A Proof of the $n!$ Conjecture for Generalized Hooks

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In [4], Garsia and Haiman [Electronic J. of Combinatorics 3, No. 2 (1996)] pose a conjecture central to their study of the Macdonald polynomials $H_{\lambda}(x; q, t)$.

For each $\lambda \vdash n$ one defines a certain determinant $A_\lambda(X, Y)$ in two sets of variables. The $n!$ conjecture asserts that the vector space given by the linear span of derivatives of $A_\lambda$, written $L[p x, q y A_\lambda(X, Y)]$, has dimension $n!$. The conjecture reduces to a well-known result about the Vandermonde determinant when $\lambda = (1^n)$ or $\lambda = (n)$ (see [1], for example). Garsia and Haiman (see [Proc. Natl. Acad. Sci. U.S.A. 90 (1993), 3607-3610]) have demonstrated the conjecture for two-rowed shapes $\lambda = (a, b)$, two-columned shapes $\lambda = (2a, 1^b)$ and hook shapes $\lambda = (a, 1^b)$. In this paper, we give an overview of the methods used by Reiner in his thesis to prove the $n!$ conjecture for generalized hooks, that is, for $\lambda = (a, 2, 1^b)$.

1. Introduction

Let $\lambda = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$ be a partition of $n$. To each cell $s$ in the Ferrers diagram of $\lambda$, in French notation, we associate a coordinate $(i, j)$ where $i$ is the row of $s$ and $j$ its column.

**Definition 1.** The set of biexponents of $\lambda$ is the set of pairs

$$\{(i - 1, j - 1) \ | \ (i, j) \in \lambda\}.$$

By ordering lexicographically we obtain $(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)$.

**Definition 2.** Let

$$A[X_n, Y_n] = A[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

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be the ring of polynomials with real coefficients in two sets of variables. Then $A_\mu \in \mathbb{A}[X_n, Y_n]$ is defined by

$$A_\mu = A_\mu(X_n, Y_n) = A_\mu(x_1, \ldots, x_n, y_1, \ldots, y_n) = \det [x_i^j y_i^j]_{i, j = 1, \ldots, n}.$$  

**Example.** Figure 1 shows the biexponents of the shape $\mu = (3, 2)$. The biexponents of $\mu = (3, 2)$ in lexicographic order are

$$(0, 0), (0, 1), (0, 2), (1, 0), (1, 1)$$

and

$$A_{(3, 2)}(X, Y) = \det \begin{pmatrix} x_1^0 y_1^0 & x_1^0 y_1^1 & x_1^1 y_1^0 & x_1^1 y_1^1 \\ x_2^0 y_2^0 & x_2^0 y_2^1 & x_2^1 y_2^0 & x_2^1 y_2^1 \\ x_1^0 y_1^0 & x_1^0 y_1^1 & x_1^1 y_1^0 & x_1^1 y_1^1 \\ x_2^0 y_2^0 & x_2^0 y_2^1 & x_2^1 y_2^0 & x_2^1 y_2^1 \end{pmatrix}.$$ 

Next, set

$$\partial_x^p = \partial_{x_1}^{p_1} \cdots \partial_{x_n}^{p_n}$$

for $p = (p_1, \ldots, p_n)$ and likewise $\partial_y^q$ or $q = (q_1, \ldots, q_n)$. We define

$$H_\mu = H_\mu[X_n, Y_n] = \mathcal{F}[\partial_x^p \partial_y^q A_\mu],$$

that is, $H_\mu$ is the linear span of all partial derivatives of $A_\mu$. The $n!$ conjecture can be stated as follows.

**Conjecture.** Let $\mu$ be a partition of $n$. Then the polynomial $A_\mu(X_n, Y_n)$ has $n!$ linearly independent derivatives; or, which is the same, $\dim H_\mu = n!$.

Garsia and Haiman [4] embed the $n!$ conjecture into the context of orbit harmonics. To understand this embedding let us briefly review some of the essential ingredients of this theory. One considers the regular orbit $[p]_G = [p]$ of a point in $\mathbb{A}^m$ under the action of a group $G$ of $m \times m$ orthogonal matrices. Set $\mathbb{R} = \mathbb{A}[x_1, \ldots, x_m]$ and for $P \in \mathbb{R}$ let $h(P)$ be the highest homogeneous component of $P$. Finally, one introduces the scalar product $\langle P, Q \rangle = P(\partial) Q|_{x_1 = \ldots = x_m = 0}$. Figure 1.

[Figure 1]
operator and evaluate the result at \( x = 0 \). We now consider the following
associated spaces.

1. The ideal associated with \( [\rho] \) is \( \mathcal{J}_{[\rho]} = \{ P \in \mathbb{R} \mid P(A\rho) = 0 \forall A \in G \} \).
2. \( R_{[\rho]} = \mathbb{R}/\mathcal{J}_{[\rho]} \), the coordinate ring of \( [\rho] \).
3. \( \text{gr} \mathcal{J}_{[\rho]} = ( \{ h(P) \mid P \in \mathcal{J}_{[\rho]} \} ) \), i.e. the ideal generated by the highest
   homogeneous components of the elements of \( \mathcal{J}_{[\rho]} \).
4. \( \text{gr} R_{[\rho]} = \mathbb{R}/\text{gr} \mathcal{J}_{[\rho]} \).
5. The space of orbit harmonics associated with \( [\rho] \) is the space
   \( H_{[\rho]} = (\text{gr} \mathcal{J}_{[\rho]})^\perp \).

Two important problems that one would like to address in this general
setup are the following:

1. Construct a homogeneous basis for \( \text{gr} R_{[\rho]} \).
2. Determine whether \( H_{[\rho]} \) is a cone, i.e., whether \( H_{[\rho]} \) is the linear
   span of derivatives of a single polynomial (referred to as the \textit{summit}
   of the cone).

In \cite{1}, an answer is provided to these questions in the form of a beau-
tiful theorem which exploits the combinatorial properties of the orbit \( [\rho] \)
and the associated space \( R_{[\rho]} \):

\textbf{Theorem 1.} Let \( a_1 < a_2 < \cdots < a_\ell \) be a total order on \( [\rho] \). Suppose
further, that \( n_0 = \max \{ \deg P \mid P \in \mathcal{J}_{[\rho]} \} \) and that \( \mathcal{B} = \{ \phi_i \}_{i=1}^{\ell} \) is a set of
polynomials satisfying:

1. \( \phi_i(x) = \begin{cases} 0 & \text{if } x \in [\rho] \text{ and } x < a_i, \\ \text{nonzero} & \text{if } x = a_i. \end{cases} \)
2. \( |\{ \phi \in \mathcal{B} \mid \deg \phi = i \} | = |\{ \phi \in \mathcal{B} \mid \deg \phi = n_0 - i \} | \), for all \( 0 \leq i \leq n_0 \).
3. \( n_0 = \max \{ \deg \phi \mid \phi \in \mathcal{B} \} \).

Then

(i) \( \{ h(\phi) \mid \phi \in \mathcal{B} \} \) is a basis for \( \text{gr} R_{[\rho]} \).
(ii) \( H_{[\rho]} \) is a cone.

Theorem 1 is often referred to as the \textit{kicking principle} and a set of poly-
nomials \( \mathcal{B} \) satisfying the conditions of the hypothesis is called a set of
\textit{kicking} polynomials. In Section 1 of this paper, we review Garsia and
Haiman’s construction in \cite{2} of a group orbit \( [\rho_\mu] \) for every partition \( \mu \).
This construction leads to a reformulation the \( n! \) conjecture as the problem
of determining whether the space of orbit harmonics \( H_{[\rho_\mu]} \) corresponding
to \( [\rho_\mu] \) is a cone with summit \( A_\mu \).
In this context, one can use the kicking principle. To this end, one needs to define an appropriate total order and weight function on \([\rho_\mu]\). In [5], Garsia and Haiman outline a method for imposing an order and a weight function on this orbit for special classes of partitions \(\mu\). In Section 1, we describe how this method works for the case of generalized hooks, that is, partitions of the form \(\mu = (a, 2, 1^b)\).

Once these initial steps are established, one needs to find for each orbit point a corresponding kicking polynomial. In general, this involves the concoction of polynomials of a prescribed bi-degree which vanish on a predetermined set of points and not on others. In our specific situation, we need to address the following problem.

**Problem.** Given alphabets \(X_1 \subseteq X_2\) and \(Y\) with \(|X_1| = r, |X_2| = s, |Y| = k\) and \(r \leq k \leq s\). Find a polynomial \(P \in K[y_1, \ldots, y_k]\) of degree \(s-r+1\) where \(K = \mathbb{F}[x_1, \ldots, x_n]\) with the properties

\[
P(y_1, \ldots, y_k) = \begin{cases} 0 & \text{if } X_1 \subseteq Y \text{ or } Y \subseteq X_2, \\ c(Y) & \text{otherwise}.\end{cases}
\]

where \(c(Y)\) is a generically non-zero polynomial in the alphabet \(Y\) with coefficients in \(K\).

Section 2 is devoted to proving that the answer to this question is given by

\[
\sum_{\mu \in \Delta} (-1)^{|\mu|} S_{\mu}([Y] S_{\lambda, \mu}^*[X_1, X_2, \ldots, X_2]),
\]

where \(\lambda = (k - r + 1, 1^{s-k})\) and \(S_{\lambda, \mu}^*[X_1, X_2, \ldots, X_2]\) is the flagged Schur function indexed by \(\lambda, \mu\) in the alphabets \(X_1\) and \(X_2\).

In Section 3, we introduce the notation of [7] and describe how such functions are used in the proof of the \(n!\) conjecture for \(\mu = (a, 2, 1^b)\).

1. The Orbit \([\rho_\mu]\), Order and Weight

In this section, we review Garsia and Haiman’s construction in [4] which embeds the \(n!\) conjecture in the theory of orbit harmonics. For each \(n\), we consider the group \(G\) consisting of all matrices of the form

\[
A_\sigma = \begin{pmatrix} P(\sigma) & 0 \\ 0 & P(\sigma) \end{pmatrix},
\]

where \(\sigma \in S_n\) and \(P(\sigma)\) is the \(n \times n\) permutation matrix corresponding to \(\sigma\).

Next, one has the following combinatorial construction of a \(G\)-orbit in \(\mathbb{R}^{2n}\) for each \(\mu \vdash n\).
Assume that $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$ is a partition of $n$ and denote by $\mathcal{I}(\mu)$ the set of injective tableaux of shape $\mu$. Let $x_1, x_2, \ldots, x_k$ and $\beta_1, \beta_2, \ldots, \beta_k$ be algebraically independent real numbers (where $h = \mu_1$) and fix $T \in \mathcal{I}(\mu)$. For each label $s$ in $T$, we set

$$r_T(s) = \text{row number of the label } s \text{ in } T$$

and

$$c_T(s) = \text{column number of the label } s \text{ in } T.$$ 

Finally, we associate to a tableau $T$ a point $\mathbb{T}_T \in \mathbb{R}^n$, given by

$$\mathbb{T}_T = (r_T(1), \ldots, r_T(n), c_T(1), \ldots, c_T(n)).$$

**Example.** Let $\mu = (3, 2, 2, 1)$ and $T$ be the injective tableau below.

$$\begin{array}{cccc}
5 & 3 & 8 & \\
1 & 4 & & \\
7 & 2 & 6 & \\
\end{array}$$

Then the associated point is

$$\mathbb{T}_T = (2, 1, 3, 2, 1, 1, 1, 2, 1, 3).$$

It is clear that $T$ may be recovered from $\mathbb{T}_T$. Moreover, defining $[\rho_T] = \{ \rho_T | T \in \mathcal{I}(\mu) \}$, we see that $[\rho_T] = [\rho_T]_G$ for any fixed $T \in \mathcal{I}(\mu)$.

With a regular $G$-orbit as a point of departure one specializes the theory of orbit harmonics to the case $m = 2n$ and $R = \mathfrak{A}[X_n, Y_n]$. One now has all the algebraic structures discussed in the introduction. Namely, $\mathcal{I}_{\{\rho_T\}}$, $R_{\{\rho_T\}}$, $\text{gr} \mathcal{I}_{\{\rho_T\}}$ and $H_{\{\rho_T\}}$.

We can now give an outline of the successive steps which, if established, would provide a proof of the $n!$ conjecture for any partition $\mu$:

1. $H_\mu \subseteq H_{\{\rho_T\}}$.

2. Impose a total order $<$ on the collection $\mathcal{I}(\mu)$. This order can naturally be extended to $[\rho_T]$ by setting $\rho_{T_1} < \rho_{T_2} \iff T_1 < T_2$.

3. Define a weight function $w : \mathcal{I}(\mu) \rightarrow \{t^rq^s | r \geq 0, s \geq 0\}$, where one sets

$$w_r(T) = r \quad \text{and} \quad w_s(T) = s \quad \text{when} \quad w(T) = t^rq^s$$

and

$$w_{r,q}(T) = w_r(T) + w_s(T) \quad \text{with} \quad n_{r,q}(\mu) = \max \{w_{r,q}(T) | T \in \mathcal{I}(\mu)\}.$$
Finally, the weight function should have the symmetry

\[ |\{ T \in \mathcal{S}(\mu) \mid w_{s,T}(T) = i \}| = |\{ T \in \mathcal{S}(\mu) \mid w_{s,T}(T) = n_0(\mu) - i \}|, \quad (1) \]

for each \( 0 \leq i \leq n_0(\mu) \).

4. For each tableau \( T \) in \( \mathcal{S}(\mu) \) construct a kicking polynomial \( \phi_T(x, y) \) satisfying the following properties:

- (i) \( \phi_T(\rho_T) = \begin{cases} \text{nonzero} & \text{if } \rho_T = \rho_0, \\ 0 & \text{if } \rho_T \in [\rho_0] \text{ and } \rho_T < \rho_T. \end{cases} \)

- (ii) \( \deg_x(\phi_T) = w(T) \) and \( \deg_y(\phi_T) = w_q(T) \).

5. \( \max\{ \deg(\phi) : \phi \in H_{(\rho_0)} \} = \deg \mu = n_0(\mu) \).

Steps 2–4 will allow us to use the kicking principle and thereby yield that \( H_{(\rho_0)} \) is a cone of dimension \( n! \). Step 1 combined with step 5 will then give that \( H_{(\rho_0)} = H_\mu \), proving the conjecture.

That \( H_\mu \subseteq H_{(\rho_0)} \) is shown in [4]. The proof though elementary is not obvious. In [5] one finds the algorithm required for steps 2 and 3 for any partition whose diagram is contained in one of the following shapes \( \mu = (a, 2, 1^b) \), \( \mu = (a, b) \) and \( \mu = (2^a, 1^b) \).

The principle idea behind this algorithm is to consider injective lattice diagrams which are equivalent to skew shapes via rearrangement of rows and columns and whose subdiagrams also have this property. Within this class of diagrams, one introduces an extraction operator \( E \) which removes the largest label from a skew shape \( D \) to give (after a sequence of row and column switches) a smaller skew shape \( D' \). We shall not look at the entire class of such shapes here, but rather focus on those which are needed for the development of the order \( < \) and weight \( w \) on \( \mathcal{S}(\mu) \) when \( \mu = (a, 2, 1^b) \).

The basic building blocks for this class are skew shapes of types 1–5 which we describe below.

**Type 1:** \( \text{shape}(T) = (a, 1^b) \).

\[ \begin{array}{cccccc}
\uparrow & & & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
& & & & & \\
& & & & & \\
a & & & & & \\
\end{array} \]

**Type 2:** \( \text{shape}(T) = (a^2, 1^b)/(a-1) \).

\[ \begin{array}{cccccccc}
\uparrow & & & & & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
a & & & & & & & \\
\end{array} \]
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Type 3: $\text{shape}(T) = (a, 2^b)/(1^b)$.  

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\downarrow \\
\text{b}
\end{array}
\end{array}
\]

Type 4: $\text{shape}(T) = (a, 2, 1^{b-1})/(1)$.  

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\downarrow \\
\text{b}
\end{array}
\end{array}
\]

Type 5: $\text{shape}(T) = (a, 2, 1^{b-1})$.  

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\downarrow \\
\text{b}
\end{array}
\end{array}
\]

The set of generalized hooks, denoted $GH$, consists of all skew shapes $\lambda$ which are a finite product of shapes of types 1–5. Next, define the set of injective generalized hook tableaux by setting

$$\mathcal{F}GH = \{ T | \text{shape}(T) \in GH \text{ and } T \text{ has filling } 1, ..., |T| \}$$

and

$$\mathcal{F}GH_r = \{ T | T \in \mathcal{F}GH \text{ and } |T| = r \}.$$

The extraction operator $E: \mathcal{F}GH_r \rightarrow \mathcal{F}GH_{r-1}$ can now be introduced. Given $T \in \mathcal{F}GH_r$, $E(T)$ is a tableau obtained from $T$ by extracting the cell containing $r$ from $T$ and rearranging the resulting shape to obtain a new skew diagram. The rules below give a systematic description of $E(T)$ according to the type of $T$ (when $T$ is the product of several tableaux one considers the type of the tableau containing the largest label).

Case 1. $s$ is in a corner cell. To obtain $E(T)$ simply remove $s$ from $T$.

Case 2. $T$ is of type 1–4 and $s$ appears in a horizontal segment or $T$ is of type 5 and $s$ appears to the right of column 2. Remove the cell of $s$ from $T$ and shift each cell to its right by one unit to the left.

Case 3. $T$ is of type 1–4 and $s$ appears in a vertical segment or $T$ is of type 5 and $s$ appears above row 2. Remove the cell containing $s$ from $T$ and shift each cell above it down one unit.
Case 4. $T$ is of type 5 and $s$ appears in the coordinate $(2, 1)$. First, remove the cell containing $s$ and interchange rows 1 and 2. Next, place column 2 to the right of the last column of $T$ and shift all columns to the right of column 2 one unit to the left.

Case 5. $T$ is of type 5 and $s$ appears in the coordinate $(1, 2)$. First, remove the cell containing $s$ and interchange columns 1 and 2. Next, place row 2 above the last row of $T$ and shift all rows above the first row down one unit.

Examples (Cases 4 and 5 for type 5)

\[
\begin{array}{cccc}
8 & 5 & 3 & 9 \ 1 \\
6 & 2 & 4 & 7 \\
\end{array}
\xrightarrow{E}
\begin{array}{cccc}
8 & 5 & 3 & 9 \ 1 \\
6 & 2 & 4 & 7 \\
\end{array}
\xrightarrow{E}
\begin{array}{cccc}
8 & 5 & 3 & 9 \ 1 \\
6 & 2 & 4 & 7 \\
\end{array}
\]

\[
\begin{array}{cccc}
5 & 3 & 9 & 4 \ 6 \\
7 & 8 & 2 & 1 \\
\end{array}
\xrightarrow{E}
\begin{array}{cccc}
1 & 2 & 3 & 4 \ 6 \\
5 & 7 & 8 & 3 \\
\end{array}
\xrightarrow{E}
\begin{array}{cccc}
1 & 2 & 3 & 4 \ 6 \\
5 & 7 & 8 & 3 \\
\end{array}
\]

Notation. The tableau resulting from the successive extraction of the $h$ largest labels in $T$ will be denoted by $T^{(h)}$. With this notation, when $|T| = n$, the result of extracting all labels larger than a given label $s$ from $T$ is written $T^{(s^{-}\infty)}$.

Finally, we are ready to define the order $<$ on $\mathcal{F}(\mu)$.

Definition 3. Let $\mu \vdash n$ and $\mu \in \mathcal{B}$. Suppose $T_1, T_2 \in \mathcal{F}(\mu)$ with $T_1 \neq T_2$ and let $s$ be the largest label such that $(r_{T_1}(s), c_{T_1}(s)) \neq (r_{T_2}(s), c_{T_2}(s))$. We say $T_1 < T_2$ if

1. $c_{T_1}^{(-s)}(s) < c_{T_2}^{(-s)}(s)$

or

2. $c_{T_2}^{(-s)}(s) = c_{T_2}^{(-s)}(s)$ and $r_{T_1}^{(-s)}(s) > r_{T_2}^{(-s)}(s)$.

Otherwise, $T_2 < T_1$. 


Let us consider an illustration of this definition.

**Example.** Let

\[ T_1 = \begin{array}{cccc}
4 & 5 & 7 & 8 \\
8 & 6 & 3 & 1 \\
7 & 5 & 3 & 1 \\
6 & 9 & 3 & 1 \\
\end{array} \quad \text{and} \quad T_2 = \begin{array}{cccc}
3 & 5 & 7 & 8 \\
8 & 4 & 9 & 5 \\
7 & 1 & 9 & 6 \\
4 & 9 & 5 & 2 \\
\end{array} \]

Since the label 6 is the largest label which has different coordinates in \( T_1 \) and \( T_2 \), we apply \( E \) successively to both tableaux to extract the labels 9, 8 and 7

\[
T_1^{(3)} = E(T_1) = \begin{array}{cc}
5 & 7 \\
4 & 8 \\
6 & 3 & 1 & 2 \\
\end{array} \\
T_2^{(3)} = E(T_2) = \begin{array}{cc}
1 & 7 \\
3 & 8 \\
4 & 5 & 2 & 6 \\
\end{array}
\]

We now see that \( c_{T_1}^{(3)}(6) = 2 < c_{T_2}^{(3)}(6) = 5 \). Therefore, \( T_1 < T_2 \).

The description of a weight function on the set \( \mathcal{F}_G \mathcal{H} \) requires an additional definition.

**Definition 4.** Let \( T \) be a skew tableau and let \( s \) be any label in \( T \). Then the co-arm of \( s \), written \( ca_T(s) \), is the number of squares to the left of \( s \) in \( T \) and the leg of \( s \), written \( l_T(s) \), is the number of squares above \( s \) in \( T \).

**Example.** Let \( T \) be the tableau below.

\[
T = \begin{array}{cccc}
5 & 2 & 3 & 1 \\
4 & 3 & 1 & 2 \\
10 & 4 & 1 & 1 \\
6 & 3 & 1 & 2 \\
\end{array}
\]

Then we have \( ca_T(4) = 1 \) and \( l_T(4) = 2 \).

The weight function has a recursive definition.
**Definition 5.** Let \( T \in \mathcal{F} \mathcal{B} \mathcal{R} \) with filling \( 1, \ldots, |T| \). Setting \( s = |T| \) we have,

\[
w(T) = \begin{cases} 
1 & \text{if } |T| = 1, \\
\ell_{r(s)} q^{\sigma r(s)} w(E(T)) & \text{if } |T| > 1.
\end{cases}
\]

**Example.** Let us compute the weight of the tableau

\[
T = \begin{array}{ccc}
1 & 2 \\
3 & 5 & 4 & 6
\end{array}
\]

Applying the recursive definition, we get

\[
w(T) = w\left( \begin{array}{ccc}
1 & 2 \\
3 & 5 & 4 & 6
\end{array} \right) = q^3 \times w\left( \begin{array}{ccc}
1 & 2 \\
3 & 5 & 4
\end{array} \right) = tq^4 \times w\left( \begin{array}{ccc}
2 & 1 \\
3 & 4
\end{array} \right)
\]

\[
= tq^4 \times w\left( \begin{array}{ccc}
2 & 1 \\
3 & 3
\end{array} \right) = t^2 q^5 \times w\left( \begin{array}{ccc}
2 & 1 \\
1 & 1
\end{array} \right) = t^2 q^3 \times w\left( \begin{array}{ccc}
1 & 1 & 1
\end{array} \right)
\]

\[
= t^2 q^5.
\]

We shall not set out here to prove that this weight function is symmetric in the sense of Eq. (1). For an indication on how this might be achieved the reader is referred to [5] where the symmetry of this function is established for 2-row and 2-column shapes.

### 2. Properties of Special Schur Functions

Some of the most interesting problems which arise out of the \( n! \) conjecture involve the concoction of symmetric and skew-symmetric polynomials with special vanishing properties. It turns out that in order to construct kicking polynomials for \( \rho_n \) when \( \mu = (a, 2, 1^b) \), we need to establish some properties of special Schur functions. We begin by reviewing some basic definitions and results.

An alphabet is a finite set of variables \( \mathcal{A} = \{a_1, \ldots, a_n\} \) which we often write for convenience as a formal sum \( \mathcal{A} = a_1 + \cdots + a_n \).

**Definition 6.** Let \( X_1, X_2, \ldots, X_m \) be any sequence of alphabets and let \( \lambda/\mu \) have \( m \) parts. Then the flagged Schur function indexed by \( \lambda/\mu \) is

\[
S_{\lambda/\mu}[X_1, X_2, \ldots, X_m] = \det \{ h_{\lambda_i - \mu_j - j}(X_s) \}_{i,j=1}^{m}.
\]

where \( h_s(X) \) is the homogeneous symmetric function of degree \( s \) in \( X \).
Flagged Schur functions have a combinatorial interpretation. Let us recall the required notation.

**Definition 7.** Given any skew partition $\lambda/\mu$ with $m$ cells the set of column strict tableaux of shape $\lambda/\mu$, denoted $CS(\lambda/\mu)$, is the set of tableaux with integer entries weakly increasing in the rows and strictly increasing in the columns.

**Definition 8.** Let $X = \{x_1, \ldots, x_m\}$, and $T$ be a tableau in $CS(\lambda/\mu)$ such that if $s \in T$ then $1 \leq s \leq m$. Then

$$wt_x(T) = \prod_{s \in T} x_s.$$

We make the convention that $wt_x(T) = 0$ if $T$ contains an entry larger than $m$.

**Definition 9.** Let $I = [i_1, i_2, \ldots, i_m]$ be a weakly increasing set of indices with $1 \leq i_j \leq m$ and let $\lambda/\mu$ be a skew partition with $m$ parts. Then we set

$$CSI(\lambda/\mu) = \{T \in CS(\lambda/\mu) \mid \text{row } r \text{ has entries from } \{1, \ldots, i_r\}\}.$$

Finally, one has the following result which is due to I. Gessel whose proof may be found in [8].

**Theorem 2.** Let $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_m = X$ be a sequence of alphabets and let $\lambda/\mu$ have $m$ parts, Further, suppose $X_r = \{x_1, x_2, \ldots, x_{i_r}\}$ for $1 \leq r \leq m$. Then

$$S_{\lambda/\mu}[X_1, \ldots, X_m] = \sum_{T \in CSI(\lambda/\mu)} wt_x(T),$$

where $I = [i_1, i_2, \ldots, i_m]$.

Next, recall that the homogeneous symmetric function of the difference of two alphabets $X$ and $Y$ is obtained from the generating function

$$\sum_{k \geq 0} h_k(X - Y) t^k = \frac{\prod_{s \in Y} (1 - ty)}{\prod_{s \in X} (1 - tx)}.$$

It is important to observe that $h_k(X - Y)$ is independent of the expansion of $X - Y$ as an algebraic difference. The following generalization of the usual addition formula for Schur functions is an easy consequence of the Cauchy–Binet theorem.
Proposition 1.

\[ S_\mu[X_1 - Y, ..., X_m - Y] = \sum_{\mu \subseteq \lambda} S_\mu[-Y] S_{\lambda\mu}[X_1, ..., X_m] \]

\[ = \sum_{\mu \subseteq \lambda} (-1)^{\mu} S_{\mu}[Y] S_{\lambda\mu}[X_1, ..., X_m], \]

where \( S_{\mu}[Y] \) denotes the ordinary Schur function indexed by \( \mu \) and \( \mu^* \) is the partition conjugate to \( \mu \).

Before proceeding with the main result of this section an observation worked out by Garsia and Haiman [6] is needed.

Lemma 1. Let \( Y \) and \( X \) be alphabets with \( |X| = s \), \( |Y| = r \) and \( s < r \). If \( m > r - s \), then

\[ h_m(X - Y) = \begin{cases} 0 & \text{if } X \subseteq Y, \\ c(X, Y) & \text{otherwise}, \end{cases} \]

where \( c(X, Y) \) is a generically non-zero polynomial in \( X \) and \( Y \).

Proof. If \( X \subseteq Y \) then \( X - Y \) may be rewritten as \( \emptyset - Y' \), where \( \emptyset \) denotes the empty alphabet and \( Y' = Y - X \) is a positive alphabet. Thus, letting \( e_k \) be the elementary symmetric function of degree \( k \),

\[ h_m(X - Y) = \sum_{k=0}^{m} (-1)^k e_k(Y) h_{m-k}(X) = \sum_{k=0}^{m} (-1)^k e_k(Y') h_{m-k}(\emptyset). \]

Clearly, all the terms involving \( h_k(\emptyset) \) vanish in this sum when \( k \neq 0 \). This leaves \( (-1)^m e_m(Y') \) which also vanishes since \( |Y'| = r - s < m \). When, on the other hand, \( X \not\subseteq Y \), we can write \( X - Y = X' - Y' \) where \( X' \not\subseteq \emptyset \), \( Y' \not\subseteq \emptyset \) and \( X' \cap Y' = \emptyset \). Hence, \( h_m(X - Y) = \sum_{k=0}^{m} e_k(Y') h_{m-k}(X') \) fails to vanish by the linear independence of the elementary symmetric functions in \( Y' \).

Theorem 3. Let \( X_1 \subseteq X_2 \) and \( Y \) be alphabets. Suppose \( |X_1| = r \), \( |X_2| = s \) and \( |Y| = k \) with \( r \leq k \leq s \). Then, setting \( \lambda = (k - r + 1, 1^{s-k}) \) one has

\[ S_\lambda[X_1 - Y, X_2 - Y] = \begin{cases} 0 & \text{if } X_1 \subseteq Y \text{ or } Y \subseteq X_2, \\ c(X, Y) & \text{otherwise}, \end{cases} \]

where \( c(X, Y) \) is a generically non-zero polynomial in \( X \) and \( Y \).
Proof. In the case $X_1 \subseteq Y$, we note that all the entries in the first row of

$$S_\mu[X_1 - Y, X_2 - Y, ..., X_s - Y]$$

are of the form $h_m(X_1 - Y)$ with $m > k - r$ which vanish by Lemma 1. A little more work is required when $Y \subseteq X_2$. Letting $A = X_2 - Y$ and $B = X_2 - X_1$ one has

$$X_1 - Y = A - B \quad \text{and} \quad X_2 - Y = A - \emptyset.$$ 

Thus, 

$$S_\mu[X_1 - Y, X_2 - Y, ..., X_s - Y] = S_\mu[A - B, A - \emptyset, ..., A - \emptyset] \quad (3)$$

where the last equality follows from Proposition 1. Now, we note that when $\mu \leq \lambda$ has $s - k + 1$ parts, the terms in Eq. (4) involving $S_\mu[A]$ will vanish because $A$ has only $|X_2 - Y| = s - k$ elements. Let us consider the other terms. When $\mu$ has less than $s - k + 1$ parts, $\mu_\lambda = 0$. As a result, the last column of $S_{\lambda/\mu}[A - B, \emptyset, ..., \emptyset]$ is

$$\begin{bmatrix}
    h_{k-\mu_{s-k+2} + (s-k)(-B)} \\
    h_{k-\mu_{s-k+2}+(s-k-1)(-B)} \\
    \vdots \\
    h_{k-\mu_{s-k+2} + (s-k+1)(-B)} \\
    \vdots \\
    h_{k-\mu_{s-k+2} + 0(\emptyset)} \\
    \vdots \\
    h_{\lambda_k(\emptyset)} \\
    \vdots \\
    e_{\lambda_k(\emptyset)} \\
  \end{bmatrix} = \begin{bmatrix}
    h_{k-\mu_{s-k+2} + (s-k)(-B)} \\
    h_{k-\mu_{s-k+2}+(s-k-1)(-B)} \\
    \vdots \\
    h_{k-\mu_{s-k+2} + (s-k+1)(-B)} \\
    \vdots \\
    h_{k-\mu_{s-k+2} + 0(\emptyset)} \\
    \vdots \\
    h_{\lambda_k(\emptyset)} \\
    \vdots \\
    e_{\lambda_k(\emptyset)} \\
  \end{bmatrix}.$$

These entries all vanish. Indeed, $|B| = |X_2| - |X_1| = s - r$ forces $e_{\lambda_k(\emptyset)}(B) = 0$; also $e_{\lambda_k(\emptyset)} = 0$ whenever $j \neq 0$. Therefore, $S_{\lambda/\mu}[A - B, \emptyset, ..., \emptyset] = 0$.

Finally, we need to show that $S_\mu[X_1 - Y, X_2 - Y, ..., X_s - Y] = c(X, Y)$ when neither $X_1 \subseteq Y$ nor $Y \subseteq X_2$. To this end, we let $A = X_1 \cap Y$. This gives $X_1' = X_1 - A$, $X_2' = X_2 - A$ and $Y' = Y - A$ and we can rewrite

$$S[X_1 - Y, X_2 - Y, ..., X_s - Y]$$

$$= \sum_{\mu \subseteq \lambda} (-1)^{\mu_1} S_{\lambda/\mu}[Y] S_{\lambda/\mu}[X_1', X_2', ..., X_s'] \quad (6).$$
We shall exhibit a monomial in the expansion of Eq. (6) whose coefficient does not vanish. Indeed, expanding the r.h.s. of Eq. (6) and applying Theorem 2 we find that when \( \mu = \emptyset \), one has \( \lambda / \mu = \lambda \) and the polynomial

\[
S_\lambda[X_1', X_2', \ldots, X_s'] = \sum_{T \in CS(\lambda)} wt_x(T)
\]

appears as a term with \( X = X_s' \) and \( I = [a, b, \ldots, b] \) where \( a = |X_1'| \) and \( b = |X_2'| \). Using the fact that \( |X_1'| \geq 1 \) and \( |X_2'| > |Y| \geq s - k + 1 \) we may find indices \( j_1 < j_2 < \cdots < j_{s-k+1} \) such that

\[
x_{j_1} \in X_1' \quad \text{and} \quad x_{j_2}, \ldots, x_{j_{s-k+1}} \in X_2'.
\]

Since the tableau of shape \( \lambda \) with \( j_i \) in the \( r \)th row belongs to \( CS(\lambda) \), the monomial

\[
M = x_{j_1}^{k-r+1} x_{j_2} x_{j_3} \cdots x_{j_{s-k+1}}
\]

appears with coefficient 1 in the expansion of the r.h.s. of Eq. (7). Now, a monomial that cancels \( M \) must come from a term in Eq. (6) in which \( \mu \neq \emptyset \). But, any term in which \( \mu \neq \emptyset \) involves \( S_\mu[Y] \) which is a sum of monomials in \( Y \) of degree \( |\mu| \neq 0 \). Since \( Y' \cap X_1' = \emptyset \) and \( Y' \cap X_2' = \emptyset \) it is impossible for any such term to cancel \( M \).

3. Kicking \([\rho_\mu]\) for \( \mu = (a, 2, 1^k) \)

To prove that \( H_\mu \) is a cone we need to construct a set of kicking polynomials for the orbit \( [\rho_\mu] \). This is a considerably intricate task and involves more detail than can be given here. The reader is referred to [7] for a complete presentation.

The goal of this section is to give an idea of how one writes down a polynomial \( \phi_T \) for each tableau corresponding to an orbit point \( \rho_T \). Since each polynomial is to be written according to the labels in \( T \), new notation is needed. To this end, it will be convenient to have a means by which to refer to particular sets of labels in a given tableau.

**Definition 10.** Let \( f \) be a label in the tableau \( T \) then:

1. \( L_{<f} = \{ s \in T \mid s < f, \ r_T(s) = r_T(f) \text{ and } c_T(s) < c_T(f) \} \), in other words, \( L_{<f} \) consists of the labels smaller than \( f \) appearing to the left of \( f \) in the same row as \( f \).
2. \( A_f = \{ s \in T \mid c_T(s) = c_T(f) \text{ and } r_T(s) > r_T(f) \} \), in other words, \( A_f \) consists of the labels appearing above \( f \) in the same column as \( f \).

3. \( A_{<f} = \{ s \in A_f \mid s < f \} \).

4. \( R_f = \{ s \in T \mid r_T(s) = r_T(f) \text{ and } c_T(s) > c_T(f) \} \), in other words, \( R_f \) consists of the labels appearing to the right of \( f \) in the same row as \( f \).

5. \( R_{<f} = \{ s \in R_f \mid s < f \} \).

Define \( A_{\geq f} \) and \( R_{\geq f} \) similarly. We also define \( \tilde{A}_{<f} \) and \( \tilde{A}_{\geq f} \) where the restriction that the labels in these sets be in the same column as \( f \) is dropped; likewise, \( \tilde{R}_{<f} \) and \( \tilde{R}_{\geq f} \) where the row restriction is dropped.

Furthermore, for any tableau \( T \), we make the following definitions.

**Definition 11.** Let \( d \) be any label appearing in \( T \). We then have the corresponding polynomials

\[
P_d = \prod_{j \in A_{<d}} (x_d - x_{r_T(j)}) \quad \text{and} \quad Q_d = \prod_{j \in R_{<d}} (y_d - \beta_{c_T(j)})^k
\]

(8)

For classification purposes the convention is made that the labels \( u, v, w \) and \( z \) are distributed in the subdiagram \((2, 2)\) as in Fig. 2.

Using this labeling and the notation \( \mathcal{M} = \max \{ u, v, w, z \} \), the set of tableaux of shape \((a, 2, 1^b)\) can be partitioned into twelve classes as in Table I.

<table>
<thead>
<tr>
<th>Class</th>
<th>Properties</th>
<th>Class</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( w = \mathcal{M} ) ( z &gt; w ), and ( w &gt; v ) ( u &gt; v ) ( v &gt; u )</td>
<td>7</td>
<td>( z = \mathcal{M} ) ( w &gt; v ), ( u &gt; v ) ( v &gt; u )</td>
</tr>
<tr>
<td>2</td>
<td>( u = \mathcal{M} ) ( z &gt; w ), and ( w &lt; v ) ( v &lt; w )</td>
<td>8</td>
<td>( z = \mathcal{M} ) and ( w &lt; v ) ( v &lt; w )</td>
</tr>
<tr>
<td>3</td>
<td>( u = \mathcal{M} ) ( z &gt; w ), and ( w &gt; v ) ( v &gt; u ) ( z &gt; w )</td>
<td>9</td>
<td>( v = \mathcal{M} ) ( z &gt; u ), ( z &gt; w )</td>
</tr>
<tr>
<td>4</td>
<td>( u = \mathcal{M} ) ( z &lt; w ), and ( w &lt; v ) ( v &lt; w )</td>
<td>10</td>
<td>( v = \mathcal{M} ) ( z &gt; u ), ( z &lt; w )</td>
</tr>
<tr>
<td>5</td>
<td>( u = \mathcal{M} ) ( z &lt; w ), and ( w &gt; v ) ( v &lt; w )</td>
<td>11</td>
<td>( v = \mathcal{M} ) ( z &lt; u ), ( z &lt; w )</td>
</tr>
<tr>
<td>6</td>
<td>( z = \mathcal{M} ) ( w &gt; v ), ( u &gt; v )</td>
<td>12</td>
<td>( v = \mathcal{M} ) ( z &lt; u ), ( z &gt; w )</td>
</tr>
</tbody>
</table>

**Figure 2.**
The simplest kicking polynomials are those that involve only linear factors. This occurs for classes 1, 2, 3 and 5. For $T$ in class 1, we have

$$\phi_T = (y_u - \beta_1) \times \prod_{i \in A_u} P_i \times \prod_{i \in R_u} Q_i \times \prod_{i \in A_{cu}} (x_i - \alpha_1).$$  

Incidentally, by setting $y_u = 0$ in the preceding formula, one gets the kicking polynomial for hook shapes with the label $u$ in the $(1, 1)$ coordinate. For $T$ in class 2, for instance, one has

$$\phi_T = \prod_{i \in A_u} P_i \times \prod_{i \in R_u} Q_i \times \prod_{i \in A_{cu}} (x_i - \alpha_1) \times \prod_{i \in A_{ci}} (x_i - \alpha_2).$$

For $T$ in class 5, one sets $|A_{<u}| = k$ and

$$\phi_T = \prod_{i \in A_u} P_i \times \prod_{i \in R_u} Q_i \times \prod_{i \in A_{cu}} (x_i - \alpha_1) \times \prod_{i \in A_{ci}} (x_i - \alpha_2)$$

$$\times \left( y_u + \sum_{i \in A_{cu}} y_i - k \beta_1 - \beta_2 \right).$$

For the remaining classes one must use the machinery developed in Section 2. Namely, the kicking polynomials will contain special Schur functions and homogeneous symmetric functions as factors. For example, for $T$ in class 6, we let $|A_{<u}| = k$ and $|A_u - A_{>u}| = p$ and set

$$\phi_T = \prod_{i \in A_u} P_i \times \prod_{i \in R_u} Q_i \times \prod_{i \in A_{<u} \cup \{u\}} (x_i - \alpha_1) \times \prod_{i \in A_{>u} - A_{<u}} (x_i - \alpha_2)$$

$$\times h_{p-k} \left( x_u + \sum_{i \in A_{<u}} x_i - \sum_{j \in A_{>u} - A_{<u}} \alpha_j \right).$$

The full strength of these special functions is needed for class 9 where we define

$$\vec{Y} = \{ y_1 \mid l \in (R_{>u} - \hat{R}_{>u}) \cup \{ w \} \} \quad \text{with} \quad |\vec{Y}| = p + 1, \quad (13)$$

$$B_1 = \{ \beta_1, \beta_2 \}, \quad (14)$$

$$B_2 = \{ \beta_{e,\{m\}} \mid m \in (R_u - \hat{R}_{>u}) \cup \{ u \} \} \quad \text{with} \quad |B_2| = r + 1, \quad (15)$$
and $k = |A_{<v}|$ to get

$$
\phi_T = \prod_{i \in A_v} P_i \times \prod_{i \in R_{<u} \cup \bar{R}_{<u}} Q_i \times \prod_{i \in A_{<u}} (x_i - \alpha_i) \\
\times \prod_{i \in R_{<u} \cup \bar{R}_{<u}} \prod_{j \in L_{<u}} (y_i - \beta_{v(j)}) \times \prod_{i \in R_{<u} \cup \bar{R}_{<u}} (y_i - \beta_2) \\
\times h_k \left( x_i + x_u + \sum_{i \in R_{<u} \cup \bar{R}_{<u}} x_i - p \alpha_1 - \sum_{j \in A_{<v} \cup \{v\}} x_{v(j)} \right) \\
\times S_{(p, v, K)} \left[ B_1 - \bar{Y}, B_2 - \bar{Y}, ..., B_2 - \bar{Y} \right].
$$

The notation is best illustrated with an explicit example. Let

$$
T = 
\begin{array}{c}
1 \\
7 \\
9 \\
5 \\
4 \\
8 \\
10 \\
6 \\
3 \\
2
\end{array}
$$

One sees that this tableau is in class 9 and writes down the following sets of labels:

$$
A_v = A_9 = \{1, 7, 11\}, \quad R_{<u} \cup \bar{R}_{>u} = R_{<4} \cup \bar{R}_{>4} = \{2, 3, 10\}, \\
R_{>u} - \bar{R}_{>u} = R_{>4} - \bar{R}_{>4} = \{6, 8\}, \quad R_{<u} - R_{<u} = R_{<4} - R_{<4} = \{6\}, \\
A_{<v} = A_{<u} = \{1, 7\}.
$$

The corresponding polynomial is then

$$
\phi_T = (x_{11} - \alpha_4)(x_{11} - \alpha_4)(x_7 - \alpha_3) \times (y_{10} - \beta_2)(y_{10} - \beta_1) \\
\times (x_1 - \alpha_1)(y_6 - \beta_2) \times h_2(x_9 + x_5 + x_6 + x_8 - 2\alpha_1 - \alpha_2 - \alpha_3 - \alpha_5) \\
\times S_{(2, 1, v)} \left[ B_1 - \bar{Y}, B_2 - \bar{Y}, B_2 - \bar{Y} \right],
$$

where $B_1 = \beta_1 + \beta_2$, $B_2 = \beta_1 + \beta_2 + \beta_4 + \beta_5 + \beta_6$ and $\bar{Y} = y_5 + y_8 + y_{10}$. One can easily check that $w_{e,q}(T) = 6 + 7 = \deg_{e,q}(\phi_T)$.

To prove that the polynomials constructed are indeed correct, one must show for each $\phi_T$ that

$$
\phi_T(p) = \begin{cases} 
\text{nonzero} & \text{if } T = \bar{T}, \\
0 & \text{if } \bar{T} < T, \quad \text{and } \deg \phi_T = w_{e,q}(T). 
\end{cases}
$$

In order to give the reader an idea of how this is done in [7], we shall show this for the polynomial given in Eq. (10) for class 2.
That $\phi_T(\rho_T) \neq 0$ follows from the algebraic independence of the $\alpha$'s and $\beta$'s. Demonstrating that $\phi_T(\rho_T) \neq 0$ when $\tilde{T} < T$ is far more cumbersome since one needs to know the coordinates of $s$, the largest label where $T$ and $\tilde{T}$ differ. To this end, we use the order $<$ to carefully build up a table of possible cases for each class. In particular, for class 2, one constructs Table II.

With this table as a guide we proceed to show that $\phi_T$ vanishes in each of these cases.

Case 1. $s \in A_s$. Since $r_T(s) > r_T(f)$, the maximality of $s$ guarantees that there is a label $f \in A_{s-f}$ in the tableau $T$ such that $r_T(s) = r_T(f)$.

Thus, $P_s$ (which appears as a factor in $\prod_{i \in s} \lambda_i$) includes the factor $(x_s - \lambda_{r_T(s)}) = (x_s - \lambda_{r_T(f)})$ and since the $s$th component of $\rho_T$ is $\lambda_{r_T(s)}$, we have $P_s(\rho_T) = 0$.

Case 2. $s \in R_{\alpha_u}$. The entry in Table II, $c_T(s) < c_T(f)$, gives rise to two possibilities. In both of these, we exhibit a factor in $Q_s$ (which appears as a factor in $\prod_{i \in s} \lambda_i$) which vanishes.

1. $r_T(s) = 1$. By the maximality of $s$, there exists a label $j < L_{<s}$ such that the coordinates of $j$ in $T$ are the same as those of $s$ in $\tilde{T}$. Thus, the factor $(y_s - \beta_{r_T(j)}) = (y_s - \beta_{r_T(f)})$ of $Q_s$ vanishes.

2. $r_T(s) > 1$. We note that $s > u > z \Rightarrow u, z \in L_{<s}$. Therefore, $Q_s$ contains the factor $(y_s - \beta_{r_T(j)}) \times (y_s - \beta_{r_T(j)}) = (y_s - \beta_1)(y_s - \beta_2)$ which must vanish because the $(n+s)$th component of $\rho_T$ is either $\beta_1$ or $\beta_2$.

### Table II

Relative Position Table for $T$ in Class 2 with $\tilde{T} < T$

<table>
<thead>
<tr>
<th>Case</th>
<th>Position of $s$ in $T$</th>
<th>Position of $s$ in $\tilde{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$s \in A_s$</td>
<td>$r_T(s) &gt; r_T(f)$</td>
</tr>
<tr>
<td>2</td>
<td>$s \in R_{&gt;u}$</td>
<td>$c_T(s) &lt; c_T(f)$</td>
</tr>
<tr>
<td>3</td>
<td>$s = u, z$</td>
<td>$r_T(s) &gt; r_T(f)$</td>
</tr>
<tr>
<td>4</td>
<td>$s = u, w$</td>
<td>$r_T(s) &gt; 2$</td>
</tr>
<tr>
<td>5</td>
<td>$s \in R_2 - R_{&gt;u}$</td>
<td>$(c_T(s) &lt; c_T(f)$ and $r_T(s) = 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>or $r_T(s) &gt; 1$</td>
</tr>
</tbody>
</table>
Case 3. $s = u, z$. Since, $r_T(s) > r_T(u) = 1$ and $s \leq u$, we may apply the maximality of $s$ and the pigeonhole principle to conclude that there is a label $l \in A_{<u}$ such that $r_T(l) = 1$. But, then the product

$$
\prod_{i \in A_{<u}} (x_i - x_1),
$$

which includes $(x_i - x_1)$ as a factor, vanishes on $p_T$.

Case 4. $s = v, w$. Here, $r_T(s) > r_T(u) = 1$ and $s \leq v < u$. Thus, we may use the pigeonhole principle and the maximality of $s$ to conclude that there is a label $l \in A_{<v} \subseteq A_{<u}$ such that $r_T(l) = e \leq 2$. But, then the product

$$
\prod_{i \in A_{<v}} (x_i - x_1) \times \prod_{i \in A_{<v}} (x_i - x_2),
$$

which involves $(x_i - x_2)$ as a factor, vanishes on $p_T$.

Case 5. $s \in R_z - R_{<u}$. The table gives two possibilities:

1. $c_T(s) < c_T(u)$ and $r_T(s) = 1$. Here, since $r_T(s) = 1$, we argue as in case 2 to show that $Q(s) = 0$.

2. $r_T(s) > 1$. In this situation, since $s < u$, we may argue exactly as in case 4 to show that factor Eq. (18) vanishes.

Next, we need to show that the $x, y$-degree of $\phi_T$ equals the $t, q$-weight of $T$. To this end, we write the disjoint unions

$$
R_u = \{ z \} \cup R_{<z} \cup (R_{>z} - R_{>u}) \cup R_{>u}
$$

and

$$
A_z = A_{<z} \cup (A_{>z} - A_{>u}) \cup A_{>u}.
$$

Using the definition of the weight function, the first labels to be removed from $T$ are those which are larger than $u$. Namely, these labels are in $A_{>u}$ and $R_{>u}$; and since removing labels of $A_{>u}$ does not affect the contribution to the weight of the labels from $R_{>u}$, and vice versa, we may write

$$
w(T) = \left( \prod_{i \in A_{>z}} \prod_{j \in A_{<z}} t \right) \left( \prod_{i \in R_{>z}} \prod_{j \in L_{<z}} q \right) w(T_1),
$$

where $T_1$ is the tableau resulting by extracting $R_{>u}$ and $A_{>u}$ from $T$. Next, observing that contributions from the sets $A_{>z} - A_{>u}$ and $R_{>z} - R_{>u}$ are

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Codes: 2561 Signs: 1455. Length: 45 pic 0 pts, 190 mm
independent, one removes $u$ followed by $A_{>u} - A_{>w}$ and $R_{>z} - R_{>u}$. The resulting tableau is named $T_2$.

$$w(T_1) = t^{d_{A,c}} \left( \prod_{v \in A_{<w} - A_{>w}} t \right) \left( \prod_{v \in R_{<z} - R_{>u}} q \right) w(T_2).$$  \hspace{1cm} (20)

The remaining labels that can contribute to the weight of $T_2$ are included in the sets $\{z\} \cup R_{>z}$ and $\{v\} \cup A_{<v}$. The contributions from these sets are independent of each other. Thus, we have

$$w(T_2) = t^{d_{A,c}} \left( \prod_{v \in A_{<v}} t \right) \left( \prod_{v \in R_{<z}} q \right).$$  \hspace{1cm} (21)

Combining the above calculations we get

$$w(T) = 1 + |A_{<w}| + |A_{<z}| + \left( \sum_{v \in A_{<w}} \sum_{j \in A_{<v}} 1 \right)$$

$$+ \left( \sum_{v \in R_{<z}} \sum_{j \in A_{<z}} 1 \right)$$

$$= 1 + |A_{<w}| + |A_{<z}| + \left( \sum_{v \in A_{<w}} \sum_{j \in A_{<v}} 1 \right)$$

$$= \deg(x)(T).$$

Likewise,

$$w(T) = \left( \sum_{v \in R_{<z}} \sum_{j \in L_{<z}} 1 \right)$$

$$= \deg(y)(T).$$

**Remark.** We should add a word regarding step 5, of Section 1, which is required to complete the proof of the conjecture. Namely, one is required to show that

$$\max \{ \deg \phi \mid \phi \in H(\{\phi\}) \} = \deg \mu = n_d(\mu).$$
This is achieved in the following manner. First, by a straightforward inductive argument one shows that if \( \mu = (a, 2, 1^b) \) then
\[
\max \{ w_{\mu, q}(T) : T \in \mathcal{F}(\mu) \} = 2 + \binom{b+2}{2} + \binom{a}{2}.
\]
Next, the construction of the kicking polynomials gives the following equality of multisets,
\[
\{ \deg \phi_T : T \in \mathcal{F}(\mu) \} = \{ w_{\mu, q}(T) : T \in \mathcal{F}(\mu) \}.
\]
Combining this with the fact that the set \( \{ h(\phi_T) : T \in \mathcal{F}(\mu) \} \) forms a basis for \( \text{gr} \mathbf{R}_{\{\mu\}} \) we conclude that
\[
\max \{ \deg P : P \in \text{gr} \mathbf{R}_{\{\mu\}} \} = 2 + \binom{b+2}{2} + \binom{a}{2}.
\]
Furthermore, since \( H_{\{\mu\}} \) and \( \text{gr} \mathbf{R}_{\{\mu\}} \) are equivalent as graded modules one has
\[
\max \{ \deg P : P \in \text{gr} \mathbf{R}_{\{\mu\}} \} = \max \{ \deg P : P \in H_{\{\mu\}} \}.
\]
Finally, a straightforward computation yields
\[
\deg A_{(a, 2, 1^b)} = 2 + \binom{b+2}{2} + \binom{a}{2}.
\]

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References


