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On the torsion of chiral bars in gradient elasticity

D. Ieşan *

Octav Mayer Institute of Mathematics (Romanian Academy), 700508 Iaşi, Romania

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ABSTRACT

This paper contains a study of the problem of torsion of chiral bars with arbitrary cross-sections in the context of the linear theory of gradient elasticity. The solution is expressed in terms of solutions of four auxiliary plane problems characterized by loads which depend only on the constitutive coefficients. It is shown that, in general, the torsion produces extension (or contraction) and bending effects. The results are used to investigate the torsion of a homogeneous circular bar. In contrast with the case of achiral circular cylinders, the torsion and extension cannot be treated independently of each other.

1. Introduction

The behavior of chiral materials is of interest in investigation of carbon nanotubes, auxetic materials, bones as well as composites with inclusions. In this paper we use the theory of gradient elasticity (Toupin, 1962; Mindlin, 1964; Mindlin and Eshel, 1968; Papanicolopulos, 2011) to study the problem of torsion of homogeneous and isotropic chiral cylinders. This work is motivated by the recent interest in using gradient elasticity to model the chiral behavior of elastic materials (see Maranganti and Sharma, 2007; Auffray et al., 2009; Papanicolopulos, 2011; Askes and Aifantis, 2011 and references therein). We note that the gradient elasticity has been recently used to investigate the behavior of carbon nanotubes (Wang and Hu, 2005; Wang and Wang, 2007; Askes and Aifantis, 2009; Aifantis, 2009; Zhang et al., 2010; Yayli, 2011). A material is called isotropic chiral if its symmetry group equals the proper orthogonal group. In gradient elasticity the torsion of a circular cylinder, subjected to displacement conditions on the ends, has been investigated by Papanicolopulos (2011). In the present paper we study the deformation of a cylinder with arbitrary cross-section which is subjected to moments on the ends. The torsion problem is reduced to the study of some generalized plane strain problems. The method is applied to study the torsion of a circular cylinder.

The paper is structured as follows. In Section 2 we present the basic equations of the linear theory of gradient elasticity. Section 3 is devoted to the formulation of the problem of torsion of chiral rods. In Section 4 we define the generalized plane strain problem and introduce some auxiliary plane problems. The solutions of these auxiliary plane problems depend only on the constitutive coefficients and the cross-section of the cylinder. Section 5 presents the solution of the torsion problem. The three-dimensional problem is reduced to the study of some plane problems. In general, the torsion of an elastic cylinder is accompanied by extension (or contraction) and bending. In Section 6 we use the solution given in the preceding section to investigate the torsion of a circular cylinder. It is shown that the torsion of a right cylinder made of an isotropic chiral elastic material is accompanied only by extension.

2. Basic equations

In this section we present the basic equations of isotropic chiral elastic solids in the first strain-gradient theory (Toupin, 1962; Mindlin, 1964; Mindlin and Eshel, 1968; Papanicolopulos, 2011). We consider a body that in undeformed state occupies the region B of euclidean three-dimensional space and is bounded by the surface ∂B. We refer the deformation of the body to a fixed system of rectangular axes Ox_k, $(k = 1, 2, 3)$. Let $\mathbf{n}$ be the outward unit normal of ∂B. Letters in boldface stand for tensors of an order $p \geqslant 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij,\dots,k}$ (p subscripts) for the components of $\mathbf{v}$ in the Cartesian coordinate system. We shall employ the usual summation and differentiation conventions: Latin subscripts (unless otherwise specified) are understood to range over the integers $(1, 2, 3)$, whereas Greek subscripts to the range $(1, 2)$, summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

We assume that B is a bounded region with Lipschitz boundary ∂B, consisting of a finite number of smooth surfaces. Let Γ_p be the

* Corresponding author. Tel.: +40 232 201060.
E-mail address: iesan@uaic.ro

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intersection of two adjoined smooth surfaces and $C = \cup\Gamma_p$. We assume that B is occupied by a homogeneous and isotropic chiral elastic solid. Let $\mathbf{u}$ be the displacement vector field on B. Throughout this paper, the strain measures are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} = u_{k,ij}. \tag{1}$$

The constitutive equations for isotropic chiral elastic solids are (Mindlin and Eshel, 1968; Papanicolopulos, 2011).

$$\begin{aligned}\tau_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + f(\varepsilon_{ikm}\kappa_{jkm} + \varepsilon_{jkm}\kappa_{ikm}),\\ \mu_{ijk} &= \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik}) + \alpha_2(\kappa_{irr}\delta_{jk} + \kappa_{jrr}\delta_{ik})\\ &\quad + 2\alpha_3\kappa_{rrk}\delta_{ij} + 2\alpha_4\kappa_{ijk} + \alpha_5(\kappa_{kji} + \kappa_{kij}) + f(\varepsilon_{iks}e_{js} + \varepsilon_{jks}e_{is}),\end{aligned} \tag{2}$$

where τ_{ij} is the stress tensor, μ_{ijk} is the double stress tensor, δ_{ij} is the Kronecker delta, ε_{ijk} is the alternating symbol and $\lambda, \mu, \alpha_s, \ (s = 1, 2, \ldots, 5)$, and f are constitutive constants. In the case of a centrosymmetric (achiral) material the coefficient f is equal to zero.

The equilibrium equations are

$$\tau_{ji,j} - \mu_{sji,sj} + F_i = 0, \tag{3}$$

where F_i is the body force per unit volume.

Following Toupin (1962) and Mindlin (1964) we introduce the functions P_i, R_i and Q_i by

$$\begin{aligned}P_i &= (\tau_{ki} - \mu_{rki,r})n_k - D_j(n_r\mu_{rji}) + (D_kn_k)n_sn_p\mu_{spi},\\ R_i &= \mu_{rsi}n_rn_s, \quad Q_i = \langle\mu_{pji}n_pn_q\rangle\varepsilon_{jrq}s_r,\end{aligned} \tag{4}$$

where D_i are the components of the surface gradient,

$$D_i = (\delta_{ik} - n_in_k)\frac{\partial}{\partial x_k},$$

s_k are the components of the unit vector tangent to C, and $\langle g\rangle$ denotes the difference of limits of g from both sides of C. We denote by $\overline{B}$ the closure of B.

We say that the vector field u_j is an admissible displacement field on $\overline{B}$ provided $u_j \in C^4(B) \cap C^3(\overline{B})$. An admissible system of stresses on $\overline{B}$ is an ordered array of function (τ_{ij}, μ_{pqr}) with the following properties: (i) $\tau_{ij} \in C^1(\overline{B}), \mu_{ikj} \in C^2(\overline{B})$; (ii) $\tau_{ij} = \tau_{ji}, \mu_{ijk} = \mu_{jik}$. By an admissible state on $\overline{B}$ we mean an ordered array of fields $A = (u_i, e_{ij}, \kappa_{ijk}, \tau_{ij}, \mu_{ijk})$ with the properties: (i) u_i is an admissible displacement field on $\overline{B}$; (ii) $e_{ij} \in C^1(\overline{B}), \kappa_{ijk} \in C^2(\overline{B}), e_{ij} = e_{ji}$, $\kappa_{ijk} = \kappa_{jik}$; (iii) (τ_{ij}, μ_{ijk}) is an admissible system of stresses on $\overline{B}$.

By an external data system on $\overline{B}$ we mean an ordered array $L = (F_i, \widetilde{P}_i, \widetilde{R}_i, \widetilde{Q}_i)$ with the properties: (i) F_i is continuous on $\overline{B}$; (ii) $\widetilde{P}_i$ and $\widetilde{R}_i$ are piecewise regular on ∂B; (iii) $\widetilde{Q}_i$ is piecewise regular on C. We say that $A = (u_i, e_{ij}, \kappa_{ijk}, \tau_{ij}, \mu_{ijk})$ is an elastic state corresponding to the body force F_k if A is an admissible state that satisfies the Eqs. (1)–(3) on B.

The traction problem of elastostatics consists in finding an elastic state that corresponds to the body force F_i and satisfies the boundary conditions

$$P_i = \widetilde{P}_i, \quad R_i = \widetilde{R}_i \quad \text{on} \quad \partial B \setminus C, \quad Q_i = \widetilde{Q}_i \quad \text{on} \quad C, \tag{5}$$

where $\widetilde{P}_i, \widetilde{R}_i$ and $\widetilde{Q}_i$ are prescribed functions.

Papanicolopulos (2011) has shown that in the case of isotropic linear gradient elasticity the chiral behavior is controlled by a single material parameter. In the case of anisotropic materials the potential energy density W is given by

$$2W = C_{ijmn}e_{ij}e_{mn} + 2F_{ijkmn}e_{ij}\kappa_{kmn} + D_{ijkmnp}\kappa_{ijk}\kappa_{mnp}.$$

In the case of isotropic chiral materials the tensor F_{ijkmn} has the form

$$\begin{aligned}F_{ijkmn} &= f_1\varepsilon_{ijk}\delta_{mn} + f_2\varepsilon_{imk}\delta_{jn} + f_3\varepsilon_{ink}\delta_{jm} + f_4\varepsilon_{jik}\delta_{mn} + f_5\varepsilon_{jnk}\delta_{im}\\ &\quad + f_6\varepsilon_{mnk}\delta_{ij},\end{aligned}$$

where $f_k, \ (k = 1, 2, \ldots, 6)$, are arbitrary coefficients. Thus, the potential energy density for isotropic chiral materials is given by

$$\begin{aligned}W &= \frac{1}{2}\lambda e_{rr}e_{jj} + \mu e_{ij}e_{ij} + \alpha_1\kappa_{iik}\kappa_{kjj} + \alpha_2\kappa_{ijj}\kappa_{irr} + \alpha_3\kappa_{iir}\kappa_{jjr}\\ &\quad + \alpha_4\kappa_{ijk}\kappa_{ijk} + \alpha_5\kappa_{ijk}\kappa_{kji} + 2f\varepsilon_{ikm}e_{ij}\kappa_{kjm}.\end{aligned} \tag{6}$$

where $f = (f_3 + f_5)/2$.

In what follows we assume that the elastic potential is a positive definite quadratic form in the variables e_{ij} and κ_{ijk}. The restrictions imposed by this assumption on the constitutive coefficients have been presented by Mindlin and Eshel (1968) and Papanicolopulos (2011).

The necessary and sufficient conditions for the existence of a solution of the traction problem are (Hlavacek and Hlavacek, 1969)

$$\begin{aligned}&\int_B F_i dv + \int_{\partial B}\widetilde{P}_i da + \int_C \widetilde{Q}_i ds = 0,\\ &\int_B \varepsilon_{ijk}x_jF_k dv + \int_{\partial B}\varepsilon_{ijk}\left(x_j\widetilde{P}_k + n_j\widetilde{R}_k\right)da + \int_C \varepsilon_{ijk}x_j\widetilde{Q}_k ds = 0.\end{aligned} \tag{7}$$

We note that the mixed problem of elastostatics has been investigated by Hlavacek and Hlavacek (1969).

3. Statement of the problem

We assume that the region B from here on refers to the interior of a right cylinder of length h with the cross-section Σ and the lateral boundary Π. Let Γ be the boundary of Σ. The Cartesian coordinate frame is supposed to be chosen in such a way that x_3-axis is parallel to the generators of B and the x_1Ox_2 plane contains one of terminal cross-sections. We denote by Σ_1 and Σ_2, respectively, the cross-section located at $x_3 = 0$ and $x_3 = h$. We denote by Γ_α the boundary of the cross-section Σ_α (Fig. 1). In view of the foregoing agreements, we have

$$\begin{aligned}&B = \{x : (x_1, x_2) \in \Sigma, 0 < x_3 < h\}, \quad \Pi = \{x : (x_1, x_2) \in \Gamma, 0 < x_3 < h\},\\ &\Sigma_1 = \{x : (x_1, x_2) \in \Sigma, x_3 = 0\}, \quad \Sigma_2 = \{x : (x_1, x_2) \in \Sigma, x_3 = h\},\\ &\Gamma_1 = \{x : (x_1, x_2) \in \Gamma, x_3 = 0\}, \quad \Gamma_2 = \{x : (x_1, x_2) \in \Gamma, x_3 = h\}.\end{aligned}$$

We assume that the lateral surface Π is smooth, so that Q_i is equal to zero on Π. The cylinder is supposed to be free from lateral loading. The conditions on the lateral boundary are

$$P_i = 0, \quad R_i = 0 \quad \text{on } \Pi. \tag{8}$$

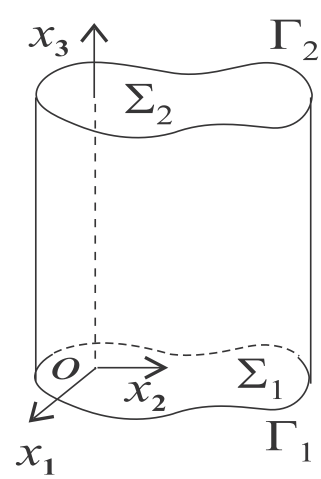

Fig. 1. A prismatic bar.

We assume that the body forces are absent and that the load on the cylinder is distributed over its ends, Σ_1 and Σ_2, in a way which fulfills the equilibrium conditions of a rigid body. Let the loading applied on Σ_1 be statically equivalent to the force $\mathcal{F} = (0, 0, \mathcal{F}_3)$ and the moment $\mathcal{M} = (M_1, M_2, M_3)$. We shall prove that the torsion of cylinder is accompanied by bending and extension. We have introduced the loads $\mathcal{F}_3$ and M_α to put in evidence these effects. For the end located at $x_3 = 0$ we have the following conditions

$$\int_{\Sigma_1} P_\alpha da + \int_{\Gamma_1} Q_\alpha ds = 0, \tag{9}$$

$$\int_{\Sigma_1} P_3 da + \int_{\Gamma_1} Q_3 ds = \mathcal{F}_3, \tag{10}$$

$$\int_{\Sigma_1} (x_\alpha P_3 + R_\alpha) da + \int_{\Gamma_1} x_\alpha Q_3 ds = \varepsilon_{\beta\alpha 3} M_\beta, \tag{11}$$

$$\int_{\Sigma_1} \varepsilon_{\alpha\beta 3} x_\alpha P_\beta da + \int_{\Gamma_1} \varepsilon_{\alpha\beta 3} x_\alpha Q_\beta ds = M_3. \tag{12}$$

On the end located at $x_3 = h$ we have the conditions

$$\int_{\Sigma_2} P_\alpha da + \int_{\Gamma_2} Q_\alpha ds = 0, \tag{13}$$

$$\int_{\Sigma_2} P_3 da + \int_{\Gamma_2} Q_3 ds = -\mathcal{F}_3, \tag{14}$$

$$\int_{\Sigma_2} (x_\alpha P_3 - R_\alpha) da + \int_{\Gamma_2} x_\alpha Q_3 ds = \varepsilon_{\alpha\beta 3} M_\beta, \tag{15}$$

$$\int_{\Sigma_2} \varepsilon_{\alpha\beta 3} x_\alpha P_\beta da + \int_{\Gamma_2} \varepsilon_{\alpha\beta 3} x_\alpha Q_\beta ds = -M_3. \tag{16}$$

Let us note that from (4) we obtain

$$\begin{aligned} &P_i = -\tau_{3i} + 2\mu_{\alpha 3i,\alpha} + \mu_{33i,3}, \quad R_i = \mu_{33i} \quad \text{on} \quad \Sigma_1, \\ &P_i = \tau_{3i} - 2\mu_{\alpha 3i,\alpha} - \mu_{33i,3}, \quad R_i = \mu_{33i} \quad \text{on} \quad \Sigma_2, \\ &Q_i = -2\mu_{\alpha 3i} n_\alpha \quad \text{on} \quad \Gamma_1, \quad Q_i = 2\mu_{\alpha 3i} n_\alpha \quad \text{on} \quad \Gamma_2, \end{aligned} \tag{17}$$

where $(n_1, n_2, 0)$ are the direction cosines of the exterior normal to Π.

The Eqs. (3) reduce to

$$\tau_{ji,j} - \mu_{rji,rj} = 0. \tag{18}$$

The problem consists in finding the functions u_i of class $C^4(B) \cap C^3(\overline{B})$ which satisfy the Eqs. (1), (2) and (18) on B, the conditions (8) on the lateral surface, and the conditions (9)–(16) on the ends, when the constants $\mathcal{F}_3$ and M_j, and the constitutive coefficients are prescribed. If $\mathcal{F}_3 = 0$ and $M_\alpha = 0$, then the problem reduces to the torsion problem.

4. Auxiliary plane problems

Let us assume that the cylinder B is subjected to the external data system $(F_i, \widetilde{P}_i, \widetilde{R}_i, \widetilde{Q}_i)$ with the properties: (i) $F_i, \widetilde{P}_i$ and $\widetilde{R}_i$ are independent of the axial coordinate; (ii) $\widetilde{Q}_i = 0$ on Π. We note that the lateral surface is smooth so that we have $Q_i = 0$.

Let $A = (u_i, e_{ij}, \kappa_{ijk}, \tau_{ij}, \mu_{ijk})$ be an elastic state on the cylinder B. Then A is a state of generalized plane strain provided

$$u_i = u_i(x_1, x_2), \quad (x_1, x_2) \in \Sigma. \tag{19}$$

The restrictions (19), in conjunction with the Eqs. (1) and (2) imply that $e_{ij}, \kappa_{ijk}, \tau_{ij}$ and μ_{ijk} are all independent of the axial coordinate. The strain measures (1) reduce to

$$2e_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha}, \quad 2e_{\alpha 3} = u_{3,\alpha}, \quad \kappa_{\alpha\beta j} = u_{j,\alpha\beta} \tag{20}$$

and

$$e_{33} = 0, \quad \kappa_{k3i} = 0.$$

The constitutive equations become

$$\begin{aligned} \tau_{\alpha\beta} &= \lambda e_{\rho\rho}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta} + f(\varepsilon_{\alpha\rho 3}\kappa_{\beta\rho 3} + \varepsilon_{\beta\rho 3}\kappa_{\alpha\rho 3}), \\ \tau_{\alpha 3} &= 2\mu e_{\alpha 3} + f\varepsilon_{\rho\beta 3}\kappa_{\alpha\rho\beta}, \\ \mu_{\alpha\beta\gamma} &= \frac{1}{2}\alpha_1(\kappa_{\rho\rho\alpha}\delta_{\beta\gamma} + 2\kappa_{\gamma\rho\rho}\delta_{\alpha\beta} + \kappa_{\rho\rho\beta}\delta_{\alpha\gamma}) \\ &\quad + \alpha_2(\kappa_{\alpha\rho\rho}\delta_{\beta\gamma} + \kappa_{\beta\rho\rho}\delta_{\alpha\gamma}) + 2\alpha_3\kappa_{\rho\rho\gamma}\delta_{\alpha\beta} + 2\alpha_4\kappa_{\alpha\beta\gamma} \\ &\quad + \alpha_5(\kappa_{\gamma\beta\alpha} + \kappa_{\gamma\alpha\beta}) + f(\varepsilon_{\alpha\gamma 3}e_{\beta 3} + \varepsilon_{\beta\gamma 3}e_{\alpha 3}), \\ \mu_{\alpha\beta 3} &= 2\alpha_3\kappa_{\rho\rho 3}\delta_{\alpha\beta} + 2\alpha_4\kappa_{\alpha\beta 3} + f(\varepsilon_{\rho\alpha 3}e_{\beta\rho} + \varepsilon_{\rho\beta 3}e_{\alpha\rho}), \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\tau_{33} = \lambda e_{\rho\rho}, \ \mu_{3\alpha\beta} = \frac{1}{2}\alpha_1\kappa_{\rho\rho 3}\delta_{\alpha\beta} + \alpha_5\kappa_{\beta\alpha 3} + f\varepsilon_{\beta\rho 3}e_{\alpha\rho}, \\ &\mu_{3\alpha 3} = \frac{1}{2}\alpha_1\kappa_{\rho\rho\alpha} + \alpha_2\kappa_{\alpha\rho\rho} + f\varepsilon_{\rho\alpha 3}e_{3\rho}, \\ &\mu_{33\alpha} = \alpha_1\kappa_{\alpha\rho\rho} + 2\alpha_3\kappa_{\rho\rho\alpha} + 2f\varepsilon_{\alpha\rho 3}e_{3\rho}, \\ &\mu_{333} = (\alpha_1 + 2\alpha_3)\kappa_{\rho\rho 3}. \end{aligned}$$

The equations of equilibrium reduce to

$$\tau_{\beta j,\beta} - \mu_{\rho\nu j,\rho\nu} + F_j = 0 \quad \text{on} \quad \Sigma. \tag{22}$$

It follows from (4) that

$$\begin{aligned} &P_i = (\tau_{\beta i} - \mu_{\rho\beta i,\rho})n_\beta - D_\rho(n_\beta\mu_{\beta\rho i}) + (D_\rho n_\rho)n_\beta n_\nu\mu_{\beta\nu i}, \\ &R_i = \mu_{\rho\nu i}n_\rho n_\nu. \end{aligned} \tag{23}$$

The conditions on the lateral surface reduce to

$$P_i = \widetilde{P}_i, \quad R_i = \widetilde{R}_i \quad \text{on} \quad \Gamma. \tag{24}$$

The generalized plane strain problem consists in finding an elastic state on $\overline{B}$ which satisfies the geometrical Eqs. (20), the constitutive Eqs. (21) and the equilibrium Eqs. (22) on Σ, and the boundary conditions (24) on Γ. We assume that $F_i, \widetilde{P}_i$ and $\widetilde{R}_i$ are functions of class C^∞, and that Σ is C^∞-smooth. The functions $\tau_{33}, \mu_{3\beta i}$ and μ_{33i} can be determined after the displacement field is found.

In view of (20) and (21) the equations of equilibrium (22) can be expressed in terms of the functions u_k in the form

$$\begin{aligned} &\mu\Delta u_\alpha + (\lambda + \mu)u_{\beta,\beta\alpha} - 2(\alpha_3 + \alpha_4)\Delta\Delta u_\alpha - 2(\alpha_1 + \alpha_2 + \alpha_5)\Delta u_{\beta,\beta\alpha} \\ &+ 2f\varepsilon_{\alpha\beta 3}\Delta u_{3,\beta} + F_\alpha = 0, \\ &\mu\Delta u_3 - 2(\alpha_3 + \alpha_4)\Delta\Delta u_3 + 2f\varepsilon_{\rho\nu 3}\Delta u_{\nu,\rho} + F_3 = 0, \quad \text{on} \quad \Sigma. \end{aligned} \tag{25}$$

In the case of achiral materials we have $f = 0$, and the Eqs. (25) reduce to two uncoupled systems: one for the functions u_α and the other for the function u_3. The next theorem can be established using the results presented by Hlavacek and Hlavacek (1969).

Theorem 1. *The generalized plane strain problem has a solution if and only if*

$$\begin{aligned} &\int_\Sigma F_k da + \int_\Gamma \widetilde{P}_k ds = 0, \\ &\int_\Sigma \varepsilon_{3\alpha\beta}x_\alpha F_\beta da + \int_\Gamma \varepsilon_{3\alpha\beta}(x_\alpha\widetilde{P}_\beta + n_\alpha\widetilde{R}_\beta)ds = 0. \end{aligned} \tag{26}$$

Following Fichera (1972), if we consider a "C^∞-theory", then (26) are necessary and sufficient conditions for the existence of a C^∞ solution of the generalized plane strain problem.

In what follows we will use four special problems of generalized plane strain, denoted by $A^{(k)}$, $(k = 1, 2, 3, 4)$. In the problem $A^{(1)}$ the external data system is $(F_i^{(1)}, \widetilde{P}_i^{(1)}, \widetilde{R}_i^{(1)})$ where

$$\begin{aligned} &F_i^{(1)} = \lambda\delta_{1i}, \quad \widetilde{P}_1^{(1)} = -\lambda x_1 n_1 + (\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1 n_2)_{,\nu}n_\alpha, \\ &\widetilde{P}_2^{(1)} = -\lambda x_1 n_2 + \frac{1}{2}(\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1^2 - n_2^2)_{,\alpha}n_\nu, \\ &\widetilde{P}_3^{(1)} = 2fn_2, \quad \widetilde{R}_1^{(1)} = 2\alpha_3 - \alpha_1 + (\alpha_1 - 2\alpha_2)n_1^2, \\ &\widetilde{R}_2^{(1)} = (\alpha_1 - 2\alpha_2)n_1 n_2, \quad \widetilde{R}_3^{(1)} = 0. \end{aligned} \tag{27}$$

The problem $A^{(2)}$ is characterized by the following loading

$$\begin{aligned} &F_i^{(2)} = \lambda\delta_{2i}, \quad \widetilde{P}_1^{(2)} = -\lambda x_2 n_1 + \frac{1}{2}(\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1^2 - n_2^2)_{,\alpha} n_\nu, \\ &\widetilde{P}_2^{(2)} = -\lambda x_2 n_2 + (\alpha_1 - 2\alpha_2)\varepsilon_{3\alpha\nu}(n_1 n_2)_{,\nu} n_\alpha, \quad \widetilde{P}_3^{(2)} = -2fn_1, \\ &\widetilde{R}_1^{(2)} = (\alpha_1 - 2\alpha_2)n_1 n_2, \quad \widetilde{R}_2^{(2)} = 2\alpha_3 - \alpha_1 + (\alpha_1 - 2\alpha_2)n_2^2, \quad \widetilde{R}_3^{(2)} = 0. \end{aligned} \tag{28}$$

In the problem $A^{(3)}$ the body force and the boundary data are given by

$$F_i^{(3)} = 0, \quad \widetilde{P}_\alpha^{(3)} = -\lambda n_\alpha, \quad \widetilde{P}_3^{(3)} = 0, \quad \widetilde{R}_i^{(3)} = 0. \tag{29}$$

The problem $A^{(4)}$ corresponds to the following external data

$$\begin{aligned} &F_i^{(4)} = 0, \quad \widetilde{P}_1^{(4)} = \frac{1}{2}f[5n_1 + D_1(x_2 n_2) + D_2(x_2 n_1 - 2x_1 n_2) \\ &- 2(x_2 n_1 n_2 - x_1 n_2^2)(D_\rho n_\rho)], \\ &\widetilde{P}_2^{(4)} = \frac{1}{2}f[5n_2 + D_1(x_1 n_2 - 2x_2 n_1) + D_2(x_1 n_1) \\ &- 2(x_1 n_1 n_2 - x_2 n_1^2)(D_\rho n_\rho)], \quad \widetilde{P}_3^{(4)} = \mu\varepsilon_{3\beta\rho} x_\rho n_\beta, \\ &\widetilde{R}_1^{(4)} = f(x_1 n_2^2 - x_2 n_1 n_2), \quad \widetilde{R}_2^{(4)} = f(x_2 n_1^2 - x_1 n_1 n_2), \quad \widetilde{R}_3^{(4)} = 0. \end{aligned} \tag{30}$$

It is easy to show that the necessary and sufficient conditions (26) for the existence of the solution are satisfied for each boundary value problem $A^{(k)}$, $(k = 1, 2, 3, 4)$.

Let us denote by $u_i^{(k)}, e_{ij}^{(k)}, \kappa_{pqr}^{(k)}, \tau_{ij}^{(k)}$ and $\mu_{pqr}^{(k)}$ the displacement, the strain measures, the stress tensor and the double stress tensor in the problem $A^{(k)}$, $(k = 1, 2, 3, 4)$, respectively. We introduce the notations

$$\begin{aligned} &P_i^{(k)} = (\tau_{\beta i}^{(k)} - \mu_{\rho\beta i,\rho}^{(k)})n_\beta - D_\rho(n_\beta \mu_{\beta\rho i}^{(k)}) + (D_\rho n_\rho)\mu_{\beta\nu i}^{(k)} n_\beta n_\nu, \\ &R_i^{(k)} = \mu_{\rho\nu i}^{(k)} n_\rho n_\nu. \end{aligned} \tag{31}$$

The functions $u_i^{(k)}, e_{ij}^{(k)}, \kappa_{pqr}^{(k)}, \tau_{ij}^{(k)}$ and $\mu_{pqr}^{(k)}$ satisfy the geometrical equations

$$2e_{\alpha\beta}^{(k)} = u_{\alpha,\beta}^{(k)} + u_{\beta,\alpha}^{(k)}, \quad 2e_{\alpha 3}^{(k)} = u_{3,\alpha}^{(k)}, \quad \kappa_{\alpha\beta j}^{(k)} = u_{j,\alpha\beta}^{(k)}, \tag{32}$$

the constitutive equations

$$\begin{aligned} &\tau_{\alpha\beta}^{(k)} = \lambda e_{\rho\rho}^{(k)}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}^{(k)} + f(\varepsilon_{\alpha\rho 3}\kappa_{\beta\rho 3}^{(k)} + \varepsilon_{\beta\rho 3}\kappa_{\alpha\rho 3}^{(k)}), \\ &\tau_{\alpha 3}^{(k)} = 2\mu e_{\alpha 3}^{(k)} + f\varepsilon_{\rho\beta 3}\kappa_{\alpha\rho\beta}^{(k)}, \\ &\mu_{\alpha\beta\gamma}^{(k)} = \frac{1}{2}\alpha_1(\kappa_{\rho\rho\alpha}^{(k)}\delta_{\beta\gamma} + 2\kappa_{\gamma\rho\rho}^{(k)}\delta_{\alpha\beta} + \kappa_{\rho\rho\beta}^{(k)}\delta_{\alpha\gamma}) + \alpha_2(\kappa_{\alpha\rho\rho}^{(k)}\delta_{\beta\gamma} + \kappa_{\beta\rho\rho}^{(k)}\delta_{\alpha\gamma}) \\ &\qquad + 2\alpha_3\kappa_{\rho\rho\gamma}^{(k)}\delta_{\alpha\beta} + 2\alpha_4\kappa_{\alpha\beta\gamma}^{(k)} + \alpha_5(\kappa_{\gamma\beta\alpha}^{(k)} + \kappa_{\gamma\alpha\beta}^{(k)}) + f(\varepsilon_{\alpha\gamma 3}e_{\beta 3}^{(k)} + \varepsilon_{\beta\gamma 3}e_{\alpha 3}^{(k)}), \\ &\mu_{\alpha\beta 3}^{(k)} = 2\alpha_3\kappa_{\rho\rho 3}^{(k)}\delta_{\alpha\beta} + 2\alpha_4\kappa_{\alpha\beta 3}^{(k)} + f(\varepsilon_{\rho\alpha 3}e_{\beta\rho}^{(k)} + \varepsilon_{\rho\beta 3}e_{\alpha\rho}^{(k)}), \end{aligned} \tag{33}$$

and the equilibrium equations

$$\tau_{\beta j,\beta}^{(k)} - \mu_{\rho\nu j,\rho\nu}^{(k)} + F_j^{(k)} = 0, \tag{34}$$

on Σ, and the boundary conditions

$$P_i^{(k)} = \widetilde{P}_i^{(k)}, \quad R_i^{(k)} = \widetilde{R}_i^{(k)} \quad \text{on} \quad \Gamma, \tag{35}$$

where $F_i^{(k)}, \widetilde{P}_i^{(k)}$ and $\widetilde{R}_i^{(k)}$, $(k = 1, 2, 3, 4)$ are defined by (27)–(30).

We note that the solutions of the problems $A^{(k)}$ depend only on the constitutive coefficients and the domain Σ.

5. Solution of the problem

It is known (Ieşan, 1986; Ieşan, 2009) that the solution of the problem of extension, bending and torsion can be found in the class of displacement vector fields $\mathbf{u}$ with the property that $\mathbf{u}_{,3}$ is a rigid displacement. This result has been established in the classical theory but it also holds in the gradient elasticity. We seek the solution of the problem formulated in Section 3 in the form

$$\begin{aligned} &u_\alpha = -\frac{1}{2}c_\alpha x_3^2 + \varepsilon_{3\beta\alpha}c_4 x_\beta x_3 + \sum_{k=1}^4 c_k u_\alpha^{(k)}, \\ &u_3 = (c_1 x_1 + c_2 x_2 + c_3)x_3 + \sum_{k=1}^4 c_k u_3^{(k)}, \end{aligned} \tag{36}$$

where $u_j^{(k)}$ are the displacements in the problem $A^{(k)}$, and c_k, $(k = 1, 2, 3, 4)$, are unknown constants. It is easy to see that the displacement vector $\mathbf{u}$ given by (36) has the property that $\mathbf{u}_{,3}$ is a rigid displacement. In view of (1)and (36) we find that

$$\begin{aligned} &e_{\alpha\beta} = \sum_{k=1}^4 c_k e_{\alpha\beta}^{(k)}, \quad e_{\alpha 3} = \frac{1}{2}\varepsilon_{3\beta\alpha}x_\beta c_4 + \sum_{k=1}^4 c_k e_{\alpha 3}^{(k)}, \\ &e_{33} = c_1 x_1 + c_2 x_2 + c_3, \quad \kappa_{\alpha\beta\gamma} = \sum_{k=1}^4 c_k \kappa_{\alpha\beta\gamma}^{(k)}, \\ &\kappa_{\alpha\beta 3} = \sum_{k=1}^4 c_k \kappa_{\alpha\beta 3}^{(k)}, \quad \kappa_{\beta 3\alpha} = \varepsilon_{3\beta\alpha}c_4, \\ &\kappa_{\alpha 33} = -\kappa_{33\alpha} = c_\alpha, \quad \kappa_{333} = 0, \end{aligned} \tag{37}$$

where $e_{ij}^{(k)}$ and $\kappa_{\alpha\beta j}^{(k)}$ are defined in (32). It follows from (2) and (37) that the stress tensor and the double stress tensor are given by

$$\begin{aligned} &\tau_{\alpha\beta} = [\lambda(c_1 x_1 + c_2 x_2 + c_3) - 2fc_4]\delta_{\alpha\beta} + \sum_{k=1}^4 c_k \tau_{\alpha\beta}^{(k)}, \\ &\tau_{\alpha 3} = \mu\varepsilon_{3\beta\alpha}c_4 x_\beta + 2f\varepsilon_{\alpha\rho 3}c_\rho + \sum_{k=1}^4 c_k \tau_{\alpha 3}^{(k)}, \\ &\tau_{33} = (\lambda + 2\mu)(c_1 x_1 + c_2 x_2 + c_3) + 4fc_4 + \lambda\sum_{k=1}^4 c_k e_{\rho\rho}^{(k)}, \\ &\mu_{111} = 2(\alpha_2 - \alpha_3)c_1 + \sum_{k=1}^4 c_k \mu_{111}^{(k)}, \quad \mu_{222} = 2(\alpha_2 - \alpha_3)c_2 + \sum_{k=1}^4 c_k \mu_{222}^{(k)}, \\ &\mu_{221} = (\alpha_1 - 2\alpha_3)c_1 - fc_4 x_1 + \sum_{k=1}^4 c_k \mu_{221}^{(k)}, \\ &\mu_{112} = (\alpha_1 - 2\alpha_3)c_2 - fc_4 x_2 + \sum_{k=1}^4 c_k \mu_{112}^{(k)}, \\ &\mu_{121} = \frac{1}{2}(2\alpha_2 - \alpha_1)c_2 + \frac{1}{2}fc_4 x_2 + \sum_{k=1}^4 c_k \mu_{121}^{(k)}, \\ &\mu_{122} = \frac{1}{2}(2\alpha_2 - \alpha_1)c_1 + \frac{1}{2}fc_4 x_1 + \sum_{k=1}^4 c_k \mu_{122}^{(k)}, \\ &\mu_{\alpha 33} = \frac{1}{2}(2\alpha_2 - \alpha_1 + 4\alpha_4)c_\alpha - \frac{1}{2}fc_4 x_\alpha \\ &\qquad + \sum_{k=1}^4 c_k\left(\alpha_2\kappa_{\alpha\rho\rho}^{(k)} + \frac{1}{2}\alpha_1\kappa_{\rho\rho\alpha}^{(k)} + \frac{1}{2}f\varepsilon_{3\rho\alpha}u_{3,\rho}^{(k)}\right), \\ &\mu_{33\alpha} = (\alpha_1 - 2\alpha_3 - 2\alpha_4 + 2\alpha_5)c_\alpha + fc_4 x_\alpha \\ &\qquad + \sum_{k=1}^3 c_k\left(\alpha_1\kappa_{\alpha\rho\rho}^{(k)} + 2\alpha_3\kappa_{\rho\rho\alpha}^{(k)} + f\varepsilon_{3\alpha\rho}u_{3,\rho}^{(k)}\right), \\ &\mu_{\alpha 3\beta} = \varepsilon_{3\alpha\beta}f(c_1 x_1 + c_2 x_2 + c_3) + \varepsilon_{3\alpha\beta}(2\alpha_4 - \alpha_5)c_4 \\ &\qquad + \sum_{k=1}^4 c_k\left(\frac{1}{2}\alpha_1\kappa_{\rho\rho 3}^{(k)}\delta_{\alpha\beta} + \alpha_5\kappa_{\beta\alpha 3}^{(k)} + f\varepsilon_{3\beta\rho}e_{\alpha\rho}^{(k)}\right), \\ &\mu_{\alpha\beta 3} = \sum_{k=1}^4 c_k \mu_{\alpha\beta 3}^{(k)}, \quad \mu_{333} = (\alpha_1 + 2\alpha_3)\sum_{k=1}^4 c_k \kappa_{\rho\rho 3}^{(k)}, \end{aligned} \tag{38}$$

where $\tau_{\alpha i}^{(k)}$ and $\mu_{\alpha\beta j}^{(k)}$ are given by (33).

The equations of equilibrium (18) and the boundary conditions (8) are satisfied on the basis of the Eqs. (34) and (35), where the functions $F_i^{(k)}, \widetilde{P}_i^{(k)}$ and $\widetilde{R}_i^{(k)}$ are given by (27)–(30).

The conditions (9) on the end Σ_1 are identically satisfied. Indeed, with the help of the divergence theorem and the Eqs. (4), (17), (18) and (8) we obtain

$$\begin{aligned}\int_{\Sigma_1} P_1 da + \int_{\Gamma_1} Q_1 ds &= -\int_{\Sigma} \tau_{31} da = -\int_{\Sigma} [\tau_{31} + x_1(\tau_{\alpha 3,\alpha} - \mu_{\alpha\beta 3,\alpha\beta})] da \\ &= -\int_{\Gamma} x_1(\tau_{3\alpha} - \mu_{\alpha\beta 3,\beta}) n_\alpha ds - \int_{\Sigma} \mu_{1\beta 3,\beta} da \\ &= -\int_{\Gamma} x_1(\mu_{\rho\nu 3} n_\alpha n_\rho - \mu_{\rho\alpha 3} n_\rho n_\nu)_{,\nu} n_\alpha ds \\ &\quad -\int_{\Sigma} \mu_{1\beta 3,\beta} da \\ &= -\int_{\Sigma} [(\mu_{\rho\nu 3} n_1 n_\rho - \mu_{\rho 13} n_\rho n_\nu)_{,\nu} + \mu_{1\beta 3,\beta}] da \\ &= -\int_{\Gamma} \mu_{\rho\nu 3} n_\rho n_\nu n_1 ds = -\int_{\Gamma} R_3 n_1 ds = 0.\end{aligned}$$

In a similar way we can prove that the second condition from (9), as well as the conditions (13) are identically satisfied. We note that the stress tensor and the double stress tensor from (38) can be written in the form

$$\tau_{ij} = \sum_{k=1}^{4} c_k t_{ij}^{(k)}, \quad \mu_{pqr} = \sum_{k=1}^{4} c_k \xi_{pqr}^{(k)}, \tag{39}$$

where

$$\begin{aligned}
&t_{\alpha\beta}^{(\rho)} = \lambda x_\rho \delta_{\alpha\beta} + \tau_{\alpha\beta}^{(\rho)}, \quad t_{\alpha\beta}^{(3)} = \lambda\delta_{\alpha\beta} + \tau_{\alpha\beta}^{(3)},\\
&t_{\alpha\beta}^{(4)} = -2f\delta_{\alpha\beta} + \tau_{\alpha\beta}^{(4)}, \quad t_{\alpha 3}^{(\rho)} = 2f\varepsilon_{\alpha\rho 3} + \tau_{\alpha 3}^{(\rho)},\\
&t_{\alpha 3}^{(3)} = \tau_{\alpha 3}^{(3)}, \quad t_{\alpha 3}^{(4)} = \mu\varepsilon_{3\beta\alpha} x_\beta + \tau_{\alpha 3}^{(4)},\\
&t_{33}^{(\nu)} = (\lambda + 2\mu) x_\nu + \lambda e_{\rho\rho}^{(\nu)}, \quad t_{33}^{(3)} = \lambda + 2\mu + \lambda e_{\rho\rho}^{(3)}, \ t_{33}^{(4)} = 4f + \lambda e_{\rho\rho}^{(4)},\\
&\xi_{111}^{(1)} = 2(\alpha_2 - \alpha_3) + \mu_{111}^{(1)}, \quad \xi_{111}^{(2)} = \mu_{111}^{(2)}, \quad \xi_{111}^{(3)} = \mu_{111}^{(3)},\\
&\xi_{111}^{(4)} = \mu_{111}^{(4)}, \quad \xi_{222}^{(1)} = \mu_{222}^{(1)}, \quad \xi_{222}^{(2)} = 2(\alpha_2 - \alpha_3) + \mu_{222}^{(2)},\\
&\xi_{222}^{(3)} = \mu_{222}^{(3)}, \quad \xi_{222}^{(4)} = \mu_{222}^{(4)}, \quad \xi_{221}^{(1)} = \alpha_1 - 2\alpha_3 + \mu_{221}^{(1)},\\
&\xi_{221}^{(2)} = \mu_{221}^{(2)}, \quad \xi_{221}^{(3)} = \mu_{221}^{(3)}, \quad \xi_{221}^{(4)} = -fx_1 + \mu_{221}^{(4)},\\
&\xi_{112}^{(1)} = \mu_{112}^{(1)}, \quad \xi_{112}^{(2)} = \alpha_1 - 2\alpha_3 + \mu_{112}^{(2)}, \quad \xi_{112}^{(3)} = \mu_{112}^{(3)},\\
&\xi_{112}^{(4)} = -fx_2 + \mu_{112}^{(4)},\\
&\xi_{121}^{(1)} = \mu_{121}^{(1)}, \quad \xi_{121}^{(2)} = \frac{1}{2}(2\alpha_2 - \alpha_1) + \mu_{121}^{(2)}, \quad \xi_{121}^{(3)} = \mu_{121}^{(3)},\\
&\xi_{121}^{(4)} = \frac{1}{2}fx_2 + \mu_{121}^{(4)}, \quad \xi_{122}^{(1)} = \frac{1}{2}(2\alpha_2 - \alpha_1) + \mu_{122}^{(1)},\\
&\xi_{122}^{(2)} = \mu_{122}^{(2)}, \quad \xi_{122}^{(3)} = \mu_{122}^{(3)}, \quad \xi_{122}^{(4)} = \frac{1}{2}fx_1 + \mu_{122}^{(4)},\\
&\xi_{\alpha 33}^{(j)} = \frac{1}{2}(2\alpha_2 - \alpha_1 + 4\alpha_4)\delta_{j\alpha} + \alpha_2\kappa_{\alpha\rho\rho}^{(j)} + \frac{1}{2}\alpha_1\kappa_{\rho\rho\alpha}^{(j)} + \frac{1}{2}f\varepsilon_{3\rho\alpha}u_{3,\rho}^{(j)},\\
&\xi_{\alpha 33}^{(4)} = -\frac{1}{2}fx_\alpha + \alpha_2\kappa_{\alpha\rho\rho}^{(4)} + \frac{1}{2}\alpha_1\kappa_{\rho\rho\alpha}^{(4)} + \frac{1}{2}f\varepsilon_{3\rho\alpha}u_{3,\rho}^{(4)},\\
&\xi_{33\alpha}^{(j)} = (\alpha_1 - 2\alpha_3 - 2\alpha_4 + 2\alpha_5)\delta_{j\alpha} + \alpha_1\kappa_{\alpha\rho\rho}^{(j)} + 2\alpha_3\kappa_{\rho\rho\alpha}^{(j)}\\
&\qquad + f\varepsilon_{3\alpha\rho}u_{3,\rho}^{(j)}, \quad (j = 1,2,3),\\
&\xi_{33\alpha}^{(4)} = fx_\alpha + \alpha_1\kappa_{\alpha\rho\rho}^{(4)} + 2\alpha_3\kappa_{\rho\rho\alpha}^{(4)} + f\varepsilon_{3\alpha\rho}u_{3,\rho}^{(4)},\\
&\xi_{\alpha 3\beta}^{(\rho)} = \varepsilon_{3\alpha\beta}fx_\rho + \frac{1}{2}\alpha_1\kappa_{\nu\nu 3}^{(\rho)}\delta_{\alpha\beta} + \alpha_5\kappa_{\beta\alpha 3}^{(\rho)} + f\varepsilon_{3\beta\nu}e_{\alpha\nu}^{(\rho)},\\
&\xi_{\alpha 3\beta}^{(3)} = \varepsilon_{3\alpha\beta}f + \frac{1}{2}\alpha_1\kappa_{\rho\rho 3}^{(3)}\delta_{\alpha\beta} + \alpha_5\kappa_{\beta\alpha 3}^{(3)} + f\varepsilon_{3\beta\rho}e_{\alpha\rho}^{(3)},\\
&\xi_{\alpha 3\beta}^{(4)} = \varepsilon_{3\alpha\beta}(2\alpha_4 - \alpha_5) + \frac{1}{2}\alpha_1\kappa_{\rho\rho 3}^{(4)}\delta_{\alpha\beta} + \alpha_5\kappa_{\beta\alpha 3}^{(4)} + f\varepsilon_{3\beta\rho}e_{\alpha\rho}^{(4)},\\
&\xi_{\alpha\beta 3}^{(k)} = \mu_{\alpha\beta 3}^{(k)}, \quad \xi_{333}^{(k)} = (\alpha_1 + 2\alpha_3)\kappa_{\rho\rho 3}^{(k)}, \quad (k = 1,2,3,4).
\end{aligned} \tag{40}$$

With the help of (17) and (39), the conditions (10)–(12) and (14)–(16) reduce to

$$\sum_{k=1}^{4} D_{\alpha k} c_k = \varepsilon_{\alpha\beta 3} M_\beta, \quad \sum_{k=1}^{4} D_{3k} c_k = -\mathcal{F}_3, \quad \sum_{k=1}^{4} D_{4k} c_k = -M_3, \tag{41}$$

where the constants D_{rs}, $(r, s = 1,2,3,4)$, are defined by

$$\begin{aligned} D_{\alpha k} &= \int_{\Sigma} [x_\alpha t_{33}^{(k)} + 2\xi_{\alpha 33}^{(k)} - \xi_{33\alpha}^{(k)}] da,\\ D_{3k} &= \int_{\Sigma} t_{33}^{(k)} da, \ D_{4k} = \int_{\Sigma} \varepsilon_{3\alpha\beta}(x_\alpha t_{3\beta}^{(k)} + 2\xi_{\alpha 3\beta}^{(k)}) da. \end{aligned} \tag{42}$$

The constants D_{rs} can be calculated after the solutions of the problems $A^{(k)}$, $(k = 1,2,3,4)$, are found.

Let us prove that the system (41) can always be solved for the constants c_1, c_2, c_3 and c_4. Following Mindlin and Eshel (1968) and Hlavacek and Hlavacek (1969), in the absence of body forces, we have

$$2\int_B W dv = \int_{\partial B} (P_i u_i + R_i u_{i,j} n_j) da + \int_C Q_i u_i ds. \tag{43}$$

We consider two elastic states $S' = (u_i', e_{ij}', \kappa_{ijk}', \tau_{ij}', \mu_{ijk}')$ and $S'' = (u_i'', e_{ij}'', \kappa_{ijk}'', \tau_{ij}'', \mu_{ijk}'')$ corresponding to zero body forces. We denote

$$2E(u', u'') = \int_{\partial B} (P_i' u_i'' + R_i' u_{i,j}'' n_j) da + \int_C Q_i' u_i'' ds, \tag{44}$$

where P_i', R_i' and Q_i' are the functions P_i, R_i and Q_i from (4) associated to the state S'. The reciprocity relation (Hlavacek and Hlavacek, 1969; Beatty and Cheverton, 1976) leads to the following equality

$$E(u', u'') = E(u'', u'). \tag{45}$$

It follows from (36) that the displacement $\mathbf{u}$ can be expressed as

$$u_i = \sum_{k=1}^{4} c_k v_i^{(k)}, \tag{46}$$

where

$$\begin{aligned} &v_\alpha^{(\beta)} = -\frac{1}{2}x_3^2\delta_{\alpha\beta} + u_\alpha^{(\beta)}, \quad v_\alpha^{(3)} = u_\alpha^{(3)},\\ &v_\alpha^{(4)} = \varepsilon_{3\beta\alpha} x_\beta x_3 + u_\alpha^{(4)}, \quad v_3^{(\alpha)} = x_\alpha x_3 + u_3^{(\alpha)},\\ &v_3^{(3)} = x_3 + u_3^{(3)}, \quad v_3^{(4)} = u_3^{(4)}. \end{aligned} \tag{47}$$

It follows from (43) and (46) that the internal energy E can be expressed in the form

$$E = \int_B W dv = \sum_{r,s=1}^{4} B_{rs} c_r c_s,$$

where

$$B_{rs} = E(v^{(r)}, v^{(s)}), \quad (r, s = 1,2,3,4).$$

Since the potential energy is positive definite, we find that $\det(B_{rs}) \neq 0$. Let us apply the relation (44) for the states $S^{(k)} = (v_i^{(k)}, \varepsilon_{ij}^{(k)}, \eta_{pqr}^{(k)}, t_{ij}^{(k)}, \xi_{pqr}^{(k)})$, $(k = 1,2,3,4)$, where $2\varepsilon_{ij}^{(k)} = v_{i,j}^{(k)} + v_{j,i}^{(k)}$ and $\eta_{pqr}^{(k)} = v_{r,pq}^{(k)}$. We obtain

$$2E(v^{(r)}, v^{(s)}) = hD_{rs}.$$

Thus, we conclude that $\det(D_{rs}) \neq 0$. From the reciprocity relation (45) we get

$$D_{rs} = D_{sr}, \quad (r, s = 1,2,3,4). \tag{48}$$

The solution of the problem is given by (36) where the constants c_1, c_2, c_3 and c_4 are determined by the system (41), and the functions $u_i^{(k)}$ are the displacements in the generalized plane strain problems $A^{(k)}$, $(k = 1,2,3,4)$.

It follows from (41) that the torsion of a chiral cylinder is accompanied, in general, by extension and bending.

Remark 1. Let us describe how the problems $A^{(j)}$, $(j = 1, 2, 3, 4)$, were selected. Following (Ieşan, 1986), the solution $\mathbf{u}$ of the problem can be found in the class of functions $\mathbf{u}$ with the property that $\mathbf{u}_{,3}$ is a rigid displacement. This fact implies that

$$u_\alpha = -\frac{1}{2}c_\alpha x_3^2 + c_4\varepsilon_{3\beta\alpha}x_\beta x_3 + g_\alpha, \quad u_3 = (c_1x_1 + c_2x_2 + c_3)x_3 + g_3,$$

where $c_k, (k = 1, 2, 3, 4)$, are arbitrary constants and g_j are arbitrary functions of x_1 and x_2 . In these relations we have neglected a rigid deformation. Let us take $g_i = c_1u_i^{(1)} + c_2u_i^{(2)} + c_3u_i^{(3)} + c_4u_i^{(4)}$, where $u_i^{(j)}$ are unknown functions. If we require that the equilibrium equations and the conditions on the lateral surface be satisfied for any constants c_1, c_2 , c_3 and c_4 , then we find that the functions $u_i^{(j)}$ satisfy the plane strain problem $A^{(j)}, (j = 1, 2, 3, 4)$. The functions $u_i^{(j)}, (j = 1, 2, 3, 4)$, are not really displacements since they do not have dimensions of length. In the classical elasticity this method has been developed by Ieşan (1986), Ieşan (2009). The solution presented in this paper for a cylinder with arbitrary cross-section is new even for achiral gradient elasticity.

6. Application

In this section we use the method presented in Section 5 to study the torsion of a circular cylinder. The torsion of a circular cylinder subjected to displacement boundary conditions on the ends has been investigated by Papanicolopulos (2011). In this case the torsion produces a dilatation in the radial direction. In what follows we study the torsion of circular cylinder subjected to moments on the ends. We assume that the cylinders B is defined by $B = \{x : x_1^2 + x_2^2 < a^2, 0 < x_3 < h\}, (a > 0)$ and suppose that the moments M_1 and M_2 are equal to zero. To obtain the solution of the problem we have to solve the auxiliary plane problems $A^{(k)}$ and to determine the constants c_k, $(k = 1, 2, 3, 4)$. We introduce the notation $r = (x_1^2 + x_2^2)^{1/2}$. Let us note that on the boundary of Σ we have $r = a$ and $n_\alpha = x_\alpha/a$.

It is easy to see that the external data system for the problem $A^{(3)}$ is $(F_i^{(3)} = 0, \widetilde{P}_\alpha^{(3)} = -\lambda n_\alpha, \widetilde{P}_3^{(3)} = 0, \widetilde{R}_i^{(3)} = 0)$. The external data system for the problem $A^{(4)}$ reduces to $(F_j^{(4)} = 0, \widetilde{P}_\alpha^{(4)} = 2fn_\alpha, \widetilde{P}_3^{(4)} = 0, \widetilde{R}_j^{(4)} = 0)$. First, we study the problem $A^{(3)}$. We seek the solution of this problem in the form

$$u_\alpha^{(3)} = x_\alpha U(r), \quad u_3^{(3)} = 0, \tag{49}$$

where U is an unknown function of class C^4. The functions $u_\alpha^{(3)}$ must satisfy the Eqs. (25) in the absence of body forces. We note that

$$\begin{aligned} &u_{\alpha,\rho}^{(3)} = \delta_{\alpha\rho}U + x_\alpha x_\rho r^{-1}U', \\ &u_{\alpha,\rho\nu}^{(3)} = x_\alpha x_\rho x_\nu r^{-2}U'' - r^{-3}x_\alpha x_\rho x_\nu U' + r^{-1}U'(\delta_{\alpha\rho}x_\nu + \delta_{\alpha\nu}x_\rho + \delta_{\rho\nu}x_\alpha), \\ &u_{\beta,\beta\alpha}^{(3)} = \Delta u_\alpha^{(3)} = x_\alpha(U'' + 3r^{-1}U'), \end{aligned} \tag{50}$$

where $U' = dU/dr, U'' = d^2U/dr^2$.

The equilibrium Eqs. (25), with $F_i = 0$, reduce to

$$(1 - \ell_1^2\Delta)\Delta(x_\alpha U) = 0, \tag{51}$$

where

$$\ell_1 = [2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)/(\lambda + 2\mu)]^{1/2}.$$

Let us note that $\Delta(x_\alpha U) = x_\alpha(U'' + 3r^{-1}U')$. It follows from (51) that the function U satisfies the equation

$$\left(\frac{d^2}{dr^2} + 3r^{-1}\frac{d}{dr} - \frac{1}{\ell_1^2}\right)\left(\frac{d^2}{dr^2} + 3r^{-1}\frac{d}{dr}\right)U = 0.$$

The solution of this equation, which is bounded at $r = 0$, is given by

$$U = C_1 + C_2I_1(r/\ell_1), \tag{52}$$

where I_1 is the modified Bessel function of the first kind and order one, and C_1 and C_2 are arbitrary constants.

Let us impose the boundary conditions (35) for $k = 3$. First, from (33), (49) and (50) we obtain

$$\mu_{\rho\nu3}^{(3)} = f[\varepsilon_{\rho3\beta}x_\nu x_\beta + \varepsilon_{\nu3\beta}x_\rho x_\beta]r^{-1}U',$$

so that

$$R_3^{(3)} = \mu_{\rho\nu3}^{(3)}n_\rho n_\nu = 0 \quad \text{on} \quad r = a.$$

As $\widetilde{R}_3^{(3)} = 0$, we conclude that the condition $R_3^{(3)} = \widetilde{R}_3^{(3)}$ on Γ is identically satisfied. In view of (33), (49) and (50) we find that on the boundary Γ we have

$$\begin{aligned} R_\alpha^{(3)} = 2x_\alpha[&(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)U'' + (3\alpha_1 + 3\alpha_2 + 3\alpha_3 \\ &+ 2\alpha_4 + 2\alpha_5)a^{-1}U']. \end{aligned} \tag{53}$$

Since for a circular cylinder we have $\widetilde{R}_\alpha^{(3)} = 0$, with the help of the relation (52) and (53) we find that the conditions $R_\alpha^{(3)} = \widetilde{R}_\alpha^{(3)}$, on $r = a$, imply that $C_2 = 0$. From (49) and (52) we get

$$u_\alpha^{(3)} = C_1x_\alpha, \quad u_3^{(3)} = 0, \tag{54}$$

where C_1 is an unknown constant. Thus, we find that

$$\begin{aligned} &e_{\alpha\beta}^{(3)} = C_1\delta_{\alpha\beta}, \quad e_{\alpha3}^{(3)} = 0, \quad \kappa_{\alpha\beta\gamma}^{(3)} = 0, \\ &\tau_{\alpha\beta}^{(3)} = 2(\lambda + \mu)C_1\delta_{\alpha\beta}, \quad \tau_{\alpha3}^{(3)} = 0, \quad \mu_{\alpha\beta\gamma}^{(3)} = \mu_{\alpha\beta3}^{(3)} = 0, \\ &P_\alpha^{(3)} = 2(\lambda + \mu)C_1n_\alpha, \quad P_3^{(3)} = 0, \quad R_i^{(3)} = 0. \end{aligned}$$

The conditions (35) for $k = 3$ are satisfied if C_1 is given by

$$C_1 = -\frac{\lambda}{2(\lambda + \mu)}. \tag{55}$$

In a similar way we find that the solution of the problem $A^{(4)}$ is

$$u_\alpha^{(4)} = \frac{f}{\lambda + \mu}x_\alpha, \quad u_3^{(4)} = 0. \tag{56}$$

It follows from (54), (56) and (40) that

$$\begin{aligned} &t_{33}^{(3)} = E, \quad t_{3\alpha}^{(3)} = 0, \quad \xi_{\alpha3\beta}^{(3)} = \varepsilon_{3\alpha\beta}f(1 - C_1), \quad \xi_{\alpha33}^{(3)} = 0, \quad \xi_{33\alpha}^{(3)} = 0, \\ &t_{33}^{(4)} = 2f\left(2 + \frac{\lambda}{\lambda + \mu}\right), \quad t_{3\beta}^{(4)} = \mu\varepsilon_{3\rho\beta}x_\rho, \quad \xi_{\alpha33}^{(4)} = -\frac{1}{2}fx_\alpha, \quad \xi_{33\alpha}^{(4)} = fx_\alpha, \\ &\xi_{\alpha3\beta}^{(4)} = \varepsilon_{3\alpha\beta}\left(2\alpha_4 - \alpha_5 - \frac{f^2}{\lambda + \mu}\right), \end{aligned}$$

where

$$E = \mu(3\lambda + 2\mu)/(\lambda + \mu).$$

By using these relations, from (42) we obtain

$$\begin{aligned} &D_{\alpha3} = 0, \quad D_{\alpha4} = 0, \quad D_{33} = \pi a^2E, \\ &D_{44} = \frac{1}{2}\pi\mu a^4 + 4(2\alpha_4 - \alpha_5 - \frac{f^2}{\lambda + \mu})\pi a^2, \\ &D_{34} = D_{43} = \frac{2f(3\lambda + 2\mu)}{\lambda + \mu}\pi a^2. \end{aligned} \tag{57}$$

From (48) and (57) we find that $D_{3\alpha} = 0$ and $D_{4\alpha} = 0$. Thus, the system (41) reduces to

$$D_{\alpha\beta}c_\beta = 0, \quad D_{33}c_3 + D_{34}c_4 = -\mathcal{F}_3, \quad D_{43}c_3 + D_{44}c_4 = -M_3. \tag{58}$$

The solution of the system (59) is

$$c_1 = c_2 = 0, \quad c_3 = (D_{34}M_3 - D_{44}\mathcal{F}_3)/d, \quad c_4 = (D_{34}\mathcal{F}_3 - D_{33}M_3)/d, \tag{59}$$

where $d = D_{33}D_{44} - D_{34}^2$. From (36), (54), (56) and (59) we see that the solution of the problem is

$$u_\alpha = \varepsilon_{3\beta\alpha} c_4 x_\beta x_3 + \left(c_3 C_1 + \frac{f}{\lambda + \mu} c_4 \right) x_\alpha, \quad u_3 = c_3 x_3,$$

where c_3, c_4 and C_1 are given by (55) and (59).

We note that for a chiral circular cylinder we have $D_{34} \neq 0$ so that the torsion and extension cannot be treated independently of each other. In the case of an achiral circular cylinder the coefficient f is equal to zero and the torsion is not accompanied by extension. In the context of the gradient elasticity the torsion problem for an achiral cylinder has been studied in various papers (see Lardner, 1971; Lomakin, 1987; Kahrobaiyan et al., 2011 and references therein). The method presented in this paper can be used to investigate the torsion problem in the theory of gradient elasticity established by Aifantis (1992), Aifantis (2003).

7. Conclusions

The results established in this paper can be summarized as follows:

(a) In the context of the gradient elasticity we establish the solution of the torsion problem for isotropic chiral cylinders with arbitrary cross-sections. As in classical theory the problem is reduced to the study of two-dimensional problems.
(b) We introduce the generalized plane strain problem and present an existence result for the traction problem.
(c) We express the solution of the torsion problem in terms of solutions of four generalized plane strain problems which depend only on the constitutive coefficients and the cross section of the bar.
(d) We show that, in general, the torsion of an isotropic chiral bar is accompanied by extension and bending.
(e) We use the method given in this paper to investigate the torsion of a circular cylinder. It is shown that the torsion of a right circular cylinder made of an isotropic chiral elastic material is accompanied only by extension. The solution could be of interest for experimental investigations.

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