Asymptotic Representation of Solutions of Linear Second Order Difference Equations*

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Using dichotomy conditions, we obtain asymptotic formulae for the solutions of perturbed linear second order difference equations. © 1992 Academic Press, Inc.

1. INTRODUCTION

In recent years, there has been great interest in studying discrete analog of qualitative results valid for ordinary differential equations, see [1–4, 6–8, 12]. Our purpose is to obtain asymptotic formulae for the solutions of the difference equation

$$\Delta(p(k) y(k+1)) + (f(k) + g(k)) y(k) = 0 \quad (1)$$

whenever we know a fundamental system of solutions \(\{\varphi_1, \varphi_2\}\) of the unperturbed equation

$$\Delta(p(k) z(k+1)) + f(k) z(k) = 0, \quad (2)$$

where \(\Delta\) is the operator defined by

$$\Delta y(k) = y(k+1) - y(k). \quad (3)$$

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Our main result establishes that Eq. (1) has a fundamental system of solutions \( \{y_1, y_2\} \) such that for \( k \to \infty \)
\[
y_1(k) = \rho_1(k) [\varphi_1(k)(1 + o(1)) + o(1) \varphi_2(k)]
\]
\[
y_2(k) = \rho_2(k) [\varphi_2(k)(1 + o(1)) + o(1) \varphi_1(k)].
\]
where \( \rho_i, i = 1, 2, \) are functions which we will precise later (see Theorem 3.1).

In Section 4 we will prove (see Theorem 4.1) that formulae (4) are also valid for the equation
\[
y(k + 2) + (f(k) + g(k)) y(k) = 0
\]
which is not included in (1). Thus we have studied the perturbations of all finite difference linear equations
\[
y(k + 2) + a(k) y(k + 1) + b(k) y(k) = 0
\]
or
\[
E^2y + aEy + by = 0,
\]
where \( E \) is the operator defined by
\[
Ey(k) = y(k + 1).
\]

The operators \( \Delta \) and \( E \) are very important in these kind of equations in special by the product rules (see [4])
\[
\Delta(a \cdot b) = a\Delta b + \Delta aEb
\]
\[
E(a \cdot b) = aEb + \Delta aEb.
\]

In this way, we obtained the discrete analog of some continuous results obtained in [5, 9–11], which do no in appear in the literature.

2. PRELIMINARIES

To facilitate the statement of the results we introduce some notation. First, all sequences considered are complex and defined for \( k \geq k_0 \). For a sequence \( \sigma \) we will denote by \( l_1(\sigma) \) the vectorial space of all sequence \( x \) such that \( x\sigma \) is summable, i.e., if \( \sum_{k \geq k_0} |x(k) \sigma(k)| < \infty \). The determinant
\[
c_k(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1(k) & \varphi_2(k) \\ \varphi_1(k + 1) & \varphi_2(k + 1) \end{vmatrix}
\]
will be denoted by $c_k$. It is called the Casorati and is the analog to the Wronskian for ordinary differential equations. We will also adopt the conventions

$$\sum_{k_0}^{k_0 - 1} f(k) = 0, \quad \prod_{k_0}^{k_0 - 1} f(k) = 1$$

and the notation $\langle f(k) \rangle^2 = f(k) f(k + 1)$.

In the sections below we will use the following discrete analogs of the Levinson's theorem [19, p. 95], due to Benzaid and Lutz [7, pp. 200–202]. Consider the linear system

$$x(k + 1) = [A(k) + R(k)] x(k), \quad (9)$$

where $A(k) = \text{diag}\{\lambda_1(k), ..., \lambda_n(k)\}$ for $k \geq k_0$ is a diagonal matrix.

**Theorem A.** Assume

(i) $\lambda_i(k) \neq 0$ for every $k \geq k_0, 1 \leq i \leq n$

(ii) $R = R(k)$ is an $n \times n$ matrix defined for all $k \geq k_0$ satisfying for $1 \leq i \leq n$

$$\sum_{k}^{\infty} \|R(k)\|_{/i,i(k)} < \infty$$

(iii) $A$ satisfies the following dichotomy condition: There exist constants $\mu > 0$ and $K > 0$ such that for each index pair $(i, j), i \neq j$, either

$$\prod_{k_0}^{k} \left|\frac{\lambda_i(l)}{\lambda_j(l)}\right| \rightarrow +\infty \quad \text{as} \quad k \rightarrow \infty$$

and

$$\prod_{k_1}^{k_2} \left|\frac{\lambda_i(l)}{\lambda_j(l)}\right| \geq \mu \quad (k_0 \leq k_1 \leq k_2)$$

or

$$\prod_{k_1}^{k_2} \left|\frac{\lambda_i(l)}{\lambda_j(l)}\right| \leq K \quad (k_0 \leq k_1 \leq k_2).$$

Then system (9) has a fundamental matrix $X$ satisfying as $k \rightarrow \infty$

$$X(k) = (I + o(1)) \prod_{l = k_0}^{k-1} A(l).$$
3. RELATED FORMULAE

In this section, we will prove formula (4) for Eq. (1). We consider that in Eq. (1) $p$, $f$, and $g$ are real sequences and $p(k) 
eq 0 
eq f(k)$ for all $k \geq k_0$.

**Theorem 3.1.** Let $\{\varphi_1, \varphi_2\}$ be a fundamental system of solutions of Eq. (2). Assume in addition

(a) The series

$$q_i(k) = \sum_{l=k}^{\infty} r(l) \langle \varphi_i(l), \varphi_i(l) \rangle^2, \quad i = 1, 2$$

are conditionally convergent series for $k \geq k_0$, where $r = g/E(pc)$.

(b) If $\sigma_1 = 1 + r \varphi_1 E \varphi_2$, $\sigma_2 = 1 - r \varphi_2 E \varphi_1$ then $\text{diag}\{\sigma_1, \sigma_2\}$ satisfies the dichotomy condition in Theorem A (see Remark 1).

(c) The sequences $\varphi_1 E \varphi_2 \cdot q_i$ $(i = 1, 2)$, $\langle \varphi_2 \rangle^2 E \varphi_i$ $(i = 1, 2)$, $\varphi_1 E \varphi_2 E \varphi_1$, $\varphi_2 E \varphi_1 \cdot q_1$, and $\langle \varphi_2 \rangle^2 q_i$ $(i = 1, 2)$ belong to $l_1(r/\sigma_j)$, $j = 1, 2$.

Then there exists a fundamental system of solutions $\{y_1, y_2\}$ of Eq. (1) such that for $k \to \infty$ we have

$$
y_1 = \rho_1 [\varphi_1 \cdot (1 + o(1)) + \varphi_2 \cdot o(1)] \quad (10)
y_2 = \rho_2 [\varphi_2 \cdot (1 + o(1)) + \varphi_1 \cdot o(1)],
$$

where

$$
\rho_1(k) = \prod_{l = k_0}^{k - 1} (1 + r(l) \varphi_1(l) \varphi_2(l + 1))
$$

and

$$
\rho_2(k) = \prod_{l = k_0}^{k - 1} (1 - r(l) \varphi_2(l) \varphi_1(l + 1)).
$$

**Proof.** Taking $z_1 = y$, $z_2 = p Ey$, and $z = (z_1, z_2)$, Eq. (1) is equivalent to the system

$$
A z(k) = (A(k) + B(k)) z(k), \quad (11)
$$

where

$$
A(k) = \begin{bmatrix} -1 & p(k)^{-1} \\ -f(k) & 0 \end{bmatrix}, \quad B = -g \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
$$
Let $\Phi = \Phi(k)$ be the fundamental matrix

$$
\Phi = \begin{bmatrix}
\varphi_1 & \varphi_2 \\
pE\varphi_1 & pE\varphi_2
\end{bmatrix}
$$

then $u = \Phi^{-1}z$ satisfies

$$
\Delta u = r \begin{bmatrix}
\varphi_1 E\varphi_2 & \langle \varphi_2 \rangle^2 \\
-\langle \varphi_1 \rangle^2 & -\varphi_2 E\varphi_1
\end{bmatrix} u,
$$

where $r = g/E(pc)$ and $c = c_k$ is the Casorati of the system $\{\varphi_1, \varphi_2\}$.

Now, let $u(k) = (I + Q(k)) v(k)$, where

$$
Q(k) = \begin{bmatrix}
0 & -q_2(k) \\
q_1(k) & -q_1(k) q_2(k)
\end{bmatrix},
$$

and $I$ is the identity matrix.

Thus, we have obtained the system

$$
\Delta v(k) = (A(k) + R(k)) v(k),
$$

where $A = r \cdot \text{diag}\{\varphi_1 E\varphi_2, -\varphi_2 E\varphi_1\}$ and $R = r(h_{ij})$ with

$$
\begin{align*}
    h_{11} &= \langle \varphi_2 \rangle^2 q_1 - \varphi_1 E\varphi_2 q_1 q_2 - \langle \varphi_2 \rangle^2 q_1^2 q_2 - \varphi_2 E\varphi_1 q_1 q_2, \\
    h_{21} &= -\varphi_1 E\varphi_2 E q_1 - \langle \varphi_2 \rangle^2 q_1 - \varphi_2 E\varphi_1 q_1, \\
    h_{12} &= \varphi_1 E\varphi_2 q_2 - 1 + q_1 q_2 - \langle \varphi_2 \rangle^2 (q_2 - q_1 q_2 - q_1^2 q_2 - E(q_1) q_2)) \\
        &\quad + \varphi_2 E\varphi_1 q_1 q_2, \\
    h_{22} &= \varphi_1 E\varphi_2 E q_1 q_2 + \langle \varphi_2 \rangle^2 (q_1 E q_1) + \varphi_2 E\varphi_1 q_1 q_2.
\end{align*}
$$

System (13) is equivalent to the system

$$
v(k + 1) = (\tilde{A}(k) + R(k)) v(k),
$$

where

$$
\tilde{A}(k) = I + A(k).
$$

Therefore since system (14) satisfies Theorem A, there exists a fundamental matrix $V$ such that for $k \to \infty$,

$$
V(k) = [I + o(1)] \prod_{k=0}^{k-1} \tilde{A}(l).
$$
Hence system (11) has a fundamental matrix $Z(k)$ such that for $k \to \infty$,

$$Z(k) = \Phi(k)(I + Q(k))(I + o(1)) \prod_{k=0}^{k-1} \bar{A}(l).$$

Thus Eq. (1) has a fundamental system of solutions $\{y_1, y_2\}$ which for $k \to \infty$, verifies the asymptotic formulae (10).

This theorem is in effect the analog to [5, Theorem 1]. From system (12) we obtain

**Corollary 3.2.** The conclusion of Theorem 3.1 remains true if condition (c) is replaced by $\langle \varphi_i \rangle^2 \in l_1(r/\sigma_i), i = 1, 2; j = 1, 2$.

**Remark 1.** Suppose that the following limits exist and verify

$$\lim_{k \to \infty} r(k) \varphi_1(k) \varphi_2(k + 1) \neq -1$$

and

$$\lim_{k \to \infty} r(k) \varphi_2(k) \varphi_1(k + 1) \neq 1$$

which holds if both limits exist and are equal to zero. Then $\text{diag}\{\sigma_1, \sigma_2\}$ satisfies the dichotomy condition of Theorem A and we can eliminate $\sigma_j$ in the summability condition (c) in Theorem 3.1.

**Example 1.** Consider the equation

$$\Delta(y(k + 1)) + \left(1 + \frac{1}{k^2}\right) y(k) = 0, \quad \frac{1}{2} < \alpha < 1. \quad (15)$$

Here $p(k) = 1 = f(k), \quad g(k) = 1/k^2$, and $\varphi_1(k) = \exp(ik\pi/3), \quad \varphi_2(k) = i \exp(-ik\pi/3)$ is a fundamental system of solutions of the unperturbed equation

$$\Delta(z(k + 1)) + z(k) = 0.$$

It is not difficult to see that $r = \text{constant} \cdot g$,

$$q_i(k) = \sum_{l=k}^{\infty} r(l) \langle \varphi_i(l) \rangle^2, \quad i = 1, 2$$

are conditionally summable, $\sigma_i \to 1, \quad i = 1, 2$ and then that the summable
condition (c) in Theorem 3.1 holds. Thus, from Theorem 3.1, Eq. (15) has a fundamental system of solutions \( y^\pm \) such that for \( k \to \infty \)
\[
  y^+(k) = \rho^+(k) [\cos(k\pi/3)(1 + o(1)) + o(1) \sin(k\pi/3)] \\
  y^-(k) = \rho^-(k) [\sin(k\pi/3)(1 + o(1)) + o(1) \cos(k\pi/3)],
\]
where
\[
  \rho^\pm(k) = (1 \pm e^{\pm i})^{k - 1 - k_0}.
\]

Theorem 3.1 is very useful if the solutions \( \phi_1 \) and \( \phi_2 \) of Eq. (2) have the same behaviour. Thus when both solutions are comparable, then \( r(\phi_1)^2 \) or \( r(\phi_2)^2 \) may not be conditionally summable, although a linear combination, say
\[
  r(\langle \phi_1 \rangle^2 - \langle \phi_2 \rangle^2) = r(\phi_1 + i\phi_2)^2 - ir(\phi_1 E\phi_2 + \phi_2 E\phi_1),
\]
may be it.

Thus, we obtain the following theorem.

**THEOREM 3.3.** Let \( \{\psi_1, \psi_2\} \) be a fundamental system of solutions. Assume
\[
  (a) \quad S_1(k) = \sum_{l=k}^{\infty} r(l)(\langle \psi_1(l) \rangle^2 - \langle \psi_2(l) \rangle^2),
  \\
  S_2(k) = \sum_{l=k}^{\infty} r(l)(\psi_1(l) \psi_2(l+1) + \psi_2(l) \psi_1(l+1))
\]
are conditionally convergents for \( k \geq k_0 \).

(b) If \( \sigma^\pm = 1 \pm r(\langle \phi_1 \rangle^2 + \langle \phi_2 \rangle^2) \pm ir(\phi_2 E\phi_1 - \phi_1 E\phi_2) \) then \( \text{diag}\{\sigma^+, \sigma^-\} \) satisfies for \( k \geq k_0 \) the dichotomy condition of Theorem A (see Remark 2).

(c) For \( i, j = 1, 2 \), \( \langle \psi_i(k) \rangle^2 \cdot S_j(k) \in l_1(r/\sigma^\pm) \).

Then Eq. (1) has a fundamental system of solutions \( \{y_1, y_2\} \) such that for \( k \to \infty \),
\[
  y_1 = \rho^+ [(\psi_1 + i\psi_2)(1 + o(1)) + o(1)(\psi_1 - i\psi_2)] \\
  y_2 = \rho^- [(\psi_1 - i\psi_2)(1 + o(1)) + o(1)(\psi_1 + i\psi_2)],
\]
where
\[
  \rho^\pm(k) = \prod_{k_0}^{k-1} \sigma^\pm(l).
\]
Proof. Taking
\[ \varphi_1(k) = \psi_1(k) + i\psi_2(k), \quad \varphi_2(k) = \psi_1(k) - i\psi_2(k) \]
this is a direct consequence of Theorem 3.1.

Remark 2. Suppose that the following limit exist and verifies
\[ \lim_{k \to \infty} r \left[ \langle \psi_1 \rangle^2 + \langle \psi_2 \rangle^2 + i(\psi_2 E\psi_1 - \psi_1 E\psi_2) \right] \neq 1 \]
which holds if this limit exists and is equal to zero. Then diag\{\sigma^+, \sigma^-\} satisfies the dichotomy condition of Theorem A and we can eliminate \sigma^+ and \sigma^- in the summability condition (c) in Theorem 3.1.

Example 2. Consider the equation
\[ A(y(k+1)) + \left( 1 + \frac{\cos 2\alpha k}{k} \right) y(k) = 0, \quad \alpha = \frac{\pi}{3}. \quad (16) \]
In this case, \( g(k) = \cos(2\alpha k)/k \) with \( \varphi_1(k) = e^{i\alpha k}, \varphi_2(k) = e^{-i\alpha k} \) does not satisfy Theorem 3.1, because
\[ \sum_{l=k}^{\infty} g(l) \langle \varphi_1(l) \rangle^2 \]
is not conditionally convergent for \( k \geq k_0 > 0 \).
However, if we take \( \psi_1(k) = \exp(i\alpha k), \psi_2(k) = \exp(-i\alpha k) \) in Theorem 3.2, we have that
\[ S_1(k) = \sum_{l=k}^{\infty} \frac{\cos 2\alpha l}{l} \left[ e^{i\alpha l} e^{i\alpha(l+1)} - e^{-i\alpha l} e^{-i\alpha(l+1)} \right] \]
\[ = M_1 \sum_{l=k}^{\infty} \frac{\cos 2\alpha l}{l} \cdot \sin(2\alpha + 1), \quad M_1 \text{ constant} \]
and
\[ S_2(k) = \sum_{l=k}^{\infty} \frac{\cos 2\alpha l}{l} \left[ e^{i\alpha l} \cdot e^{-i\alpha(l+1)} + e^{-i\alpha l} \cdot e^{i\alpha(l+1)} \right] \]
\[ = M_2 \sum_{l=k}^{\infty} \frac{\cos 2\alpha l}{l}, \quad M_2 \text{ constant.} \]
are conditionally summable for \( k \geq k_0 > 0 \). The other conditions in
Theorem 3.2 are also satisfied. Then Eq. (16) has a fundamental system of solutions \( \{ y_1(k), y_2(k) \} \) such that for \( k \to \infty \),

\[
\begin{align*}
y_1(k) & \sim \rho^+(k)(\cos \alpha k + \sin \alpha k), \\
y_2(k) & \sim \rho^-(k)(\cos \alpha k - \sin \alpha k),
\end{align*}
\]

where

\[
\rho^\pm(k) = \prod_{k_0}^{k-1} \sigma^\pm(l), \quad \sigma^\pm = 1 \pm O(\epsilon).
\]

From Corollary 3.2 we obtain

**Corollary 3.4.** The conclusion of Theorem 3.3 remains true if condition (c) is replaced by \( \langle \psi_1 \rangle^2 - \langle \psi_2 \rangle^2, \psi_1 E\psi_2 + \psi_2 E\psi_1 \in l_1(r/\sigma_i), i = 1, 2. \)

4. A **Special Case**

The results in the section above are valid for the equation in difference

\[
y(k + 2) + a(k) y(k + 1) + b(k) y(k) = 0
\]

which may be transformed in an equation as

\[
\Delta(p(k) y(k + 1)) + q(k) y(k) = 0.
\]

That is to say when

\[
p(k) = \prod_{k_0}^{k-1} (-a(l))^{-1}, \quad q(k) = b(k) p(k).
\]

Thus, the results in Section 3 do not apply when \( a(k) = 0 \) for \( k \geq k_0 \).

Consider the equation

\[
y(k + 2) + (f(k) + g(k)) y(k) = 0,
\]

where \( f(k) \) and \( g(k) \) are real valued sequences defined for \( k \geq k_0 \) and \( f(k) \neq 0 \) for \( k \geq k_0 \).

In this case, although Eq. (17) cannot be transformed in Eq. (1) we are able to determinate the asymptotic behaviour of the solutions of Eq. (17) if we know a fundamental system of solutions of the unperturbed equation

\[
y(k + 2) + f(k) y(k) = 0.
\]
The results and proofs in this case, except by small modifications, are similar to those in Section 3.

**Theorem 4.1.** Let \( \{ \varphi_i(k) \}; i = 1, 2 \), be a fundamental system of solutions of Eq. (18). Suppose in addition that (a), (b), and (c) of Theorem 3.1. hold, with \( r = g/Ec \) instead of \( r = g/E(\bar{pc}) \).

Then Eq. (17) has a fundamental system of solutions \( \{ y_i(k) \}; i = 1, 2 \), such that for \( k \to \infty \),

\[
\begin{align*}
y_1 &= \rho_1 [\varphi_1(1 + o(1)) + o(1) \varphi_2] \\
y_2 &= \rho_2 [\varphi_2(1 + o(1)) + o(1) \varphi_1],
\end{align*}
\]

where

\[
\begin{align*}
\rho_1(k) &= \prod_{k_0}^{k-1} (1 + g(l) \varphi_1(l) \varphi_2(l + 1)/c_{l+1}), \\
\rho_2(k) &= \prod_{k_0}^{k-1} (1 - g(l) \varphi_2(l) \varphi_1(l + 1)/c_{l+1}).
\end{align*}
\]

**Proof.** Taking \( z_1 = y \), \( z_2 = E y \), and \( z = (z_1, z_2) \), Eq. (17) is equivalent to the system

\[
z(k + 1) = (A(k) + B(k)) z(k),
\]

where

\[
A(k) = \begin{bmatrix} 0 & 1 \\ -f(k) & 0 \end{bmatrix}, \quad B(k) = \begin{bmatrix} 0 & 0 \\ g(k) & 0 \end{bmatrix}.
\]

Let \( \Phi(k) \) be the fundamental matrix

\[
\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 \\ E\varphi_1 & E\varphi_2 \end{bmatrix}.
\]

Then \( u(k) = \Phi^{-1}(k) z(k) \) satisfies

\[
Au = r \begin{bmatrix} \varphi_1 E\varphi_2 & \varphi_2^2 \\ -\varphi_2^2 & -\varphi_2 E\varphi_1 \end{bmatrix} u,
\]

where \( r = g/Ec \). Thus we have obtained a system as (12) in Theorem 3.1. Then from now on the proof follows as in Theorem 3.1.

In the same way, we may obtain results for Eq. (17) which are analogs to Theorem 3.2 and Corollaries 3.1 and 3.2.
COROLLARY 4.2. Assume

(i) $q_1(k) = \sum_{l=k}^{\infty} g(l) \sin \pi l$, and $q_2(k) = \sum_{l=k}^{\infty} g(l) \cos \pi l$
are conditionally convergents for $k \geq k_0$.

(ii) $q_i, E_{q_i} \in l_1(g/2 + g), i = 1, 2.$

Then the equation

$$y(k + 2) + (1 + g(k)) y(k) = 0$$

has a fundamental system of solutions $\{y_1(k), y_2(k)\}$ such that for $k \to \infty$,

$$y_1(k) = \rho(k) \left[ \sin \frac{k\pi}{2} \cdot (1 + o(1)) + o(1) \cdot \cos \frac{k\pi}{2} \right],$$

$$y_2(k) = \rho(k) \left[ \cos \frac{k\pi}{2} \cdot (1 + o(1)) + o(1) \cdot \sin \frac{k\pi}{2} \right],$$

where

$$\rho(k) = \prod_{k_0}^{k-1} \left( 1 + \frac{g(l)}{2^l} \right).$$

If in the transformed system (10) $|R(k)|$ is not summable but the summability condition of $g$ has improved, then we can apply the process again.

REFERENCES


