \) mod 4.

In (b), \( \lfloor (t+1)/2 \rfloor \) means the integral part of \( (t+1)/2 \). The anticommutativity of Whitehead products implies that \( 2[i_{n+k,k}, i_{n+k,k}] = 0 \) always. The first significant occurrence in the complex case arises when \( t = 3 \). If \( t = 0,1,2 \) the Whitehead squares are zero using naturality properties and the fact that Whitehead products vanish on the \( H \)-spaces \( S^1, S^3 \) and \( S^5 \). Proposition 1.2 shows that when \( t = 3 \) Theorem 1.1(a) gives the best possible result.

PROPOSITION 1.2. In the complex Stiefel manifold \( W_{7+k,k} \), the square \( [i_{7+k,k}, i_{7+k,k}] \) is of order 2 for \( 1 \leq k \leq 4 \) and is zero for \( k \geq 5 \).

The proof of Theorem 1.1 utilizes the three theorems given below on the action of the Steenrod algebra in the mod 2 cohomology of a topological space \( Y \) whose integral homology is finitely generated in each dimension and is free of 2-torsion. The first theorem is taken from [5] but a proof is implicit in those given for Theorems 1.4 and 1.5. We will use the notation that \( v_2(q) \) is the exponent of 2 in the primary decomposition of the integer \( q \).

THEOREM 1.3. Assume that \( H^{m+2(q-j)}(Y; \mathbb{Z}_2) = 0 \) for \( 1 \leq j \leq 2 + v_2(q) \) if \( q \) is even and for \( j = 1 \) if \( q \) is odd. Then

\[ Sq^{2q} : H^m(Y; \mathbb{Z}_2) \to H^{m+2q}(Y; \mathbb{Z}_2) \]

is zero.

We strengthen this theorem in two ways.

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THEOREM 1.4. Assume that \( H^{m+2j-1}(Y, Z_2) = 0 \) for \( 1 \leq j \leq 1 + v_2(q) \) and \( v_2(q) \equiv 1 \mod 2 \). Then
\[
S^qZ_2^ullet : H^m(Y, Z_2) \to H^{m+2q}(Y, Z_2) \text{ is zero.}
\]

THEOREM 1.5. Assume that \( H^{m+2j-1}(Y, Z_2) = 0 \) for \( 1 \leq j \leq 3 + 2v_2(q) \) and \( v_2(q) \equiv 0 \mod 4 \). Then
\[
S^qZ_2^ullet : H^m(Y, Z_2) \to H^{m+2q}(Y, Z_2) \text{ is zero.}
\]

The relationship between Theorem 1.1 and Theorems 1.3, 1.4, 1.5 is explained in detail in [10]. The idea briefly is that first one replaces \( Q_{n+k,k} \) by \( Q_{n+k,k} \), the stunted quasi-projective space which it contains [6]. That this is possible depends upon \( (n+1-k, Q_{n+k,k}) \) being \((2dn + 3d - 3)\)-connected. Also \( Q_{n+k,k} \) has a cellular structure with cells lying only in dimensions \( d_i \) for \( n + 1 \leq i \leq n + k \). Most significantly \( Q_{n+k,k} \) is homeomorphic to the Thom space of a certain oriented real \( d(n + 1) - 1 \) dimensional vector bundle over complex or quaternionic projective space and \( i_{n+k,k} : S^{d(n+1)-1} = Q_{n+1,1} \to Q_{n+k,k} \) can be interpreted as a homotopy Thom class. One proves the following. Let \( \xi \) be an \( r \)-dimensional real vector bundle over a connected finite CW-complex \( B \) and let \( B^q \) be the Thom complex of \( \xi \). The homotopy Thom class of \( \xi \) is the class of the embedding \( \iota : S^r = \ast^q \to B^q \) where \( \ast \) is any point of \( B \).

THEOREM 1.6. Let \( \xi \) be orientable and assume that \( H_* (B; Z) \) is free of 2-torsion and \( \Theta_1(B; Z) \) is finite. Assume also that one of the following conditions is satisfied:

(a) \( r = 1 \mod 4 \) \( \geq 5 \) and \( H^{-1}(B; Z_2) = 0 \),
(b) \( r = 3 \mod 4 \) \( \geq 11 \) and \( H^2(B; Z_2) = 0 \) for \( r - 2v_2(r+1) \leq j \leq r \),
(c) \( r = 3 \mod 4 \) \( \geq 11 \), \( v_2(r+1) \equiv 1 \mod 4 \) and \( H^2(B; Z_2) = 0 \) for \( r - 2v_2(r+1) \leq j \leq r \),
(d) \( r = 3 \mod 4 \) \( \geq 11 \), \( v_2(r+1) \equiv 0 \mod 2 \) and \( H^2(B; Z_2) = 0 \) for \( r - 2v_2(r+1) \leq j \leq 2r - 1 \).

Then the Whitehead square \( [i, i] \in \pi_{2r-1}(B^q) \) has order 2.

This result provides a supplement to Propositions 2.2 and 2.3 of [8]. The idea behind the proof, again the details of which can be found in [10], is to investigate \( [i, i] \) by comparing the EHP-sequences associated with \( \ast^q \) and \( B^q \). It follows that provided all maps \( f : S^{2r+1} \to EB^q \) induce zero homomorphisms in reduced cohomology, then \( [i, i] \) cannot have odd order if \( S^{r+1} : H^{r+1}(C_j; Z_2) \to H^{r+2}(C_j; Z_2) \) is trivial where \( C_j \) is the mapping cone of any \( f \). As \( 2[i, i] = 0 \), the result follows from Theorems 1.3, 1.4 and 1.5.

By modifying slightly the above method, we prove

THEOREM 1.7. Let \( \xi \) be orientable and assume that \( H_* (B; Z) \) is free of 2-torsion and \( \Theta_1(B; Z) \) has rank 1. Suppose that \( d^q(\xi) = 0 \) or \( d^q(\xi) \neq 0 \) and \( d(\xi)/d^q(\xi) \) is odd. Assume also that one of the conditions (a), (b) and (d) in Theorem 1.6 is satisfied. Then \( [i, i] \in \pi_{2r-1}(B^q) \) has order 2.

The notations \( d^q(\xi) \) and \( d(\xi) \) will be explained in §3. In particular we may take \( B = M \) to be a smooth, closed, connected, orientable \( r \)-dimensional manifold and \( \xi = \nu \) the normal bundle to an embedding of \( M \) in Euclidean space \( R^{2r} \). In [11] Sutherland proved that if \( r + 1 \) is not a power of 2 then \( [i, i] \in \pi_{2r-1}(M') \) is not zero. (He considered only the case when \( r \) is odd, but his method works in the case when \( r \) is even.) He also gave examples in which \( r + 1 \) is a power of 2 and \( [i, i] = 0 \). As a corollary of Theorem 1.7 we prove

THEOREM 1.8. Let \( r + 1 = 2^t \) for \( t \geq 4 \). Assume that \( H_* (M; Z) \) is free of 2-torsion and \( H^2(M; Z_2) = 0 \) for \( r - 2t - 1 \leq j \leq r - 1 \) if \( t \) is odd and for \( r - 2t + 1 \leq 2t - 1 \) if \( t \) is even. Then \( [i, i] \in \pi_{2r-1}(M') \) has order 2.

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We prove Theorems 1.5 and 1.6 using the generalized cohomology operations of [5]. We recall the notations and some results and refer to this latter for details.

Let $X$ be a connected finite CW-complex with $H_\ast(X; \mathbb{Z})$ free of 2-torsion. As usual we write $Q$ for the field of rational numbers and $\mathbb{Z}_{(2)}$ rather than $\mathbb{Q}_2$ for the subring of rationals with odd denominators. It follows from the Universal Coefficient Theorem that $H_\ast(X; \mathbb{Z}_{(2)})$ is a free $\mathbb{Z}_{(2)}$-module so that the mod 2 reduction $H_\ast(X; \mathbb{Z}) \to H_\ast(X; \mathbb{Z}_2)$ is surjective and $H_\ast(X; \mathbb{Z}_2) = 0$ if and only if $H_\ast(X; \mathbb{Z}_2) = 0$. (2.1)

Let $\tilde{x}$ denote the mod 2 reduction of $x$.

The inclusion $i: \mathbb{Z}_{(2)} \to Q$ induces a monomorphism $i_\ast: H_\ast(X; \mathbb{Z}_{(2)}) \to H_\ast(X; Q)$ and we identify $H_\ast(X; \mathbb{Z}_{(2)})$ with its image in $H_\ast(X; Q)$ under $i_\ast$.

The complex K-theory is denoted by $K(-)$ and $K(-; A)$ by $K(-; A)$ for $A = \mathbb{Z}_{(2)}$ or $Q$. In the proof of Theorem 1.5 we will use symplectic K-theory and will adopt similar notations. As usual $K(X; A)$ is filtered with the CW-filtration. The Atiyah–Hirzebruch spectral sequence for $K^\ast(X; A)$ collapses. Thus $K^\ast(X; A) \cong K_{2n}(X; A)/K_{2n+2}(X; A)$ for $A = \mathbb{Z}_{(2)}$ or $Q$. We choose an isomorphism $T: H^{even}(X; \mathbb{Z}_{(2)}) \to K(X; \mathbb{Z}_{(2)})$ such that the composition $K_{2n}(X; \mathbb{Z}_{(2)}) \to K_{2n}(X; \mathbb{Z}_{(2)}) \to K_{2n}(X; \mathbb{Z}_{(2)})/K_{2n+2}(X; Z_{(2)}) \cong K^\ast(X; \mathbb{Z}_{(2)})$ is the identity.

We define $S_\ast: H^{even}(X; \mathbb{Z}_{(2)}) \to H^{even}(X; \mathbb{Z}_{(2)})$ to be $T^{-1} \psi T$, where $\psi$ is the Adams operation.

Theorem 2.8 of [5] says that if $x_0 \in H^{2n}(X; \mathbb{Z}_{(2)})$, then there exist unique elements $x_i = x_i(T, x_0) \in H^{2n+2i}(X; \mathbb{Z}_{(2)})$ such that $x = \sum_{i \geq 0} x_i$ satisfies $\Phi_\ast(x) = k^q x$ in $H^{even}(X; \mathbb{Z}_{(2)}) \otimes Q = H^{even}(X; Q)$. The homomorphism $S_\ast: H^{2n}(X; \mathbb{Z}_{(2)}) \to H^{2n+2q}(X; \mathbb{Z}_{(2)})$ is defined by setting $S_\ast(x_0) = x_q$. We call $S_\ast$ the generalized cohomology operation. Related to these are the homomorphisms $Q_\ast: H^{2n}(X; \mathbb{Z}_{(2)}) \to H^{2n+2q}(X; \mathbb{Z}_{(2)})$ defined by the recurrence relations $Q_\ast = \sum_{i=0}^q Q_{\ast-i} \Phi_{\ast-i}$. (2.2)

Lemma 2.12 of [S] implies that restricted to $H^{2n}(X; \mathbb{Z}_{(2)})$, $S_\ast = k^q \sum_{i=0}^q R_\ast(k)$. (2.3)

When there can be no confusion, we omit the suffix $T$ in all the above notations.

In applying (2.4) it is usually desirable to choose $k$ so that $v_2(1 - k^q)$ is small. From Lemma 2.12 of [1] we deduce that

$$v_2(1 - k^q) = \begin{cases} 1 & \text{if } q \text{ is odd} \\ 2 + v_2(q) & \text{if } q \text{ is even} \end{cases}$$

$$v_2(1 - s^q) = \begin{cases} 2 & \text{if } q \text{ is odd} \\ 2 + v_2(q) & \text{if } q \text{ is even} \end{cases}$$

(2.5)

We now turn to the proofs of Theorems 1.4 and 1.5. There are some assumptions which can make about $Y$ without loss of generality. As Steenrod squares commute with the suspension isomorphism in cohomology, we can assume that $Y$ is simply connected and so for cohomological purposes is a CW-complex with finite skeleton. Also as the homology is free of 2-torsion, all 2-primary Bocksteins vanish and so we can build $Y$ out of 2-local cells so that if $Y'$ is the $r$-skeleton then the inclusion $i: Y' \to Y$ inducing $i^*: H^*(Y'; Z_{(2)}) \to H^*(Y'; Z_{(2)})$ is an isomorphism for $q \leq r$ and is of course trivial for $q > r$. We therefore assume that $\dim Y = \ldots$
$m + 2q$ and by suspending if necessary that in Theorem 1.4, $m \equiv 0 \mod 2$ and in Theorem 1.5, $m \equiv 0 \mod 8$. Strictly speaking $Y$ is no longer a CW-complex of finite type but the discussion above taken from [5] applies equally well to finite 2-local CW-complexes or alternatively we can replace $Y$ by a finite complex with the same mod 2 homotopy type. We will write $m = 2n$ and set $p = 2 + v_2(q)$.

**The proof of Theorem 1.4.**

We suppose that $Sq^{2q}: H^{2n}(Y; Z_2) \to H^{2n+2q}(Y; Z_2)$ is non zero and obtain a contradiction. Let $x \in H^{2n}(Y; Z_{2q})$ be such that $Sq^{2q}x \neq 0$. It follows from (2.2) that $S^p x \neq 0 \mod 2$ for any choice of $T$ and so by (2.5) it follows that $(1 - k^p)S^p x \equiv 0 \mod 2^p$ but $(1 - k^p)S^p x \not\equiv 0 \mod 2^{p+1}$ if $k$ is either 3 or 5. By (2.1) and the hypothesis that $H^{2n+2q}(Y; Z_{2q}) = 0$ for $q - p + 1 \leq i \leq q - 1$, we deduce that $S^p x = 0$ with $i$ in this range. Therefore (2.4) implies that

$$(1 - k^p)S^p x \equiv \Sigma_{i=0}^{2q} k^i 2^q - 1 \cdot R^i (-k) \sigma_{i} x \mod 2^{p+1} \quad (*)$$

The last equation arises from the definition of $R^i(k)$ since $Q^i S^p x \in H^{2n+2q-2p+2i}(Y; Z_{2q})$ which is zero for $1 \leq i \leq p - 1$. Therefore setting $k = 3$ or 5, $(1 - k^p)S^p x \equiv 0 \mod 2^p$ but $(1 - k^p)S^p x \not\equiv 0 \mod 2^{p+1}$. This is clearly impossible as by (2.5), $v_2(1 - 3^p) = 1$ but $v_2(1 - 5^p) = 2$ since $p$ is odd under the hypotheses of Theorem 1.5. This contradiction establishes the result.

**Remark.** If $q$ is even, Theorem 1.3 follows from equation $(\ast)$. If $q$ is odd Theorem 1.3 can be proved similarly or deduced from the Adem relation $Sq^{2q} = Sq^{2q-1}S^q + Sq^2Sq^{2q-2}$.

**The proof of Theorem 1.5.**

Again we suppose that $Sq^{2q}: H^{2n}(Y; Z_2) \to H^{2n+2q}(Y; Z_2)$ is non zero and seek a contradiction. We follow the proof of Theorem 1.5 as far as equation $(\ast)$ but now it is sufficient to set $k = 3$,

$$(1 - 3^q)S^p x \equiv \Sigma_{i=0}^{2q} 3^q 2^q - 1 \cdot R^i (3) \sigma_{i} x \mod 2^{p+1}.$$ 

As $(1 - 3^q)S^p x \equiv 0 \mod 2^{p+1}$, we deduce that $R^i(3)y \equiv 0 \mod 2$, where $y = S^p x$. As dim $Y = 2n + 2q$, by (2.3) $\Phi^3(Y) = 3^q 2^q - 1 \cdot R^i (3) w$ and so $\psi^3(3) = 3^q 2^q - 1 \cdot \xi$ and $\eta$ where $T (3) = \xi \in K_{2n+q-p}(Y; Z_{2q})$ and $T(3^q 2^q - 1 \cdot R^i (3) y) = \eta \in K_{2n+q-p}(Y; Z_{2q}) \approx H^{2n+2q}(Y; Z_{2q})$. So $\psi^3(3) \equiv 0 \mod 2$ in $\Phi(3) = K_{2n+q-p}(Y; Z_{2q}) = K(Y/Y^{2n+q-p-1}; Z_{2q})$ as $R^i(3)y \equiv 0 \mod 2$. We set $W = Y/Y^{2n+q-p-1}$. At the prime 2 we have the 2-local cofibration

$$\nu S^{7(2n+q-p)} \approx W \approx S^{7(2n+q)}.$$ 

We will obtain a contradiction using symplectic $K$-theory. Let $c^r: KSp(-; Z_{2q}) \to K(-; Z_{2q})$ be the complexification or forgetful functor. The cofibration induces a commutative diagram with exact rows,

**Diagram 1**

$$\begin{array}{cccc}
\tilde{K} \left( \sqrt{S^{2(n+q-p)}}, Z_{2q} \right) & \xrightarrow{\nu} & \tilde{K} \left( W; Z_{2q} \right) & \xrightarrow{\nu} & \tilde{K} \left( \sqrt{S^{2(n+q)}}, Z_{2q} \right) \\
\uparrow c & & \uparrow c & & \uparrow c \\
0 & \xrightarrow{\nu} & KSp \left( \sqrt{S^{2(n+q-p)}}, Z_{2q} \right) & \xrightarrow{c} & KSp \left( \sqrt{S^{2(n+q)}}, Z_{2q} \right) \\
\end{array}$$
The hypotheses of Theorem 1.5 and the fact that we chose \( m = 2n \equiv 0 \mod 8 \) imply that
\[
2(n + q - p) = 4 \mod 8 \quad \text{and} \quad 2(n + q) = 0 \mod 8
\]
so the left hand \( c' \) is an isomorphism while any element in the image of the right hand \( c' \) is divisible by 2.

We now choose \( \xi \in KSp(W; Z_{2}) \) such that \( c' \) and \( i^* \) for odd \( k \) commute, we deduce that
\[
\psi^3(\xi) = 3^{n+q-p}\xi + i^*c'(\xi) \quad \text{and} \quad \psi^3(\xi) = 3^n - 3^{n+q-p}i^*c'(\xi)
\]
which establishes Theorem 1.5.

\section{The Proofs of Theorems 1.1, 1.6, 1.7 and 1.8}

Let \( EX \) denote the reduced suspension of a space \( X \) with a prescribed base point, \( E: 
\pi_i(X) \rightarrow \pi_i+1(EX) \) the suspension homomorphism and \( C_f \) the mapping cone of a map \( f \).

The proof of Theorem 1.6.

By Lemmas (3.3) and (3.4) of [10], it suffices to show that for every continuous map \( f: S^{2r+1} \rightarrow EB^2 \),
\[
f^* = 0: \quad H^{2r+1}(EB^2; Z_2) \rightarrow H^{2r+1}(S^{2r+1}; Z_2)
\]
and
\[
Sq^* = 0: \quad H^{r+1}(C_f; Z_2) \rightarrow H^{2r+2}(C_f; Z_2).
\]
The first condition is satisfied since \( H^{2r+1}(EB^2; Z_2) \cong H^*(B; Z_2) \cong H_*(B; Z_2) = 0 \). Also it follows easily that \( H_*(C_f; Z) \) is free of 2-torsion. Parts (a) and (b) of the theorem now follow from Theorem 1.2, part (c) from Theorem 1.5 and part (d) from Theorem 1.4.

The proof of Theorem 1.1.

Part (a) follows in this case since \( B \) is complex projective space of dimension \( k - 1 \) for which all cohomology groups are trivial in dimensions greater than \( 2k - 2 \) and the \( r \) of Theorem 1.6 is \( 2n+1 \). Part (b) follows as \( B \) becomes quaternionic projective space of dimension \( k - 1 \) whose cohomology groups vanish in dimension greater than \( 4k - 4 \) and the \( r \) of Theorem 1.6 is \( 4n+3 \).

Definitions of \( d^i(\xi) \) and \( d(\xi) \).

By \( A/Tor \) we denote the quotient group of an abelian group \( A \) by its torsion subgroup. Let \( \xi \) be an \( r \)-dimensional real vector bundle over a connected finite CW-complex \( B \) such that
\[
H_2(B; Z) \text{ is free of 2-torsion and } H_2(B; Z)/Tor \cong Z.
\]
Let \( d(\xi) \) and \( d^i(\xi) \) be the non negative generators of the images of
\[
\tilde{h}: \pi_{2r}(B^i) \rightarrow H_2(B^i; Z) \rightarrow H_2(B; Z)/Tor \cong Z
\]
and
\[
\pi_{2r}(B^i) \rightarrow H_2(B; Z) \rightarrow H_2(B; Z)/Tor \cong Z
\]
respectively, where \( \pi^h_*(\ ) \) is the stable homotopy functor, \( h \) is the Hurewicz homomorphism and \( \rightarrow \) is the quotient map. Note that \( d(\xi) \) is a multiple of \( d^i(\xi) \) and that \( d^i(\xi) \) is the non negative generator of the image of
\[
\tilde{h}: \pi_{2r+1}(EB^i) \rightarrow H_{2r+1}(EB^i; Z) \rightarrow H_{2r+1}(EB^i; Z)/Tor \cong Z
\]
since \( B^i \) is \( (r - 1) \)-connected.
The proof of Theorem 1.7.

We choose \( g \in \pi_{2r}(B^2) \) with \( h'(g) = d_{(5)}(f) \) and define an endomorphism \( \alpha_x \) of \( \pi_{2r+1}(EB^2) \) by

\[
\alpha_x(f) = \begin{cases} 
0 & \text{if } d'(5) = 0 \\
(d(5)/d'(5))f - (h'(f)/d'(5))e & \text{if } d'(5) \neq 0.
\end{cases}
\]

Let \( H_2: \pi_{2r+1}(EB^2) \to Z \) be the mod 2 reduction of the Hopf invariant \( H: \pi_{2r+1}(EB^2) \to \pi_{2r+1}(EB^2 \wedge B^2) \equiv Z \) (see [10] or [11]). It follows that for any \( f \in \pi_{2r+1}(FR^2) \),

\[
H_2(\alpha_x(f)) = \begin{cases} 
H_2(f) & \text{if } d'(5) = 0 \\
(d(5)/d'(5))H_2(f) & \text{if } d'(5) \neq 0.
\end{cases}
\]

so that

\[
(\alpha_x(f))^* = 0; \quad H^{2r+1}(EB^2; Z) \to H^{2r+1}(S^{2r+1}; Z).
\]

Hence (3.3) and (3.4) of [10] imply

**Lemma 3.2.** If \( d'(5) = 0 \) or \( d'(5) \neq 0 \) and \( d(5)/d'(5) \) is odd, then the order of \( \psi \) is even or infinite if and only if

\[
Sq^{r+1} = 0: \quad H^{r+1}(C_{at(f)}; Z) \to H^{2r+2}(C_{at(f)}; Z)
\]

for every \( f \in \pi_{2r+1}(EB^2) \).

By (3.1) it follows that \( H_2(\pi_2(B^2); Z) \) is free of 2-torsion, so that Theorem 1.7 follows from Theorems 1.3, 1.4 and Lemma 3.2.

The proof of Theorem 1.8.

It is well known that the Pontrjagin–Thom construction gives a map \( a: S^{2r} \to M^* \) such that \( a*: \pi_2(S^{2r}; Z) \to \pi_2(M^*; Z) \) is an isomorphism. Hence \( d(v) \) is defined and 1, so that \( d'(v) = 1 \). Therefore the result follows from Theorem 1.7.

**Remarks.** (1) The mod \( m \) lens space of dimension \( 2t - 1 \) with \( t \leq 4 \) satisfies the hypotheses of Theorem 1.8 if \( m \) is odd. (2) Let \( \tau M \) be the tangent bundle of a smooth, closed, connected and orientable manifold \( M \). It is not difficult to show that \( d'(\tau M) = 1 \) if \( M \) is a \( \pi \)-manifold. Hence \( d'(\tau S^r) = 1 \). It seems difficult in general to calculate \( d(\tau M) \), though we can show that \( d(\tau S^r) = 0 \) if \( r = 0 \) mod 2, 1 if \( r = 1, 3 \) or 7, and 2 otherwise. Let \( w \) be the Whitehead square of the identity map of \( S^r \). Since the Thom complex of \( \tau S^r \) is \( S^r \vee e^{2r} \) and the homotopy Thom class of \( \tau S^r \) is the natural inclusion \( S^r \to S^r \vee e^{2r} \), it follows that the Whitehead square of the homotopy Thom class of \( \tau S^r \) is zero. Therefore the hypothesis on \( d(5)/d'(5) \) in Theorem 1.7 is necessary in general.

§4. WHITEHEAD PRODUCTS IN \( W_{n+k} \)

In this section, we restrict our attention to the complex case. First we prove Proposition 1.2 and then a general lemma about the orders of Whitehead products. Finally we give a few fragmentary results on Samelson products in the unitary group.

We will denote by \( \Delta_k: \pi_k(W_{n+k,k}) \to \pi_{k-1}(U(n)) \) the boundary homomorphism associated with the homotopy exact sequence of the fibration \( U(n) \to U(n+k) \vee W_{n+k,k} \).

The proof of Proposition 1.2.

The result for \( k \leq 4 \) follows from Theorem 1.1. Thus it suffices to show that \([i_{12,5}, i_{12,5}] = 0\).
Let $\pi_i(X; 2)$ denote the 2-primary component of $\pi_i(X)$; $\sigma_6 \in \pi_{15}(S^6; 2)$ the element defined and named in [12]; $\sigma_6 = E^{n=8}\sigma_6$; $i_m: S^n \to S^n$ the identity map and $w_m = [1_m, 1_m] \in \pi_{2n-1}(S^n)$. From [12] and [9] we note that $\pi_{21}(U(7); 2) \cong \pi_{21}(U(8); 2) \cong \pi_{21}(S^{15}; 2) \cong Z_2$; $\pi_{22}(S^{15}; 2) \cong Z_{16}$ generated by $\sigma_{15}; \pi_{20}(U(7)) \cong \pi_{20}(U(8))$ and $\pi_{20}(S^{15}) = 0$.

From the exact sequence of $U(7) \to U(8) \to S^{15}$ it follows that $\Delta_1: \pi_{22}(S^{15}; 2) \to \pi_{21}(U(7); 2)$ is surjective. So there exists $x_0 \in \pi_{22}(U(8))$ with $p_*(x_0) = 2\sigma_{15}$. We consider the commutative square

Diagram 2

and deduce that $i_{12, 5*}(2\sigma_{15}) = 0$. Since $w_{15} = 2\sigma_{15} \circ \sigma_{22}$ by (10.10) of [12], it follows that $[i_{12, 5}, i_{12, 5}] = i_{12, 5*}(w_{15}) = (i_{12, 5*}(2\sigma_{15})) \circ \sigma_{22} = 0$. This completes the proof.

**Lemma 4.1.** Every Whitehead product in $W_{n+k,k}$ has finite order.

**Proof.** Let $x \in \pi_{i+1}(W_{n+k,k})$ and $y \in \pi_{j+1}(W_{n+k,k})$. As $W_{n+k,k}$ is $2n$-connected, we may take $i, j \leq 2n$. Therefore $\pi_i(U(n))$ and $\pi_j(U(n))$ are finite groups. Let the orders of $\Delta_k x$ and $\Delta_k y$ be $\alpha$ and $\beta$ respectively and $p_*(x') = \alpha x, p_*(y') = \beta y$. Whitehead products vanish in an $H$-space and so $\alpha \beta [x, y] = [\alpha x, \beta y] = [p_*(x'), p_*(y')] = p_*(x', y') = 0$, as required.

We recall that if $G$ is a topological group, $G \to X \to B$ is a principal fibration, $a \in \pi_{p+1}(B)$, $b \in \pi_{q+1}(B), p, q \geq 1$, then $\Delta[a, b] = (-1)^p \left\langle \Delta(a), \Delta(b) \right\rangle$ where $\Delta$ is the boundary homomorphism in the exact homotopy sequence and $\left\langle \cdot, \cdot \right\rangle$ denotes the Samelson product (see 5.2 of [10]).

**Proposition 4.2.** Let $x \in \pi_{n+i}(W_{n+k,k}), y \in \pi_{n+j}(W_{n+k,k})$ and $2n + i + j + 2 \leq 2k$. Then the order of $[x, y]$ equals the order of $\left\langle \Delta_k(x), \Delta_k(y) \right\rangle$.

**Proof.** By [2], $\pi_{n+i+j+k}(U(n+k))$ is $Z$ or $0$ and $\pi_{n+i+j+k}(U(n))$ is finite. Thus $p_*: \pi_{n+i+j+k}(U(n+k)) \to \pi_{n+i+j+k}(W_{n+k,k})$ is injective and so is $\Delta_k$ restricted to the torsion subgroup of $\pi_{n+i+j+k}(W_{n+k,k})$ which by Lemma 4.1 contains $[x, y]$. Therefore the result follows.

In [3], Furukawa has computed some Whitehead products for $k = 2$ or 3. From his results we notice the following.

**Proposition 4.3.** Let $x \in \pi_{2n}(U(n)) \cong Z_n$ and let $x$ be any element of (a) $\pi_{2n+2}(U(n))$ for $n \equiv 1 \mod 2$, (b) $\pi_{2n+4}(U(n))$ for $n \equiv 0 \mod 2 > 0$ or (c) the two component of $\pi_{2n+3}(U(n))$ for $n \equiv 1 \mod 4$. Then $[x, \Delta] = 0$.

**Proof.** We prove this with $x$ in (a) only because the other cases can be proved similarly. Now $\pi_{2n+2}(W_{n+2,2}) \cong Z$ is generated by $[2i_{2n+3}]$ where $p_*(2i_{2n+3}) = 2i_{2n+3}$. It follows from Theorem 3 of [3] that $[2i_{2n+3}] = 0$ so that $0 = \Delta_2 [2i_{2n+3}].$ Since $\Delta_2 [2i_{2n+3}]$ generates the cyclic group $\pi_{2n+2}(U(n))$ and $\Delta_1 i_{2n+3}$ is a generator of the cyclic group $\pi_{2n}(U(n))$, the result follows.
**Proposition 4.4.** Let \( n \equiv 3 \mod 4 \).

(a) The Samelson product \( \langle , \rangle : \pi_{2n+3}(U(n)) \times \pi_{2n}(U(n)) \to \pi_{4n-3}(U(n)) \) is zero.

(b) The Samelson square \( \pi_{2n+3}(U(n)) \to \pi_{4n+6}(U(n)) \) is zero.

**Proof.** We deduce from the homotopy exact sequence of the fibration \( U(n) \to U(n+1) \to S^{2n+1} \) that \( \Delta_1 : \pi_{2n+4}(S^{2n+1}) \to \pi_{2n+3}(U(n)) \) is surjective. Hence \( \pi_{2n+3}(U(n)) \) is generated by \( \Delta_1(\gamma_{2n+1}) \), where \( \gamma_{2n+1} = E^{2n-3}g_4 \) and \( g_4 : S^7 \to S^4 \) is the Hopf map. Now since \( n \equiv 3 \mod 4 \), by (4.7) of [4] it follows that \( [\gamma_{2n+1}, \gamma_{2n+1}] = 0 \) and therefore by (2.1) of [4] that \( \langle \Delta_1\gamma_{2n+1}, \Delta_1\gamma_{2n+1} \rangle = 0 \) and \( \langle \Delta_1\gamma_{2n+1}, \Delta_1\gamma_{2n+1} \rangle = 0 \). This completes the proof.

Finally, we note that the Jacobi identity for Samelson products (see [6]) implies

**Proposition 4.5.** Let \( x \in \pi_{4}(U(n)) \) and \( z \in \pi_{2n}(U(n)) \). Then \( \langle x, \langle x, z \rangle \rangle = 0 \).

**References**