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Generalized stable shape and the Whitehead theorem

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Abstract

In this paper we define a stable shape category based on the category of CW-spectra. Then we formulate and prove a Whitehead-type theorem in this category.

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0. Introduction

Lima [6] constructed the Čech stable homotopy theory for compacta. This construction gives the stable shape theory on the category of metric compacta. Dold and Puppe [4] and Henn [5] defined a stable shape category for compacta, which is based on the Spanier–Whitehead category and studied a duality in this stable shape category. More results on duality and complements were obtained in Nowak [12,13] and Mrozik [10].

In this paper we define a stable shape category for arbitrary spaces, which is based on the category of CW-spectra and show that the earlier stable shape category embeds in our stable shape category. This approach allows us to work on cells of CW-spectra. We then formulate and give a proof of a stable shape version of a Whitehead-type theorem (see Theorems 6.1 and 6.2).

In the next section we recall the stable categories and in Section 2 we define our stable shape category. In Section 3 we prove a finite-dimensional version of a

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Whitehead-type theorem for CW-spectra, and in Section 4 we generalize the statement in Section 3 over the pro-CW-spectra. In Section 5 we introduce a notion of dimension in our stable shape category, and finally in Section 6 we obtain a stable shape version of a Whitehead-type theorem.

For CW-spectra, we refer to Switzer [14, Chapter 8] and Adams [2, Part III], and for this shape theory, we refer to Mardešić and Segal [9]. *All spaces dealt with in paper are assumed to be based.*

1. Stable categories

Let **Top** denote the category of all spaces and pointed (continuous) maps, and let **CG**, **Cpt**, **CW**, and **fCW** denote the full subcategories of **Top** consisting of all compactly generated spaces, all compacta, all CW complexes, and all finite CW complexes, respectively. Let **HTop**, **HCG**, **HCpt**, **HCW** and **HfCW** denote the homotopy categories of the corresponding categories. If $\mathscr{C}$ is a category, then *Ob* $\mathscr{C}$ denotes the set of objects and for objects X and Y of $\mathscr{C}$, $\mathscr{C}(X, Y)$ denotes the set of all morphisms from X to Y in $\mathscr{C}$.

The *Spanier–Whitehead category* $\mathbf{HCG}_{sw}$ is defined by $Ob\mathbf{HCG}_{sw} = Ob\mathbf{CG}$ and for any two compactly generated spaces X and Y, $\mathbf{HCG}_{sw}(X, Y) = \{X, Y\}(= \operatorname{colim}_k[S^kX, S^kY])$, where $S^kX = S^k \wedge X$ for $k \geqslant 0$ and $S^1X = SX$. Then $\mathbf{HCpt}_{sw}$, $\mathbf{HCW}_{sw}$, $\mathbf{HfCW}_{sw}$ denote the full subcategories of $\mathbf{HCG}_{sw}$ whose objects are all compacta, all CW complexes, all finite CW complexes, respectively.

The *category of CW-spectra* $\mathbf{CW}_{spec}$ is the category whose objects are all CW-spectra and whose morphisms are all maps of CW-spectra. Let $\mathbf{fCW}_{spec}$ denote the full subcategory of $\mathbf{CW}_{spec}$ whose objects are all finite CW-spectra. The *suspension spectrum* $E(X)$ of a space X is the spectrum defined by

$$(E(X))_n = \begin{cases} S^nX, & n \geqslant 0, \\ *, & n < 0, \end{cases}$$

and if X is a CW complex, $E(X)$ is called the *suspension CW-spectrum*. Let $\mathbf{CW}^s_{spec}$ denote the full subcategory of $\mathbf{CW}_{spec}$ whose objects are all suspension CW-spectra. Let $\mathbf{HCW}_{spec}$ denote the homotopy category of $\mathbf{CW}_{spec}$. That is, $Ob\mathbf{HCW}_{spec}$ is the set of all CW-spectra, and for any two CW-spectra E and F, $\mathbf{HCW}_{spec}(E, F)$ is the set of homotopy classes $[E, F]$. Let $\mathbf{HCW}^s_{spec}$ denote the full subcategory of $\mathbf{HCW}_{spec}$ whose objects are the suspension CW-spectra.

There is functor $F_{sw}: \mathbf{HCW} \to \mathbf{HCW}_{sw}$ defined by $X \mapsto X$ for each $X \in Ob\mathbf{HCW}$ and $[f] \mapsto \{f\}$ for each $[f] \in [X, Y]$ where $\{f\} \in \operatorname{colim}_k[S^kX, S^kY]$ is the element represented by $[f] \in [X, Y]$. Also, there is a functor $F_{spec}: \mathbf{HCW} \to \mathbf{HCW}_{spec}$ defined by $X \mapsto E(X)$ for each CW complex X and $[f] \mapsto [E(f)]$ for each $[f] \in [X, Y]$ where $E(f): E(X) \to E(Y)$ is the map defined as

$$(E(f))_n = \begin{cases} S^nf: S^nX \to S^nY, & n \geqslant 0, \\ * : * \to *, & n < 0. \end{cases}$$

For each $[f] \in [X, Y]$, we write $E([f])$ for the unique class $[E(f)]$. Moreover, there is an embedding of a category $R: \mathbf{HfCW}_{sw} \to \mathbf{HCW}_{spec}$ defined by $X \mapsto E(X)$ for each finite CW complex X and

$$(Rf)_n = \begin{cases} *, & n < k, \\ S^{n-k} f_k, & n \geqslant k, \end{cases}$$

if an element $f \in \{X, Y\}$ is represented by a map $f_k : S^k X \to S^k Y$. Then we have the commutativity of functors $R \circ F_{sw} | \mathbf{HfCW} = F_{spec} | \mathbf{HfCW}$.

2. Generalized stable shape categories

The stable shape category based on the Spanier–Whitehead category which is defined by Dold and Puppe [4] and Henn [5] is essentially the abstract shape theory (in the sense of Mardešić and Segal [9, I, §2]) for the pair of categories $(\mathbf{HCpt}_{sw}, \mathbf{HfCW}_{sw})$ as shown in the following theorem.

Theorem 2.1. $\mathbf{HfCW}_{sw}$ *is a dense subcategory of* $\mathbf{HCpt}_{sw}$.

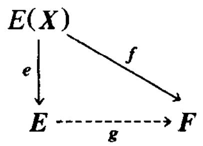
Proof. Let X be a compactum. Then there exists an **HCW**-expansion $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of X with each X_λ being a finite polyhedron. Then $S(\boldsymbol{p}) = (Sp_\lambda): SX \to S\boldsymbol{X} = (SX_\lambda, Sp_{\lambda\lambda'}, \Lambda)$ is an **HPol**-expansion of SX (see Ungar [15, Theorem 1.3]). On the other hand $\boldsymbol{p}$ induces a morphism $\bar{\boldsymbol{p}} = (\bar{p}_\lambda): X \to \bar{\boldsymbol{X}} = (X_\lambda, \bar{p}_{\lambda\lambda'}, \Lambda)$ in $\mathbf{pro\text{-}HCpt}_{sw}$, where $\bar{p}_\lambda = F_{sw}(p_\lambda)$ and $\bar{p}_{\lambda\lambda'} = F_{sw}(p_{\lambda\lambda'})$. We claim that this is an $\mathbf{HCW}_{sw}$-expansion of X. First note that each X_λ is a finite CW complex. Let $h: X \to P$ be a stable map (an S-map) to any CW complex, and let h be represented by an H-map $h_k: S^k X \to S^k P$. Then since $S^k P$ is a CW complex, by Ungar's result, there exist $\lambda \in \Lambda$ and an H-map $g_k: S^k X_\lambda \to S^k P$ such that $h_k = g_k(S^k p_\lambda)$. Let $g: X_\lambda \to P$ be the S-map represented by g_k. Then $h = g\bar{p}_\lambda$ in $\mathbf{HCG}_{sw}$. Moreover, let $g, h: X_\lambda \to P$ be two S-maps to any CW complex such that $g\bar{p}_\lambda = h\bar{p}_\lambda$ in $\mathbf{HCG}_{sw}$. Then there exists $k \geqslant 0$ such that $g_k(S^k p_\lambda) = h_k(S^k p_\lambda)$ where $g_k: S^k X_\lambda \to S^k P$ and $h_k: S^k X_\lambda \to S^k P$ are H-maps representing g and h, respectively. Again by Ungar's result, there exists $\lambda' \geqslant \lambda$ such that $g_k(S^k p_{\lambda\lambda'}) = h_k(S^k p_{\lambda\lambda'})$. Thus $g\bar{p}_{\lambda\lambda'} = h\bar{p}_{\lambda\lambda'}$ in $\mathbf{HCW}_{sw}$ as required. Hence this proves our claim and completes the proof. □.

We denote by $\mathbf{Sh}_{sw}$ the abstract shape category for the pair $(\mathbf{HCpt}_{sw}, \mathbf{HfCW}_{sw})$ and call it the *stable shape category* (or *Spanier–Whitehead stable shape category* or *SW-shape category*) for compacta.

Now we wish to define a *generalized shape theory* for the suspension spectra $E(X)$ of any space X, using the suspension CW-spectra. Here we should note there is no map defined between spectra unless the domain is a CW-spectrum. Thus such a shape theory will not be an abstract shape theory for a pair of categories in the sense of Mardešić and Segal but an "extension" of the homotopy theory on the category $\mathbf{HCW}_{\text{spec}}$.

Let $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an **HCW**-expansion of a space X, and let $E(\boldsymbol{X}) = (E(X_\lambda), E(p_{\lambda\lambda'}), \Lambda)$ be the inverse system in $\mathbf{HCW}_{\text{spec}}$ induced from the inverse system $\boldsymbol{X}$ in **HCW** by the functor F_{spec}. A morphism $\boldsymbol{e}: E(\boldsymbol{X}) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ in $\textbf{pro-HCW}_{\text{spec}}$ is said to be *a generalized expansion* of X in $\mathbf{HCW}_{\text{spec}}$ provided the following universal property is satisfied:

(U) if $\boldsymbol{f}: E(\boldsymbol{X}) \to \boldsymbol{F}$ is a morphism in $\textbf{pro-HCW}_{\text{spec}}$ then there exists a unique morphism $\boldsymbol{g}: \boldsymbol{E} \to \boldsymbol{F}$ in $\textbf{pro-HCW}_{\text{spec}}$ such that $\boldsymbol{f} = \boldsymbol{ge}$.



One should note here that the definition of a generalized expansion does not depend on the choice of the **HCW**-expansion $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$. Also note that for any two generalized expansions $\boldsymbol{e}: E(\boldsymbol{X}) \to \boldsymbol{E}$ and $\boldsymbol{e}': E(\boldsymbol{X}) \to \boldsymbol{E}'$ in $\mathbf{HCW}_{\text{spec}}$ there exists a unique isomorphism $\boldsymbol{i}: \boldsymbol{E} \to \boldsymbol{E}'$ in $\textbf{pro-HCW}_{\text{spec}}$ (which we call the *natural isomorphism*) such that $\boldsymbol{ie} = \boldsymbol{e}'$. It is easy to see that the identity induced morphism $E(\boldsymbol{X}) \to E(\boldsymbol{X})$ is a generalized expansion of X in $\mathbf{HCW}_{\text{spec}}$.

The following theorem gives a characterization of generalized expansions in $\mathbf{HCW}_{\text{spec}}$.

Theorem 2.2. *Let $\boldsymbol{e}: E(\boldsymbol{X}) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ be a morphism in $\textbf{pro-HCW}_{\text{spec}}$ which is represented by a morphism (e_a, φ) of inverse systems where $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an* **HCW***-expansion of any space X. Then $\boldsymbol{e}$ is a generalized expansion in $\mathbf{HCW}_{spec}$ if and only if the following two conditions are satisfied*:

(GE1) *Every morphism $h: E(X_\lambda) \to F$ in $\mathbf{HCW}_{\text{spec}}$ admits $a \in A$ and a morphism $g_a: E_a \to F$ in $\mathbf{HCW}_{\text{spec}}$ such that $hE(p_{\lambda\lambda'}) = g_a e_a E(p_{\varphi(a)\lambda'})$ for some $\lambda' \geqslant \lambda, \varphi(a)$.*

(GE2) *If $g_a, h_a: E_a \to F$ are two morphisms in $\mathbf{HCW}_{\text{spec}}$ such that $g_a e_a E(p_{\varphi(a)\lambda}) = h_a e_a E(p_{\varphi(a)\lambda})$ for some $\lambda \geqslant \varphi(a)$, then there exists $a' \geqslant a$ such that $g_a e_{aa'} = h_a e_{aa'}$.*

Proof. Suppose $\boldsymbol{e}$ is a generalized expansion, and let $h: E(X_\lambda) \to F$ be a morphism in $\mathbf{HCW}_{\text{spec}}$. Then h represents a morphism $\boldsymbol{h}: E(\boldsymbol{X}) \to (F)$ in $\textbf{pro-HCW}_{\text{spec}}$, and there exists a morphism $\boldsymbol{g}: \boldsymbol{E} \to (F)$ in $\textbf{pro-HCW}_{\text{spec}}$ such that $\mathbf{h} = \boldsymbol{ge}$. If $\boldsymbol{g}$ is represented by a morphism $g_a: E_a \to F$ in $\mathbf{HCW}_{\text{spec}}$, then $\boldsymbol{h} = \boldsymbol{ge}$ implies the assertion in (GE1). For (GE2), let $g_a, h_a: E_a \to F$ be as in the hypothesis. Let g_a, h_a represent morphisms in $\textbf{pro-HCW}_{\text{spec}}$, $\boldsymbol{g}, \boldsymbol{h}: \boldsymbol{E} \to (F)$, respectively. Then $\boldsymbol{ge} = \boldsymbol{he}$. By the uniqueness in the definition of a generalized expansion, $\boldsymbol{g} = \boldsymbol{h}$, which implies $g_a e_{aa'} = h_a e_{aa'}$ for some $a' \geqslant a$.

Conversely, assume that the assertions in (GE1) and (GE2) hold. Let $\boldsymbol{e}$ be represented by (e_a, φ), and let $\boldsymbol{f}: E(\boldsymbol{X}) \to \boldsymbol{F} = (F_b, f_{bb'}, B)$ be a morphism in

pro-HCW$_{\text{spec}}$, which is represented by (f_b, ψ). Then by (GE1) for each $b \in B$ there exist $\eta(b) \in A$ and a morphism $g_b : E_{\eta(b)} \to F_b$ such that

$$f_b E(p_{\psi(b)\lambda}) = g_b e_{\eta(b)} E(p_{\varphi(\eta(b))\lambda}) \quad \text{for some } \lambda \geqslant \psi(b), \eta(b). \tag{1}$$

We claim that (g_b, η) defines a morphism $\boldsymbol{g} : \boldsymbol{E} \to \boldsymbol{F}$ in **pro-HCW**$_{\text{spec}}$ such that $\boldsymbol{f} = \boldsymbol{ge}$. Indeed, let $b \leqslant b'$ in B. Then

$$f_{b'} E(p_{\psi(b')\lambda'}) = g_{b'} e_{\eta(b')} E(p_{\varphi(\eta(b'))\lambda'}) \quad \text{for some } \lambda' \geqslant \psi(b'), \eta(b'). \tag{2}$$

Choose $a \in A$ such that $a \geqslant \eta(b), \eta(b')$, and then take $\lambda'' \geqslant \lambda, \lambda', \varphi(a)$ in Λ such that the following three equalities hold:

$$f_{bb'} f_{b'} E(p_{\psi(b')\lambda''}) = f_b E(p_{\psi(b)\lambda''}), \tag{3}$$

$$e_{\eta(b')a} e_a E(p_{\varphi(a)\lambda''}) = e_{\eta(b')} E(p_{\varphi(\eta(b'))\lambda''}), \tag{4}$$

$$e_{\eta(b)a} e_a E(p_{\varphi(a)\lambda''}) = e_{\eta(b)} E(p_{\varphi(\eta(b))\lambda''}). \tag{5}$$

Then (2), (3) and (1) imply

$$f_{bb'} g_{b'} e_{\eta(b')} E(p_{\varphi(\eta(b'))\lambda''}) = f_b E(p_{\psi(b)\lambda''}). \tag{6}$$

On the other hand (4) and (5) imply

$$f_{bb'} g_{b'} e_{\eta(b')a} e_a E(p_{\varphi(a)\lambda''}) = f_{bb'} g_{b'} e_{\eta(b')} E(p_{\varphi(\eta(b'))\lambda''}) \tag{7}$$

and

$$g_b e_{\eta(b)a} e_a E(p_{\varphi(a)\lambda''}) = g_b e_{\eta(b)} E(p_{\varphi(\eta(b))\lambda''}). \tag{8}$$

Then (6), (7), (8) and (GE2) imply

$$f_{bb'} g_{b'} e_{\eta(b')a'} = g_b e_{\eta(b)a'} \quad \text{for some } a' \geqslant a. \tag{9}$$

This shows that (g_b, η) represents a morphism $\boldsymbol{g}$ in **pro-HCW**$_{\text{spec}}$, and (1) implies $\boldsymbol{f} = \boldsymbol{ge}$.

It remains to show the uniqueness of $\boldsymbol{g}$. Suppose that $\boldsymbol{h} : \boldsymbol{E} \to \boldsymbol{F}$ is a morphism in **pro-HCW**$_{\text{spec}}$ such that $\boldsymbol{ge} = \boldsymbol{he}$, and let (g_b, η) and (h_b, ξ) represent $\boldsymbol{g}$ and $\boldsymbol{h}$, respectively. Now, fix $b \in B$. Then $\boldsymbol{ge} = \boldsymbol{he}$ implies

$$g_b e_{\eta(b)} E(p_{\varphi(\eta(b))\lambda}) = h_b e_{\xi(b)} E(p_{\varphi(\xi(b))\lambda}) \quad \text{for some } \lambda \geqslant \varphi(b), \xi(b). \tag{10}$$

Choose $a \geqslant \eta(b), \xi(b)$ in A, and then take $\lambda' \geqslant \lambda, \varphi(a)$ in Λ such that

$$e_{\eta(b)a} e_a E(p_{\varphi(a)\lambda'}) = e_{\eta(b)} E(p_{\varphi(\eta(b))\lambda'}) \tag{11}$$

and

$$e_{\xi(b)a} e_a E(p_{\varphi(a)\lambda'}) = e_{\xi(b)} E(p_{\varphi(\eta(b))\lambda'}). \tag{12}$$

Put $g'_a = g_b e_{\eta(b)a}$ and $h'_a = h_b e_{\xi(b)a}$. Then by (11), (10), and (12),

$$\begin{aligned} g'_a e_a E(p_{\varphi(\eta(b))\lambda'}) &= g_b e_{\eta(b)} E(p_{\varphi(\eta(b))\lambda}) E(p_{\lambda\lambda'}) h_b e_{\xi(b)} E(p_{\varphi(\xi(b))\lambda}) E(p_{\lambda\lambda'}) \\ &= h'_a e_a E(p_{\varphi(a)\lambda'}). \end{aligned}$$

This and (GE2) imply $g'_a e_{aa'} = h'_a e_{aa'}$ for some $a' \geqslant a$. Thus $g_b e_{\eta(b)a'} = h_b e_{\xi(b)a'}$, so that $\boldsymbol{g} = \boldsymbol{h}$. □.

Theorem 2.3. *A morphism in* **pro-HCW**$_{\text{spec}}$, $\boldsymbol{e}: E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$, *where* $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *is an* **HCW**-*expansion of any space* X, *is a generalized expansion in* **HCW**$_{\text{spec}}$ *if and only if* $\boldsymbol{e}$ *is an isomorphism in* **pro-HCW**$_{\text{spec}}$.

Proof. Suppose that $\boldsymbol{e}$ is a generalized expansion in **HCW**$_{\text{spec}}$. Then the identity induced morphism $\boldsymbol{i}: E(X) \to E(X)$ is a generalized morphism in **HCW**$_{\text{spec}}$, so that $\boldsymbol{e}$ is a natural isomorphism. The converse is obvious. □

Theorem 2.4. *Every* **HfCW**$_{\text{sw}}$-*expansion* $\boldsymbol{q} = (q_\mu): X \to \boldsymbol{Z} = (Z_\mu, q_{\mu\mu'}, M)$ *of a compactum* X *induces a generalized expansion* $\boldsymbol{e}: E(X) \to R\boldsymbol{Z} = (E(Z_\mu), Rq_{\mu\mu'}, M)$ *in* **HCW**$_{\text{spec}}$ *where* $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *is any* **HCW**-*expansion of* X.

Proof. Without loss of generality we can assume that each X_λ is finite. Since $\boldsymbol{p}: X \to \boldsymbol{X}$ is an **HCW**-expansion of X, it induces an **HfCW**$_{\text{sw}}$-expansion $\bar{\boldsymbol{p}} = (\bar{p}_\lambda): X \to \bar{\boldsymbol{X}} = (X_\lambda, \bar{p}_{\lambda\lambda'}, \Lambda)$ (see the proof of Theorem 2.1), so that there is a natural isomorphism $\boldsymbol{e}: \bar{\boldsymbol{X}} \to \boldsymbol{Z}$ in **pro-HfCW**$_{\text{sw}}$. Thus it induces a natural isomorphism $E(\boldsymbol{X}) \to E(\boldsymbol{Z})$ in **pro-HCW**$_{\text{spec}}$. Hence by Theorem 2.3 this is a generalized expansion in **HCW**$_{\text{spec}}$. □

Using generalized expansions, we define the *generalized stable shape category* **Sh**$_{\text{spec}}$ for spaces as follows: First let *Ob***Sh**$_{\text{spec}}$ be the set of all spaces. For any two spaces X and Y, let $\mathscr{E}_{(X,Y)}$ be the set of all morphisms $\boldsymbol{g}: \boldsymbol{E} \to \boldsymbol{F}$ in **pro-HCW**$_{\text{spec}}$ where $\boldsymbol{e}: E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ and $\boldsymbol{f}: E(X) \to \boldsymbol{F} = (F_b, f_{bb'}, B)$ are generalized morphisms in **HCW**$_{\text{spec}}$. Then we define an equivalence relation $\sim$ on $\mathscr{E}_{(X,Y)}$ as follows: for $\boldsymbol{g}: \boldsymbol{E} \to \boldsymbol{F}$ and $\boldsymbol{g}': \boldsymbol{E}' \to \boldsymbol{F}'$ in $\mathscr{E}_{(X,Y)}$, $\boldsymbol{g} \sim \boldsymbol{g}'$ if and only if the following diagram commutes in **pro-HCW**$_{\text{spec}}$:

$$\begin{array}{ccc} \boldsymbol{E} & \xrightarrow{\boldsymbol{g}} & \boldsymbol{F} \\ {\scriptstyle \boldsymbol{i}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{j}} \\ \boldsymbol{E}' & \xrightarrow{\boldsymbol{g}'} & \boldsymbol{F}' \end{array}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the natural isomorphisms. It is easy to verify that $\sim$ is an equivalence relation on $\mathscr{E}_{(X,Y)}$. We define a morphism from X to Y as each equivalence class of $\mathscr{E}_{(X,Y)}$. Thus $\mathbf{Sh}_{\text{spec}}(X, Y) = \mathscr{E}_{(X,Y)}/\sim$. We write $Sh_{\text{spec}}(X) \leqslant Sh_{\text{spec}}(Y)$ (respectively, $Sh_{\text{sw}}(X) \leqslant Sh_{\text{sw}}(Y)$) provided X is dominated by Y in **Sh**$_{\text{spec}}$ (respectively, in **Sh**$_{\text{sw}}$). Also we write $Sh_{\text{spec}}(X) = Sh_{\text{spec}}(Y)$ (respectively, $Sh_{\text{sw}}(X) = Sh_{\text{sw}}(Y)$) provided X is equivalent to Y in **Sh**$_{\text{spec}}$ (respectively, in **Sh**$_{\text{sw}}$).

Theorem 2.5. *There exists an embedding of categories* $\Theta: \mathbf{Sh}_{\text{sw}} \to \mathbf{Sh}_{\text{spec}}$.

Proof. For each compactum X, we define $\Theta: X \mapsto X$. Let $G: X \to Y$ be a morphism in **Sh**$_{\text{sw}}$. Let $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{q} = (q_\mu): Y \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be **HCW**-expansions of X and Y, respectively, such that X_λ and Y_μ

are finite polyhedra. By the proof of Theorem 2.1 these induce $\mathbf{HCW}_{sw}$-expansions $\bar{\boldsymbol{p}} = (\bar{p}_\lambda): X \to \bar{\boldsymbol{X}} = (X_\lambda, \bar{p}_{\lambda\lambda'}, \Lambda)$ and $\bar{\boldsymbol{q}} = (\bar{q}_\mu): Y \to \bar{\boldsymbol{Y}} = (Y_\mu, \bar{q}_{\mu\mu'}, M)$. So, G is represented by a morphism $\boldsymbol{g}: \bar{\boldsymbol{X}} \to \bar{\boldsymbol{Y}}$ of **pro-HCW**$_{sw}$. But then $\boldsymbol{g}$ induces a morphism $R\boldsymbol{g}: R\bar{\boldsymbol{X}} = (E(X_\lambda), R\bar{p}_{\lambda\lambda'}, \Lambda) \to R\bar{\boldsymbol{Y}} = (E(Y_\lambda), R\bar{q}_{\mu\mu'}, M)$. Since the X_λ and Y_μ are finite, $R\bar{\boldsymbol{X}} = E(X)$ and $R\bar{\boldsymbol{Y}} = E(Y)$. Since the identity induced morphisms $E(X) \to E(X)$ and $E(Y) \to E(Y)$ are generalized expansions in $\mathbf{HCW}_{spec}$ of X and Y, respectively, $\boldsymbol{g}$ represents a morphism $\Theta(G)$ in $\mathbf{Sh}_{spec}$. It is easy to see that the function $\Theta: G \mapsto \Theta(G)$ is well defined. That is, $\Theta(G)$ does not depend on the choice of the representative $\boldsymbol{g}$. It is a routine to check Θ defines a functor. To see this is an embedding, we define a functor $\Theta': \mathbf{Sh}_{spec} | \mathbf{Cpt} \to \mathbf{Sh}_{sw}$ as follows, where $\mathbf{Sh}_{spec} | \mathbf{Cpt}$ denotes the full subcategory of $\mathbf{Sh}_{spec}$ whose objects are all compacta: First for each compactum X, define $\Theta': X \mapsto X$. Let $G: X \to Y$ be a morphism in $\mathbf{Sh}_{spec}$ represented by a morphism in **pro-HCW**$_{spec}$, $\boldsymbol{g}: E(X) \to E(Y)$. Since X_λ and Y_μ are finite, $\boldsymbol{g}$ induces a morphism $\bar{\boldsymbol{X}} \to \bar{\boldsymbol{Y}}$ in **pro-HfCW**$_{sw}$, so that it represents a morphism $\Theta'(G)$ in $\mathbf{Sh}_{sw}$. Note that $\Theta'(G)$ does not depend on the choice of such $\boldsymbol{g}$. Thus $\Theta': G \mapsto \Theta'(G)$ is well defined, and it is again a routine to check Θ' is a functor and that $\Theta \circ \Theta'$ and $\Theta' \circ \Theta$ are the identities on $\mathbf{Sh}_{spec} | \mathbf{Cpt}$ and $\mathbf{Sh}_{sw}$, respectively. □

Let **Sh** denote the pointed shape category for spaces in the sense of Mardešić and Segal [9].

Theorem 2.6. *There exists a functor $\Xi: \mathbf{Sh} \to \mathbf{Sh}_{spec}$ such that the restriction of Ξ on $\mathbf{Sh} | \mathbf{Cpt}$ factors through the embedding $\Theta: \mathbf{Sh}_{sw} \to \mathbf{Sh}_{spec}$ in Theorem 2.5.*

Proof. For each space X, we define $\Xi: X \mapsto X$. Let $G: X \to Y$ be a morphism in **Sh** represented by a morphism in **pro-HCW**, $\boldsymbol{g}: \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ where $\boldsymbol{p}: X \to \boldsymbol{X}$ and $\boldsymbol{q}: Y \to \boldsymbol{Y}$ are **HCW**-expansions of X and Y, respectively. Then $\boldsymbol{g}$ induces a morphism in **pro-HCW**$_{spec}$, $E(\boldsymbol{g}): E(X) \to E(Y)$ where if $\boldsymbol{g}$ is represented by a morphism (g_μ, φ) of inverse systems then $E(\boldsymbol{g})$ is the unique morphism represented by $(E(g_\mu), \varphi)$. Thus $\boldsymbol{g}$ represents a morphism $\Xi(G)$ in $\mathbf{Sh}_{spec}$. It is a routine to check that Ξ is a functor, and it is easy to see that Ξ is the desired functor. □

Theorem 2.7. *For any spaces X and Y, $\mathbf{Sh}_{spec}(X, Y)$ has the structure of an Abelian group.*

Proof. Let $\boldsymbol{e}: E(X) \to \boldsymbol{E} = (E_a, E_{aa'}, A)$ and $\boldsymbol{f}: E(Y) \to \boldsymbol{F} = (F_b, f_{bb'}, B)$ be generalized expansions in $\mathbf{HCW}_{spec}$ of X and Y, respectively. Then there is a one-to-one correspondence between $\mathbf{Sh}_{spec}(X, Y)$ and **pro-HCW**$_{spec}(\boldsymbol{E}, \boldsymbol{F}) = \lim_a \operatorname{colim}_b [E_a, F_b]$. Since the $[E_a, F_b]$ are Abelian groups and the bonding maps $e_{aa'}$ and $f_{bb'}$ induce group homomorphisms, **pro-HCW**$_{spec}(\boldsymbol{E}, \boldsymbol{F})$ has an Abelian group structure. □

Theorem 2.8. *For any spaces X and Y, if $Sh(S^kX) = Sh(S^kY)$ for some $k \geqslant 0$ then $Sh_{\mathrm{spec}}(X) = Sh_{\mathrm{spec}}(Y)$.*

Proof. Let $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{q} = (q_\mu): Y \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be **HCW**-expansions. Then there exists an isomorphism $\boldsymbol{f}_k : S^k\boldsymbol{X} \to S^k\boldsymbol{Y}$ of **pro-HCW** which represents the isomorphism of **Sh**, $S^kX \to S^kY$. Without loss of generality, we can assume that $\Lambda = M$ and that $\boldsymbol{f}_k$ is represented by a level morphism $(f_{k,\lambda}): S^k\boldsymbol{X} = (S^kX_\lambda, S^kp_{\lambda\lambda'}, \Lambda) \to S^k\boldsymbol{Y} = (S^kY_\lambda, S^kq_{\lambda\lambda'}, \Lambda)$. We define a level morphism $(f_\lambda): E(\boldsymbol{X}) \to E(\boldsymbol{Y})$ by

$$(f_\lambda)_q = \begin{cases} S^{q-k}f_{k,\lambda}, & q \geqslant k, \\ *, & q < k. \end{cases}$$

This induces a morphism in **pro-HCW**$_{\mathrm{spec}}$, $\boldsymbol{f} : E(\boldsymbol{X}) \to E(\boldsymbol{Y})$. Let $\lambda \in \Lambda$. Then there exist $\lambda' \geqslant \lambda$ and a morphism of **HCW** $g_{k,\lambda} : Y_{\lambda'} \to X_\lambda$ such that the following diagram commutes in **HCW**:

$$\begin{array}{ccc} S^kX_\lambda & \xleftarrow{S^kp_{\lambda\lambda'}} & S^kX_{\lambda'} \\ {\scriptstyle f_{k,\lambda}}\downarrow & \nwarrow^{g_{k,\lambda}} & \downarrow{\scriptstyle f_{k,\lambda'}} \\ S^kY_\lambda & \xleftarrow[S^kq_{\lambda\lambda'}]{} & S^kY_{\lambda'} \end{array}$$

We define a morphism of **HCW**$_{\mathrm{spec}}$, $g_\lambda : Y_{\lambda'} \to X_\lambda$ by

$$(g_\lambda)_q = \begin{cases} S^{q-k}g_{k,\lambda}, & q \geqslant k, \\ *, & q < k. \end{cases}$$

Then we obtain a commutative diagram in **HCW**$_{\mathrm{spec}}$

$$\begin{array}{ccc} E(X_\lambda) & \xleftarrow{E(p_{\lambda\lambda'})} & E(X_{\lambda'}) \\ {\scriptstyle f_\lambda}\downarrow & \nwarrow^{g_\lambda} & \downarrow{\scriptstyle f_{\lambda'}} \\ E(Y_\lambda) & \xleftarrow[E(q_{\lambda\lambda'})]{} & E(Y_{\lambda'}) \end{array}$$

so that the morphism $\boldsymbol{f} : E(\boldsymbol{X}) \to E(\boldsymbol{Y})$ is an isomorphism. Hence X and Y are equivalent in **Sh**$_{\mathrm{spec}}$. □

Let sd denote the shape dimension for pointed spaces (see Mardešić and Segal [9, II, §1]).

Theorem 2.9. *Let X and Y be compacta such that* sd $X = n$ *and* sd $Y = m$ *are finite. Then $Sh_{\mathrm{spec}}(X) = Sh_{\mathrm{spec}}(Y)$ implies $Sh(S^kX) = Sh(S^kY)$ for some $k \geqslant 0$.*

Proof. Let $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{q} = (q_\mu): Y \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be **HCW**-expansions such that the X_λ and Y_μ are finite polyhedra of dimension at most n and m, respectively. Suppose that $\boldsymbol{f} : E(\boldsymbol{X}) \to E(\boldsymbol{Y})$ is an isomorphism in

pro-HCW which represents the isomorphism in $\mathbf{Sh}_{\mathrm{spec}}$. Without loss of generality, we can assume that $\Lambda = M$ and that f is represented by a level morphism $(f_\lambda): E(X) \to E(Y)$. Let $\lambda \in \Lambda$. Then there exists $\lambda' \geqslant \lambda$ and a morphism $g_\lambda : E(Y_{\lambda'}) \to E(X_\lambda)$ such that the following diagram commutes in $\mathbf{HCW}_{\mathrm{spec}}$:

$$\begin{array}{ccc} E(X_\lambda) & \xleftarrow{E(p_{\lambda\lambda'})} & E(X_{\lambda'}) \\ {\scriptstyle f_\lambda}\downarrow & \nwarrow^{g_\lambda} & \downarrow{\scriptstyle f_{\lambda'}} \\ E(Y_\lambda) & \xleftarrow[E(q_{\lambda\lambda'})]{} & E(Y_{\lambda'}) \end{array}$$

Since all X_λ and Y_λ are finite, there exist $k \geqslant 0$ and maps $f_{\lambda,k} : S^k X_\lambda \to S^k Y_\lambda$, $f_{\lambda',k} : S^k X_{\lambda'} \to S^k Y_{\lambda'}$, and $g_{\lambda,k} : S^k Y_{\lambda'} \to S^k X_\lambda$ representing f_λ, $f_{\lambda'}$ and g_λ, respectively, such that the following diagram commutes in **HCW**:

$$\begin{array}{ccc} S^k X_\lambda & \xleftarrow{S^k p_{\lambda\lambda'}} & S^k X_{\lambda'} \\ {\scriptstyle f_{\lambda,k}}\downarrow & \nwarrow^{g_{\lambda,k}} & \downarrow{\scriptstyle f_{\lambda',k}} \\ S^k Y_\lambda & \xleftarrow[S^k q_{\lambda\lambda'}]{} & S^k Y_{\lambda'} \end{array}$$

By the Freudenthal suspension theorem, we can assume that k is independent of the choice of λ. Thus the morphism $f_k : S^k X \to S^k Y$ represented by the level morphism $(f_{\lambda,k})$ is an isomorphism of **pro-HCW**, so that $Sh(S^k X) = Sh(S^k Y)$ as $S^k \boldsymbol{p} : S^k X \to S^k \boldsymbol{X}$ and $S^k \boldsymbol{q} : S^k Y \to S^k \boldsymbol{Y}$ are **HCW**-expansions. □

Example 2.10. Let X be the 1-dimensional acyclic continuum ("figure-eight"-like continuum) described by Case and Chamberlin [3]. Mardešić and Segal [8] showed that X is nonmovable, so that $Sh(X) \neq Sh(*)$. However, its suspension SX is of trivial shape, i.e., $Sh(SX) = Sh(*)$ (see Mardešić [7]). Hence by Theorem 2.8 we conclude that $Sh_{\mathrm{spec}}(X) = Sh_{\mathrm{spec}}(*)$.

3. Whitehead theorem for CW-spectra

The *nth homotopy group* $\pi_n(E)$ of CW-spectrum E is defined as the group $[\Sigma^n S^0, E] \approx \mathrm{dirlim}_k \pi_{n+k}(E_k)$. Here for all $n \in \mathbb{Z}$, Σ^n denote the suspension functors on $\mathbf{HCW}_{\mathrm{spec}}$, and S^0 denotes the suspension spectrum $E(S^0)$. Now we recall the well-known Whitehead-type theorem for CW-spectra.

Theorem 3.1. *Let $f : E \to F$ be a map of CW-spectra such that $\pi_q(f) : \pi_q(E) \to \pi_q(F)$ is an isomorphism for all $q \in \mathbb{Z}$. Then f is a homotopy equivalence of CW-spectra.*

Proof. See Adams [2, Corollary 3.5, p. 150] or Switzer [14, Theorem 8.25, p. 144]. □

Note that every CW-spectrum E consists of cells $e = \{e_n^d, Se_n^d, S^2e_n^d, \dots\}$ where e_n^d is a d-cell in the CW complex E_n and is not a suspension of any cell in E_{n-1}. Then we define the *dimension* of e (denoted dim e) as $d-n$, $n \in \mathbb{Z}$, and the *dimension* of the CW-spectrum E (denoted dim E) is defined as sup{dim $e : e$ is a cell of E}. Similarly, if (E, F) is a pair of CW-spectra, then the *dimension* of (E, F) (denoted dim(E, F)) is defined as sup{dim $e : e$ is a cell of (E, F)}. For the base point spectrum $*$, we define dim $* = -\infty$. If E is a CW-spectrum, we write $E^{(i)}$ for the ith skeleton of E.

In this section we prove a variation of Theorem 3.1 below (see Theorem 3.2). A map of CW-spectra $f : E \to F$ is an *n-equivalence* provided $\pi_q(f) : \pi_q(E) \to \pi_q(F)$ is an isomorphism for $q \leqslant n-1$ and an epimorphism of $q = n$. A pair of CW-spectra (E, F) is said to be *n-connected* provided $\pi_q(E, F) = 0$ for $q \leqslant n$.

Theorem 3.2. *Let* $n \in \mathbb{Z} \cup \{\infty\}$, *let* $f : E \to F$ *be a map of CW-spectra, which is an n-equivalence, and suppose* dim $E \leqslant n-1$ *and* dim $F \leqslant n$. *Then f is a homotopy equivalence of CW-spectra.*

The proof follows that for Theorem 3.1. in Adams [2, III.3].

The *nth homotopy group* $\pi_n(E, F)$ of a pair of CW-spectra (E, F) is defined as the group $[\Sigma^{n-1}(D^1, S^0), (E, F)]$ where the pair of CW-spectra (D^1, S^0) is defined by

$$(D^1, S^0)_n = \begin{cases} *, & n < 0, \\ (D^{n+1}, S^n), & n \geqslant 0. \end{cases}$$

Then $\pi_n(E, F) \approx \text{dirlim}_k[(D^{n+k}, S^{n+k-1}), (E_k, F_k)] = \text{dirlim}_k \pi_{n+k}(E_k, F_k)$.

Lemma 3.3. *For any pair of CW-spectra* (E, F) *there is a natural exact sequence*

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(E, F) \to \pi_{n-1}(F) \to \cdots .$$

Lemma 3.3 immediately implies the following.

Lemma 3.4. *A pair of CW-spectra* (E, F) *is n-connected if and only if the inclusion induced map of CW-spectra* $i : E \to F$ *is an n-equivalence.*

Lemma 3.5. *Let* (E, F) *be a pair of CW-spectra with* dim$(E, F) \leqslant n$, *and let* (G, H) *be an n-connected pair of CW-spectra. Suppose that there are a map of CW-spectra* $f : E \to G$ *and a homotopy* $h : F \wedge I^+ \to G$ *from* $f \mid F$ *to a map* $g : F \to H \subseteq G$. *Then h extends to a homotopy* $k : E \wedge I^+ \to G$ *such that* $k_0 = f$ *and* k_1 *is a map of E into H.*

Proof. Let $f : E \to G$ and $h : F \wedge I^+ \to G$ be represented by functions $f' : E' \to G$ and $h' : F' \wedge I^+ \to G$ where F' and E' are cofinal subspectra of F and E, respectively, such that F' is a CW-subspectrum of E'.

First of all, let $\mathscr{C}$ be the set of all the pairs (U, k') such that U is a CW-spectrum with $F' \subseteq U \subseteq E'$ and $k' : U \wedge I^+ \to G$ is a function with $k'_0 = f' \mid U$

and $k_1'(U) \subseteq H$. Define an order $<$ on $\mathscr{C}$ by $(U_1, k_1') < (U_2, k_2')$ if and only if $U_1 \subseteq U_2$ and $k_2' | U_1 = k_1'$. Then $(F', h') \in \mathscr{C}$, so that $\mathscr{C} \neq \emptyset$. Suppose that $\{(U_\alpha, k_\alpha')\}$ is a chain in $\mathscr{C}$. Then $(\cup U_\alpha, k')$ where $k' : (\cup U_\alpha) \wedge I^+ \to G$ is defined by $k' | U_\alpha \wedge I^+ = k_\alpha'$, belongs to $\mathscr{C}$ and is an upper bound of the chain. Thus, Zorn's lemma implies the existence of a maximal element $(U, k') \in \mathscr{C}$. We claim that U is cofinal in E'. Suppose to the contrary this is not the case. Then there is a subspectrum V such that $U \subseteq V \subseteq E'$ where V consists of U and just one more cell-spectrum, say

$$V_q = \begin{cases} U_q, & q < p, \\ U_q \cup S^{q-p} e_p^m, & q \geqslant p, \end{cases}$$

where e_p^m is an m-cell in E_p'. Then the restricted maps

$$\begin{cases} f_p' \mid e_p^m \wedge \{0\}^+ : e_p^m \wedge \{0\}^+ \to G_p, \\ k_p' \mid \partial e_p^m \wedge I^+ : \left(\partial e_p^m \wedge I^+, \partial e_p^m \wedge \{1\}^+\right) \to (G_p, H_p) \end{cases}$$

define an element of $\pi_m(G_p, H_p)$ hence an element of $\pi_{m-p}(G, H) \approx \operatorname{dirlim}_q \pi_{m-p+q}(G_q, H_q)$. But since $m - p = \dim e_p^m \leqslant \dim(E, F) \leqslant n$, by the n-connectedness of (G, H), we have $\pi_{m-p}(G, H) = 0$. Thus there exists $r \geqslant 0$ such that the maps

$$\begin{cases} f_{p+r}' \mid S^r e_p^m \wedge \{0\}^+ : S^r e_p^m \wedge 0^+ \to G_{p+r}, \\ k_{p+r}' \mid \partial(S^r e_p^m) \wedge I^+ : \left(\partial(S^r e_p^m) \wedge I^+, \partial(S^r e_p^m) \wedge 1^+\right) \to (G_{p+r}, H_{p+r}) \end{cases}$$

represent the trivial element in $\pi_{m+r}(G_{p+r}, H_{p+r})$. So this extends to a map $l_{p+r}' : (S^r e_p^m \wedge I^+, S^r e_p^m \wedge \{1\}^+) \to (G_{p+r}, H_{p+r})$. Now we define a map $k_{p+r}'' : V_{p+r} \wedge I^+ \to G_{p+r}$ by

$$\begin{cases} k_{p+r}'' \mid U_{p+r} \wedge I^+ = k_{p+r}', \\ k_{p+r}'' \mid S^r e_p^m \wedge I^+ = l_{p+r}'. \end{cases}$$

Then $(k_{p+r}'')_0 = f_{p+r}' \mid V_{p+r}$ and $(k_{p+r}'')_1(V_{p+r}) \subseteq H_{p+r}$. Define a CW-spectrum V'' by

$$V_q'' = \begin{cases} U_q, & q < p + r, \\ V_q, & q \geqslant p + r, \end{cases}$$

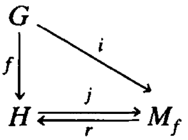
and a function $k'' : V'' \wedge I^+ \to G$ by

$$k_q'' = \begin{cases} k_q', & q < p + r, \\ S^{q-p-r} k_q'', & q \geqslant p + r. \end{cases}$$

Then $(V'', k'') \in \mathscr{C}$ and $(U, k') < (V'', k'')$, contradicting the maximality of (U, k'). Thus U must be a cofinal subspectrum of E'. So, the function $k' : U \wedge I^+ \to G$ defines a map of CW-spectra $k : E \wedge I^+ \to G$ which extends $h : F \wedge I^+ \to G$. $\square$

Let $f : G \to H$ be a function of CW-spectra. Then we define a CW-spectrum M_f called the *mapping cylinder* of the function f by $(M_f)_n = M_{f_n}$(= the usual mapping cylinder of f_n).

Lemma 3.6. *Let $f: G \to H$ be a function of CW-spectra, and let $i: G \to M_f$ and $j: H \to M_f$ be the functions such that $i_n: G_n \to M_{f_n}$ and $j_n: H_n \to M_{f_n}$ are the natural embeddings of CW complexes. Moreover, let $r: M_f \to H$ be the function such that $r_n: M_{f_n} \to H_n$ are the usual retractions. Then we have $ri = f$, $jf \simeq i$, $rj = 1_H$, and $jr \simeq 1_{M_f}$.*



Lemma 3.7. *Let $f: G \to H$ be a function of CW-spectra. Then there is an exact sequence*

$$\cdots \to \pi_n(G) \to \pi_n(H) \to \pi_n(M_f, H) \to \pi_{n-1}(G) \to \cdots.$$

Theorem 3.8. *Let $f: G \to H$ be a function of CW-spectra, which is an n-equivalence. Then for every CW-spectrum E the induced function $f_*: [E, G] \to [E, H]$ is a bijection if* $\dim E \leqslant n-1$, *and a surjection if* $\dim E \leqslant n$.

Proof. In view of Lemmas 3.6 and 3.7, we can assume that $f: G \to H$ is an inclusion function and (H, G) is an n-connected pair. First, let $\dim E \leqslant n$. Take any map $g: E \to H$ of spectra. Consider g as a map of pairs of CW-spectra $g: (E, *) \to (H, G)$ where $*$ is the base point spectrum. Then since $\dim(E, *) \leqslant n$ and (H, G) is n-connected, by Lemma 3.4, there exists a map of CW-spectra $g': E \to G \subseteq H$ such that $g' \simeq g$ as maps of CW-spectra $E \to H$. This shows the surjectivity.

On the other hand, let $\dim E \leqslant n-1$. Then $\dim E \wedge I^+ \leqslant n$. Suppose that g_1, $g_2: E \to G$ are two maps of CW-spectra such that $fg_1 \simeq fg_2$ as maps into H. We define a map of pairs of CW-spectra

$$h: (E \wedge I^+, E \wedge \{0\}^+ \cup * \wedge I^+ \cup E \wedge \{1\}^+) \to (H, G)$$

as the homotopy from fg_1 to fg_2. Then since $\dim E \wedge I^+ \leqslant n$ and (H, G) is n-connected, by Lemma 3.5, there exists a map of CW-spectra $h': E \wedge I^+ \to G$ such that

$$h' \mid E \wedge \{0\}^+ \cup * \wedge I^+ \cup E \wedge \{1\}^+ = h \mid E \wedge \{0\}^+ \cup * \wedge I^+ \cup E \wedge \{1\}^+.$$

This shows that $g_1 \simeq g_2$ as maps of E into G, so f_* is a monomorphism. □

Now Theorem 3.2 easily follows from Theorem 3.8.

4. Whitehead theorem for pro-CW-spectra

A morphism $f: \boldsymbol{E} = (E_a, e_{ee'}, A) \to \boldsymbol{F} = (F_b, f_{bb'}, B)$ in **pro-HCW**$_{\text{spec}}$ is an *n-equivalence* provided the induced morphism in pro-groups $\pi_q(f): \pi_q(\boldsymbol{E}) =$

$(\pi_q(E_a), \pi_q(e_{ee'}), A) \to \pi_q(F) = (\pi_q(F_b), \pi_q(f_{bb'}), B)$ is an isomorphism for $q \leqslant n - 1$ and an epimorphism for $q = n$.

Now we state a Whitehead-type theorem for pro-CW-spectra.

Theorem 4.1. *Let $k, n \in \mathbb{Z}$ with $k \leqslant n$, let $f: \boldsymbol{E} = (E_a, e_{ee'}, A) \to \boldsymbol{F} = (F_b, f_{bb'}, B)$ be a morphism of* **pro-HCW**$_{\text{spec}}$ *such that* $\dim E_a \leqslant n-1$ *and* $\dim F_b \leqslant n$ *for all $a \in A$ and $b \in B$, and suppose that for each $b \in B$,* $\dim e \geqslant k$ *for all cells $e \neq *$ of F_b. If f is an n-equivalence, then f is an isomorphism of* **pro-HCW**$_{\text{spec}}$.

Before proving the theorem we need to establish a series of lemmas.

Lemma 4.2. *Let $f: E \to F$ be a map of CW-spectra, which is an n-equivalence, and let R be a CW-spectrum. Then*

(i) *if* $\dim R \leqslant n$, *then every map of CW-spectra $h: R \to F$ admits a map of CW-spectra $k: R \to E$ such that $h \simeq fk$; and*

(ii) *if* $\dim R \leqslant n-1$ *and $k_1, k_2: R \to E$ are maps of CW-spectra such that $fk_1 = fk_2$, then $k_1 \simeq k_2$.*

Proof. There exists a cofinal subspectrum E' of E and a function $f': E' \to F$ which represents f. Then f' is an n-equivalence. Then the assertions for f' hold because of Theorem 3.8. Hence the assertions hold for f. □

Lemma 4.3. *Let $(\boldsymbol{E}, \boldsymbol{F}) = ((E_a, F_a), e_{aa'}, A)$ be an inverse system in pairs of CW-spectra. Then there is an exact sequence of pro-groups*

$$\cdots \to \pi_n(\boldsymbol{F}) \to \pi_n(\boldsymbol{E}) \to \pi_n(\boldsymbol{E}, \boldsymbol{F}) \to \pi_{n-1}(\boldsymbol{F}) \to \cdots.$$

Proof. Lemma 3.3 implies the following commutative diagram with the row being exact for all $a \leqslant a'$ in A:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_n(F_a) & \longrightarrow & \pi_n(E_a) & \longrightarrow & \pi_n(E_a, F_a) & \longrightarrow & \pi_{n-1}(F_a) & \longrightarrow \cdots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \cdots \longrightarrow & \pi_n(F_{a'}) & \longrightarrow & \pi_n(E_{a'}) & \longrightarrow & \pi_n(E_{a'}, F_{a'}) & \longrightarrow & \pi_{n-1}(F_{a'}) & \longrightarrow \cdots \end{array}$$

which implies the exactness of the above sequence (see Mardešić and Segal [9, Theorem 10, p. 119]). □

Lemma 4.4. *Suppose that there is a diagram in* **CW**$_{\text{spec}}$ *which is commutative up to homotopy*

$$\begin{array}{ccc} E & \xleftarrow{p} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ F & \xleftarrow{q} & H \end{array}$$

Then there exist cofinal subspectra E', G', H' of E, G, H, respectively, and functions $f' : E' \to F$, $p' : G' \to E'$, $q' : H' \to F$, $g' : G' \to H'$ and $r' : M_{g'} \to M_{f'}$ such that the following diagram commutes in $\mathbf{CW}_{\text{spec}}$:

$$\begin{array}{ccc} E' & \xleftarrow{\;p'\;} & G' \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle k'} \\ M_{f'} & \xleftarrow{\;r'\;} & M_{g'} \\ {\scriptstyle j'}\uparrow & & \uparrow{\scriptstyle l'} \\ F & \xleftarrow{\;q'\;} & H' \end{array} \tag{13}$$

where i', j', k' and l' are the inclusion induced functions of CW-spectra.

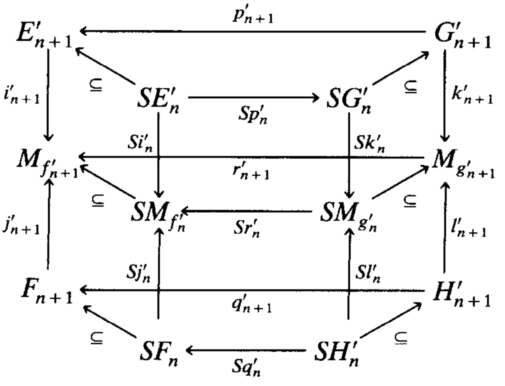
Proof. There exist cofinal subspectra E', G', H' of E, G, H, respectively, and functions $f' : E' \to F$, $p' : G' \to E'$, $q' : H' \to F$ such that for each $n \in \mathbb{Z}$ the following diagram is commutative up to pointed homotopy:

$$\begin{array}{ccc} E'_n & \xleftarrow{\;p'_n\;} & G'_n \\ {\scriptstyle f'_n}\downarrow & & \downarrow{\scriptstyle g'_n} \\ F_n & \xleftarrow{\;q'_n\;} & H'_n \end{array}$$

We choose a pointed map $r'_n : M_{g'_n} \to M_{f'_n}$ such that the following diagram commutes in **Top** for each $n \in \mathbb{Z}$ as in Mardešić and Segal [9, Lemma 3, p. 145]:

$$\begin{array}{ccc} E'_n & \xleftarrow{\;p'_n\;} & G'_n \\ {\scriptstyle i'_n}\downarrow & & \downarrow{\scriptstyle k'_n} \\ M_{f'_n} & \xleftarrow{\;r'_n\;} & M_{g'_n} \\ {\scriptstyle j'_n}\uparrow & & \uparrow{\scriptstyle l'_n} \\ F_n & \xleftarrow{\;q'_n\;} & H'_n \end{array}$$

Then the following diagram also commutes for each $n \in \mathbb{Z}$:



Then $r' = \{r'_n\} : M_{g'} \to M_{f'}$ is a function of CW-spectra which makes the diagram (13) commutative. □

Lemma 4.5. *Let $k \in \mathbb{Z}$ and $n \geqslant 1$, and for $i = 0, 1, \ldots, n-1$, let $e_i : (E_i, F_i) \to (E_{i+1}, F_{i+1})$ be maps of pairs of CW-spectra such that if $n \geqslant 2$, the induced homomorphisms $(e_i)_\# : \pi_{i+k}(E_i, F_i) \to \pi_{i+k}(E_{i+1}, F_{i+1})$ are trivial for $i = 1, 2, \ldots, n-1$. Furthermore suppose that for every $i = 0, 1, \ldots, n-1$, E_i contains no cells $e \neq *$ of dimension less than k. Then the composite of maps of CW-spectra $e_{n-1} \cdots e_1 e_0 : (E_0, F_0) \to (E_n, F_n)$ factors through an $(n+k-1)$-connected pair of CW-spectra.*

Proof. (*Special case.*) First, we assume that E_i, $i = 0, 1, \ldots, n-1$, contain no cells of negative dimension. For $i = 0, 1, \ldots, n-1$, we put

$$\begin{cases} H_i = (F_0 \wedge I^+) \cup \left(E_0^{(i)} \wedge \{1\}^+\right) \subseteq E_0' \wedge I^+, \\ G_i = (F_0 \wedge I^+) \cup \left(E_0^{(i)} \wedge I^+\right) \cup \left(E_0 \wedge \{0\}^+\right) \subseteq E_0 \wedge I^+. \end{cases}$$

We wish to define maps of pairs of CW-spectra $g_i : (G_i, H_i) \to (E_{i+1}, F_{i+1})$ such that $g_0 \mid E_0 \wedge \{0\}^+ = e_0$ and $g_i \mid G_{i-1} = e_i g_{i-1}$ (see the diagram below).

$$\begin{array}{ccccccccc} (E_0 \wedge \{0\}^+, F_0 \wedge \{0\}^+) & \xrightarrow{\subseteq} & (G_0, H_0) & \xrightarrow{\subseteq} & (G_1, H_1) & \xrightarrow{\subseteq} & \cdots & \xrightarrow{\subseteq} & (G_{n-1}, H_{n-1}) \\ \downarrow{=} & & \downarrow{g_0} & & \downarrow{g_1} & & & & \downarrow{g_{n-1}} \\ (E_0, F_0) & \xrightarrow{e_0} & (E_1, F_1) & \xrightarrow{e_1} & (E_2, F_2) & \xrightarrow{e_2} & \cdots & \xrightarrow{e_{n-1}} & (E_n, F_n) \end{array} \qquad (14)$$

Initial step: Consider the case $i = 0$. Put $g_0 \mid E_0 \wedge \{0\}^+ = e_0$ and $g_0 \mid F_0 \wedge I^+ = e_0 q$ where $q : F_0 \wedge I^+ \to F_0$ is the map of CW-spectra represented by the natural projections $q_n : (F_0 \wedge I^+)_n = (F_0)_n \wedge I^+ \to (F_0)_n$, $n \in \mathbb{Z}$. Then we wish to extend the map

$$g_0 \mid E_0 \wedge \{0\}^+ \cup F_0 \wedge I^+ : \left(E_0 \wedge \{0\}^+ \cup F_0 \wedge I^+, F_0 \wedge I^+\right) \to (E_1, F_1)$$

over (G_0, H_0). Note that there exist cofinal subspectra E_0' and F_0' of E_0 and F_0, respectively, such that $F_0' \subseteq E_0'$ and $g_0 \mid E_0 \wedge \{0\}^+ \cup F_0 \wedge I^+$ is represented by a function

$$g_0' : \left(E_0' \wedge \{0\}^+ \cup F_0' \wedge I^+, F_0' \wedge I^+\right) \to (E_1, F_1).$$

Let $e_\alpha^0 = \{e_{q_\alpha}^{q_\alpha}, Se_{q_\alpha}^{q_\alpha}, \ldots\}$ be a 0-cell in E_0', where $e_{q_\alpha}^{q_\alpha}$ is a q_α-cell in the CW complex $(E_0')_{q_\alpha}$. Then we define a map

$$\varphi_{\alpha,q_\alpha} : e_{q_\alpha}^{q_\alpha} \wedge \{0, 1\}^+ \to (E_1)_{q_\alpha}$$

by the formulas

$$\begin{cases} \varphi_{\alpha,q_\alpha} \mid e_{q_\alpha}^{q_\alpha} \wedge \{1\}^+ = *_{q_\alpha} (= \text{the base point of } (E_1)_{q_\alpha}), \\ \varphi_{\alpha,q_\alpha} \mid e_{q_\alpha}^{q_\alpha} \wedge \{0\}^+ = (g_0')_{q_\alpha} \mid e_{q_\alpha}^{q_\alpha}. \end{cases}$$

We put $\varphi_{\alpha,q_\alpha+1}=S\varphi_{\alpha,q_\alpha}$. Then since $Se^{q_\alpha}_{q_\alpha}\wedge\{0,\ 1\}^+\approx S(e^{q_\alpha}_{q_\alpha}\wedge\{0,\ 1\}^+)$, $\varphi_{\alpha,q_\alpha+1}$ is considered as a map

$$\varphi_{\alpha,q_\alpha+1}: Se^{q_\alpha}_{q_\alpha}\wedge\{0,\ 1\}^+\to S(E_1)_{q_\alpha}\subseteq(E_1)_{q_\alpha+1}.$$

Since $S(E_1)_{q_\alpha}$ is path-connected and $Se^{q_\alpha}_{q_\alpha}\wedge\{0,\ 1\}^+\subseteq Se^{q_\alpha}_{q_\alpha}\wedge I^+$ is a cofibration, $\varphi_{\alpha,q_\alpha+1}$ extends to a map

$$\tilde\varphi_{\alpha,q_\alpha+1}: Se^{q_\alpha}_{q_\alpha}\wedge I^+\to S(E_1)_{q_\alpha}\subseteq(E_1)_{q_\alpha+1}.$$

We put

$$\tilde\varphi_{\alpha,m}=\begin{cases}S^{m-q_\alpha-1}\tilde\varphi_{\alpha,q_\alpha+1}, & m\geqslant q_\alpha+1,\\ *, & m<q_\alpha+1.\end{cases}$$

Then since $S^{m-q_\alpha}e^{q_\alpha}_{q_\alpha}\wedge I^+\approx S^{m-q_\alpha}(e^{q_\alpha}_{q_\alpha}\wedge I^+)$, $\tilde\varphi_{\alpha,m}$ is considered as a map

$$\tilde\varphi_{\alpha,m}: S^{m-q_\alpha}e^{q_\alpha}_{q_\alpha}\wedge I^+\to S^{m-q_\alpha}(E_1)_{q_\alpha}\subseteq(E_1)_m.$$

We obtain CW-spectra E''_0 and F''_0 with $F''_0\subseteq E''_0$ by replacing each j-cell ($j\geqslant 0$)

$$e^j_\alpha=\left\{e^{q_\alpha+j}_{q_\alpha},\ Se^{q_\alpha+j}_{q_\alpha},\dots\right\}$$

in E'_0 by a j-cell

$$\tilde e^j_\alpha=\left\{Se^{q_\alpha+j}_{q_\alpha},\ S^2e^{q_\alpha+j}_{q_\alpha},\dots\right\}.$$

Then we define a function $g''_0:(G''_0,\ H''_0)\to(E_1,\ F_1)$ by

$$\begin{cases}g''_0\mid E''_0\wedge\{0\}^+\cup F''_0\wedge I^+=g'_0\mid E''_0\wedge\{0\}^+\cup F''_0\wedge I^+,\\ g''_0\mid\tilde e^0_\alpha\wedge I^+=\left(\tilde\varphi_{\alpha,q}:q\in\mathbb{Z}\right)\quad\text{for each 0-cell }\tilde e^0_\alpha\text{ of }E''_0,\end{cases}$$

where

$$\begin{cases}G''_0=E''_0\wedge\{0\}^+\cup F''_0\wedge I^+\cup E''^{(0)}_0\wedge I^+,\\ H''_0=F''_0\wedge I^+\cup E''^{(0)}_0\wedge\{1\}^+.\end{cases}$$

Since G''_1 and H''_0 are cofinal in G_0 and H_0, g''_0 defines a map $g_0:(G_0,\ H_0)\to(E_1,\ F_1)$ such that $g_0\mid E_0\wedge\{0\}^+=e_0$.

Inductive step: Suppose $n\geqslant 2$ and suppose that a map $g_{i-1}:(G_{i-1},\ H_{i-1})\to(E_i,\ F_i)$ has been defined as required. We wish to define a map $g_i:(G_i,\ H_i)\to(E_{i+1},\ F_{i+1})$ such that $g_i\mid G_{i-1}=e_ig_{i-1}$. Let e_i and g_{i-1} be represented by functions $e'_i:(E'_i,\ F'_i)\to(E_{i+1},\ F_{i+1})$ and $g'_{i-1}:(G'_{i-1},\ H'_{i-1})\to(E'_i,\ F'_i)$ where E'_i and F'_i are cofinal subspectra of E_i and F_i with $F'_i\subseteq E'_i$, and

$$\begin{cases}H'_{i-1}=(F'_0\wedge I^+)\cup\left(E'^{(i-1)}_0\wedge\{1\}^+\right),\\ G'_{i-1}=(F'_0\wedge I^+)\cup\left(E'^{(i-1)}_0\wedge I^+\right)\cup\left(E'_0\wedge\{0\}^+\right)\end{cases}$$

with E_0' and F_0' are cofinal subspectra of E_0 and F_0 with $F_0' \subseteq E_0'$. We first put $g_i' \mid G_{i-1}' = e_i' g_{i-1}' : (G_{i-1}', H_{i-1}') \to (E_{i+1}, F_{i+1})$. Let $e_\alpha^i = \{e_{q_\alpha}^{q_\alpha+i}, Se_{q_\alpha}^{q_\alpha+i}, \dots\}$ be an i-cell of E_0' where $e_{q_\alpha}^{q_\alpha+i}$ is a $(q_\alpha+i)$-cell in $(E_0')_{q_\alpha}$. Then the map

$$(g_{i-1}')_{q_\alpha} \mid \partial e_{q_\alpha}^{q_\alpha+i} \wedge I^+ \cup e_{q_\alpha}^{q_\alpha+i} \wedge \{0\}^+ :$$

$$\left(\partial e_{q_\alpha}^{q_\alpha+i} \wedge I^+ \cup e_{q_\alpha}^{q_\alpha+i} \wedge \{0\}^+, \partial e_{q_\alpha}^{q_\alpha+i} \wedge \{1\}^+\right) \to \left((E_i')_{q_\alpha}, (F_i')_{q_\alpha}\right)$$

defines an element of $\pi_{q_\alpha+i}((E_i')_{q_\alpha}, (F_i')_{q_\alpha})$ hence an element of

$$\pi_i(E_i, F_i) \approx \operatorname*{dirlim}_m \pi_{i+m}\left((E_i')_{i+m}, (F_i')_{i+m}\right).$$

Note that

$$\begin{aligned}&\left(S\left(\partial e_{q_\alpha}^{q_\alpha+i} \wedge I^+ \cup e_{q_\alpha}^{q_\alpha+i} \wedge \{0\}^+\right), S\left(\partial e_{q_\alpha}^{q_\alpha+i}\right) \wedge \{1\}^+\right)\\ &\quad\approx \left(\partial\left(Se_{q_\alpha}^{q_\alpha+i}\right) \wedge I^+ \cup \left(Se_{q_\alpha}^{q_\alpha+i}\right) \wedge \{0\}^+, \partial\left(Se_{q_\alpha}^{q_\alpha+i}\right) \wedge \{1\}^+\right)\\ &\quad\approx \left(D^{q_\alpha+i+1}, S^{q_\alpha+i}\right).\end{aligned}$$

Thus the pair of CW-spectra (B, B') defined by

$$(B, B')_k = \begin{cases}\left(\partial\left(S^{k-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge I^+ \cup \left(S^{k-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge \{0\}^+, \partial\left(S^{k-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge \{1\}^+\right),\\ \quad k \geqslant q,\\ *, \quad k < q\end{cases}$$

is isomorphic to the pair of CW-spectra $(\Sigma^i D^1, \Sigma^i S^0)$. Hence, since $(e_i)_\# : \pi_i(E_i, F_i) \to \pi_i(E_{i+1}, F_{i+1})$ is trivial, there exists $m_\alpha \geqslant q_\alpha$ such that the map

$$\begin{aligned}&\psi_{\alpha,m_\alpha} = (e_i')_{m_\alpha}(g_{i-1}')_{m_\alpha} \mid \partial\left(S^{m_\alpha-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge I^+ \cup \left(S^{m_\alpha-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge \{0\}^+ :\\ &\quad\left(\partial\left(S^{m_\alpha-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge I^+ \cup \left(S^{m_\alpha-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge \{0\}^+, \partial\left(S^{m_\alpha-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge \{1\}^+\right)\\ &\quad\to \left((E_{i+1})_{m_\alpha}, (F_{i+1})_{m_\alpha}\right)\end{aligned}$$

defines the trivial element of $\pi_{i+m_\alpha}((E_{i+1})_{i+m_\alpha}, (F_i)_{i+m_\alpha})$. Thus it extends to a map

$$\tilde{\psi}_{\alpha,m_\alpha} : \left(\left(S^{m_\alpha-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge I^+, \left(S^{m_\alpha-q_\alpha} e_{q_\alpha}^{q_\alpha+i}\right) \wedge \{1\}^+\right) \to \left((E_{i+1})_{m_\alpha}, (F_{i+1})_{m_\alpha}\right).$$

We put

$$\tilde{\psi}_{\alpha,q} = \begin{cases} S^{q-m_\alpha}\tilde{\psi}_{\alpha,m_\alpha}, & q \geqslant m_\alpha,\\ *, & q < m_\alpha.\end{cases}$$

We define a cofinal pair of CW-spectra (E_i'', F_i'') of (E_i', F_i') in the following way: First we put $(E_i'')^{(i-1)} = (E_i')^{(i-1)}$, and then we obtain $(E_i'')^{(i)}$ by replacing each i-cell

$$e_\alpha^i = \left\{e_{q_\alpha}^{q_\alpha+i}, Se_{q_\alpha}^{q_\alpha+i}, \dots\right\}$$

of E_i' by the i-cell

$$\left(e_\alpha^i\right)^{m_\alpha}=\left\{S^{m_\alpha-q_\alpha}e_{q_\alpha}^{q_\alpha+i},\ S^{m_\alpha-q_\alpha+1}e_{q_\alpha}^{q_\alpha+i},\dots\right\}.$$

Accordingly, using induction on dimension, we define $(E_i'')^{(j)}$ for $j \geqslant i$ by replacing each j-cell

$$e_\beta^j=\left\{e_{q_\beta}^{q_\beta+j},\ Se_{q_\beta}^{q_\beta+j},\dots\right\}$$

by a j-cell

$$\left(e_\beta^j\right)^{m_\beta}=\left\{S^{m_\beta-q_\beta}e_{q_\beta}^{q_\beta+j},\ S^{m_\beta-q_\beta+1}e_{q_\beta}^{q_\beta+j},\dots\right\}$$

for some appropriate $m_\beta \geqslant q_\beta$. Also a subspectrum F_i'' of F_i' is similarly defined, and (E_i'', F_i'') is a cofinal pair of CW-spectra of (E_i', F_i'). Then we can define a function $g_i'':(G_i'', H_i'') \to (E_{i+1}, F_{i+1})$ by

$$\begin{cases} g_i'' \mid G_{i-1}''=e_i'g_{i-1}' \mid G_{i-1}'', \\ g_i'' \mid \tilde{e}_\alpha^i \wedge I^+=\left(\tilde{\psi}_{\alpha,q}\colon q\in\mathbb{Z}\right) \quad \text{for each } i\text{-cell } \tilde{e}_\alpha^i \text{ of } E_i'', \end{cases}$$

where

$$\begin{cases} G_{i-1}''=E_0''\wedge\{0\}^+\cup F_0''\wedge I^+\cup E_0''^{(i-1)}\wedge I^+, \\ G_i''=E_0''\wedge\{0\}^+\cup F_0''\wedge I^+\cup E_0''^{(i)}\wedge I^+, \\ H_i''=F_0''\wedge I^+\cup E_0''^{(i)}\wedge\{1\}^+. \end{cases}$$

Since (G_i'', H_i'') is cofinal in (G_i, H_i), g_i'' defines a map $g_i:(G_i, H_i) \to (E_{i+1}, F_{i+1})$ such that $g_i \mid G_{i-1}=e_ig_{i-1}$.

Now, having completed the diagram (14) for $n \geqslant 1$, we put $(G, H)=(G_{n-1}, H_{n-1})$. Then, since $(G, H) \approx (E_0, F_0 \cup E_0^{(n-1)})$ in $\mathbf{HCW}_{\text{spec}}^2$(= the category of the pairs of CW-spectra),

$$\begin{aligned}\pi_q(G', H') &\approx \pi_q\left(E_0', F_0'\cup E_0'^{(n-1)}\right)\\ &=\operatorname*{dirlim}_k\left[(D^{q+k}, S^{q+k-1}), \left((E_0')_k, \left(F_0'\cup E_0'^{(n-1)}\right)_k\right)\right]\\ &=\operatorname*{dirlim}_k\left[(D^{q+k}, S^{q+k-1}), \left((E_0')_k, (F_0')_k\cup (E_0')_k^{(n+k-1)}\right)\right].\end{aligned}$$

So $\pi_q(G', H')=0$ for $q \leqslant n-1$. Thus the map $e_{n-1}\cdots e_1e_0:(E_0, F_0) \to (E_n, F_n)$ factors through an $(n-1)$-connected pair of CW-spectra (G', H') in $\mathbf{HCW}_{\text{spec}}^2$.

(*General case.*) Now we assume that $\dim e \geqslant k$ for all cells $e \neq *$ in E_i, $i=0, 1,\dots,n-1$. Then $\Sigma^{-k}E_i$ contains no cells $e \neq *$ of negative dimension for each $i=0, 1,\dots,n-1$. Consider the maps of CW-spectra $\Sigma^{-k}e_i:(\Sigma^{-k}E_i, \Sigma^{-k}F_i) \to (\Sigma^{-k}E_{i+1}, \Sigma^{-k}F_{i+1})$. Then $(\Sigma^{-k}e_i)_\#:\pi_i(\Sigma^{-k}E_i, \Sigma^{-k}F_i) \to \pi_i(\Sigma^{-k}E_{i+1}, \Sigma^{-k}F_{i+1})$ are trivial. Thus by the first part of the proof, $(\Sigma^{-k}e_{n-1})\cdots(\Sigma^{-k}e_1)(\Sigma^{-k}e_0)$: $(\Sigma^{-k}E_0, \Sigma^{-k}F_0) \to (\Sigma^{-k}E_n, \Sigma^{-k}F_n)$ factors through an $(n-1)$-connected pair of CW-spectra (G, H) in $\mathbf{HCW}_{\text{spec}}^2$. So, $e_{n-1}\cdots e_1e_0:(E_0, F_0) \to (E_n, F_n)$ factors through an $(n+k-1)$-connected pair of CW-spectra (Σ^kG, Σ^kH) in $\mathbf{HCW}_{\text{spec}}^2$ as required. □

An inverse system $(\boldsymbol{E}, \boldsymbol{F})$ of CW-spectra is said to be *n-connected* if $\pi_q(\boldsymbol{E}, \boldsymbol{F}) = 0$ for all $q \leqslant n$.

Lemma 4.6. *Let k, $n \in \mathbb{Z}$ with $k \leqslant n$, and suppose that $(\boldsymbol{E}, \boldsymbol{F}) = ((E_a, F_a), e_{aa'}, A)$ be an inverse system in $\mathbf{HCW}_{\text{spec}}$, which is n-connected. Suppose that for every $a \in A$, E_a contains no cells $e \neq *$ of dimension less than k. Then every $a \in A$ admits $a' \geqslant a$ such that the map $e_{aa'}: (E_{a'}, F_{a'}) \to (E_a, F_a)$ factors through an n-connected pair of CW-spectra (G, H).*

Proof. Let $a_0 = a \in A$. Since $(\boldsymbol{E}, \boldsymbol{F})$ is n-connected, there exists $a_1 \geqslant a_0$ such that $(e_{a_0a_1})_{\#}: \pi_k(E_{a_1}, F_{a_1}) \to \pi_k(E_{a_0}, F_{a_0})$ is trivial. Continuing this process, we obtain $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{n-k+1}$ such that $(e_{a_ia_i+1})_{\#}: \pi_{i+k}(E_{a_{i+1}}, F_{a_{i+1}}) \to \pi_{i+k}(E_{a_i}, F_{a_i})$ are trivial for $i = 0, 1, \ldots, n-k$. Thus since $e_{a_0a_{n-k+1}} = e_{a_{n-k}a_{n-k+1}} \cdots e_{a_0a_1}$, by Lemma 4.5, $e_{a_0a_{n-k+1}}: (E_{a_{n-k+1}}, F_{a_{n-k+1}}) \to (E_{a_0}, F_{a_0})$ factors through an n-connected pair of CW-spectra (G, H). □

Lemma 4.7. *Let $n \in \mathbb{Z}$, and let $(f_a): \boldsymbol{E} = (E_a, e_{aa'}, A) \to \boldsymbol{F} = (F_a, f_{aa'}, A)$ be a level morphism of inverse systems in $\mathbf{HCW}_{\text{spec}}$ which represents an n-equivalence $f: \boldsymbol{E} \to \boldsymbol{F}$. Then every $a \in A$ admits an increasing subsequence $A' = (a_m)$ of A with $a_1 = a$ such that the restriction (f_{a_m}) to A' also represents an n-equivalence.*

Proof. This can be proven as in Mardešić and Segal [9, Lemma 4, p. 148]. □

Lemma 4.8. *Let n, $k \in \mathbb{Z}$ with $k \leqslant n$, and let $(g_a): \boldsymbol{E} = (E_a, e_{aa'}, A) \to \boldsymbol{F} = (F_a, f_{aa'}, A)$ be a level morphism of inverse systems in $\mathbf{HCW}_{\text{spec}}$. Suppose that every F_a contains no cells $e \neq *$ of dimension less than k. Then $(g_a): \boldsymbol{E} \to \boldsymbol{F}$ induces an n-equivalence f if and only if every $a \in A$ admits $a' \geqslant a$ such that $e_{aa'}$ and $f_{aa'}$ factor in $\mathbf{HCW}_{\text{spec}}$ through CW-spectra P and Q, and there is an n-equivalence of CW-spectra $g: P \to Q$ which makes the following diagram commute in $\mathbf{HCW}_{\text{spec}}$:*

$$\begin{array}{ccccc}
E_a & & \xleftarrow{\quad e_{aa'} \quad} & & E_{a'} \\
 & \overset{p}{\nwarrow} & P & \overset{p'}{\swarrow} & \\
{\scriptstyle g_a}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_{a'}} \\
 & \underset{q}{\swarrow} & Q & \underset{q'}{\nwarrow} & \\
F_a & & \xleftarrow[\quad f_{aa'} \quad]{} & & F_{a'}
\end{array} \tag{15}$$

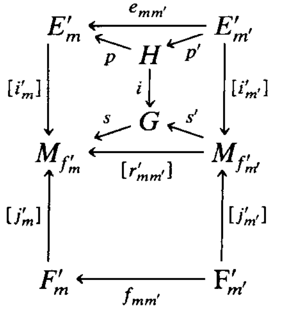
Proof. Sufficiency is proven just as in Mardešić and Segal [9, Theorem 1, p. 145]. For necessity, it suffices to assume $A = \mathbb{N}$ by virtue of Lemma 4.7. By Lemma 4.4 there exists a commutative diagram in $\mathbf{CW}_{\text{spec}}$

$$\begin{array}{ccccccccc}
E'_1 & \xleftarrow{e'_{1,2}} & E'_2 & \xleftarrow{e'_{2,3}} & \cdots & \xleftarrow{e'_{m-1,m}} & E'_m & \longleftarrow & \cdots \\
{\scriptstyle i'_1}\downarrow & & \downarrow{\scriptstyle i'_2} & & & & \downarrow{\scriptstyle i'_m} & & \\
M_{f'_1} & \xleftarrow{r'_{1,2}} & M_{f'_2} & \xleftarrow{r'_{2,3}} & \cdots & \xleftarrow{r'_{m-1,m}} & M_{f'_m} & \longleftarrow & \cdots \\
{\scriptstyle j'_1}\uparrow & & \uparrow{\scriptstyle j'_2} & & & & \uparrow{\scriptstyle j'_m} & & \\
F'_1 & \xleftarrow{f'_{1,2}} & F'_2 & \xleftarrow{f'_{2,3}} & \cdots & \xleftarrow{f'_{m-1,m}} & F'_m & \longleftarrow & \cdots
\end{array}$$

where E'_m and F'_m are cofinal subspectra of E_m and F_m, respectively, $e'_{m-1,m}: E'_m \to E'_{m-1}$ and $f'_{m-1,m}: F'_m \to F'_{m-1}$ are functions representing maps representing the homotopy classes $e_{m-1,m}$, $f_{m-1,m}$, respectively, and $r'_{m-1,m}$, i'_m, j'_m are the corresponding functions in Lemma 4.4. Put $\boldsymbol{E}' = (E'_m, e'_{m-1,m}, \mathbb{N})$, $\boldsymbol{Z}' = (M_{f'_m}, r'_{m-1,m}, \mathbb{N})$ and $\boldsymbol{F}' = (F'_m, f'_{m-1,m}, \mathbb{N})$. These are objects of **pro-HCW**$_{\text{spec}}$. Also, $([i'_m])$, $([j'_m])$, and $([f'_m])$ represent morphisms of **pro-HCW**$_{\text{spec}}$, $\boldsymbol{i}': \boldsymbol{E}' \to \boldsymbol{Z}'$, $\boldsymbol{j}': \boldsymbol{Z}' \to \boldsymbol{F}'$, and $\boldsymbol{f}': \boldsymbol{E}' \to \boldsymbol{F}'$, respectively. Then, by Lemma 3.6, $\boldsymbol{i}' = \boldsymbol{j}'\boldsymbol{f}'$, and $\boldsymbol{j}'$ is an isomorphism of **pro-HCW**$_{\text{spec}}$. So, since $\boldsymbol{f}'$ is an n-equivalence, $\boldsymbol{i}'$ is also an n-equivalence. By Lemma 3.7 and the commutativity in diagram (13), there is an exact sequence of pro-groups

$$\cdots \to \pi_q(\boldsymbol{E}') \to \pi_q(\boldsymbol{Z}') \to \pi_q(\boldsymbol{Z}', \boldsymbol{E}') \to \pi_{q-1}(\boldsymbol{E}') \to \cdots.$$

[9, Theorem 9, p. 119] implies $\pi_q(\boldsymbol{Z}', \boldsymbol{E}') = 0$ for $q \leqslant n$. Then, Lemma 4.6 implies that for every m, there is $m' \geqslant m$ such that $r'_{mm'}$ factors through an n-connected pair of CW-spectra (G, H) in $\mathbf{HCW}^2_{\text{spec}}$ since for each $m \in \mathbb{Z}$, M'_{f_m} contains no cells $e \neq *$ of dimension less than k. Let $i: H \to G$ be the homotopy class of the inclusion induced map of CW-spectra. Then we have the following commutative diagram in $\mathbf{HCW}_{\text{spec}}$:



Since (G, H) is n-connected, $i: H \to G$ is an n-equivalence by Lemma 3.4. Thus we put $(P, Q) = (G, H)$, $q' = s'[j'_{m'}]$, and $q = [j'_m]^{-1}s$. □

Lemma 4.9. *Let $n, k \in \mathbb{Z}$ with $k \leqslant n$, and $(g_a): \boldsymbol{E} = (E_a, e_{aa'}, A) \to \boldsymbol{F} = (F_a, f_{aa'}, A)$ be a level morphism of inverse systems in* $\mathbf{HCW}_{\text{spec}}$, *which is an n-equivalence $\boldsymbol{g}: \boldsymbol{E} \to \boldsymbol{F}$. Suppose that every F_a contains no cells $e \neq *$ of dimension less than k. Then every $a \in A$ admits $a' \geqslant a$ such that the following two statements hold:*

(i) *if R is a CW-spectrum of* dim $R \leqslant n$, *then every morphism $h: R \to F_{a'}$ in* $\mathbf{HCW}_{\text{spec}}$ *admits a morphism $k: R \to E_a$ such that $g_a k = f_{aa'}h$;*

(ii) *if R is a CW-spectrum of* dim $R \leqslant n - 1$ *and $k_1, k_2: R \to E_{a'}$ are morphisms in* $\mathbf{HCW}_{\text{spec}}$ *such that $g_{a'}k_1 = g_{a'}k_2$, then $e_{aa'}k_1 = e_{aa'}k_2$.*

Proof. This immediately follows from Lemmas 4.8 and 4.2. □

Now we can easily prove Theorem 4.1, following Mardešić and Segal [9, Theorem 3, p. 149].

5. Dimensions in stable shape categories

In order to state Whitehead theorems in $\mathbf{Sh}_{\text{spec}}$ and $\mathbf{Sh}_{\text{sw}}$, we need notions of dimension in these categories.

For k, $n \in \mathbb{Z}$ with $k \leqslant n$ and for every space X, we say the *stable shape dimension* $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$ if whenever $\boldsymbol{e}: E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ is a generalized expansion in $\mathbf{HCW}_{\text{spec}}$, then every $a \in A$ admits $a' \geqslant a$ such that $e_{aa'}$ factors in $\mathbf{HCW}_{\text{spec}}$ through a CW-spectrum F such that (i) dim $F \leqslant n$ and (ii) whenever $e \neq *$ is a cell of F, dim $e \geqslant k$. For k, $n \in \mathbb{Z}$, we say the *stable shape dimension* $k \leqslant \text{sd}_{\text{spec}} X \leqslant \infty$ (respectively, $-\infty \leqslant \text{sd}_{\text{spec}} X \leqslant n$) if whenever $\boldsymbol{e}: E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ is a generalized expansion in $\mathbf{HCW}_{\text{spec}}$, then every $a \in A$ admits $a' \geqslant a$ such that $e_{aa'}$ factors in $\mathbf{HCW}_{\text{spec}}$ through a CW-spectrum F such that whenever $e \neq *$ is a cell of F, dim $e \geqslant k$ (respectively, dim $F \leqslant n$).

For k, $n \in \mathbb{Z}$ with $k \leqslant n$ and for every compactum X, we say the *SW-shape dimension* $k \leqslant \text{sd}_{\text{sw}} X \leqslant n$ provided whenever $\boldsymbol{r} = (r_a): X \to \boldsymbol{Z} = (Z_a, r_{aa'}, A)$ is an $\mathbf{HCW}_{\text{sw}}$-expansion of X, then every $a \in A$ admits $a' \geqslant a$, $m \in [-n, -k]$ and a CW complex P of dim $P \leqslant m + n$ such that for some $l \geqslant -m$ and H-map $(r_{aa'})_{m+l}: S^{m+l} Z_{a'} \to S^{m+l} Z_a$ representing $r_{aa'}$ factors in $\mathbf{HCW}$ through $S^l P$. For k, $n \in \mathbb{Z}$ and for every compactum X, we say the *SW-shape dimension* $k \leqslant \text{sd}_{\text{sw}} X \leqslant \infty$ (respectively, $-\infty \leqslant \text{sd}_{\text{sw}} X \leqslant n$) provided whenever $\boldsymbol{r} = (r_a): X \to \boldsymbol{Z} = (Z_a, r_{aa'}, A)$ is an $\mathbf{HCW}_{\text{sw}}$-expansion of X, then every $a \in A$ admits $a' \geqslant a$, $m \in (-\infty, -k]$ (respectively, $m \in [-n, \infty)$) and a CW complex P (respectively, a CW complex P of dim $P \leqslant m + n$) such that for some $l \geqslant -m$ an H-map $(r_{aa'})_{m+l}: S^{m+l} Z_{a'} \to S^{m+l} Z_a$ representing $r_{aa'}$ factors in $\mathbf{HCW}$ through $S^l P$.

For $-\infty < k \leqslant n < \infty$, it is obvious that $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$ implies $k \leqslant \text{sd}_{\text{spec}} X \leqslant n + 1$ and $k - 1 \leqslant \text{sd}_{\text{spec}} X \leqslant n$, and that $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$ implies $k \leqslant \text{sd}_{\text{spec}} X \leqslant \infty$ and $-\infty \leqslant \text{sd}_{\text{spec}} X \leqslant n$. Analogous facts also hold for sd_{sw}.

For convenience we assume that for any space (respectively, compactum) X, $-\infty \leqslant \text{sd}_{\text{spec}} X \leqslant \infty$ (respectively, $-\infty \leqslant \text{sd}_{\text{sw}} X \leqslant \infty$) is a true statement.

Proposition 5.1. *For any k, $n \in \mathbb{Z}$ with $k \leqslant n$ and for every space X, the following are equivalent:*

(i) $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$;

(ii) *there exists a generalized expansion in* $\mathbf{HCW}_{\text{spec}}$, $\boldsymbol{e}: E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ *with the property that every $a \in A$ admits $a' \geqslant a$ such that $e_{aa'}$ factors in* $\mathbf{HCW}_{\text{spec}}$ *through a CW-spectrum F of* dim $F \leqslant n$ *with* dim $e \geqslant k$ *for all cells $e \neq *$ of F;*

(iii) *there exists a generalized expansion in* $\mathbf{HCW}_{\text{spec}}$, $\boldsymbol{e}: E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ *such that for each $a \in A$,* dim $E_a \leqslant n$ *and* dim $e \geqslant k$ *for all cells $e \neq *$ of E_a.*

Proof. (i) $\Rightarrow$ (ii) is trivial. We show the implication (ii) $\Rightarrow$ (iii). For each $a \in A$ we take $a' \geqslant a$ as in (ii). Let Ω be the set of pairs (a, a'), $a \in A$, and define an order $\leqslant^*$ on Ω by $(a, a') \leqslant^* (b, b')$ if $(a, a') = (b, b')$ or $a' \leqslant b$ in A. Then it is easy to see that $(\Omega, \leqslant^*)$ forms a directed set. For each $\alpha = (a, a') \in \Omega$, there exist a

CW-spectrum F_α and morphisms $p_\alpha: E_{a'} \to F_\alpha$ and $q_a: F_\alpha \to E_a$ in $\mathbf{HCW}_{\text{spec}}$ such that $e_{aa'} = q_a p_a$ and dim $F_\alpha \leqslant n$ and dim $e \geqslant k$ for all cells $e \neq *$ of F_α. For $\alpha = (a, a') \leqslant^* \beta = (b, b')$ in Ω, we define $f_{\alpha\beta} = p_a e_{a'b} q_b : F_\beta \to F_\alpha$. Then $\boldsymbol{F} = (F_\alpha, f_{\alpha\beta}, \Omega)$ forms an inverse system in $\mathbf{HCW}_{\text{spec}}$. Let $\boldsymbol{e}: E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ be represented by a morphism (e_a, φ) of inverse systems. Then we define a function $\psi: \Omega \to \Lambda$ by $\psi(\alpha) = \varphi(a')$ for each $\alpha = (a, a') \in \Omega$ and a morphism $f_\alpha = p_a e_a : E(X_{\psi(\alpha)}) \to E_a$ for each $\alpha = (a, a')$. Then (f_α, ψ) forms a morphism of inverse systems in $\mathbf{HCW}_{\text{spec}}$ and hence represents a morphism in $\mathbf{pro\text{-}HCW}_{\text{spec}}$, $\boldsymbol{f}: E(X) \to \boldsymbol{F}$. It is a routine to check that $\boldsymbol{f}$ satisfies the conditions (GE1) and (GE2) in Theorem 2.2. Hence $\boldsymbol{f}$ is a desired generalized expansion.

It remains to show the implication (iii) ⇒ (i). Let $\boldsymbol{e}: E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ be a generalized expansion in $\mathbf{HCW}_{\text{spec}}$ as in the condition (iii), and let $\boldsymbol{f}: E(X) \to \boldsymbol{F} = (F_b, f_{bb'}, B)$ be any generalized expansion in $\mathbf{HCW}_{\text{spec}}$. Fix $b \in B$. There is a natural isomorphism $\boldsymbol{g}: \boldsymbol{F} \to \boldsymbol{E}$ and let $\boldsymbol{h}: \boldsymbol{E} \to \boldsymbol{F}$ be its inverse morphism. Also let $\boldsymbol{g}$ and $\boldsymbol{h}$ be represented by (g_α, φ) and (h_b, ψ), respectively. Then there is $b' \geqslant b$, $\varphi(\psi(b))$ such that $f_{bb'} = h_b g_{\psi(b)} p_{\varphi(\psi(b))b'}$. Thus $f_{bb'}: F_{b'} \to F_b$ factors through E_a as desired. □

Proposition 5.2. *For every k, $n \in \mathbb{Z}$ with $k \leqslant n$ and for every compactum X, the following are equivalent*:

(i) $k \leqslant \operatorname{sd}_{\text{sw}} X \leqslant n$;

(ii) *there exists an $\mathbf{HCW}_{\text{sw}}$-expansion $\boldsymbol{r} = (r_a): X \to \boldsymbol{Z} = (Z_a, r_{aa'}, A)$ with the property that every $a \in A$ admits $a' \geqslant a$, $m \in [-n, -k]$ and a CW complex p of* dim $P \leqslant m + n$ *such that for some $l \geqslant -m$, the H-map $(r_{aa'})_{m+l}: S^{m+l} Z_{a'} \to S^{m+l} Z_a$ representing $r_{aa'}$ factors in* **HCW** *through $S^l P$.*

Proof. (i) ⇒ (ii) is trivial, and (ii) ⇒ (i) is proven just as the implication (iii) ⇒ (i) of Proposition 5.1. □

Theorem 5.3. *For every compactum X and for each k, $n \in \mathbb{Z} \cup \{\infty\}$ with $k \leqslant n$, $k \leqslant \operatorname{sd}_{\text{spec}} X \leqslant n$ if and only if $k \leqslant \operatorname{sd}_{\text{sw}} X \leqslant n$.*

Proof. First, we assume k, $n \in \mathbb{Z}$ with $k \leqslant n$. Suppose that $k \leqslant \operatorname{sd}_{\text{sw}} X \leqslant n$, and let $\boldsymbol{p} = (p_a): X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$ be an **HCW**-expansion of X such that the X_a are finite CW complexes. Fix $a \in A$. Choose $a' \geqslant a$, $m \in [-n, -k]$ and a CW complex P of dim $P \leqslant n + m$ such that $(r_{aa'})_{m+l} = h_{m+l} g_{m+l}$ for some $l \geqslant -m$ and H-maps $g_{m+l}: S^{m+l} X_{a'} \to S^l P$ and $h_{m+l}: S^l P \to S^{m+l} X_a$ are H-maps. Since $X_{a'}$ is finite, we can choose P so that P is a finite CW complex. Let F be the CW-spectrum defined by

$$F_i = \begin{cases} S^{i-m} P, & i \geqslant m + l, \\ *, & i < m + l. \end{cases}$$

Then $\dim F \leqslant n$ and $\dim e \geqslant k$ for all cells $e \neq *$ of F, and the map $E(p_{aa'}): E(X_{a'}) \to E(X_a)$ factors in $\mathbf{HCW}_{\text{spec}}$ through the CW-spectrum F. Hence $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$.

Conversely, suppose that $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$. Let $\boldsymbol{p} = (p_a): X \to \boldsymbol{X} = (X_a, p_{aa'}, A)$ be as above, and fix $a \in A$. Then there exists $a' \geqslant a$ such that $E(p_{aa'}) = hg$ where $g: E(X_{a'}) \to F$ and $h: F \to E(X_a)$ are morphisms in $\mathbf{HCW}_{\text{spec}}$, and F is a CW-spectrum of $\dim F \leqslant n$ and $\dim e \geqslant k$ for all cells $e \neq *$ of F. Again, since $X_{a'}$ is a finite CW complex, we can assume F to be a finite CW-spectrum. Let $m = -\inf\{\dim e : e \text{ is a cell of } F\}$ and let P be the CW complex F_m. Then $m \leqslant -k$. Since F_m and $X_{a'}$ are finite CW complexes, there exists $l \geqslant -m$ such that $S^{m+l} p_{aa'} = h_{m+l} g_{m+l}$ where $g_{m+l}: S^{m+l} X_{a'} \to S^l F_m$ and $h_{m+l}: S^l F_m \to S^{m+l} X_a$ are H-maps representing g and h, respectively. Also $\dim F_m \leqslant n + m$ since $\dim F \leqslant n$. Hence $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$. The cases where $k = -\infty$ or $n = \infty$ can be proven similarly. □

Theorem 5.4. *Suppose that X and Y are two spaces (compacta) with $Sh_{\text{spec}}(X) = Sh_{\text{spec}} Y$ ($Sh_{\text{sw}}(X) = Sh_{\text{sw}}(Y)$). Then for k, $n \in \mathbb{Z} \cup \{\infty\}$ with $k \leqslant n$, $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$ ($k \leqslant \text{sd}_{\text{sw}} X \leqslant n$) if and only if $k \leqslant \text{sd}_{\text{spec}} Y \leqslant n$ ($k \leqslant \text{sd}_{\text{sw}} Y \leqslant n$).*

Proof. Let X and Y be spaces, and $Sh_{\text{spec}}(X) \leqslant Sh_{\text{spec}}(Y)$. We wish to show $k \leqslant \text{sd}_{\text{spec}} Y \leqslant n$ implies $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$. We assume k, $n \in \mathbb{Z}$, and the case where $k = -\infty$ or $n = \infty$ is proven similarly. Suppose $k \leqslant \text{sd}_{\text{spec}} Y \leqslant n$. Suppose that $G: X \to Y$ and $G': Y \to X$ be morphisms in $\mathbf{Sh}_{\text{spec}}$ such that $G'G = 1$, and let G and G' be represented by morphisms in **pro-HCW**$_{\text{spec}}$, $\boldsymbol{g}: \boldsymbol{E} = (E_a, e_{ee'}, A) \to \boldsymbol{F} = (F_b, f_{bb'}, B)$ and $\boldsymbol{g}': \boldsymbol{F} \to \boldsymbol{E}$ where $\boldsymbol{e}: E(X) \to \boldsymbol{E}$ is any generalized expansion in $\mathbf{HCW}_{\text{spec}}$ and $\boldsymbol{f}: E(X) \to \boldsymbol{F}$ is a generalized expansion in $\mathbf{HCW}_{\text{spec}}$ such that for each $b \in B$, $\dim F_b \leqslant n$ and $\dim e \geqslant k$ for all cells $e \neq *$ of F_b. Fix $a \in A$. Also let $\boldsymbol{g}$ and $\boldsymbol{g}'$ be represented by (g_b, φ) and (g'_a, ψ), respectively. Then choose $a' \geqslant a$, $\varphi(\psi(a))$. Then $e_{aa'} = g'_a g_{\psi(a)} e_{\varphi(\psi(a))a'}$, so that $e_{aa'}: E_{a'} \to E_a$ factors through a CW-spectrum $F_{\psi(a)}$ of dimension at most n and $\dim e \geqslant k$ for all cells $e \neq *$ of $F_{\psi(a)}$. Thus $k \leqslant \text{sd}_{\text{spec}} X \leqslant n$. This implies that sd_{spec} is invariant in $\mathbf{Sh}_{\text{spec}}$. That sd_{sw} is invariant in $\mathbf{Sh}_{\text{sw}}$ follows from the first part of the proof, Theorem 5.3, and Theorem 2.5. □

Theorem 5.5. *For every space X of* sd $X < \infty$, $0 \leqslant \text{sd}_{\text{spec}} X \leqslant \text{sd } X$.

Proof. Suppose that sd $X \leqslant n$. Then X admits an **HCW**-expansion $\boldsymbol{p} = (p_\lambda): X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ such that $\dim X_\lambda \leqslant n$, which induces a generalized $\mathbf{HCW}_{\text{spec}}$-expansion of X, $\boldsymbol{e}: E(X) \to E(\boldsymbol{X})$. □

Example 5.6. Let X be the 1-dimensional acyclic continuum of Case and Chamberlin [3]. Then sd $X = 1$, but $0 \leqslant \text{sd}_{\text{spec}} X \leqslant 0$ as $Sh_{\text{spec}}(X) = Sh_{\text{spec}}(*)$.

Example 5.7. The referee has pointed out that there exists a compactum X such that

$$\operatorname{sd} X = \infty \quad \text{and} \quad -\infty \leqslant \operatorname{sd}_{\text{spec}} X \leqslant n \quad \text{for some } n \in \mathbb{Z}.$$

The reader should see [11, p. 46] where a movable continuum X with infinite sd such that the suspension of X has trivial shape is given. More specifically, $X = \prod_{i=1}^{\infty} P_i$ where P_i is the complement of an open ball in the Poincaré manifold.

6. Whitehead theorems in stable shape

Now we wish to Čech-extend the definition of π_n on $\mathbf{HCW}_{\text{spec}}$ over $\mathbf{Sh}_{\text{spec}}$. For each space X, the *nth stable pro-homotopy group pro-*$\pi_n^S(X)$ is defined as the inverse system $\pi_n(E(X)) = (\pi_n(E_a), \pi_n(e_{aa'}), A)$, where $\boldsymbol{e} : E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ is a generalized $\mathbf{HCW}^{\text{s}}_{\text{spec}}$-expansion of $E(X)$. This is well defined up to an isomorphism in pro-groups. Then the *nth stable shape group* $\check{\pi}_n^S(X)$ is defined as the limit group lim *pro-*$\pi_n(\boldsymbol{E})$.

For each morphism $G : X \to Y$ in $\mathbf{Sh}_{\text{spec}}$, we define the morphism in pro-groups *pro-*$\pi_n^S(G)$: *pro-*$\pi_n^S(X) \to$ *pro-*$\pi_n^S(Y)$ as *pro-*$\pi_n(\boldsymbol{g}) : \pi_n(\boldsymbol{E}) \to \pi_n(\boldsymbol{F})$, where $\boldsymbol{e} : E(X) \to \boldsymbol{E}$ and $\boldsymbol{f} : E(Y) \to \boldsymbol{F}$ are $\mathbf{HCW}_{\text{spec}}$-expansions of X and Y, respectively, and $\boldsymbol{g} : \boldsymbol{E} \to \boldsymbol{F}$ is a representative of G. This is well defined up to an isomorphism in pro-groups. It is a routine to check *pro-*π_n^S is a functor from $\mathbf{Sh}_{\text{spec}}$ to **pro-Gp** and that $\check{\pi}_n^S$ is a functor from $\mathbf{Sh}_{\text{spec}}$ to **Gp**.

A morphism $G : X \to Y$ in $\mathbf{Sh}_{\text{spec}}$ is said to be an *n-equivalence* if the induced morphism in pro-groups *pro-*$\pi_k^S(G)$: *pro-*$\pi_k^S(X) \to$ *pro-*$\pi_k^S(Y)$ is an isomorphism for $k = 0, \ldots, n-1$ and an epimorphism for $k = n$.

By Theorem 2.5, *pro-*π_n^S and $\check{\pi}_n^S$ can also be considered as functors from $\mathbf{Sh}_{\text{sw}}$ to the category of pro-groups **pro-Gp** and the category of groups **Gp**, respectively.

Now we are ready to state the Whitehead theorems in $\mathbf{Sh}_{\text{spec}}$ and $\mathbf{Sh}_{\text{sw}}$.

Theorem 6.1. *Let $G : X \to Y$ be a morphism in $\mathbf{Sh}_{\text{spec}}$, which is an n-equivalence. Suppose that $-\infty \leqslant \operatorname{sd}_{\text{spec}} X \leqslant n-1$ and $k \leqslant \operatorname{sd}_{\text{spec}} Y \leqslant n$ ($k, n \in \mathbb{Z}$). Then G is an isomorphism in $\mathbf{Sh}_{\text{spec}}$.*

Proof. There is a morphism $\boldsymbol{g} : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathbf{pro\text{-}HCW}_{\text{spec}}$ which represents G, where $\boldsymbol{e}$: $E(X) \to \boldsymbol{E} = (E_a, e_{aa'}, A)$ is a generalized expansion of X in $\mathbf{HCW}_{\text{spec}}$ such that $\dim E_a \leqslant n-1$ for all $a \in A$, and $\boldsymbol{f} : E(Y) \to \boldsymbol{F} = (F_b, f_{bb'}, B)$ is a generalized expansion of Y in $\mathbf{HCW}_{\text{spec}}$ such that for each $b \in B$, $\dim F_b \leqslant n$ and $\dim e \geqslant k$ for all cells $e \neq *$ of F_b. Then the theorem follows from Theorem 4.1. □

Theorem 6.2. *Let $G : X \to Y$ be a morphism in $\mathbf{Sh}_{\text{sw}}$, which is an n-equivalence. Suppose that $-\infty \leqslant \operatorname{sd}_{\text{sw}} X \leqslant n-1$ and $k \leqslant \operatorname{sd}_{\text{sw}} Y \leqslant n$ ($k, n \in \mathbb{Z}$). Then G is an isomorphism in $\mathbf{Sh}_{\text{sw}}$.*

Proof. The theorem follows from Theorems 6.1, 5.3, and 2.5. □

Example 6.3. The finite-dimensionality of Theorems 6.1 and 6.2 cannot be omitted. Recall the example in Mardešić and Segal [9, Example 1, p. 153]. Specifically, Adams [1, Theorem 1.7] constructed a finite polyhedron Y, $r \in \mathbb{N}$, and a map $a: S^r Y \to Y$ such that for each $m \in \mathbb{N}$, the composition

$$a(S^r a)(S^{2r} a) \cdots (S^{(m-1)r} a): S^{mr} Y \to Y$$

is essential. Then consider the inverse sequence of finite CW complexes

$$Y \xleftarrow{a} S^r Y \xleftarrow{S^r a} S^{2r} Y \longleftarrow \cdots.$$

Let A be its inverse limit. Then A is a metric compactum. It is easy to see that $-\infty \leqslant \mathrm{sd}_{\mathrm{spec}} A \leqslant n$ (equivalently, $-\infty \leqslant \mathrm{sd}_{\mathrm{sw}} A \leqslant n$) is false for all $n \in \mathbb{Z}$. We claim that $Sh_{\mathrm{spec}}(A) \neq Sh_{\mathrm{spec}}(*)$ but $pro\text{-}\pi_k^S(A) = 0$ for all $k \in \mathbb{Z}$. Indeed, for each $m \in \mathbb{N}$, whenever $k \in \mathbb{N}$, the morphism in $\mathbf{HCW}_{\mathrm{spec}}$ represented by the map

$$S^{mr} Y \xleftarrow{S^{(m+k-1)r} a} S^{(m+k)r} Y$$

is not trivial. For, for any $l \in \mathbb{N}$, the map

$$S^{mr+l} Y \xleftarrow{S^{(m+k-1)r+l} a} S^{(m+k)r+l} Y$$

is essential since for $N \in \mathbb{N}$ with $(m+k)r + l \leqslant (m+N)r$ the composition

$$a(S^r a)(S^{2r} a) \cdots (S^{(m+N-1)r} a): S^{(m+N)r} Y \to Y$$

is essential. This shows the first assertion. Also, $pro\text{-}\pi_k^S(A) = (\pi_k^S(S^{mr} Y), \pi_k^S(S^{mr} a), \mathbb{N})$ but for each $k \in \mathbb{Z}$, $\pi_k^S(S^{mr} Y) = \mathrm{dirlim}_i \pi_{k+i}(S^{mr+i} Y) = 0$ for m with $mr > k$, so that $pro\text{-}\pi_k^S(A) = 0$ for all $k \in \mathbb{Z}$.

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