ON THE JACOBIAN CONJECTURE

Raymond C. HEITMANN

Department of Mathematics, The University of Texas at Austin, Austin, TX 78712, USA

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The Jacobian conjecture in two variables is studied. It is shown that if \( f, g \in \mathbb{C}[x, y] \) have unit Jacobian but \( \mathbb{C}[f, g] \neq \mathbb{C}[x, y] \), then necessarily \( \gcd(\deg(f), \deg(g)) \geq 16 \). Other restrictions on counterexamples are also obtained. The conjecture is equivalent to "If \( C \) is the generic curve given by the polynomial \( f(x, y) \) and \( dx/f \) is an exact differential on this curve, then \( C \) has genus zero".

Introduction

This article is a description of the approach, in some respects novel, used by this researcher in attempts to resolve the Jacobian conjecture. The focus has been directed at the two variable case. While some aspects of the approach generalize to more variables, sufficient obstacles remain to make the generalization unpromising at this time.

The underlying strategy is the minimal counterexample approach. We assume the Jacobian conjecture is false and derive properties which a minimal counterexample must satisfy. The ultimate goal is either a contradiction (proving the conjecture) or an actual counterexample. A more immediate goal is to produce two polynomials \( f, g \) such that \( \mathbb{C}[f, g] \neq \mathbb{C}[x, y] \) while the properties do not rule out \( J(f, g) = 1 \). Such an example might help us to obtain additional properties or to get some idea what a true counterexample looks like.

The primary results of this paper are Theorems 2.5 and 2.23. The first, which I believe is new, gives an equivalent formulation of the Jacobian conjecture in two variables. (An analogue for more variables should exist.) The new formulation is: *If \( C \) is the generic curve given by the polynomial \( f(x, y) \), the differential \( dx/f \) can only be an exact differential on this curve if \( C \) is a genus zero curve.* The second is: *If \( (f, g) \) comprise a counterexample to the Jacobian conjecture, then \( \gcd(\deg(f), \deg(g)) \geq 16 \).* Nothing like this appears in the literature but results of this type are known by Abhyankar and Moh and are easily inferred from their published work. Consequently, the primary contribution of this paper is the method employed and not the specific results.

The primary work to date on lower bounds of degrees for counterexamples is that
of Moh [3]. He proved \( \sup \{\deg(f), \deg(g)\} > 100 \). He begins with a number of computational results, employs a computer search to reduce the problem to four problem cases and then develops a reduction of degree trick to rule out these four cases by hand. This paper attacks the same problem with a number of computational results and a computer search which reduces the problem to exactly the same four cases. We do not have an analogue for the reduction of degree trick and do not reprove Moh's result. The similarity in results suggests that the two methods might really be the same but they look rather different and I cannot find a translation.

In Section 1, we explore algebraically the relation between two polynomials whose Jacobian is a unit. A number of the results appear in notes by Abhyankar [1]. In Section 2, we switch to a geometric approach and derive the results. Theorem 2.23 is proved by hand but the results after that require a computer search. Section 3 is a brief consideration of a particular example.

The beginning of Section 2 (through Theorem 2.5) is independent of Section 1. As this portion of the paper contains most of the new ideas in this paper, including Theorem 2.5, it should be the starting place for most readers.

It is intended that this paper be self-contained. However, we shall not state the Jacobian conjecture in full generality nor give an expository presentation. An excellent presentation appears in the article by Bass, Connell, and Wright [2] and anyone interested in the subject should at least read the introduction to that article.

1. The algebraic approach

The Jacobian conjecture, as we shall approach it, is the following: Let \( x, y \) be algebraically independent over \( \mathbb{C} \). If \( f, g \in \mathbb{C}[x, y] \) and \( f_x g_y - f_y g_x = 1 \), then \( \mathbb{C}[f, g] = \mathbb{C}[x, y] \), or equivalently \( \mathbb{C}(f, g) = \mathbb{C}(x, y) \).

We shall let \( L \) denote the field \( \mathbb{C}(x, y) \) throughout this paper.

**Definition.** A monomial valuation on \( L \) is a triple \((u, z, w)\) such that \( L = \mathbb{C}(z, w) \) and \( u \) is a valuation on \( L \) satisfying (i) \( u \) is trivial on \( \mathbb{C} \), and (ii) if \( h \in \mathbb{C}[z, w] \), then \( u(h) = \inf \{ u(z^i w^j) \mid z^i w^j \in \text{supp}(h) \} \).

**Definition.** A polynomial \( h \in \mathbb{C}[z, w] \) is called homogeneous (with respect to \((u, z, w)\)) if \( u(z^i w^j) \) is constant for all \( z^i w^j \in \text{supp}(h) \).

The leading form of a polynomial is the homogeneous summand of lowest value. This generalizes trivially to elements of \( \mathbb{C}[z, w, z^{-1}, w^{-1}] \). In general, the leading form of an element of \( \mathbb{C}(z, w) \) will be the quotient of the leading form of its numerator by the leading form of its denominator.

**Remark.** Many of the results in this section can be found in [1] and the ideas behind the proofs are the same. However, as our notation is quite different and we need a more general form of the results, the presentation here will be thorough. The
weights used by Abhyankar are the same as monomial valuations \((u, z, w)\) where \(\mathbb{C}[z, w] = \mathbb{C}[x, y]\).

**Notation.** \(J_{x,y}(f, g)\) will denote the determinant of the Jacobian matrix \((-f_y g_x - f_x g_y)\). We will write \(J(f, g)\) when the variables are clear.

We will use capital letters to denote leading forms; e.g., \(F\) will be the leading form of \(f\).

**Lemma 1.1.** Let \((u, z, w)\) be a monomial valuation and \(f, g \in L\).

1. \(v(J_{z,w}(f, g)) \geq v(fg z^{-1}w^{-1})\).
2. Equality holds in (i) \(\iff J_{z,w}(F, G) \neq 0\). In this case, \(J_{z,w}(F, G)\) is the leading form of \(J_{z,w}(f, g)\).

**Proof.** First we claim that if \(h \in L\), \(v(h_z) \geq v(hz^{-1})\). Write \(h = h_1/h_2\) with \(h_1, h_2 \in \mathbb{C}[z, w]\). Then \(h_z = (h_2)^{-2}(h_2(h_1)_z - (h_2)_zh_1)\). So it suffices to prove the claim for \(h \in \mathbb{C}[z, w]\). As \(v\) is defined by taking the infimum over monomials in the support of \(h\), it suffices to consider the case where \(h\) is a monomial. Clearly here either \(h_z = 0\) or \(v(h_z) = v(hz^{-1})\). It should be remarked that if \(h\) is homogeneous, strict inequality is possible only if \(h_z = 0\).

If \(F, G\) are the leading forms of \(f, g\), then \(v(f-F) > v(f)\) and \(v(g-G) > v(g)\). Considering the fact \(J(f, g) = J(F, G) + J(f-F, G) + J(f, g - G)\), the lemma follows immediately from the claim.

**Lemma 1.2.** Let \((u, z, w)\) be a monomial valuation and let \(F, G\) be nonzero homogeneous elements of \(\mathbb{C}[z, w, z^{-1}, w^{-1}]\) such that \(J_{z,w}(F, G) = 0\). Then there exists \(\alpha \in \mathbb{C}\) such that \(F_{u} = \alpha G_{u}\).

**Proof.** As the case \(v(F) = v(G) = 0\) is trivial, we may assume \(v(G) \neq 0\). This rules out the case \(v(z) = v(w) = 0\); by symmetry, we may assume \(v(w) \neq 0\). Let \(i = \inf \{s \mid z^iw' \in \text{supp}(F)\}\) and \(j = \inf \{s \mid z^iw' \in \text{supp}(G)\}\). Let \(u = z^aw^{-b}\) where \((b/a) = v(z)/v(w)\), \(\gcd(a, b) = 1\), and \(a > 0\). Then \(v(u) = 0\) and we can write \(F = z^i w^k \Theta(u)\) and \(G = z^j w^l \Psi(u)\) where \(\Theta(0) \Psi(0) = 0\).

Now
\[
0 = J(F, G) = \Theta \Psi J(z^iw^k, z^jw^l) + \Theta \Psi' J(z^iw^k, u) + \Theta' \Psi J(U, z^jw^l)
\]
\[
= z^{i+j-1}w^{k+l-1}[(il-jk)\Theta \Psi + (-ib-ka)u\Theta \Psi' + (al+jb)u\Theta' \Psi].
\]

Since \(\Theta(0) \Psi(0) = 0\), we get \(il-jk = 0\) and \((-ib+ka)\Theta \Psi' + \Theta(G)\Theta' \Psi = 0\). The second condition says \(-v(F)\Theta \Psi' + v(G)\Theta' \Psi = 0\) and so \((d/du)(\Theta \Psi(G) / \Psi(u)) = 0\). Thus \(\Theta(G) = \alpha \Psi(F)\) for some \(\alpha\). The condition \(il-jk = 0\) yields \((x^iy^k)^{v(G)} = (x^iy^l)^{v(F)}\) and so the result follows.

With a proof in the same spirit as this one, we now present a technical proposition, the value of which shall become clear later.
Proposition 1.3. Let \( s, t, i, j, k, l, a, b \in \mathbb{Z} \); \( u = z^n w^m \). Suppose \( \Theta(u) \), \( \Psi(u) \in \mathbb{C}[u] \) with \( \Theta(0) \Psi(0) \neq 0 \) and \( \Theta \in \mathbb{C} \). Further assume \( J_{2,1}(z^iw^j \Theta, z^kw^j \Psi) = w^j(z^i w^j \Theta)^j \). Let \( \gamma = ib - ju \), \( \sigma_1 = u(1 + s) - b \), \( \sigma_2 = i(1 + s) - j \), \( M = \deg \Theta \), \( N = \deg \Psi \), \( \alpha = il - jk \), and \( \alpha^* = (i + Ma)(l + Nb) - (j + Mb)(k + Na) \). If \( \gamma \neq 0 \), then one of the following holds:

(a) \( \sigma_1 \gamma \geq 0 \). Also, if \( \sigma_1 \neq 0 \), \( \Theta^{-1} \) does not divide \( \Psi \).

(b) \( \alpha \neq 0, \alpha^* = 0, \Theta^{-1} \big| \Psi, \sigma_2 \neq 0 \), and \( \sigma_2 + \sigma_1 M = 0 \), and \( \sigma_2 / \gamma = C \in \mathbb{Z}^+ \).

(c) \( \alpha = \alpha^* = 0, \Theta^{-1} \big| \Psi, \sigma_2 / \gamma \in \mathbb{Z}^+ \), \( (\sigma_2 + \sigma_1 M)/(\gamma) \in \mathbb{Z}^+ \) and \( \sigma_1 M/(\gamma) = C \in \mathbb{Z}^+ \).

(d) \( \alpha = \alpha^* = 0, \Theta^{-1} \big| \Psi, \sigma_2 / \gamma \in \mathbb{Z}^+ \), \( (\sigma_2 + \sigma_1 M)/(\gamma) \in \mathbb{Z}^+ \) and \( \sigma_1 M/(\gamma) = C \in \mathbb{Z}^+ \).

In cases (b), (c), (d), \( \Theta \) has at most \( C \) linear factors (excluding repetition).

Proof. Let \( \beta = al - bk \). Using straightforward computation, as in the proof of Lemma 1.2, the Jacobian condition in the hypothesis can be rewritten

\[
\begin{align*}
&z^{-i-k-l}w^{-i-l-1}[a \Theta \Psi + yu \Theta \Psi' + \beta u \Theta' \Psi] = z^{-i-l-s} \Theta \Psi' \\
&\text{If we let } D = \text{the largest power such that } u^D \text{ divides } [a \Theta \Psi + yu \Theta \Psi' + \beta u \Theta' \Psi], \text{ we obtain (i) } i+k-l+aD = it \\
&\text{and so } k = (t-1) + 1 - aD, \text{ (ii) } j+l-1+bD = j+s \text{ and so } l = j(t-1) + 1 + s - bD, \text{ and (iii) } a \Theta \Psi + yu \Theta \Psi' + \beta u \Theta' \Psi = u^D \Theta \Psi'. \text{ Of course, } D = 0 \Leftrightarrow \alpha \neq 0. \text{ Using our first two equations, we can compute }
\end{align*}
\]

\[
\beta = al - bk = a(j(t-1) + 1 + s - bD) - b(i(t-1) + 1 - aD) = -\gamma(t-1) + \sigma_1.
\]

The proof neatly divides into four cases depending on the values of \( \alpha \) and \( \alpha^* \).

Case (a). We first consider the case \( a \alpha^* \neq 0 \). Since \( D = 0 \), we have \( a \alpha^* = \gamma^2 \). Consider all cases in which \( \alpha^* \neq 0 \) and \( \Theta'^{-1} \) does not divide \( \Psi \). Now

\[
\begin{align*}
\alpha^* &= (i + Ma)(l + Nb) - (j + Mb)(k + Na) = \alpha + \gamma N + \beta M.
\end{align*}
\]

Up to multiplication by the lead coefficients of \( \Theta \) and \( \Psi; \alpha + \gamma N + \beta M \) will be the coefficient of \( u^{N+M} \) on the left-hand side of our Jacobian condition and so \( \alpha^* \neq 0 \) says precisely that \( \deg(\Theta') = N + M \), i.e., \( N = (t-1)M \). From this, we may conclude that either \( \Psi \) is a scalar multiple of \( \Theta'^{-1} \) or \( \Theta'^{-1} \) does not divide \( \Psi \). Consider the first case. There, \( \Theta \Psi' = (t-1) \Theta' \Psi \). Substituting this and the value of \( \beta \) into the Jacobian condition, we obtain \( \alpha \Theta \Psi' + \sigma_1 \Theta \Psi = \Theta \Psi' \), and so \( \alpha \Theta + \sigma_1 \Theta' = \delta \Theta \) for some scalar \( \delta \). This is satisfied when \( \sigma_1 = 0 \) and so \( \sigma_1 \gamma \geq 0 \), but otherwise yields \( \Theta' / \Theta = (\delta - \alpha) / \sigma_1 \). This indicates \( \Theta \) is a power of \( u \), a contradiction. So \( a \alpha^* \neq 0 \) and consider all cases in which \( \Theta'^{-1} \) does not divide \( \Psi \). Here \( \Theta \) has a linear factor \( \phi \) such that \( \phi \deg(\Psi) < (t-1)(\phi \deg(\Theta)) \). As \( \phi \deg(\Psi') + \phi \deg(\Theta) \) divides \( u^D \Theta' - a \Theta \Psi \), it must also divide \( \gamma \Theta \Psi' + \beta \Theta' \Psi \). This occurs only when \( \gamma(\phi \deg(\Psi)) + \beta(\phi \deg(\Theta)) = 0 \) and so \( \gamma^2(\phi \deg(\Psi)) + \gamma \beta(\phi \deg(\Theta)) = 0 \). Our inequality now yields \( \gamma^2(t-1)(\phi \deg(\Theta)) + \gamma \beta(\phi \deg(\Theta)) > 0 \) and so \( \gamma^2(t-1) + \gamma \beta > 0 \). Finally, plugging in for \( \beta \), we get \( \alpha_1 \gamma \geq 0 \) as desired.

Simplification for subsequent cases: It remains to show that if \( a \alpha^* - 0 \) and \( \Theta'^{-1} \big| \Psi \), then one of cases (b), (c), (d) holds. Here we may reduce to the \( t = 1 \) case...
by replacing $z^k w^j$ by $z^{k-(t-1)} w^{j-(t-1)}$. The quantities $s, i, j, a, b, c, \sigma, \sigma_2, M, \gamma$ are unchanged by this replacement, and those which are do not appear in the conclusion. We will denote the new value of $N$ by $C$. It is easy to see from the Jacobian condition $a\theta'Y + \gamma u\theta'Y' + \beta u\theta'Y' = uD\theta$ that $\theta$ must divide $\theta'Y$ and so each linear factor of $\theta$ must divide $Y$. Thus $C = \deg Y$ bounds the number of distinct linear factors of $\theta$. Of course, $C \in \mathbb{Z}^+$.  

Case (b). $\alpha \neq 0, \alpha^* = 0$. With $t=1$, $\beta = \sigma_1$ and $\alpha = il - jk = i(1 + s - bD) - j(1 - aD) = \sigma_2 - \gamma D$. Since $\alpha \neq 0$, $D = 0$ and so $\sigma_2 = \alpha \neq 0$. Finally $0 = \alpha^* = \alpha + \gamma N + \beta M = \sigma_2 + \gamma C + \sigma_1 M$ and so $C = (\sigma_2 + \sigma_1 M)/(-\gamma)$.

Case (c). $\alpha = 0, \alpha^* \neq 0$. Considering the equation $a\theta'Y + \gamma u\theta'Y' + \beta u\theta'Y' = uD\theta$, $\alpha^* \neq 0$ says the highest degree terms on the left side do not sum to zero and so $C = \deg Y = D$. Plugging in $\beta = \sigma_1$ and $\alpha = \sigma_2 - \gamma D = \sigma_2 - \gamma C$, $0 = \alpha^* = \alpha + \gamma N + \beta M = \sigma_2 - \gamma C + \gamma C + \sigma_1 M$ and so $\sigma_2 + \sigma_1 M \neq 0$. Also $0 = \alpha = \sigma_2 - \gamma D = \sigma_2 - \gamma C$ says $C = \sigma_2/\gamma$.

Case (d). $\alpha = \alpha^* = 0$. $0 = \alpha^* = \alpha + \gamma N + \beta M = \gamma C + \sigma_1 M$. So $C = \sigma_1 M/\gamma$. Also $\alpha^* = 0 \Rightarrow C > D$. As $\alpha = 0$, $D = \sigma_2/\gamma \in \mathbb{Z}^+$ and so $-\sigma_1 M/\gamma > \sigma_2/\gamma$; thus $(\sigma_2 + \sigma_1 M)/(-\gamma) \in \mathbb{Z}^+$.

Remark. The idea for using Proposition 1.3 will be to ask if $Y$ can be found satisfying the Jacobian condition if all other quantities are fixed. If none of the four cases is satisfied, the answer is no (the useful answer in the sequel). On the other hand, we have not proved the converse and in fact have used so little of our hypothesis that a stronger result would seem probable at first glance. However, after considerable study, I now suspect that Proposition 1.3 tells most of the story.

It will be useful to have a topology on the monomial valuations $(u, z, w)$ for fixed $z, w$. Associate to $v$ the point $(-v(z), -v(w))$. Every integral point is thus associated to a monomial valuation. The origin corresponds to the trivial valuation and so we discard it. Also note that the valuations corresponding to two points on the same half-line emanating from the origin induce the same valuation ring. We will call such valuations equivalent. Then the equivalence classes of non-trivial valuations correspond in a 1-1 manner to the points on the unit circle where the slope is rational (a dense set). The usual topology on the circle induces a topology on monomial valuations. The points on the circle where the slope is irrational also correspond to valuation rings (not discrete ones) in a natural fashion. For the irrational valuations, the leading form of any element of $L$ will be a monomial; this means they will not be of any real interest. However, including them makes our topological space compact. We also shall delete those valuations corresponding to the open third quadrant. We do not need them and the deletion yields the property that the intersection of two intervals will again be an (possibly empty) interval. Hereafter, we will always assume our monomial valuations are equipped with this topology; we will interchangeably refer to the points of this space as valuations, valuation rings, and equivalence classes.
For any open interval $U$, there is a ring $\Gamma(U)$ which is the intersection of all valuation rings in $U$. Each valuation ring can be regarded as a topological ring and so can be completed. Let $\Gamma(U)^*$ denote the intersection of the completions of the valuation rings in $U$. Finally let $A(U)$ be the algebraic closure of $L$ in the quotient field of $\Gamma(U)^*$. We let $A_v$ denote the stalk at $v$ of the presheaf $A$, i.e., those elements in the quotient field of the completion of the valuation ring for $v$ which are algebraic over $L$ and which lie in the quotient field of the completion of all valuation rings in some neighborhood of $v$.

**Lemma 1.4.** Let $(v, z, w)$ be a monomial valuation and let $h_1, \ldots, h_n$ be a finite set of elements of $L$ with $v$-leading forms $H_1, \ldots, H_n$ respectively. Then there is an open neighborhood $U$ containing $v$ such that for any $(v', z, w) \in U$ and any $h_i \in L$, the $v'$-leading form of $h_i$ will be the $v'$-leading form of $H_i$.

**Proof.** We can write each $h_i$ as a quotient of two elements of $C[z, w]$. It suffices to prove the lemma using these $2n$ elements in place of our original set. Further, if we can prove the result for a single element of $C[z, w]$, we may find $U$ by taking a finite intersection of the neighborhoods $U_i$. So we only need to prove the lemma for a single $h \in C[z, w]$. If $M_1$ is a monomial in $\text{supp}(H)$ and $M_2$ is a monomial in $\text{supp}(h) - \text{supp}(H)$, then we note $v'(M_1) < v'(M_2)$ for all $v'$ in an open half-circle which contains $v$. By taking the intersection of the half-circles corresponding to all pairs $(M_1, M_2)$, we find our open set $U$. 

**Lemma 1.5.** Let $(v, z, w)$ be a monomial valuation, $h \in L$, $H$ the leading form of $h$, $q \in \mathbb{Z}^+$, $\gcd(p, q) = 1$. Then $h^{(p/q)} \in A_v \iff H^{(1/q)} \in L$.

**Proof.** Elements of $A_v$ can be regarded as convergent infinite sums of elements of $L$. As such, the concept of leading forms still makes sense. Clearly if $g = h^{(p'/q')}$, then $G = H^{(p'/q')}$ and so $H^{(1/q')} \in L$. So one implication is proved. Conversely, if $H^{(1/q')} \in L$, we will show $h^{(1/q')} \in A_v$ and so $h^{(p/q')} \in A_v$.

We choose an interval $U$ by Lemma 1.4, so that for all $v' \in U$, the $v'$-leading form of $h$ is the $v'$-leading form of $H$. Now $h = H + h^* = H(1 + H^{-1}h^*) = H(1 + \phi)$ where $v'(\phi) > 0$. Since $H$ has a qth root, it suffices to prove $(1 + \phi)$ has a qth root. However, simply expand $(1 + \phi)^{1/q}$ in a binomial expansion $1 + (1/2)\phi + (1/4)\phi^2 + \cdots$. As $v'(\phi^n) \to \infty$, this series is an element of $A_v$. 

In general, of course, $h$ will actually have $q$ qth roots, and the notation $h^{(1/q)}$ is ambiguous. We shall need to make this choice in a consistent manner. Choosing $h^{(1/q)}$ is the same as selecting an embedding $L(h^{1/q}) \to A_v$. Then if $U$ is the largest interval containing $v$ for which $h$ has a qth root in $A(U)$, this embedding induces an embedding $L(h^{1/q}) \to A(U)$. This gives a consistent choice for $h^{(1/q)}$ for all $v' \in U$. We will assume all roots are chosen in this manner.

**Lemma 1.6.** If $(v, z, w)$ is a monomial valuation and $\phi, \theta \in L(v)$ are defined by convergent sequences $\phi_n \to \phi$ and $\theta_n \to \theta$, then we may define $J_{z, w}(\phi, \theta) = \lim J_{z, w}(\phi_n, \theta_n)$. 

Proof. Since \( \{\phi_n\}, \{\theta_n\} \) converge, so do \( \{(\phi_n)z\}, \{(\phi_n)w\}, \) etc.; the result follows trivially. \( \square \)

**Lemma 1.7.** \( J(h, h^{(p/q)}) = 0. \)

**Proof.** If \( \phi_n \to h^{(p/q)} \), then \( \phi_n^q \to h^p \). So as \( J(h, h^p) = 0 \), \( J(h, \phi_n^q) \to 0. \) However, \( J(h, \phi_n^q) = qh^{q-1}J(h, \phi_n) \to qh^{(p-p/q)}J(h, h^{(p/q)}) \) and so the lemma follows. \( \square \)

**Proposition 1.8.** Let \((v, z, w)\) be a monomial valuation. Assume \( f, g \) are limits of sequences of elements in \( \mathbb{C}[z, w, z^{-1}, w^{-1}] \); in particular the leading coefficients \( F, G \in \mathbb{C}[z, w, z^{-1}, w^{-1}] \). Assume \( J(f, g) \neq 0 \) and \( v(f) \neq 0 \). Let \( F \) be a \( q \)-th power in \( L \) (and no higher power) and let \( K \) denote the leading form of \( J_{v, w}(f, g) \). Then there exists \( g_0 \in \mathbb{C}[f^{(1/q)}, f^{-1/(1/q)}] \) such that if \( h = g - g_0 \) has leading form \( H \), we will have \( J(F, H) = K = 0 \). If we additionally require \( v(f^{(r/q)}) < v(h) \) for all \( f^{(r/q)} \in \text{supp}(g_0) \), there will be a unique \( g_0 \), which we call an approximation to \( g \).

**Proof.** We weaken the hypothesis slightly to allow \( g \) to be a limit of a sequence of elements in \( \mathbb{C}[z, w, z^{-1}, w^{-1}, F^{-1}] \). We prove the proposition by induction on \( n = v(K) - v(F) \). If \( n = 0, g_0 = 0 \) works. So assume \( n > 0. \) Then \( J(F, G) = 0. \)

There exists an integer \( t \) such that \( GF^t \in \mathbb{C}[z, w, z^{-1}, w^{-1}] \). As \( J(F, GF^t) = 0, \) we can apply Lemma 1.2 to obtain \( (GF^t)^{v(F)} = aF^{v(G) + v(F)} \). Thus \( G^{v(F)} = aF^{v(G)} \) and so \( G = a^{v(F)/v(G)}F^{v(G)/v(F)} \). (Recall \( v(F) = v(f) \neq 0. \) As \( F \) is a \( q \)-th power, and no more, \( qv(G)/v(F) \) will be an integer \( p \) and so \( G \) has the form \( \beta F^{(p/q)} \). Now let \( \tilde{g} = \beta F^{(p/q)} \) and \( g^* = g - \tilde{g} \).

Observe the following: \( J(f, g^*) = J(f, g), v(g^*) = v(g), \) and \( v(K) - v(FG^*z^{-1}w^{-1}) < n. \) Moreover, as \( \tilde{g} \) is a limit of a sequence of elements in \( \mathbb{C}[z, w, z^{-1}, w^{-1}, F^{-1}] \), \( g^* \) will also have this property. So we may apply the induction assumption starting with \( f, \tilde{g} \) and obtain \( g_0 \in \mathbb{C}[f^{(1/q)}, f^{-1/(1/q)}] \). Then \( g_0 = \tilde{g} + g^* \) is the desired element.

Concerning the support property, we first note that the property \( J(F, H) = K \) is the same as \( v(h) = v(Kzw^{-1}) - v(F) \). This property is unaffected by terms in \( \mathbb{C}[f^{1/(q)}, f^{-1/(1/q)}] \) with value \( \geq v(h) \) and so we need not include these terms. So there exists at least one \( g_0 \) satisfying the support property. If there is a second \( g' \), we note that \( v(g_0 - g') < v(h) \) and so \( v(h) = v(g - g') = v((g_0 - g_0) + (g_0 - g')) < v(h) \). This contradiction gives uniqueness. \( \square \)

**Remark.** While the assumption on \( f, g \) was necessary for the existence part of this proof of Proposition 1.8, approximations can exist at other times and the uniqueness property always holds.

The next proposition tells when we can employ Proposition 1.8.

**Proposition 1.9.** Suppose \((v, z, w)\) is a monomial valuation and \( h = h_1/h_2 \) with \( h_1, h_2 \in \mathbb{C}[z, w] \). If \( H_2 \) is the leading form of \( h_2 \), then \( h \) is the limit of a sequence of
elements of \( \mathbb{C}[z, w, H_2^{-1}] \). This sequence will converge on an open neighborhood, specifically for all valuations for which the leading form of \( h_2 \) is the leading form of \( H_2 \). If \( H_2 \) is a monomial, \( h \) is the limit of a sequence of elements of \( \mathbb{C}[z, w, z^{-1}, w^{-1}] \).

**Proof.** We need to show that for every \( M \), there exist \( h_M \in \mathbb{C}[z, w, H_2^{-1}] \) with \( v(h - h_M) > M \). We can actually construct \( h_M \) via a recursive procedure provided we can find \( h^* \in \mathbb{C}[z, w, H_2^{-1}] \) with \( v(h - h^*) > v(h) \) and \( h = h^* / p_1 / p_2 \) with \( p_1, p_2 \in \mathbb{C}[z, w] \) and \( p_2 \) a power of \( H_2 \).

Let \( h^* = h_1 / H_2 \). Then

\[
h - h^* = h_1 / h_2 - h_1 / H_2 = h_1 (H_2 - h_2) / h_2 H_2 = h ((H_2 - h_2) / H_2)\]

and so

\[
v(h - h^*) = v(h) + v(H_2 - h_2) - v(H_2) > v(h)
\]

since \( v(H_2 - h_2) > v(H_2) \). Also \( p_2 \), the leading form of \( H_2 h_2 \) is \( H_2^2 \). So we have our desired \( h^* \). Continuing, we find an element \( h_p^* \) to approximate \( p_1 / p_2 \) and so forth. This produces our convergent sequence \( (h^*, h^* + p^*, ...) \).

In order to guarantee that our sequence converges for \( v \), we used only one specific property of \( v \), the inequality \( v(H_2 - h_2) > v(H_2) \). This is precisely the assumption that the leading form of \( h_2 \) equals the leading form of \( H_2 \). So we have convergence on the desired set of valuations. By Lemma 1.4, this set is open. The last statement is now obvious. \( \square \)

**Definition.** Using the notation of Proposition 1.9, we say that \( h \) has property LDM for \( v \) if \( H_2 \) is a monomial.

Thus, Proposition 1.9 says we can apply Proposition 1.8 if \( f, g \) have property LDM for \( v \).

We now wish to explore how \( g_0 \) will depend on \( v \). The statement of the next proposition is rather technical. This is unfortunately necessary to deal with the situations that shall arise in the next section. We will get some more immediate results from this and the next proposition at the end of this section. The somewhat contrived hypotheses, especially in part (3), are those needed to get the desired results. They will frequently be satisfied in the situations which occur.

**Proposition 1.10.** (1) Suppose \( (v, z, w) \) is a monomial valuation and \( \tau \) is the approximation in \( \mathbb{C}[f^{(1/q)} f^{-(1/q)}] \) to \( g \). Then there is a neighborhood \( U \) of \( v \) such that for all \( (v', z, w) \in U \), the corresponding approximation \( \tau' = \tau + \tau^0 \), where \( v'(\tau^0) > v'(f^{(p/q)}) \) for all \( f^{(p/q)} \in \text{supp}(\tau) \). In other words, the approximation sum \( \tau' \) begins with \( \tau \) but may contain additional terms.

(2) Using the same notation, if we further assume that \( v(f) < 0 \), \( w^j \) is the \( v \)-leading form of \( J(f, g) \), \( z^i w^j \) is the \( v' \)-leading form of \( f \), and \( [i(s+1)-j][v'(w)v(z) - v'(z)v(w)] > 0 \), then \( \tau^0 = 0 \).
(3) Use the same notation as (1). Assume \( f \) has property LDM for \( v \), \( v(f) < 0 \), \( w^i \) is the \( v \)-leading form of \( J(f, g) \), \( z^j w^l \) is the \( v \)'-leading form of \( f \), \( |(s+1) - j| v'(w) u(z) - v'(z) u(w) | < 0 \), \( |(s+1) - j| u(w) u(z) + u(z) < 0 \), and \( |u(w)| \neq 1 \). Also assume \( F \) is not a monomial and \( i^* \geq 0 \) whenever \( z^i w^j \in \text{supp}(F) \). Then \( \tau^o \neq 0 \). Moreover, if \( \zeta f^{(p/q)} \) is the leading term of \( \tau^o \), \( F \) does not have a \( q \)th root in \( \mathbb{C}[z, w, z^{-1}, w^{-1}] \).

(4) Assume we delete the hypothesis \( |u(w)| \neq 1 \) from part (3) and the conclusion fails. Then \( F \) has the form \( v w^i (z w^{-b} - \omega)^i \).

Proof. (1) Let \( F, G \) denote the \( v \)-leading forms of \( f, g \) respectively. We choose our initial \( U \) by Lemma 1.4 so that the \( v \)'-leading forms \( F', G' \) of \( f, g \) are the \( v \)-leading forms of \( F, G \), so that \( v'(f) \neq 0 \), and so that \( f, g \) have property LDM for all \( v' \neq v \in U \).

Then \( \tau \in \tilde{A}(U) \) and so \( g - \tau \in \tilde{A}(U) \). Restricting \( v' \) to get the LDM property enables us to apply Proposition 1.8 to the pair \( f, g - \tau \). This yields \( \tau^o \). If \( \tau^o = 0 \), then we are done; otherwise \( v'(\tau^o) = v'(g - \tau) \). Now we further restrict \( U \) to insure that the \( v \)'-leading form of \( g - \tau \) will be the \( v \)-leading form of the \( v \)-leading form of \( g - \tau \). Let \( \alpha f^{(r/s)} \) be the leading term of \( \tau^o \), and let \( H \) be the \( v \)'-leading form of \( g - \tau \), which is also the \( v \)-leading form of \( \alpha f^{(r/s)} \). Then \( H^s \) is the \( v \)-leading form of \( \alpha f^r \), which is the same as the \( v \)-leading form of the \( v \)-leading form of \( \alpha f^r \). So \( v(g - \tau) = v(H) = (r/s) v(f) \). Suppose \( f^{(p/q)} \in \text{supp}(\tau) \). Since \( v'(f) \) cannot be zero, \( v(f) \) and \( v'(f) \) must have the same sign. As \( (p/q)v(f) < (r/s)v(f) \), \( (p/q)v(f) < (r/s)v(f) \). Thus \( (p/q)v(f) < (r/s)v(f) \).

(2) We assume \( \tau^o \neq 0 \) and derive a contradiction. From the proof of (1), we see that \( v(g - \tau) = (r/s) v(f) \) and \( v'(g - \tau) = (r/s) v'(f) \). Thus \( v(g - \tau)/v(f) = v'(g - \tau)/v'(f) \). Now let \( H \) denote the \( v \)-leading form of \( g - \tau \). So \( J(F, H) = w^s \). As \( \tau^o \neq 0 \), \( J(F', H') = 0 \). So, using Lemma 1.1, \( v(w^s) = v(f(g - \tau) z^{-1} w^{-1}) \) and \( v'(w^s) > v'(f(g - \tau) z^{-1} w^{-1}) \). Now we simply rearrange to contradict the inequality in the hypothesis. \( v(z w^{s+1}) = v(f(g - \tau)) = [(r/s) + 1] v(f) \) while \( v'(z w^{s+1}) > [(r/s) + 1] v'(f) \). Thus, since \( v(f) < 0 \), \( v(z w^{s+1}) > v'(z w^{s+1}) v(f) \). Next the assumption that \( z^j w^l \) is the \( v \)-leading form of \( f \) allows us to replace \( f \) by \( z^j w^l \) in this inequality and we obtain

\[
[v(z) + (s+1) v(w)] [iv'(z) + jv'(w)] > [v'(z) + (s+1) v'(w)] [iv(z) + jv(w)].
\]

So

\[
[jv(z) v'(w) + i(s+1) v(w) v'(z)] > [jv(w) v'(z) + i(s+1) v(z) v'(w)],
\]

and this is just

\[
[i(s+1) - j] [u(w) v'(z) - v(z) v'(w)] > 0.
\]

This is the desired contradiction.

(3) Let \( H \) be the \( v \)-leading form of \( g - \tau \). By Proposition 1.8, \( H \in \mathbb{C}[z, w, z^{-1}, w^{-1}] \). Letting \( K = F^t H \) for some appropriate \( t \neq 0 \), we get \( K \in \mathbb{C}[z, w, z^{-1}, w^{-1}] \).

As \( J(F, H) = w^s \), \( J(F, K) = w^s F^t \). This will enable us to apply Proposition 1.3, but first we must synchronize notation. Let \( u^* = z^u w^{-v(z)} \), \( \delta = v'(u^*)/|v'(u^*)| \), and \( u = \)
(u*)^2. So \( v'(u) > 0 \). Thus \( F = z' w^j \Theta(u) \) with \( \Theta(u) \in \mathbb{C}[u] - \mathbb{C} \). In the setting of Proposition 1.3, \( a = \delta v(w) \) and \( b = -\delta v(z) \). Thus \( \sigma_j = a(1 + s) - b = \delta [(1 + s)v(w) + v(z)] \) and \( \gamma = \delta(v(z) + v(w)) = -\delta v(f) \). Our hypothesis then says \( \sigma_j \delta < 0, \gamma \delta > 0, \) and \( a = \delta v(w) < 0 \). We see immediately that \( \sigma_j \gamma < 0 \) and so case (a) is impossible. To rule out case (b), we shall show \( (\sigma_2 + \sigma_1 M)/(\gamma) < 1 \) and so cannot be a positive integer. Since \( z' + aM w^j + bM \in \text{supp}(F) \), \( i + aM \geq 0 \); as \( a < 0 \), this yields \( M \leq i/(-a) \). Thus

\[
\delta(\sigma_2 + \sigma_1 M) = \delta \sigma_2 + \delta \sigma_1 M \geq \delta \sigma_2 + \delta \sigma_1 i/(-a)
\]

\[
\delta(i(l + s) - j + (a(l + s) - b)i/(-a))
\]

\[
\delta(i(l + s) - j - i(l + s) + bi/a)
\]

\[
= \delta(-j + bi/a) = \delta \gamma/a > -\delta \gamma.
\]

[Note: the last inequality is the only place we have used the hypothesis \( |v(w)| \neq 1 \), i.e., \( a \neq -1 \). We can still salvage strict inequality for the entire chain if the inequality \( M \leq i \) is strict.] Finally, as \( \gamma \delta < 0 \), dividing both sides by this quantity reverses the inequality. So \( (\sigma_2 + \sigma_1 M)/(\gamma) < 1 \) and case (b) cannot occur. Thus we must have case (c) or (d). Thus \( i(l + s) = 0 \) and since \( \Theta^{t-1} | \Psi \), we may assume \( t = 1 \). Letting \( e = k/l, \ K = (z' w^j)^e \Phi(u) \) with \( \Phi(0) \neq 0 \). Then \( H = K F^{-1} = (z' w^j)^{-1} \Psi(u)(\Theta(u))^{-1} \). So \( H' = (z' w^j)^{-1} \Psi(0)(\Theta(0))^{-1} \) and \( F' = z' w^j \Theta(0) \). Thus \( J(F', H') = 0 \) and so \( \tau \neq 0 \).

We will prove the second statement by contradiction. Assume \( F \) is a \( q \)th power; then \( g - \tau - \zeta f^{(p/q)} \in A_v \). As \( v(g - \tau) = v(H) = v(H') = v(f^{(p/q)}) \), \( v(g - \tau - \zeta f^{(p/q)}) \geq v(g - \tau) \). However, by Lemma 1.1,

\[
v(g - \tau - \zeta f^{(p/q)}) \leq v(J(f, g - \tau - \zeta f^{(p/q)}) z w f^{-1})
\]

\[
= v(J(f, g) z w f^{-1}) = v(g - \tau).
\]

So \( v(g - \tau - \zeta f^{(p/q)}) = v(g - \tau) \). If we let \( H^* = \text{the } v \text{-leading form of } g - \tau - \zeta f^{(p/q)} \), we again obtain \( J(F, H^*) = w^j \). Exactly as in the previous paragraph, we define \( K^* = F^* H^* \) and determine the form of \( K^* \). As before, we can take \( t = 1 \) and \( K^* = (z' w^j)^e \Phi(u) \) with \( \Phi(0) \neq 0 \). Because \( v(K) = v(K^*) \), we get the same \( e \). Therefore, \( (H^*)^e = (z' w^j)^{-1} \Phi(0)(\Phi(0))^{-1} \). But then \( v'(H^*)^e = v'(H') \). This yields \( v'(g - \tau - \zeta f^{(p/q)}) = v'(g - \tau) \), which cannot happen. So \( F \) is not a \( q \)th power.

(4) By the note contained in the proof of (3), the proof will succeed unless \( i = M \). However, in that situation, case (b) is a possibility with \( C = (\sigma_2 + \sigma_1 M)/(\gamma) = 1 \). So \( \Theta(u) \) must be the \( \alpha \)th power of a single linear polynomial. Then we see \( (z' w^j u^i)(u^{-1} \Theta(u)) \) will have the desired form. 

Proposition 1.11. Let \( f, g \in L \). Suppose \( U \) is an open interval of monomial valuations such that for \( (u, z, w) \in U \), \( v(f) < 0 \), \( f, g \) have property LDM for \( v \), and the leading forms of \( f, g, \) and \( J(f,g) \) are independent of the choice of \( v \). Let \( \tau_k \in \mathbb{C}[f^{(1/q)}, f^{(-1/q)}] \) denote the \( v_k \)-approximation to \( g \). If \( w^j, z' w^j \) are the \( v \)-leading forms of \( J(f,g) \), \( f \) respectively, and if \( \tau_1(s + 1 - j)[v_1(w) v_2(z) - v_1(z) v_2(w)] > 0 \) for some \( v_1, v_2 \in U \), then \( \tau_2 = \tau_1 + \tau^c \), where \( v(\tau^c) > v(f^{(p/q)}) \) for all \( f^{(p/q)} \in \text{supp}(\tau_1) \).
On the Jacobian conjecture

Proof. Since \( \tau_2 \in A(U) \subset A_v \), the conclusion of the proposition simply states that \( \tau_2 \) satisfies all the properties of a \( v_1 \)-approximation to \( g \) except that it may have extra terms. If we let \( h = g - \tau_2 \) and \( F_k, G_k, H_k \) denote the appropriate leading forms, this is precisely the condition \( J(F_1, H_1) \neq 0 \). This is what we shall prove.

The LDM property allows us to express \( f \) as a limit of a sequence of elements in \( \mathbb{C}[z, w, z^{-1} w^{-1}] \). The LDM property also forces the leading form of the denominator to remain constant on \( U \). Thus the sequence does not depend on the choice of \( v \). The same holds true for \( g \), fractional powers of \( f \), and so also \( h \). The point is that the support of \( h \) really makes sense and we can compute values by taking the infimum over all elements in the support. Since \( J(F_2, H_2) = w^s \) and \( F_1 = F_2 = z^i w^j \), \( z^{i-1} w^{s+1-j} \in \text{supp}(H_2) \subset \text{supp}(h) \). So \( v_1(h) = (1 - i) v_1(z) + (1 + s - j) v_1(w) \). Now we shall assume \( J(F_1, H_1) = 0 \) and derive a contradiction. Necessarily, \( \text{supp}(H_1) - (z^i w^j)^e \) with \( e \in \mathbb{Q} \), and so there is no equality above. As \( v_1(H_1) = v_1(h) \), we have \( \text{supp}(H_1) - (1 - i) v_1(z) + (1 + s - j) v_1(w) \). Thus \( e > [(1 - i) v_1(z) + (1 + s - j) v_1(w) / v_1(f)] \).

However, since \( v_2(h) \leq v_2(H_1) \), \( (1 - i) v_2(z) + (1 + s - j) v_2(w) \leq e v_2(f) \), and so \( e \leq [(1 - i) v_2(z) + (1 + s - j) v_2(w) / v_2(f)] \). We combine these two inequalities and multiply through by the positive quantity \( v_1(f) v_2(f) \). This yields

\[
[ (1 - i) v_1(z) + (1 + s - j) v_1(w) ] v_2(f) < [(1 - i) v_2(z) + (1 + s - j) v_2(w) ] v_1(f).
\]

Now just plug in \( v_k(f) = iv_k(z) + jv_k(w) \) and simplify. We get

\[
[(1 - i) j - i(1 + s - j)][v_1(z) v_2(w) - v_1(w) v_2(z)] < 0;
\]

thus

\[
[(1 + s - j)][v_1(w) v_2(z) - v_1(z) v_2(w)] < 0.
\]

But this contradicts our hypothesis. \( \square \)

The next theorem seems a little more tangible. It is yet another combination of Propositions 1.3 and 1.8.

**Theorem 1.12.** Let \( (u, x, y) \) be a monomial valuation, \( u(x) < 0, u(y) < 0 \). Assume \( f, g \in \mathbb{C}[x, y] \) with \( J(f, g) = 1 \). Then either

1. \( F = \sigma^m \theta^n \) where \( \sigma, \theta \) are linear polynomials, or
2. \( F = \delta x^m (y + \xi x^{u(y)/(u(x))})^n \) where \( \xi \in \mathbb{C} \) and \( u(y)/u(x) \in \mathbb{Z} \), or
3. \( F = \delta y^m (x + \zeta y^{u(x)/(u(y))})^n \) where \( \zeta \in \mathbb{C} \) and \( u(x)/u(y) \in \mathbb{Z} \).

Moreover, \( m \neq n \).

**Proof.** First apply Proposition 1.8. This enables us to find \( H \in \mathbb{C}[x, y, x^{-1}, y^{-1}, F^{-1}] \) with \( J(F, H) = 1 \). For some integer \( t \), \( F^t H \in \mathbb{C}[x, y, x^{-1}, y^{-1}] \) and \( J(F, F^t H) = F^t \).

We may assume \( \text{gcd}(u(x), u(y)) = 1 \). Choose \( j = \sup \{ d \mid x^d y^d \in \text{supp}(F) \} \) and \( l = \sup \{ d \mid x^d y^d \in \text{supp}(F^t H) \} \). Then we have \( x^i y^j \in \text{supp}(F) \) for some \( i \). If we let \( a = -u(y), b = u(x), \) and \( u = x^i y^j \), then \( F = x^i y^j \hat{\Theta}(u) \) where \( \hat{\Theta} \in \mathbb{C}[u] \) and \( \hat{\Theta}(0) \neq 0 \). Likewise \( F^t H = x^{k_l} y^j \Psi(u) \). If \( \Theta \in \mathbb{C} \), then the result holds (Case 1) and so we may assume \( \Theta \in \mathbb{C} \). Then we may apply Proposition 1.3 with \( s = 0 \).
Now \( y = ib - ja = iv(x) + jv(y) = v(F) < 0 \) and \( \sigma_1 = a - b = -v(y) - v(x) > 0 \). Thus \( \sigma_1 y < 0 \) and case (a) fails. Also \( \sigma_2 = (i - j)/(iv(x) + jv(y)) \). This can be a positive integer only if \( i = 0 \) and \( v(y) = -1 \), in which case it equals 1. Also 
\[
(\sigma_2 + \sigma_1 M)/(-y) = [(i + Ma) - (j + Mb)]/[-(i + Ma)b + (j + Mb)a]
\]
can be a positive integer only if \( j + Mb = 0 \) and \( v(x) = b = -1 \). (Note that \( i + Ma, j + Mb \) must be nonnegative.) Again that positive integer must be one. Finally \( \sigma_1 M/(-y) \) is the sum of these two integers and so will equal 2 if both are positive integers. Thus we have either case (b) or case (c) with \( C = 1 \) or case (d) with \( C = 2 \).

First assume case (c). We have \( i = 0 \) and \( v(y) = -1 \), so \( v(y) \mid v(x) \). Since \( C = 1 \), 
\[
F = \delta y^i(xy^{u(x)} + \zeta)^d.
\]
This gives possibility (3). Case (b) is the same as case (c) with the variables revrsed and leads to possibility (2). Next consider case (d). Since both quantities computed above must be positive integers in this case, we must have \( v(x) = v(y) = -1, \ i = 0, \) and \( j + Mb = 0 \). Since \( C = 2 \), 
\[
F = \delta y^i(xy^{u(x)} + \zeta)^m(xy^{u(x)} + \zeta_2)^n.
\]
But \( m + n = M = j \) and so we get 
\[
F = \delta(x + \zeta_1 y)^m(x + \zeta_2 y)^n
\]
and possibility (1).

Finally observe that in all cases, up to change of variable, we have \( F = x^m y^n \). The solution \( H \) to \( J(F, H) = 1 \) is necessarily \( H = x^{1-m} y^{1-n}/(m-n) \). There can be no solution when \( m = n \).

**Notation.** Hereafter, we shall assume \( J(f, g) = 1 \) but \( C[f, g] \neq C[x, y] \). Thus we are now assuming the existence of a counterexample to the Jacobian conjecture.

We are now ready to discuss the support of \( f \) and the Newton polygon discussed by Abhyankar [1]. Let \( (u, v, y) \) be the monomial valuation with \( v(x) = -1, \ v(y) = -1 \) (the degree valuation) and let \( F \) be the leading form of \( f \). Consider the case where \( F \) has only one linear factor. Changing variables if necessary, \( F = \delta x^n \). Now \( f \) cannot be a function of \( x \) alone clearly and so if we increase the ratio \( v(y)/v(x) \), the leading form of \( f \) will change. This can be regarded as a continuous process and for a particular choice of valuation \( v_1 \), the leading form \( F_1 = \delta x^n + \Theta(x, y), \Theta \neq 0 \). If \( a = -v_1(y) \) and \( b = -v_1(x) \), \( F_1 = x^a \Phi(x^a, y^b) \) and \( a > b \). As \( \Phi \) is not homogeneous, we must be in case (2) of Theorem 1.12. So \( F_1 = \beta x^i(y + \zeta x^a)^j \). Replacing \( x, y \) by new variables \( x, y + \zeta x^a \), we reduce the degree of \( f \) from \( n = i + aj \) to \( i + j \). Thus, if \( F \) has only one linear factor, we can make a change of variable reducing the degree of \( f \) (and likewise \( g \)). In a minimal counterexample, \( F \) will have two distinct linear factors. Changing variables if necessary, we can assume \( F = ax^D y^E \). Next we claim that \( y \)-deg of \( f = E \) (and symmetrically \( x \)-deg of \( f = D \)). For otherwise, as before, we may increase the ratio \( v(y)/v(x) \) until we change the leading form to one given in case (2) of Theorem 1.12. However, this leading form will be divisible by \( y^E \), and this is not allowed in case (2).

We can plot the support of \( f \) in the Cartesian plane, plotting the point \((k, l)\) if and only if \( x^k y^l \in \text{supp}(f) \). The constant term will not affect the Jacobian but for convenience we will assume that \((0, 0)\) is on the graph. The Newton polygon, denoted \( N(f) \), is the smallest convex polygon containing the entire graph. Thus far, we know
the polygon is contained in the rectangle with vertices $(0, 0), (0, E), (D, E), (D, 0)$. We also know from Theorem 1.12 that $D \neq E$ and so we assume (by symmetry) $D > E$.

Next we note that for every monomial valuation $(u, x, y)$, the support of the leading form of $f$ will either be a vertex of $N(f)$ or the points along an edge of $N(f)$. Conversely, for each vertex or edge, there will be a monomial valuation making the vertex or edge correspond to the support of the leading form. If $v$ corresponds to an edge, the slope of the edge will be $-v(x)/v(y)$.

**Convention.** Our primary concern with $N(f)$ will be with the clockwise path from the $y$-axis to the $x$-axis. It seems natural to think of this as travelling from left to right (although upon reaching $(D, E)$ we are no longer travelling leftward). In this manner we will interpret the idea of an edge being left of a vertex or another edge. We will refer to those edges and vertices left of $(D, E)$ as the left side of $N(f)$ and those to the right as the right side.

**Proposition 1.13.** If $(f, g)$ is a minimal counterexample to the Jacobian conjecture, we may assume $N(f)$ is contained in the quadrangle with vertices $(0, 0), (0, E), (D, E), (D - E, 0)$. In fact the slope of any edge on the right side is $\leq 1$.

**Proof.** First consider the monomial valuation $(u', x, y)$ given by $u'(x) = -1$ and $u'(y) = 0$. The leading form of $f, f'$, equals $x^D \phi(y)$. If $y$ does not divide $\phi(y)$, we may make a change of variable, replacing $y$ by $y - \zeta$, to make $\phi(y)$ divisible by $y$. With these new variables, we will prove $N(f)$ has the desired form.

The first implication follows from the second. So we shall assume there is an edge on the right side with slope $> 1$ (possibly infinite) and derive a contradiction. As the edge immediately to the right of $(D, E)$ has the greatest slope, we may use this one. Let $(u, x, y)$ be the monomial valuation corresponding to this edge. We have $-u(x) > u(y) \geq 0$. Now

$$v(f) = Du(x) + Eu(y) = -D + D(u(x) + 1) + Eu(y) \leq -D.$$ 

So $v(f) < 0$ and we can apply Proposition 1.8 to find $H \in \mathbb{C}[x, y, x^{-1}, y^{-1}, F^{-1}]$ with $J(F, H) = 1$. As in the proof of Theorem 1.12, we can replace $H$ by $HF'$ so that $J(F, H) = F'$ and $H \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$. Now we will employ Proposition 1.3 with $z = x, w = y, a = -v(y), b = v(x), i = D, j = E, s = 0, u = x^a y^b, F = x^i y^j \Theta(u), H = x^{k} y^{l} \Psi(u)$ where $k, l, \Theta, \Psi$ are defined by these equations. $y = ib - ja = Du(x) + Ev(y) = v(f) < 0$. $\sigma_1 = a - b = -v(y) - v(x) > 0$; so case (a) fails. $\sigma_2 = i - j > 0$, so $\sigma_2 / y < 0$ and cases (c), (d) fail.

It remains only to rule out case (b). First suppose $a \neq 0$. Then

$$(\sigma_2 + \sigma_1 M) / (\gamma) = [(i + Ma) - (j + Mb)] / [(i + Ma) - (j + Mb) - a]$$

$$= 1 / (-a) \leq 1.$$
so \( (\sigma_2 + \sigma_1 M)/(\gamma) < 1 \) and cannot be a positive integer. So case (b) cannot occur here. It only remains to consider \( a = -u(\gamma) = 0 \). So \( (\sigma_2 + \sigma_1 M)/(\gamma) = [1 - (j-M)]/i \). This can be a positive integer only if \( j-M = 0 \) and then \( C = 1 \). Here \( F = \delta x^D y^E (y^{-1} - \omega)^E = \delta x^D (1 - \omega y)^E \). However, \( v \) is the valuation \( v' \) considered in the first paragraph of this proof and we note that \( \phi(\gamma) = \delta (1 - \omega y)^I \). This contradicts the assumption that \( y \) divides \( \phi(\gamma) \).

**Proposition 1.14.** Let \((f, g)\) be a minimal counterexample to the Jacobian conjecture with \( N(f) \) as in Proposition 1.13. Then \( N(g) \) is a rational multiple of \( N(f) \). It is not an integral multiple and so both are integral multiples of a smaller polygon \( P \).

**Proof.** First we shall prove the proportionality. Since \( J(f, g) = 1 \), \( \{(0,1),(1,0)\} \subseteq \text{supp}(f) \cup \text{supp}(g) \). This means \( \{(0,1),(1,0)\} \subseteq \text{supp}(h) \) where \( h \) is one of \( f, g, f + g \). The Jacobian of any two of these three is a unit and so we will assume \( f = h \) and \( g \) is either of the other two. Then, when we show \( N(g) \) is proportional to \( N(f) \), we will actually have shown all three polygons are proportional. In this way, we can assume \( \{(0,1),(1,0)\} \subseteq \text{supp}(f) \).

Let \((u,x,y)\) be a monomial valuation which does not correspond to the origin or an edge along one of the axes for \( N(f) \). This guarantees either \( v(x) \) or \( v(y) < 0 \) and since we have \( (0,1),(1,0) \in \text{supp}(f) \), \( v(f) < 0 \). Assume \( v(y) < 0 \). (For \( v(x) < 0 \), the proof is similar.) First consider the case \( v(x) > 0 \). Since \( v(fgx^{-1}y^{-1}) = v(fgy^{-1}) + v(g) \leq -v(x) < 0 \), we obtain proportionality. If \( u(x) < 0 \), then we know \( F = \delta x^D y^E \), with \( D > E \), and \( v(fgx^{-1}y^{-1}) \leq (D-1)v(x) + (E-1)v(y) \leq v(x) < 0 \) and again we obtain proportionality. Since these edges intersect the constant of proportionality cannot change. Finally, as we have proportionality for the boundaries of the polygons along a path clockwise from the \( y \)-axis to the \( x \)-axis and the polygons are contained in the first quadrant and contain the origin, the entire polygons must be proportional.

Finally, if \( N(g) \) were in fact an integral multiple of \( N(f) \), then the highest degree term of \( g \) would be \( \zeta (x^D y^E)^n \). Then we contradict the minimality of \( (f, g) \) by producing a counterexample of lower degree, namely \( (f, g - \beta f^n) \), where \( \beta = \zeta \delta^{-n} \).

**Remark.** For the monomial valuation \((u,x,y)\) with \( u(x) = 0 \) and \( u(y) = -1 \), \( F = y^E \phi(x) \) where \( \phi(x) \) is a polynomial of degree \( D \). Unlike the situation with \( x \) and \( y \)
reversed, we cannot conclude $\varphi(x)$ is a power of a single linear factor by combining Propositions 1.3 and 1.8. In fact, we cannot conclude anything at all about $\varphi(x)$. If $x$ does not divide $\varphi(x)$, the left side of $N(f)$ will simply be a horizontal line and will provide us with no information. On the other hand, if $i > 0$ is the highest power of $x$ which divides $\varphi(x)$, the horizontal segment directly to the left of $(D, E)$ will only extend from $(i, E)$ to $(D, E)$ and we will be able to exploit the path from the $y$-axis to $(i, E)$. (If $i = D$, there is no upper horizontal segment.) Consequently, we shall want to assume $x | \varphi(x)$ and this can be accomplished via a change of variable. In fact, if $\varphi(x)$ has more than one linear factor, it may be useful to consider one Newton polygon for each factor. Each one will contain different information.

**Notation.** For the remainder of this section and the second half of Section 2, we will assume $\{f, g\}$ is a fixed minimal counterexample as given in Proposition 1.13. The quantities $x, y, 0, E$ will also be fixed.

In Section 2, we will use the full strength of Propositions 1.10 and 1.11. However, the case $\{z = x, w = y\}$ will yield some immediate results about $N(f)$ which we would like to present here.

First we discuss the connection between the circle of valuations $(u, x, y)$ and $N(f)$. The edges of $N(f)$ correspond to points on the circle; an edge corresponds to that point whose tangent line has the same slope as the edge and lies on the same side of the figure. The vertex between two edges corresponds to the arc between the two points corresponding to the edges. For example, $(0, 0)$ corresponds exactly to the (open) third quadrant of the circle while $(D, E)$ corresponds to an arc which includes the first quadrant and half of the fourth quadrant. The left side of $N(f)$ will correspond to all or part of the second quadrant.

Now we regard $\tau(u)$ as a function on our space of valuations which assigns to each valuation the appropriate approximation to $g$. (Deleting the third quadrant eliminates those valuations for which $\nu(f) > 0$.) In this setting, Proposition 1.10(1) is a semicontinuity condition. Let us say that $u_2 < u_1$ if $u_2$ is to the right of $u_1$. (So $u_1$ corresponds to the larger angle in the usual circular measure.)

**Lemma 1.15.** If $u_2 < u_1$ and they are separated by less than $180^\circ$, then $u_1(y)u_2(x) - u_1(x)u_2(y) > 0$.

**Proof.** We prove this using case by case analysis. First note that the slope diminishes as you travel clockwise around the circle except at the two points where the slope is undefined. If $u_1(y)u_2(y) > 0$, this gives $-u_1(x)/u_1(y) > -u_2(x)/u_2(y)$. Multiplying through by $u_1(y)u_2(y)$ yields $-u_1(x)u_2(y) > -u_2(x)u_1(y)$ and the desired inequality follows. If $u_1(y)u_2(y) < 0$, the interval crosses one of the discontinuities of the slope function and so the slope inequality is reversed. (We need the $180^\circ$ assumption here.) However, multiplying through by $u_1(y)u_2(y)$ reverses the inequality and so we again get the desired result. If $u_1(y) = 0$, we are either at the point where
\[ u_1(x) > 0, \text{ in which case } u_2(y) < 0, \text{ or at the point where } u_1(x) < 0, \text{ in which case } u_2(y) > 0. \text{ Either way, } u_1(y)v_2(x) - u_1(x)v_2(y) = -u_1(x)v_2(y) > 0. \text{ The case } u_2(y) = 0 \text{ is similar.} \]

Any vertex \((i, j)\) of \(N(f)\), except the origin, corresponds to an open interval \(U\) of valuations to which we can apply Proposition 1.11. Note \(s = 0\) here. If \(i > j\), \(\tau(v)\) can add terms as \(v\) increases but not as \(v\) decreases. If \(i < j\), the situation is reversed. Of course, by Theorem 1.12, \(i \neq j\). Employing Proposition 1.10 to compare \(\tau(v)\) at an edge with the nearest valuations corresponding to the vertex \((i, j)\) at one end, we see that \(\tau(v)\) can be shorter but only if either \(i > j\) and the vertex is on the right (clockwise) or \(i < j\) and the vertex is on the left. Let \((m, n)\) be the first vertex \((i, j)\) for which \(i > j\) and let \(e\) be the edge immediately to the left of \((m, n)\). Let \(v_e\) be the corresponding valuation. Putting together our observations, we see that \(\tau(v_e)\) is the shortest approximation and as we proceed in either direction from \(e\), \(\tau(v)\) may occasionally add terms. There is no reason to suspect there is any relation between the terms we add going left and the terms we add going right.

So far we have not ruled out the case \(\tau(v)\) is constant. Here is where Proposition 1.10(3) comes into play. Let \(v^*_0, \ldots, v^*_k\) be valuations corresponding to the vertices \((m, n), \ldots, (D, E)\). (We are numbering from left to right. Note we may have \(k = 0\).) Let \(v_0, \ldots, v_k\) be the valuations corresponding to the edges immediately to the left of these vertices. For any \(v_i, v_i(x) \geq 0\) and \(v_i(y) < 0\). Since we have \(v_i(x' y') < 0\) and \(i > j\), we must have \(v_i(x) + v_i(y) < 0\). Hence, if we let \(v = v_i\) and \(v' = v^*_i\), the full hypothesis of Proposition 1.10(3) holds. Thus \(\tau(v^*_i)\) contains more terms than does \(\tau(v_i)\). More importantly, the last conclusion of Proposition 1.10(3) forces \(F^*_i\) to be a \(q\)th power for some \(q\) for which \(F_i\) is not a \(q\)th power where \(F_i, F^*_i\) denote the obvious leading forms. Of course, all leading forms corresponding to valuations to the right of \(v_i\) must now be \(q\)th powers. This is a strong restriction on \(N(f)\) but its full strength does not lend itself to a nice statement. There is one nice immediate theorem, however.

**Theorem 1.16.** \(\text{gcd}(\deg f, \deg g)\) is not a prime number or 4.

**Proof.** Let \(b = \text{gcd}(\deg f, \deg g)\) and let \(\deg f = ab\). Certainly \(F_k\) must be an \(a\)th power. As \(x^D y^E = F^*_k\) is a \(q\)th power for some \(q\) for which \(F_k\) is not, \(F^*_k\) is an \((ad)\)th power for some \(d > 1\). Thus \(F^*_k = (x^i y^j)^{ad}\), where \(i > j \geq 1\). Then \(ab = \deg f = D + E = (i + j)ad\) and so \(b = (i + j)d\). So \(b\) cannot be a prime or 4. \(\Box\)

Actually, it is easy to use a proof like this one to show that \(b\) must have at least \((k + 2)\) factors, but for some Newton polygons, \(k\) will equal zero.

**Remark.** Abhyankar has communicated the observation that he earlier developed a stronger version of Theorem 1.16 — namely, \(\text{gcd} \neq p\) or \(2p\) for any prime \(p\). This strengthening can be proved in the manner used to prove Proposition 2.21 below.
2. The differential \( \frac{dy}{f_x} \)

In this section, we take a more geometric approach and build on the results of the previous section. Let \( f \) be any nonconstant element of \( \mathbb{C}[x, y] \). (Eventually we shall assume \( f \) is as in Section 1.) Let \( K \) denote the field \( \mathbb{C}(f) \). Thus \( L \) will be a field of algebraic functions in one variable over \( K \). Let \( C \) denote the abstract nonsingular curve associated to this function field. Now there is a natural birational equivalence between \( C \) and a curve \( C^* \subset \mathbb{P}^2(K) \) defined as follows: write \( K = \mathbb{C}(t) \) where \( t \) is an indeterminate, then homogenize \( f(x, y) - t = 0 \) and let \( C^* \) be the curve defined by this equation. The (natural) affine coordinate ring of \( C^* \) is \( \mathbb{C}(f)[x, y] \). As this ring is a localization of \( \mathbb{C}[x, y] \), it is regular and so \( C^* \) is nonsingular in affine space. The points at infinity will usually be singular. By abuse of notation, we will refer to affine points and points at infinity of \( C \). There is of course a 1-1 correspondence between affine points of \( C \) and affine points of \( C^* \). Hereafter, we shall mostly concern ourselves with \( C \). We shall let \( dx, dy, \) etc. stand for differentials on \( C \) and we will denote partial derivatives by using subscripts, e.g., \( f_y \).

Lemma 2.1. If \( g \in \mathbb{C}[x, y] \), \( dg = J(f, g) \frac{dy}{f_x} = -J(f, g) \frac{dx}{f_y} \). If \( \frac{dy}{f_x} \) denotes the divisor associated to the differential \( \frac{dy}{f_x} \), \( \text{supp}(\frac{dy}{f_x}) \) is contained in the set of points at infinity. So, if \( J(f, g) = 1 \), \( \text{supp}(\frac{dg}{f_y}) \) is contained in the set of points at infinity.

Proof. Since \( f \in K \), \( df = 0 \). By the chain rule, \( g, dx + g_y dy = dg \) and \( f_x, dx + f_y dy = 0 \). We can solve these two equations simultaneously to obtain \( dx \) and \( dy \) in terms of \( dg \). The solutions are \( dy = (J(f, g))^{-1} f_x dg \) and \( dx = -(J(f, g))^{-1} f_y dg \). The first statement follows immediately. Next we will show \( \frac{dy}{f_x} = \frac{dx}{f_y} \) has no poles in affine space. As \( \frac{dx}{} \) and \( \frac{dy}{} \) have no poles in affine space, it suffices to show \( f_x \) and \( f_y \) have no common zeros. But this is immediate since \( C^* \) is nonsingular in affine space. To show that \( \frac{dy}{f_x} \) has no affine zeros, we first note that \( f_x \) and \( f_y \) have no affine poles and so it suffices to prove that \( dx \) and \( dy \) have no common affine zeros. As \( C^* \) is nonsingular in affine space, this is also immediate. The third sentence is a direct consequence of the first two. \( \square \)

Remark. We can specialize \( C \) by replacing \( t \) by any complex number \( \gamma \). Then, the curve can have affine singular points. In fact, if \( f_y(\alpha, \beta) = f_x(\alpha, \beta) = 0 \) and we choose \( \gamma = f(\alpha, \beta) \), the curve will have an affine singular point at \( (\alpha, \beta) \). It is easy to see that \( \frac{dy}{f_x} \) must have a pole at any affine singularity; yet \( \frac{dg}{f_y} \) can never have an affine pole. Thus \( J(f, g) = 1 \) prohibits the specialization from having affine singularities. This suggests the following conjecture: If \( f(x, y) - \gamma = 0 \) has no affine singularities for every \( \gamma \in \mathbb{C} \), then there exists \( g \in L \) with \( L = \mathbb{C}(f, g) \). This conjecture implies the Jacobian conjecture though it may be stronger.

Actually, this new conjecture can easily be generalized to any number of variables. If \( f(x_1, \ldots, x_n) - \gamma = 0 \) has no affine singularities for every \( \gamma \in \mathbb{C} \), then there
exists \( g_1, \ldots, g_{n-1} \in \mathbb{C} \) with \( L = \mathbb{C}(f, g_1, \ldots, g_{n-1}) \). The conjecture is true for \( n = 1 \) since \( f_1 = 0 \) has a solution unless \( f \) is linear. This approach will not be considered further in this article.

**Lemma 2.2.** Suppose \( J(f, g) = \Theta(g) \in \mathbb{C}[g] - \mathbb{C} \). Then \( \text{supp}(\Theta) \) is contained in the set of points at infinity.

**Proof.** Let \( P \) be an affine point and let \( v \) be the corresponding valuation. Since \( \text{deg}(\Theta) = \text{deg}(g) \), we have \( v(\text{deg}(g)) = v(\Theta) + v(dg) \). Also, letting \( \Theta' \) denote the usual derivative of \( \Theta(g) \), \( d\Theta = \Theta' \cdot dg \) and so \( v(d\Theta) = v(\Theta') + v(dg) \). Now assume \( P \in \text{supp}(\Theta) \). Then \( v(d\Theta) = v(\Theta) - 1 \) and so

\[
-1 = v(d\Theta) - v(\Theta) = v(\Theta') + v(dg) - v(\Theta) = v(\Theta') + (v(dg) - v(dy/f_1)).
\]

By Lemma 2.1, \( v(dy/f_1) = 0 \); so \( v(\Theta') = -1 \). But \( \Theta' \in \mathbb{C}[x, y] \) is contained in the affine coordinate ring of \( C^* \) and so \( \Theta' \geq 0 \). This contradiction proves the lemma. \( \square \)

**Theorem 2.3.** Assume \( f, g \in \mathbb{C}[x, y] \) such that \( g - y \) is irreducible for each \( y \in \mathbb{C} \) and assume \( J(f, g) = \Theta(g) \neq 0 \). Then we may find \( f^* \in \mathbb{C}[x, y] \) with \( J(f^*, g) = 1 \).

**Proof.** We induct on the degree of \( \Theta(g) \). If \( \text{deg}(\Theta) = 0 \), the result is immediate. Otherwise we employ Lemma 2.2 to obtain \( \text{supp}(\Theta) \) is contained in the set of points at infinity. Since \( C^* \) is nonsingular in affine space, this says \( \Theta \) is a unit in \( \mathcal{O}(C^*) \), the coordinate ring. Thus, for some \( h \neq 0 \in \mathbb{C}[x, y] \), \( \Theta h \in \mathbb{C}[f] \). So \( \Theta h = f_1 \). Let \( \phi \) be an irreducible factor of \( \Theta \). Necessarily \( \phi = g - y \) by hypothesis. By unique factorization, \( \phi | (f - \alpha_1) \) for some \( \alpha_1 \). Let \( f_1 = (f - \alpha_1)/\phi \); then \( J(f_1, g) = \phi^{-1} J(f, g) = \Theta \). Since \( \text{deg}(\Theta/\phi) = \text{deg}(\Theta) - 1 \), we are done by induction. \( \square \)

**Theorem 2.4.** Suppose \( f - y \) is irreducible for each \( y \in \mathbb{C} \). Then \( dy/f_x \) is exact \( \Leftrightarrow \) there exists \( g \in \mathbb{C}[x, y] \) such that \( J(f, g) = 1 \).

**Proof.** \( (\Leftarrow) \) is immediate from Lemma 2.1. To prove \( (\Rightarrow) \), first note that \( dy/f_x \) exact implies \( \exists h \in L \) with \( J(f, h) = 1 \). Since \( dh \) has no affine poles in \( C^* \), \( h \in \mathcal{O}(C^*) \). Thus there exists \( \Theta(f) \neq 0 \in \mathbb{C}[f] \) with \( h\Theta(f) \in \mathbb{C}[x, y] \). Now we have \( J(f, h\Theta(f)) = \Theta(f) \). Letting \( f \) play the role of \( g \) in Theorem 2.3 and \( h\Theta(f) \) play the role of \( f_1 \), \( -f^* \) is the desired element. \( \square \)

The next result seems best placed here for continuity although the proof requires Theorem 2.19 below. This does not lead to a circular argument as Lemma 2.2 and Theorems 2.3-2.5 will not be considered again in Section 2.

**Theorem 2.5.** Let \( f, g \in \mathbb{C}[x, y] \). Then the following are equivalent:

1. If \( J(f, g) = 1 \), then \( \mathbb{C}[f, g] = \mathbb{C}[x, y] \).
2. If \( dy/f_x \) is exact, then \( C \) has genus zero.
Proof. First consider the case where \( f - y \) is irreducible for all \( y \in \mathbb{C} \).

(1) \( \Rightarrow \) (2). \( \frac{dy}{f} \), exact \( \Rightarrow \exists g \) with \( J(f, g) = 1 = \mathbb{C}[f, g] = \mathbb{C}[x, y] \). So \( L = K(g) \) and \( C \) is rational, hence genus zero.

(2) \( \Rightarrow \) (1). \( J(f, g) = 1 = \frac{dy}{f} \), is exact \( \Rightarrow C \) has genus zero. Assume for the moment that \( C \) has a rational point; then genus zero \( \Rightarrow C \) is rational. So we have \( \exists h \) with \( K(h) = L \) and we can choose \( h \) such that \( J(f, h) = 1 \). Then \( J(f, g - h) = 0 \) and so \( g - h \) is a constant. Since \( C \) is rational, \( g - h \in K \) and so \( L = K(g) \). It is well known that proving equality of quotient fields gives the Jacobian conjecture. So it only remains to justify the assumption that \( C \) has a rational point. If \( \frac{dy}{f} \) is exact, it must have poles and it is a consequence of Theorem 2.19 below that each of these poles must be a rational point of \( C \).

Now suppose \( f - y \) is reducible for some \( y \). Statement (1) is vacuously true since it is known \( J(f, g) \) cannot equal 1. On the other hand, if \( \frac{dy}{f} \) is exact, as above, \( C \) must have a rational point. Thus \( K \) is its field of constants, contradicting the reducibility of \( f - y \). So (2) is also vacuously true.

Now we want to determine when the differential \( \frac{dy}{f} \) will be an exact differential. This is a very strong condition on \( f \). One very simple observation is that all exact differentials have poles (since nonconstant elements do) while not all differentials have poles and in most cases of interest to us, \( \frac{dy}{f} \) will not have one. To study this differential, we shall need to examine the points at infinity of \( C \). This will be accomplished via an algebraic procedure. However, in essence, we are simply desingularizing \( C^* \) by blowing up.

Definition. We say a monomial valuation \((u, z, w)\) is positive if \( u(z) > 0 \) and \( u(w) > 0 \).

Definition. If \((u, z, w)\) is a monomial valuation and \( u^* \) is any valuation on \( L \) such that \( u^*(z) = u(z) \) and \( u^*(w) = u(w) \), we say \( u^* \) dominates \((u, z, w)\).

Lemma 2.6. If \( u^* \) dominates \((u, z, w)\) and \( h \in \mathbb{C}[z, w] \), \( u^*(h) \geq u(h) \).

Proof. Trivial.

Notation. We let \( \mathcal{V}_i \) denote the monomial valuation \((u_i, x_i, y_i)\).

Definition. Let \( \mathcal{V}_i \) be a monomial valuation and let \( \mathcal{V}_{i+1} \) be one of the following monomial valuations:

- (A) \( (u_i, x_i/y_i, y_i) \);
- (B) \( (u_i, x_i, y_i/x_i) \);
- (C) \( (u_{i+1}, x_i + \gamma y_i, y_i) \) where \( \gamma \neq 0 \in \mathbb{C} \) and \( u_{i+1}(x_i + \gamma y_i) \geq u_{i+1}(y_i) = u_i(y_i) = u_i(x_i) \).

Then \( \mathcal{V}_{i+1} \) is an alteration of \( \mathcal{V}_i \) (of type (A), (B), (C) respectively).

Lemma 2.7. Suppose \( v \) dominates a positive monomial valuation \( \mathcal{V}_i \). If \( v \neq u_i \),
then there is a unique positive monomial valuation $v_{i+1}$ such that $v$ dominates $v_{i+1}$ and $v_{i+1}$ is an alteration of $v_i$.

**Proof.** There is always precisely one alteration each of type (A) and (B) respectively. As $v_{i+1} = v_i$ in these cases, $v$ will dominate $v_{i+1}$. However, the type (A) alteration will not be positive unless $v_i(x_i) > v_i(y_i)$ and the type (B) alteration will not be positive unless $v_i(y_i) > v_i(x_i)$. If either inequality holds, a type (C) alteration is impossible. So it only remains to show the existence of a unique type (C) alteration dominated by $v$ in the case $v_i(x_i) = v_i(y_i)$.

Since $v \neq v_i$, there exists $h \in \mathbb{C}[x_i, y_i]$, with $v_i(h) < v(h)$. As $v_i$ depends only on the degrees of monomials, we can replace $h$ by its homogeneous component of least degree. Then $h$ may be factored into a product of linear polynomials. This allows us to conclude $v_i(h^*) < v(h^*)$ for some linear factor $h^*$ of $h$. As $h^*$ cannot equal $x_i$ or $y_i$, we may assume $h^* = x_i + y y_i$ for some nonzero $y$. Thus we may construct our type (C) alteration by choosing $x_i+1 = x_i + y y_i$ and $v_{i+1}(x_i+1) = v(x_i+1)$.

Finally, if $y^* \neq y$, $v(x_i + y^* y_i) = v(x_i+1 + (y^* - y) y_i) = v(y_i)$ since $v(y_i) < v(x_i+1)$. Thus no other type (C) alteration will be dominated by $v$.  

**Definition.** A resolution for a valuation $v$ is a sequence of positive monomial valuations $v_1, v_2, \ldots$ such that $v$ dominates each $v_i$, $v_{i+1}$ is an alteration of $v_i$ for each $i$, and the sequence is infinite or $v = v_n$ for some $n$. (We assume $v \neq v_{n-1}$.)

**Remark.** By Lemma 2.7, $v_1$ will determine a unique resolution for $v$.

**Lemma 2.8.** Let $v_1, v_2, \ldots$ be a resolution for a valuation $v$. Let $h \in L$. Then, for some $N$, $v_n(h) = v(h)$ for all $n \geq N$.

**Proof.** First consider the case $h \in \mathbb{C}[x_i, y_i]$. As $\mathbb{C}[x_i, y_i] \subseteq \mathbb{C}[x_{i+1}, y_{i+1}]$ for each $i$, $h \in \mathbb{C}[x_i, y_i]$ for each $i$. If the resolution is finite, the lemma is immediate. Otherwise, note that for alterations of types (A) and (B), we have $0 < v_{i+1}(x_{i+1}, y_{i+1}) < v_i(x_i, y_i)$ and so there cannot be infinite sequences which do not include type (C) alterations. So we may assume there are infinitely many alterations of type (C). Next we claim that if $v_{i+1}$ is a type (C) alteration of $v_i$, then $v_i(h) \leq v_{i+1}(h)$ with equality precisely when $v_i(h) = v(h)$. Assuming the claim, as $v(h) - v_i(h)$ is finite, we must eventually achieve equality and moreover get the condition $v_i(h) = v(h)$ for all $i > m$ for some $m$. So, for any $h \in L$, we can write $h$ as a quotient of two polynomials in $\mathbb{C}[x_i, y_i]$ and choose $N$ large enough to get the correct values for both numerator and denominator.

It only remains to prove the claim. As $v_{i+1}$ dominates $v_i$, the inequality is immediate. As $v$ dominates $v_{i+1}$, we must get equality in the desired case. So it remains to assume $v_i(h) < v(h)$ and prove strict inequality. Write $h$ as a polynomial in $\mathbb{C}[x_i, y_i]$ and let $h_0$ be the homogeneous component of least degree. So $v_i(h) = v_i(h_0) < v_i(h - h_0) \leq v_{i+1}(h - h_0)$. Necessarily, $v_i(h_0) < v(h_0)$. As in the proof
of Lemma 2.7, we factor \( h_0 \) into a product of linear polynomials and obtain a linear factor \( h^* \) such that \( \nu_i(h^*) < \nu(h^*) \). As \( x_{i+1} \) is the only linear polynomial with this property (up to constant multiple), we can assume \( h^* - x_{i+1} \). So \( \nu_i(h_0) < \nu_{i+1}(h_0) \) and therefore \( \nu_i(h) < \nu_{i+1}(h) \).

**Theorem 2.9.** Let \( \nu \) be an essential valuation of \( C \) such that \( \nu(x) > 0 \) and \( \nu(y) < 0 \). Suppose \( \nu_1, \nu_2, \ldots \) is a resolution of \( \nu \) with \( x_1 = x \) and \( y_1 = y^{-1} \). Then this resolution is finite. In fact, it terminates at the first integer \( n \) for which \( \nu_n(f) = 0 \). We shall call this the natural resolution of \( \nu \). The analogous result holds with \( \nu(x) < 0 \) and \( x_1 = x^{-1} \).

**Proof.** First we write \( f = (y_1^{-m})h \) for some positive integer \( m \) and some \( h \in C[x_1, y_1] \). By Lemma 2.8, there exists a least \( n \) for which \( \nu_n(h) = \nu(h) \). We will obtain the theorem by showing \( \nu = \nu_n \). Since \( \nu \) is an essential valuation of \( C \), it is trivial on \( K \) and so \( \nu(f) = 0 \). Of course, \( \nu(y_i) = \nu_1(y_i) \leq \nu_i(y_i) \leq \nu(y_i) \) for every \( i \) and so this is also the least \( n \) for which \( \nu_n(f) = 0 \). Suppose \( \nu \neq \nu_n \). Then the resolution continues beyond \( \nu_n \) and as in the proof of Lemma 2.8, there must be another type (C) alteration \( \nu_{n+1} \) of \( \nu_n \) with \( k = n \). Assuming \( k \) is chosen minimally, \( \nu_{n+1} = \nu_n \) and so we may assume \( k = n \). Since \( y_1, h \in C[x_{n+1}, y_{n+1}] \) and \( (x_{n+1}, y_{n+1}) \) is a monomial valuation, we can write \( y_1^m = x_{n+1}^{\alpha_1} + x_{n+1}^{\alpha_2} \sigma_1 + \varphi_3 \) and \( h = \alpha_2 y_{n+1}^{\sigma_2} + \alpha_2 + \varphi_2 \) where \( \alpha_1, \alpha_2 \in C \) and \( \sigma_1, \sigma_2, \varphi_1, \varphi_2 \in C[x_{n+1}, y_{n+1}] \). \( \nu_n(\varphi_1), \nu_n(\varphi_2) > m

\( \nu(y_i) \), and \( c

\nu_n(y_{n+1}) = m\nu(y_i) \). (Here \( \alpha_1 y_{n+1}^{\sigma_1} + x_{n+1}^{\alpha_2} \sigma_1 \) are the \( \nu_n \)-leading forms of \( y_1^m \) and \( h \).) Since \( \nu_{n+1}(x_{n+1}^{\alpha_1} + \varphi_1) > m\nu(y_i) \) and \( \nu_{n+1}(y_i) = \nu_n(y_i) \), we must have \( \alpha_1 \neq 0 \). Thus there exists a complex number \( \beta \) such that \( \alpha_1 \beta + \alpha_2 = 0 \). Then \( \nu_{n+1}(\beta y_1^m + h) > m\nu(y_i) \). However, \( \nu(f + \beta) = 0 \), so \( \nu((f + \beta)) = m\nu(y_i) \). As \( y_1^m(f + \beta) = h + \beta y_1^m \), we have \( \nu(h + \beta y_1^m) < \nu_{n+1}(h + \beta y_1^m) \), which contradicts the assumption that \( \nu \) dominates \( \nu_{n+1} \). The theorem is thus proved.

Our next objective will be to show how this resolution can be used to compute \( \nu(dy/f) \).

**Proposition 2.10.** Suppose the monomial valuation \( (v, z, w) \) is an essential valuation of \( C \) with \( \nu(z) \neq 0 \). Then \( \nu(f_w) = -\nu(w) \).

**Proof.** Express \( f = \phi/\theta \) with \( \phi, \theta \in C[z, w] \). Let \( \Phi, \Theta \) be the \( \nu \)-leading forms of \( \phi, \theta \) respectively; so we can write \( \phi = \Phi + \phi^* \) and \( \theta = \Theta + \theta^* \). If \( h \) is any monomial in \( C[z, w] \), clearly either \( h_w = 0 \) or \( \nu(h_w) = \nu(h) - \nu(w) \). Thus \( \nu(\phi^*) > \nu(\phi) - \nu(w) \) and \( \nu(\theta^*) > \nu(\theta) - \nu(w) \). We compute

\[
\begin{align*}
f_w &= \theta^{-2}(\theta \phi_w - \phi \theta_w) \\
&= \theta^{-2}[(\Theta \Phi_w - \Phi \Theta_w) + (\Theta \varphi_w^* - \varphi^* \Theta_w) + (\theta^* \varphi_w - \varphi \theta_w^*)] \\
&= \theta^{-2}[(\psi_1 + \psi_2 + \psi_3)]
\end{align*}
\]
for simplicity. Clearly both \( v(\psi_2) \) and \( v(\psi_3) \) are greater than \( v(\theta) + v(\varphi) + v(w) \) while either \( \psi_1 = 0 \) or \( v(\psi_1) = v(\theta) + v(\varphi) - v(w) \). We claim \( \psi_1 \neq 0 \). In that case,

\[
v(f_w) = v(\theta^{-2} \psi_1) = -v(\theta) + v(\varphi) - v(w) = v(f) - v(w) = -v(w).
\]

It remains to prove the claim. If \( \Theta \Phi_\omega - \Phi \Theta_\omega = 0 \), then \( (\Phi/\Theta)_\omega = 0 \) and so \( (\Phi/\Theta) \in \mathbb{C}(z) \). Write \( (\Phi/\Theta) = (h_1/h_2) \) with \( h_1, h_2 \in \mathbb{C}[z] \). So \( h_2 \Phi = h_1 \Theta \). This equation gives equality between the \( \nu \)-homogeneous components of each side and since \( \Phi, \Theta \) are homogeneous, we can assume \( h_1, h_2 \) are homogeneous. As \( v(z) \neq 0 \), they must be monomials of the same degree and so \( (\Phi/\Theta) \in \mathbb{C} \). So we can write \( \Phi = \gamma \Theta \). Then \( v(\varphi - \gamma \theta) > v(\theta) \) and so \( v(f - y) > 0 \). But \( v(f - y) \) must equal zero. This contradiction proves the claim and so the proposition.

\[ \blacksquare \]

**Theorem 2.11.** Let \( v \) be an essential valuation of \( C \) with \( v(x) \neq 0 \) and \( v(y) < 0 \). Let \( \gamma_1, \ldots, \gamma_n \) be the natural resolution of \( v \) and let

\[
e = \sum_{k=1}^{n-1} v_{k+1}(x_{k+1}) - n_k(x_{k+1}).
\]

Then \( v(dy/f_r) = v(x) + v(y) + e - 1 \).

**Proof.** We assume \( v(x) > 0 \); the alternative case can be proved via an almost identical proof. By Lemma 2.1, for any \( g \), we have \( dg = J_{x,y}(f,g) dy/f_r \) and \( dg = J_{x,y}(f,g) dy/f_r \). So by the chain rule, we get \( dy/f_r = J_{x,y}(x,y) dy/f_r \). Since \( v(y_n) \neq 0 \), \( v(dy_n) = v(y_n) - 1 \). By Proposition 2.10, \( v(f_x) = -v(x_n) \).

The next step is to compute \( J_{x,y}(x,y) \) by the chain rule and note that the value of a product is the sum of the values. Thus \( u(J_{x,y}(x,y)) = u(J_{x_1,y_1}(x,y)) + \sum u(J_{x_k+1,y_k+1}(x_{k+1}, y_{k+1})) \). Since \( y_1 = y^{-1} \) and \( x_1 = x \), \( v(J_{x_1,y_1}(x,y)) = v(-y_1^{-2}) = 2v(y) \). Now \( J_{x_k+1,y_k+1}(x_{k+1}, y_{k+1}) \) for a type (A) alteration, \( x_k \) for a type (B) alteration, and 1 for a type (C) alteration. In all cases we have \( u(J_{x_k+1,y_k+1}(x_{k+1}, y_{k+1})) = u_k(x_{k+1} y_{k+1}) - n_k(x_{k+1} y_{k+1}) \). Noting that \( n_k(x_{k+1}) = n_{k+1}(x_{k+1}) \) for each \( k \), we obtain

\[
\sum u(J_{x_k+1,y_k+1}(x_{k+1}, y_{k+1})) = \sum [v_k(x_{k+1} y_{k+1}) - v_{k+1}(x_{k+1} y_{k+1})] = v_1(x_1 y_1) - v_n(x_n y_n) + \sum [v_{k+1}(x_{k+1}) - v_k(x_{k+1})] = v(x) - v(y) - v(x_n) + e.
\]

So we combine to get

\[
v(dy/f_r) = 2v(y) + v(x) - v(y) - v(x_n) + e + v(y_n) - 1 - (-v(x_n)) = v(y) + v(x) + e - 1
\]

as desired. \[ \blacksquare \]

Next we want to relate the points at infinity of \( C \) to \( N(f) \), the Newton polygon of \( f \). If \( v \) is an essential valuation of \( C \) corresponding to a point at infinity, then \( v \) dominates a unique monomial valuation of the form \((v_0, x, y)\) where either \( v_0(x) \)
or \( u_0(y) \) is negative. Let \( F \) be the \( u_0 \)-leading form of \( f \). If \( F \) were a monomial, we would have \( u(F) = u_0(F) \), so \( u_0(f) = u(f) = 0 \). As we assume \((0,0) \in N(f)\), we have \( F \in \mathbb{C} \). But then \( u(f - f(0,0)) > 0 \), contradicting the assumption that \( u \) is an essential valuation of \( C \). Consequently, \( \text{supp}(F) \) must be an edge of \( N(f) \), not a vertex. The slope of the edge is always given by \(-u(x)/u(y)\). We will assume \( N(f) \) is not a straight line. (Otherwise, \( f \) is a function of a single monomial and none of this is very interesting.) It is then clear from the polygon whether \( u(x) \) or \( u(y) \) or both are negative. Precisely, if \( i = x - \deg(f) \), the clockwise path from \((0,p)\) to \((i,q)\) will account for the poles of \( y \) and if \( j = y - \deg(f) \), the counterclockwise path from \((r,0)\) to \((s,j)\) will account for the poles of \( x \). Note that \([L : K(y)] = i\) and so \( y \) will have precisely \( i \) poles and \( i \) zeros, counting in the usual fashion. What we shall ultimately show is that these occur exactly where the polygon suggests they should occur. Precisely, if \((a,b) - (c,d)\) with \( a < c \) is an edge suitable for a pole of \( y \), it accounts for exactly \((c-a)\) poles of \( y \). Necessarily, it then accounts for \((d-b)\) zeros of \( x \) (or \((b-d)\) poles if \( d < b \)). The plan for the proof is to give \((c-a)\) poles as an upper bound and then to notice that we can only get \( i \) poles if every edge attains the maximum.

**Lemma 2.12.** Let \( \mathcal{V}_{k+1} \) be an alteration of \( \mathcal{V}_k \) and \( h \in \mathbb{C}[x_k, y_k] \). Let \( H_i \) denote the \( u_i \)-leading form of \( h \) for \( i = k, k+1 \). Assume \( H_k \) is a monomial. Then \( H_{k+1} \) is a monomial. Moreover, if the alteration is of type (C), \( \text{supp}(H_{k+1}) = \{ y^d_{k+1} \} \) for some \( d \).

**Proof.** Suppose \( H_k = \alpha x^m_k y^n_k \) and \( h^* = h - H_k \). Since \( u_{k+1}(H_k) = u_k(H_k) < u_k(h^*) \leq u_{k+1}(h^*) \), \( H_{k+1} \) will be the \( u_{k+1} \)-leading form of \( H_k \). Thus we reduce to the case \( h = H_k \). For alterations of types (A) and (B), monomials in \( \mathbb{C}[x_k, y_k] \) are also monomials in \( \mathbb{C}[x_{k+1}, y_{k+1}] \) and so the result is clear. So consider a type (C) alteration with \( x_{k+1} = x_k + \gamma y_k \). Then \( h = \alpha x^m_k y^n_k = \alpha (x_{k+1} - \gamma y_k + 1)^m y^n_k \). As \( u_{k+1}(y_{k+1}) < u_{k+1}(x_{k+1}) \), \( H_{k+1} = \alpha (-y)^m y_{k+1}^m \).

**Lemma 2.13.** Let \( v \) be an essential valuation of \( C \) with \( v(x) \neq 0 \) and \( v(y) < 0 \). Let \( \mathcal{V}_1, \ldots, \mathcal{V}_n \) be the natural resolution of \( u \) and let \( K^* \) be the residue field corresponding to \( v \). Suppose \( h \in \mathbb{C}[x_1, y_1] \) is \( y^n_1 f \) if \( v(x) > 0 \) and \( (x_1, y_1)^m f \) if \( v(x) < 0 \). Let \( H \) be the \( v_n \)-leading form of \( h \) and \( r = x_n - \deg(H) \). Then \([K^*: K] v_n(y_n) \leq r \). (We will eventually see, Theorem 2.14, that equality holds.)

**Proof.** Let \( \Phi \) denote the \( v_n \)-leading form of \( hf^{-1} \) (either \( y^m_1 \) or \( (x_1, y_1)^m \)). First consider the case \( n \neq 1 \). Since \( v \neq v_{n-1} \), \( \mathcal{V}_n \) is a type (C) alteration of \( \mathcal{V}_{n-1} \). By Lemma 2.12, \( \Phi = \alpha y^d_n \). If \( n = 1 \), \( v(f) = 0 \) and so necessarily \( v(x) > 0 \). So \( \Phi = y^m_1 \). In either case, since \( F = \Phi^{-1} H \), \( r = x_n - \deg(F) \).

We turn our attention to \( K^* \). We will use the same symbols for elements in \( L \) and their images in \( K^* \), but will use \( = \) for equality in \( L \) and \( \equiv \) for equality in \( K^* \) to avoid any confusion. The equation defining \( C, f(x, y) = t, \) implies \( F = t \). Let \( \phi \) be any non-zero element in \( K^* \). Then there are \( v_n \)-homogeneous elements \( \phi_1, \phi_2 \in \mathbb{C}[x_n, y_n] \) with
Multiplying numerator and denominator by some power of $x_r$, if necessary, we may assume $u_n(y_n)$ divides $u_n(y_1)$. Let $q = u_n(y_1)/u_n(y_n)$ and set $\theta_1 = y_n^{-q} \phi_1$ and $\theta_2 = y_n^{q} \phi_2$. Then $\theta_1, \theta_2$ are nonzero elements of $K^*$ and $\phi = \theta_1/\theta_2$. Now each $\theta_i$ (and also $F$) is a $u_n$-homogeneous element of $C[x_n, y_n, y_n^{-1}]$.

Next let $z = x_n^{\nu_n(y_n)}y_n^{-\nu_n(x_n)}$. The only monomials in $C[x_n, y_n, y_n^{-1}]$ with value zero are nonnegative powers of $z$ and so each $\theta_i \in C[z]$. Clearly then $K^* = C(z)$. Now $z$-deg$(F) = r/u_n(y_n)$ and so $[K^*: K]u_n(y_n) < r$.

**Theorem 2.14.** Let $(u, x, y)$ be a monomial valuation with $v(y) < 0$ which corresponds to an edge $(i, b)-(i, j)$ of $N(f)$ with $i > a$. Then the number of poles of $y$ (counted in the usual way) whose valuations dominate $v$ or a positive multiple of $v$ is precisely $(i-a)$. Moreover, the inequality deduced in Lemma 2.13 is an equality.

**Proof.** We shall actually prove that the number of poles is less than or equal to $(i-a)$. Then, to get the correct total number of poles, we must have the maximum for each edge. If $b = j$, we cannot use the techniques developed in this section. In this case, $u(x) = 0$. Here $F = y^b x^a \phi(x)$ where $\phi(x)$ has degree $(i-a)$. Suppose $\gamma$ is a root of $\phi(x)$ and $K$ is the highest power of $(x - \gamma)$ which divides $\phi(x)$. Regarding $f$ as a polynomial in $x - \gamma$ and $y$, we get a new Newton polygon which we denote $N^*(f)$. It is easy to see that the perimeter, travelling clockwise, follows a path upward from $(0,0)$ to $(k,j)$, then along a horizontal edge to $(i,j)$, and finally returns to $(0,0)$ along the same path as $N(f)$. By showing that $y$ can have only $k$ poles for which $u(x - \gamma) > 0$, which is what we accomplish by handling the non-horizontal edges, we see that the horizontal edge can account for at most $(i-a)$ poles. Thus we may assume $b \neq j$ and $u(x) \neq 0$.

If $u(f) = 0$, $v$ is an essential valuation of $C$ and so the only one which dominates itself (or a positive multiple). It accounts for $[K^*: K](-u(y))$ poles. By Lemma 2.13, as $-u(y) = u(y)$, this product is bounded by $x$-deg$H = x$-deg$F = i$. Also, since $u(f) = 0$, the origin must lie on this edge and so $(a,b) = (0,0)$. So $a = 0$ and $i - a = i$; this case is proved.

Now assume $u(f) < 0$. Let $v^*$ be an essential valuation which dominates $v$ (up to positive multiple) and let $\gamma_1, ..., \gamma_n$ be the natural resolution of $v^*$. Choose $k$ minimal such that $\gamma_{k+1}$ is a type (C) alteration of $\gamma_k$. Observe that $\gamma_1, ..., \gamma_k$ is a sequence of alterations of types (A) and (B) and so is independent (up to positive multiple) of the choice of $v^*$. Choosing $h = y_1^m f$ or $(x_1 y_1)^m f$ as in the earlier proofs, and letting $H_k$ be its $v_k$-leading form, $H_k = ax_k^p y_k^q \phi(x_k, y_k)$ with $a \in C, p, q \in \mathbb{Z}$, and $\phi$ homogeneous and not divisible by either $x_k$ or $y_k$. It is relatively easy to see that the degree of $\phi$ is $(i-a)/(u(y))$. (The idea is that $\phi$ corresponds to the change along the edge, the total $x$-change is $(i-a)$, and it occurs in increments of $(-u(y))$ while $y$ is changing in increments of $u(x)$.) Necessarily, $x_{k+1}$ must be a root of $\phi$. Suppose $m$ is the highest power of $x_{k+1}$ which divides $\phi$. In order to obtain our result, it suffices to show that the number of poles of $y$ corresponding to this particular choice of $x_{k+1}$ is bounded by $m(-u(y))$. With $y_{k+1} = y_k$, $v^*$ dominates a uni-
que monomial valuation \((v_k+1, x_k+1, y_k+1)\). Write \(f\) as a quotient of two polynomials in \(\mathbb{C}[x_k+1, y_{k+1}]\). By Lemma 2.12, we can assume the leading form of the denominator is a power of \(y_{k+1}\) (for \(v^*\) or any other valuation giving \(x_{k+1}\) a higher value than \(y_{k+1}\)). By Proposition 1.9, \(f\) is a limit of elements of \(\mathbb{C}[x_k+1, y_{k+1}, x_{k+1}^{-1}, y_{k+1}^{-1}]\) and so, inasmuch as \(y\) has only finitely many poles, there is a single element \(f^*\) which approximates \(f\) up to an element of positive value for \(v^*\) and any similar valuation. As the leading form of the denominator was a power of \(y_{k+1}\), we actually have \(f^* \in \mathbb{C}[x_{k+1}, y_{k+1}, y_{k+1}^{-1}]\). We can safely choose \(f^*\) so that \(\theta \in \text{supp}(f^*) = v_k(\theta) \leq 0\).

Then, as both \(x_{k+1}\) and \(y_{k+1}\) have positive values, we must have \(f^* \in \mathbb{C}[x_{k+1}, y_{k+1}^{-1}]\). Letting \(z = x_{k+1}\) and \(w = (y_{k+1})^{-1}, f^* \in \mathbb{C}[z, w]\) is a polynomial. Since \(v^*(f-f^*) > 0\), we observe that \(v^*\) is also an essential valuation of the generic curve of \(f^* (f^*(z, w) = t)\). So \(v^*\) corresponds to one of the edges of \(N(f^*)\) which yields poles of \(w\). In fact \(v^*\) will correspond to one of the edges along the path from \((0, c)\) to \((m, d)\) for some \(c, d\) as those edges to the right of \((m, d)\) correspond to valuations \(v'\) for which \(v'(z) < v'(w)\). Now \(v^*(y) = v^*(y_k)u(y)\) since \(v(y_k) = 1\). Also \(v^*(w) = -v^*(y_k)\); thus \(v^*(y) = v^*(w)(-v(y))\). In order to show \(y\) has at most \(m(-u(y))\) poles of this type, it is enough to show that \(w\) has at most \(m\) poles corresponding to the path from \((0, c)\) to \((m, d)\). So, if the theorem holds for the appropriate edges of \(N(f^*)\), it holds for \(N(f)\). So we simply repeat the process until we reach the \(v(f) = 0\) case. As resolutions are finite and we are actually resolving the same valuations throughout, the process really does terminate. To see the last statement in the theorem, simply note that unless every inequality is actually an equality, we will get too few poles.

We now have the ability to analyze the points at infinity of \(C\). We restrict \(f\) so that \((f, g)\) is the minimum counterexample to the Jacobian conjecture discussed in Section 1. There are infinitely many choices for the pair \((D, E)\), which seems to preclude a case-by-case proof of the Jacobian conjecture. However, if we are willing to accept a bound on the degree of \(f\), the number of pairs becomes finite. For each such pair, there are only finitely many possible Newton polygons for \(f\). For each \(N(f)\), there are only finitely many ways to resolve the points at infinity. What we wish to learn is what constraints the counterexample property places on \(N(f)\) and the information we obtain via the resolution process. These constraints rule out almost all possibilities. Unfortunately, this requires an ad hoc case-by-case procedure and a computer search. We shall find some possibilities which are not inconsistent with the counterexample property and these will be considered in Section 3.

Some constraints on \(N(f)\) were developed in Section 1. The primary constraint we shall consider here is the requirement that \([dg] = [dy/f_i]\) must have a pole. We will focus on the resolution of those valuations corresponding to poles; we will learn more about \(\tau(v)\) during this process. We begin with a result both nice and easy.

**Theorem 2.15.** If \(v\) is an essential valuation of \(C\) with \(v(x) < 0\), then \(v(dg) \geq 0\).
Proof. On the right side of $N(f)$, the slope is always positive and $\leq 1$. So $v(y) \geq -v(x)$. Applying Theorem 2.11 with the variables reversed, we obtain

$$v(dg) = v(x) + v(y) + e - 1 \geq e - 1 = 0.$$ 

The last inequality follows since $e$ is the sum of (at least one) positive integers. □

Our next objective is to learn how the approximations $\tau(u_i)$ vary along a resolution. Since our topology depended on the choice of variables, the notation $\tau(v)$ is a bit ambiguous. ($\tau(V)$ would actually be more precise.) However, note that the change of variables $\{x_k, y_k\} \to \{x_{k+1}, y_{k+1}\}$ which we encounter in the resolution process preserves leading forms of polynomials for any of our monomial valuations. Thus $A_v$ will be the same for $(u, x_k, y_k)$ and $(u, x_{k+1}, y_{k+1})$. So it makes sense to say $\tau(u_k) = \tau(u_{k+1})$ if $u_k = u_{k+1}$. It remains to consider the situation when $Y_k+1$ is a type (C) alteration of $Y_k$. To understand this case, we think of $v_k$ as the monomial valuation $(v_k, x_{k+1}, y_{k+1})$. In this way, $v_k$ and $v_{k+1}$ live in the same topological space; we have some hope of applying Propositions 1.10 and 1.11. Now we analyze the resolution process and determine when the hypotheses of these theorems hold.

In light of Theorem 2.15, we restrict our attention to those essential valuations for which $v(y) < 0$.

**Lemma 2.16.** If $Y_{k+1}$ is a type (C) alteration of $Y_k$, then $J_{x_{k+1}, y_{k+1}}(f, g)$ has leading form $\beta(y_{k+1})^2$ where $\beta \neq 0 \in \mathbb{C}$ and

$$s + 2 = (v(x) + v(y) + e^o)/v(y_k) \quad \text{for } e^o = \sum_{i=1}^{k-1} v_{i+1}(x_{i+1}) - v_i(x_{i+1}).$$

**Proof.** Let $x_0 = x$ and $y_0 = y$. By the chain rule,

$$J_{x_{k+1}, y_{k+1}}(f, g) = J(f, g) \left[ \prod_{i=1}^{k} J_{x_i, y_i}(x_{i-1}, y_{i-1}) \right] J_{x_{k+1}, y_{k+1}}(x_k, y_k)$$

$$= (1)(\chi)(1),$$

where $\chi$ is a product of monomials, each occurring in $\mathbb{C}[x_i, y_i]$ for some $i \leq k$. By Lemma 2.12, the leading form of $\chi$ in $\mathbb{C}[x_k, y_k]$ is a monomial and so, applying Lemma 2.12 again, the leading form in $\mathbb{C}[x_{k+1}, y_{k+1}]$ will be a constant multiple of a power of $y_{k+1}$. It remains to compute $s$. Certainly, $sv(y_{k+1}) = v(J_{x_{k+1}, y_{k+1}}(f, g))$. The right-hand side equals $v(x) + v(y) - v(x_k) - v(y_k) + e^o$. (See proof of Theorem 2.11 for computation.) As $v(y_{k+1}) = v(y_k) = v(x_k)$, the formula follows. □

**Remark.** It is harmless to assume $\beta = 1$.

**Notation.** For the next few results (Lemma 2.17 and Theorems 2.18, 2.19), we let $z = x_{k+1}$ and $w = y_{k+1}$. We restrict our attention to those monomial valuations $(v', z, w)$ which have the property $v'(z), v'(w) > 0$ and we define $v_1 < v_2$ if $v_1(w)v_2(z) - v_1(z)v_2(w) < 0$. This coincides with the notion of order we get going counterclockwise around $N(f^*)$ where $f^*$ is as in the proof of Theorem 2.14 and
is consistent with Lemma 1.15. So \( u_k \) is maximal among all valuations in the space for which \( v(z) \geq v(w) \). The number \( s \) will be that given by Lemma 2.16.

**Lemma 2.17.** Let \( (v', z, w) \) be a monomial valuation such that \( v_{k+1} \leq v' < u_k \) where \( v_{k+1} \) is dominated by \( v \), an essential valuation of \( C \) with \( v(dg) < 0 \). Suppose \( z^iw^j \in \text{supp}(F') \), where \( F' \) is the \( v' \)-leading form of \( f \). If \( v' = v_{k+1} \), assume \( z^iw^j \) is chosen so that \( i \) is maximal. Then \( i(s+1) - j < 0 \) and \( (s+1)v'(w) + v'(z) < 0 \).

**Proof.** First we consider the case \( v' = v_{k+1} \). In general, for any type (C) alteration \( \gamma_{i+1} \) of \( \gamma_i \), if \( m \) is the highest power of \( x_{i+1} \) which divides the leading form of \( f \), then \( v_{i+1}(f) - v_i(f) \leq m[v_{i+1}(x_{i+1}) - v_i(x_{i+1})] \). For an alteration in the resolution of \( v \) with \( t > k \), necessarily \( m \leq i \). (This last statement necessitated the maximality condition on \( i \).) As \( v(f) = 0 \), this yields

\[
i[e - e' - (v_{k+1}(z) - v_k(z))] \geq -v_{k+1}(f) = -iv_{k+1}(z) - jv_{k+1}(w).
\]

Noting \( v_{k+1}(w) = v_k(w) \) and simplifying, we have \( i(e - e') \geq -iv_k(z) - jv_k(w) \). As \( v(dg) \) cannot equal \(-1\), \(-1 > v(dg) = v(x) + v(y) + e - 1 \) by Theorem 2.11 and so \( e < -v(x) - v(y) \). Combining inequalities,

\[
i(-v(x) - v(y) - e') > i(e - e') \geq -iv_k(z) - jv_k(w).
\]

By Lemma 2.16, \( s + 2 = \frac{v(x) + v(y) + e'}{v(y)} \). Substituting, \( i(s + 2)v(y) < iv_k(z) + jv_k(w) \). Now \( v(y) = v_k(z) = v_k(w) > 0 \) and so \( i(s + 2) < i + j \). Thus \( i(s + 1) - j < 0 \). (The first desired inequality!) This implies \( s + 1 < (j/i) \) and so

\[
(s + 1)v_{k+1}(w) + v_{k+1}(z) < (j/i)v_{k+1}(w) + v_{k+1}(z) = (1/i)v_{k+1}(f) \leq 0,
\]
giving the second inequality.

Now we consider \( v' \) with \( v_{k+1} < v' < u_k \). Let \( F_{k+1} \) be the \( v_{k+1} \)-leading form of \( f \) and let \( x^ay^b \) be the element in \( \text{supp}(F_{k+1}) \) used in the first paragraph. From the first paragraph, we know \( s + 1 < (b/a) \) and \( s + 1 < (v_{k+1}(z)/v_{k+1}(w)) \). The desired inequalities will follow by transitivity if we can show \( (b/a) \leq (j/i) \) and \(-v_{k+1}(z)/v_{k+1}(w)) \leq (v'(z)/v'(w)) \). Since \( v_{k+1} < v' \), the second holds by definition. We must prove the first. If \( a = i \) and so \( b = j \), it is obvious. Otherwise, as \( z^aw^b \in \text{supp}(F_{k+1}) \), \( av_{k+1}(z) + bv_{k+1}(w) \leq iv_{k+1}(z) + jv_{k+1}(w) \); so \( (a - i)v_{k+1}(z) + (b - j)v_{k+1}(w) \leq 0 \). As \( (a - i) < 0 \) and \( v_{k+1}(w) > 0 \), \( (b - j)/(a - i) \geq -v_{k+1}(z)/v_{k+1}(w) \). Similarly, as \( av_{k+1}(z) + bv_{k+1}(w) \leq 0 \), \(-v_{k+1}(z)/v_{k+1}(w) \geq b/a \). So \( (b - j)/(a - i) \geq b/a \). As \( a(a - i) < 0 \), \( a(b - j) \leq b(a - i) \); so \( -aj \leq -bi \). Finally, as \( a, i > 0 \), we have \( j/i \geq b/a \), which is the desired inequality. \( \Box \)

**Theorem 2.18.** Assume \( v_{k+1} \) is dominated by \( v \), an essential valuation of \( C \) for which \( v(dg) < 0 \). Let \( F_{k+1} = \Phi_1, \Phi_2, \ldots, \Phi_r = F_k \) be the leading forms of \( f \) for valuations corresponding to a path of edges from \( F_{k+1} \) to \( F_k \) (possibly \( r = 2 \)). For each \( t \geq 2 \), there exists an integer \( q \) such that \( \Phi_{t+1} \) is a \( q \)-th power and \( \Phi_t \) is not. If \( v_{k+1} \neq v \) and \( F_{k+1} \), does not have the form \( vw^d(zw^d - \omega)^i \), the result also holds for \( t = 1 \).
Proof. The idea is to apply Propositions 1.10 and 1.11. Certainly \( f, g \) have the LDM property for \( v \), and this property is preserved by alterations. Thus \( f, g \) have the LDM property for \((u_k, x_k, y_k)\) and also for \((u', z, w)\) when \( u' < u_k \). The property actually fails at the point \((u_k, z, w)\) but is not required to apply Proposition 1.10 at this point. We also have \( w^* \) is the \( u' \)-leading form of \( J(f, g) \) for all \( u' < u_k \). Using the inequalities we proved in Lemma 2.17, Proposition 1.10(1),(2), and Proposition 1.11 tell us that if \( u_{k+1} < u' < u' < u_k \), then \( \tau(u') = \tau(u') + \tau^e \), where \( \tau^e \) may contain additional higher value terms. We could also apply Proposition 1.10(3) except that we do not know if \( \tau'(w) \neq 1 \) or \( \tau'(f) < 0 \). (In Proposition 1.10, it is assumed that \( \tau'(z) \) and \( \tau'(w) \) are relatively prime. So we must interpret the condition \( \tau'(w) \neq 1 \) to mean \( \tau'(w) \) does not divide \( \tau'(z) \).) As \( \tau'(z) \)/\( \tau'(w) \) increases and \( \tau'(f)/\tau'(w) \) increases. Thus it attains its maximum when \( \tau'(w) = u_{k+1} \). Since \( u_{k+1}(f) \leq 0 \), we get \( \tau'(f) < 0 \) for all \( u' > u_{k+1} \). Also, for \( u' > u_{k+1} \), if \( z^* w^{*'} \) is the support of the \( u' \)-leading form of \( f \), necessarily \( i^* > 0 \). So, by Proposition 1.10(4), the hypothesis \( \tau'(w) \neq 1 \) is unnecessary here. Thus, for \( t \geq 2 \), the theorem holds. Now we must handle the \( t = 1 \) case. Since \( u_{k+1} \neq u \), we have \( u_{k+1}(f) < 0 \). Then Proposition 1.10(4) yields the desired result. 

Theorem 2.19. As usual, \( \mathcal{V}, \ldots, \mathcal{V}_n \) is the standard resolution of \( v \). Let

\[
e^o(t) = \sum_{i=1}^{t-1} v_{i+1}(x_{i+1}) - v_i(x_{i+1}).
\]

Then

1. \( \tau(v_i) = 0 \Leftrightarrow e - e^o(t) + v_i(f) = 0 \) and \( v(g) < 0 \). In this case, if \( t < k \), \( z^2 \) does not divide \( F_k \).

2. In particular, \( \tau(v) = 0 \Leftrightarrow v(dg) < 0 \).

3. Let \( \deg(f)/\deg(g) = p/q \) with \( \gcd(p, q) = 1 \). Assume \( \tau(v_k) + 0 \) and \( \tau(v_{k+1}) = 0 \). Let \( v^* \), with \( v_{k+1}(f) \leq v^* < u_k \), be maximal with respect to \( \tau(v^*) = 0 \). Then \( F^* = z^{e^o} \Theta(z^{e^o} w^b) \) where \( N = a \deg(\Theta) + r \) is a multiple of \( p \), \( \Theta \) is a product of unrepeatable linear factors, \( r = 0 \) or \( 1 \), \( c < 0 \), and \( v^* \neq v \). Moreover, if \( v^* = u_{k+1} \), \( v(g) = [q/(p + q)]a(s + 1) - b \) while if \( v^* \neq u_{k+1} \), \( v(g) = [q/(p + q)]a(s + 1) - b)/a \). Also, \( a \mid p \) or \( a \mid q \), and if \( a \) does not divide \( q \), then \( r = 0 \) and \( u_{k+1} = v^* \).

Proof. 1. Note that \( e^o(k) = e^o \) and \( e^o(n) = e \). Looking at the resolution process, we see that if \( z^m \) is the highest power of \( z \) which divides \( F_k \), \( e^o(k + 1) - e^o(k) \leq u_{k+1}(f) - u_k(f) \) \( \leq m(e^o(k + 1) - e^o(k)) \). The first inequality is an equality precisely when \( m = 1 \). This inequality says that the function \( \mu(t) = e - e^o(t) + v_i(f) \) is non-decreasing. Also \( \mu(k + 1) > \mu(k) \) if \( z^2 \) divides \( F_k \). Since \( \sup\{\mu(t)\} = \mu(n) = 0 \), if \( \mu(t) = 0 \) and \( k > t \), \( z^2 \) cannot divide \( F_k \). This gives the second statement of (1). Next note \( \tau(v) = 0 \Leftrightarrow J_{x, y}(f, g) = v_i(fg_{x_i^{-1}} y^{-1}) \). Expanding the left hand side as in the proof of Theorem 2.11, this equality is just

\[
v(x) + v(y) - v(x_i) - v(y_i) + e^o(t) = v_i(f) + v_i(g) - v(x_i) - v(y_i).
\]
(We use \( \nu \) in place of \( \nu_t \) for elements where valuations must coincide.) By Theorem 2.11, \( \nu(dg) = \nu(x) + \nu(y) + e - 1 \) and so \( \tau(v_t) = 0 \Leftrightarrow \nu(dg) = e - e^2(t) + v_t(f) + v_t(g) - 1 - \mu(t) + v_t(g) = 1 \). The right-hand side of this equation is a nondecreasing function of \( t \) and equals \( \nu(g) - 1 \) when \( t = n \). Now \( \nu(dg) \geq \nu(g) - 1 \) with equality precisely when \( \nu(g) \neq 0 \). So \( \tau(v_t) = 0 \Leftrightarrow v_t(g) = \nu(g) \neq 0 \) and \( \mu(t) = 0 \).

Now suppose \( t \) is minimal so that \( \tau(v_t) = 0 \). Necessarily \( \tau' \) is a type \((C)\) alteration of \( \tau_{k-1} \) and we let \( t = k+1 \) and use our familiar notation. Now choose \( v' < v_k \) maximal with respect to \( \tau(v') = 0 \). We can choose \( v'' > v' \) such that \( z^i w^j = \text{supp}(F^*) \subseteq \text{supp}(F') \) and \( \text{supp}(G^*) \subseteq \text{supp}(G') \). Noting that \( \tau(v') \) is a truncation of \( \tau(v_t) \), which is itself a truncation of \( \tau(v_t) \), we see that \( d = \frac{q}{p} \). Thus \( \nu'(g) = \nu'(f) \leq 0 \). Now for any \( v^* \), \( v_{k+1} \leq v^* \leq v' \), we have \( v^*(J_{z,w}(f,g)) = v^*(fgz^{-1}w^{-1}) \). In considering \( v^* \), we hold \( v^*(w) \) constant, allowing \( v^*(z) \) to not be an integer. Then, as \( v^*(J_{z,w}(f,g)) \) and \( v^*(w) \) are constant, \( v^*(fz^{-1}) + v^*(g) \) must be constant. Both of these quantities are nondecreasing as \( v^* \) decreases; hence they must be constant. So \( v_{k+1}(g) = v^*(g) \leq 0 \). Since \( v_k(g) = v(g) \neq 0 \), we obtain \( v(g) < 0 \) and so one direction of (1).

Conversely, suppose \( v(g) < 0 \) and \( \mu(t) = 0 \). If \( t = n \), we have \( v_t(g) = \nu(g) \) and we are done by the first paragraph. Otherwise, choose \( k \geq t \) minimal with respect to \( \tau_{k-1} \) is a type \((C)\) alteration of \( \tau_k \). Since \( \tau(v_t) = \tau(v_k) \), it suffices to show \( \tau(v_k) = 0 \). Since \( \mu(k) = 0 \), we know that \( z \) divides \( F_k \) but \( z^2 \) does not. Thus \( F_k \) cannot have a \((q/p)\)th power since \( q/p \) is not an integer. But if \( \tau(v_k) \neq 0 \), \( G_k \) must be a \((q/p)\)th power of \( F_k \). So \( \tau(v_k) = 0 \).

(2) This is just the \( t = n \) case of (1).

(3) \( v^* \) is the valuation \( v' \) discussed in (1). Let \( z' \) be the highest power of \( z \) which divides \( F^* \), the leading form, and let \( z^N \) be the highest power which divides a term of \( F^* \). As seen in the earlier proof, the highest power of \( z \) which divides a term of \( G^* \) is \( z^{Na/p} \) and so \( p \mid N \). On the other hand, we can write \( F^* = z^r w^c \Theta(z^a w^b) \) with \( r \geq 0 \), \( a > 0 \), and \( b < 0 \). Similarly we write \( G^* = z^r w^c \Psi(z^a w^b) \) and exploit the equation \( J(F^*, G^* ) = w^s \). Expanding, and letting \( u = z^u w^b \), we have

\[
\begin{align*}
z^{i-r-1} w^{j+c-1} [(rj-ci) \Theta(u) \Psi(u) + (rb-ca) u \Theta'(u) \Psi(u) + (aj-bi) u \Theta(u) \Psi'(u)] = w^s.
\end{align*}
\]

So

\[
\begin{align*}
(rj-ci) \Theta(u) \Psi(u) + (rb-ca) u \Theta'(u) \Psi(u) + (aj-bi) u \Theta(u) \Psi'(u)
= z^{1-i-r} w^{s+1-j-c}
\end{align*}
\]

and we may consider the exponent of \( z \). If \( rj-ci = 0 \), \( r = i = 0 \) and \( a = 1 \). If \( rj-ci \neq 0 \), \( r+i = 1 \). Either way, we have \( r = 0 \) or \( 1 \). As no linear polynomial in \( u \), except \( u \) itself, divides the right-hand side of the equation, the same is true of the left-hand side. So \( \Theta(u) \) has no repeated roots. Next we show that \( c < 0 \) and \( v^* \neq v \). If \( r \neq 0 \), then there is another edge below this one and so there is a valuation \( \hat{\nu} < v^* \) such that \( \hat{\nu} \) is dominated by an essential valuation of \( C \). Since \( \hat{\nu}(f) \leq 0 \), \( v^*(f) < 0 \) and so \( c < 0 \) and \( v^* \neq v \). If \( r = 0 \), then \( c < 0 \) implies \( v^* \neq v \) and so we need only rule out the case
\( r = c = 0 \). In that case, \( rb - ca = 0 \) and so \( \Theta(u) \) divides the left-hand side of the equation given by the Jacobian condition. This means \( \Theta(u) \) is a constant and then so is \( F^* \), which is absurd.

It is clear from the proof of (1) that

\[
v(g)/v(w) = v^*(g)/v^*(w) = (q/p)v^*(f)/v^*(w).
\]

Then, as \( v^*(J(f, g)) = v^*(fgz^{-1}w^{-1}) \),

\[
v^*(fg) = v^*(zw^{i+1}) = (s + 1)v^*(w) + v^*(z) = [(s + 1)a - b][v^*(w)/a].
\]

So \( v(g) = [q/(p+q)][(s+1)a-b][v(w)/a] \). If \( v_{k+1} = v^* \), then \( v \) is one of \( \text{deg}(\Theta) \) essential valuations which dominate \( v^* \) and must account for \( \text{deg}(\Theta) \) zeros of \( w \). Since \( v(w) \) must be a multiple of \( a \), we must have \( v(w) = a \). If \( v_{k+1} \neq v \), then \( r = 1 \) and \( z\text{-deg}(F_{k+1}) = 1 \). Thus \( v_{k+1} \) accounts for a single zero of \( w \) and so \( v(w) = 1 \). Since \( v(g) \) is an integer and \( \gcd(a, b) = 1 \), \( a \mid q \). Finally, note that if \( r = 1 \), there is a valuation corresponding to this bottom edge (whether or not that valuation is \( v_{k+1} \)) and so \( a \mid q \) anyway. There is a symmetry involving \( F^* \) and \( G^* \) and if \( i = 1 \), \( a \mid p \). Of course, if \( i = r = 0 \), then \( a = 1 \); so \( a \mid p \) and \( a \mid q \). The proof is now complete. 

**Definition.** A \( \pi \)-resolution for \( v \) is obtained as follows: Begin with a resolution \( \mathcal{V}_1, \ldots, \mathcal{V}_n \) for \( v \). If \( v_{k+1} \) is a type (A) or (B) alteration of \( v_k \), delete \( \mathcal{V}_{k+1} \). Then rescale each remaining valuation \( v_k \) so that the value group is \( \mathbb{Z} \); i.e., divide \( v_k \) by \( \gcd(v_k(x_k), v_k(y_k)) \). Then reindex the sequence of valuations so that we can describe it as \( \mathcal{V}_1, \ldots, \mathcal{V}_n \) as before (new \( n \)).

Hereafter, we will work with \( \pi \)-resolutions. They contain all of the information of a full resolution and can be computed more naturally. In order to determine the correct value group of \( v \) in the usual resolution, one must know \( v \) beforehand. In computing the \( \pi \)-resolution of an essential valuation corresponding to a particular edge, \( v_1 \) will be the natural valuation corresponding to that edge.

We can abstract one step further and consider possible resolutions (really \( \pi \)-resolutions) of an edge of a Newton polygon without specifying the actual polynomial. To see how to do this, let us recall how the resolution process works. The edge determines the valuation \( v_1 \). After a suitable change of variables, we find a polynomial \( f^* \) as in the proof of Theorem 2.14. There is a path \( \Pi^* \) along \( N(f^*) \) from the \( w \)-axis to a particular vertex and the essential valuations of the original curve \( C \). \( v_2 \) is determined by one of the edges along this path. We continue by finding another polynomial \( f^{**} \), an appropriate edge of \( N(f^{**}) \), and then \( v_3 \). After a finite number of steps, the process terminates. Thus the resolution is determined by \( N(f), N(f^*), N(f^{**}) \), etc. and requires no direct knowledge of \( f \).

So we could begin with \( N(f) \) and select an edge; this determines \( v_1 \). Up to a permutation of \( \mathbb{C} \), which we can ignore, there are only finitely many possible factorizations of the leading form of \( f \). For each root of this leading form, the multiplicity determines the crucial vertex of \( N(f^*) \). The number of choices for \( \Pi^* \) is now finite.
And so on. Though there is much branching, the entire process is ultimately finite.

Now suppose \{f, g\} is our minimal counterexample. Let \(\deg(f) = p[\gcd(\deg(f), \deg(g))]\) and \(\deg(g) = q[\gcd(\deg(f), \deg(g))]\). Then, by Proposition 1.14, there is a polygon \(P\) such that \(N(f) = pP\) and \(N(g) = qP\). There is no difference whatsoever between resolving an edge of \(P\) and resolving the corresponding edge of \(N(f)\) or \(N(g)\) until we reach a valuation \(v_k\) such that \(\tau(v_k) = 0\). For example, there will be a path \(P^*\) such that \(P^* = pP^*\). We will work with \(P\). At each step, we will also need to know the leading form of \(J_{z,w}(f^*, g^*)\), which is the same as the leading form of \(J_{z,w}(f, g)\); this is provided by Lemma 2.16 and does not depend on \((p, q)\).

We are now ready to describe the search procedure. Fix a vertex \((A, B)\), \(A > B\). The nonhorizontal or ascending portion of the left side of \(P\) will be a path from the \(y\)-axis to \((A, B)\). We consider all possible convex paths and try all possible ways to resolve the edges. For a given resolution, three things can happen: We can find a pole of \([dg]\), we can get a valid resolution without a pole, or an attempted resolution can fail by contradicting one of our results. The counterexample requires a complete family of resolutions, all of which are allowed and at least one of which gives a pole of \([dg]\). We do not actually carry out every possible resolution; we simply go as far as we need to see which of the three possibilities occur.

Next we will offer a few propositions needed in our search which are consequences of what has gone before. Then we shall prove \(\gcd(\deg(f), \deg(g)) > 15\) by hand and conclude with more substantial results obtained via a computer search.

**Proposition 2.20.** Let \(v\) be an essential valuation of \(C\) and suppose \(\hat{f}, \hat{g} \in C[z, w]\) occur in the resolution process for \(v\). (Notation as in Proposition 2.14.) Assume \(\tau(v_{k-1}) \neq 0\) and \((pi, pj)\) is the critical vertex of \(N(\hat{f})\) needed to resolve \(v_{k-1}\) and \((qi, qj)\) is the corresponding vertex of \(N(\hat{g})\) with \(\gcd(p, q) = 1\). Assume \(w^s\) is the \(v_k\)-leading form of \(J_{z,w}(\hat{f}, \hat{g})\) and \(v_k(\hat{f}) < 0\). Let \(\sigma_1 = -(1+s)v_k(w)+v_k(z)\) and \(\sigma_2 = l(1+s)-j\). Then

1. If \(\tau(v_k) = 0\), \(\sigma_2 - i > 0\).
2. If \(\sigma_1 \leq 0\), \([dg]\) will not have a pole at \(v\).
3. Let \(z^{pc}w^{pd}\) be the highest degree monomial in the \(v_k\)-leading form of \(f^*\). If \(\sigma_1 > 0\) and either \(v_k(w) \neq 1\) or \(v_k(z)\) does not correspond to the leftmost edge of \(N(\hat{f})\), then \((d - c(1+s))/v_k(\hat{f}) \in \mathbb{Z^+}\).

**Proof.** Let \(a = -v_k(w)\) and \(b = v_k(z)\); so \(a, b > 0\), \(\sigma_1 = a(1+s) - b\), and \(v_k(\hat{f}) = ib - ja\). We let \(\gamma = v_k(\hat{f})\). In the notation of Proposition 1.3, \(\sigma_2\) will be \(\sigma_2 + m\gamma\) since it represents the large degree end of the leading form we will generally consider.

1. We employ Theorem 2.19. As \(\sigma_2 - i = sl - j\), the claim is \(j/l < s\). If \((pc, pd)\) is a vertex of \(N(\hat{f})\) to the left of \((pi, pj)\), \(d/c > j/i\) and so if we can prove \(d/c < s\), necessarily \(j/i < s\). So we can safely assume \(z^{pd}w^{pj}\) is the highest degree term of \(F^*\) (notation as in Theorem 2.19). There are monomials in \(F^*\) and \(G^*\) whose product is \(z^{(p+q)i}w^{(p+q)j}\), namely the highest degree terms, and also monomials whose product is \(zw^{s+1}\) since \(J(F^*, G^*) = w^s\). Consequently, there is an integer \(T\) such that
\[(p+q)^i = aT^i + (p+q)^j = bT^j \quad \text{and} \quad (p+q)^i = aT^i + (p+q)^j = bT^j.\] Multiply the first equation by \(j\), the second by \(i\), and then subtract. This yields \(0 = -yT - \sigma_2\). Since \(a \mid p\) or \(a \mid q\), \(a < p+q-1\) and so \((p+q)^i = aT^i + (p+q)^j = bT^j\). So \(\sigma_2 - i > \sigma_2 - T > \sigma_2 + yT = 0.\)

(2) Recall \(z = x_k, w = y_k^{-1}\), and \(J(f, g) = J(f, g)\). As \(J_{x_k, y_k}(z, w) = -w^2, J_{x_k, y_k}(f, g) = -w^{s+2} = y_k^{-s-2}\). By Lemma 2.16, this last equation yields \(-u(y_k) = v(x) + v(y) + e^0(k)\), where \(e^0(k)\) is as in Theorem 2.19. Since we have a \(\pi\)-resolution, \(u(y_k) = au(y_k+1)\) and \(e^0(k+1) = e^0(k) + (b-a)v(y_k+1)\). So
\[v(x) + v(y) + e^0(k+1) = (-s+1)a + b) v(y_k+1) = -\sigma_1 v(y_k+1) \geq 0\]
since \(\sigma_1 \leq 0\). By Theorem 2.11, \(v(dg) \geq 1\). Then \(v(g) \geq 0\) and \([dg]\) cannot have a pole at \(v\).

(3) First we employ Proposition 1.8 to find an approximation \(g_0\) to \(g\). As in the proof of Proposition 1.10, we can find a \(t\) such that \((f')G_0\) is a polynomial and \(J(f, f', G_0) = -w^2(f')\). Now Proposition 1.3 applies and the result will follow immediately if we can show that case (b) is the only possibility. Since \(\sigma_1 > 0\), case (a) fails. Letting \(i, j\) be the \(i, j\) of Proposition 1.3,
\[\sigma_2 \geq \sigma_2 - i\sigma_1 / a\]
\[= i(1+s) - j - (i(1+s) - i/b/a) = i(b/a - j) = y/a \geq y.\]
As \(y < 0\), \(\sigma_2 / y \in \mathbb{Z}^+\) can only happen if \(\sigma_2 = y\). Looking at the inequality chain, we see that this happens only when \(i = 0\) and \(a = 1\). However, this is ruled out by hypothesis and so cases (c) and (d) also fail.

**Notation.** We now wish to consider a \(\pi\)-resolution \(\mathcal{R}_1, \ldots, \mathcal{R}_n\) so that \(n\) is minimal with respect to \(\tau(v_k) = 0\). As we want to consider the entire sequence rather than just one \(v_k\) at a time, we will distinguish between quantities by using subscripts. We let \(e_k = (d_k, e_k)\) be the edge corresponding to \(v_k\). Recall the variables here are \(x_k\) and \(y_k^{-1}\). Let \(a_k = u_k(x_k)\) and \(b_k = u_k(y_k)\). Set \(M_k = (E_k - e_k)/a_k = (D_k - d_k)/b_k > 0\). We let \(s_k\) be the exponent of \(y_k^{-1}\) in the leading form of \(J_{x_k, y_k}(f, g)\). Also, \(\sigma_{1k} = a_k(1+s_k) - b_k, \sigma_{2k} = d_k(1+s_k) - e_k, \sigma_{2k} = D_k(1+s_k) - E_k, \) and \(\gamma_k = D_k b_k - E_k a_k = d_k b_k - e_k a_k\).

Since we are starting with \(P\), rather than \(N(f)\), \((d_m, e_m)\) may not be defined; this poses no problem. Note \(a_1 > b_1\) but for \(k > 1, a_k < b_k\). Also, \(s_1 = 0\) and, as noted in the proof of Proposition 2.20(2), \(s_k = -(s+2)\), where \(s\) is the exponent which appears in Lemma 2.16. The next proposition is really a sequence of lemmas culminating in the result we want—Proposition 2.21(10). Statement (2) will also be re-used.

**Proposition 2.21.** With the above notation, we have for all appropriate \(k,\)
(1) \(\sigma_{1k} > 0,\)
(2) \(s_{k+1} = \sigma_{1k},\)
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(3) \( \delta_{2k} = \sigma_{1k} M_k \),

(4) \( \gamma_k = -d_k \sigma_{1k} + a_k \sigma_{2k} \),

(5) \( \delta_{2,k+1} \leq \delta_{2k} - \sigma_{2k} + \gamma_k \),

(6) \( \delta_{2,k+1} = \delta_{2k} \),

(7) \( \delta_{2k} > 1 \),

(8) If \( \sigma_{2k} > 0 \) and \( \gamma_k = -1 \), then \( d_k s_k > e_k \),

(9) If \( s_k = 1 \), then \( \delta_{2k} > 5 \).

(10) \( \delta_{21} > 7 \).

**Proof.** (1) This is just Proposition 2.20(2).

(2) As in the proof of Proposition 2.20(2), \( -s_k \nu(y_k) = \nu(x) + \nu(y) + \nu^e(k) \) and \( -s_{k+1} \nu(y_{k+1}) = \nu(x) + \nu(y) + \nu^e(k+1) \). Since \( \nu^e(k+1) = \nu^e(k) + (b_k - a_k) \nu(y_{k+1}) \), \( -s_{k+1} \nu(y_{k+1}) = -s_k \nu(y_k) + (b_k - a_k) \nu(y_{k+1}) \). As \( \nu(y_k) = a_k \nu(y_{k+1}) \), \( s_{k+1} = a_k s_k - (b_k - a_k) = \sigma_{1k} \).

(3) \( \delta_{2k} - \sigma_{2k} = D_k(s_k + 1) - E_k - (d_k(s_k + 1) - e_k) = (D_k - d_k)(s_k + 1) - (E_k - e_k) = M_k \sigma_{1k} \).

(4) \( \gamma_k = b_k d_k - a_k e_k = b_k d_k - a_k(s_k + 1)d_k + a_k(s_k + 1)d_k - a_k e_k = -d_k \sigma_{1k} + a_k \sigma_{2k} \).

(5) \( \sigma_{k+1} \) is an edge of a polygon \( P_{k+1} \). Since \( \sigma_{1k} M_k > 0 \), (3) shows that \( \sigma_{2,k+1} \) is maximized if \( \varepsilon_{k+1} \) is the rightmost permissible edge. So we assume \( \varepsilon_{k+1} \) is that edge. Thus \( (D_{k+1}, E_{k+1}) \) is the critical vertex of \( P_{k+1} \). Then \( D_{k+1} \leq M_k \) and, as \( u_{k+1} \) would equal \( u_k \) if \( a_k = b_k = 1 \), \( D_{k+1} - E_{k+1} = u_k(f) = \gamma_k \). Thus, \( \delta_{2,k+1} = D_{k+1}(1 + s_{k+1}) - E_{k+1} = D_{k+1}s_{k+1} + \gamma_k \leq M_k \sigma_{1k} + \gamma_k = \delta_{2k} - \sigma_{2k} + \gamma_k \).

(6) Since \( \gamma_k < 0 \), (4) yields \( \sigma_{2k} \geq \gamma_k / a_k \geq \gamma_k \). So \( -\sigma_{2k} + \gamma_k \leq 0 \) and the inequality now follows from (5).

(7) By Proposition 2.20(1), we must have \( \delta_{2n} - D_n > 0 \) and so \( \delta_{2n} > 1 \). So, by (6), \( \delta_{2k} > 1 \) for all \( k \).

(8) If \( \sigma_{2k} > 0 \), \( d_k \neq 0 \) and so there is an edge \( \varepsilon^* \) to the left of \( \varepsilon_k \). The lower end of \( \varepsilon^* \) must have a \( u_k \)-value greater than \( \gamma_k \). So, if \( \gamma_k = -1 \), the lower end must be the origin. Let \( u^* \) be the valuation corresponding to \( \varepsilon^* \) and suppose \( u^* \) occurs in the resolution process for an essential valuation of the curve. Then \( u^*(f) = 0 = u^* \) is an essential valuation. As \( \sigma_{2k} > 0 \), \( \sigma_1(v^*) = -u^*(v^*)(1 + s_k) - u^*(y_k^{-1}) > 0 \) and computing as in the proof of Proposition 2.20(2), we obtain \( u^*(dg) < 0 \). Necessarily, by Theorem 2.19, \( \tau(v^*) = 0 \). But then, \( \tau(v_{k-1}) \neq 0 \) and \( \tau(v^*) = 0 = u^*(f) \neq 0 \), a contradiction. So \( \varepsilon^* \) cannot be part of the resolution process for an essential valuation. (This is conceivable because \( P_k \) is only proportional to \( N(f^*) \).) In this case, consider the portion of \( N(f^*) \) corresponding to \( \varepsilon^* \). Since we do not have proportionality, we can apply Proposition 2.20(1) and obtain the desired result.

(9) Assume false and choose \( \delta_{2k} \) minimal among counterexamples. Since \( b_k > a_k \) \( (k \neq 1) \) and \( b_k D_k - a_k E_k < 0 \), \( \delta_{2k} = D_k - E_k < 0 \); thus \( k \neq n \). Noting \( \delta_{2,k+1} \leq \delta_{2k} \), we can replace \( k \) by \( k+1 \) if \( \sigma_{1k} = 1 \); so we may assume \( \sigma_{1k} > 1 \). This forces \( a_k \geq 3 \).

By Proposition 2.20(3), \( \gamma_k \left| \delta_{2k} \right. \) and so by (4), \( \gamma_k \geq -5 \), \( \sigma_{2k} \geq -1 \). If \( \gamma_k = -\delta_{2k} \), (5) yields \( \delta_{2,k+1} \leq 1 \), contradicting (7). So \( \gamma_k \) must be a proper divisor of \( \delta_{2k} \); thus \( \gamma_k \geq -2 \) and \( \sigma_{2k} \geq 0 \).
First consider \( y_k = -1 \). Since \( d_k/e_k < D_k/E_k \), \( d_k - e_k < 0 \) and (8) yields \( \sigma_{2k} \leq 0 \). Thus \( \sigma_{2k} = 0 \); then \( -1 = y_k = -d_k \sigma_{1k} \) contradicts \( \sigma_{1k} > 1 \). So we must have \( y_k = -2 \) and \( \tilde{\sigma}_{2k} = 4 \). Now \( \tilde{\sigma}_{2,k+1} = 2 - \sigma_{2k} \). Necessarily \( \tilde{\sigma}_{2,k+1} = 2 \), which forces \( \sigma_{2k} = 0 \), \( \sigma_{1k} = 2 \), \( M_k = 2 \), \( D_{k+1} = 2 \), and \( E_{k+1} = 4 \). Since \( \tilde{\sigma}_{2k} - D_{k+1} = 0 \), \( k+1 \neq n \) and we must proceed further. As we need \( \tilde{\sigma}_{2,k+2} = 2 \), \( \sigma_{2,k+1} = y_k \); so \( a_{k+1} = 1 \). But then \( b_{k+1} > a_{k+1} \) implies \( y_{k+1} \geq 0 \) — a contradiction.

(10) We assume \( \tilde{\sigma}_{21} \leq 7 \) and derive a contradiction. Since a portion of \( P_1 \) lies along the \( y \)-axis, \( y_1 \leq -a_1 \leq -2 \). Also, by Proposition 2.20(3), \( y_k \mid \tilde{\sigma}_{2k} \). First we claim \( a_1 < 4 \). For, if \( a_1 \geq 4 \), \( y_1 \leq -4 \) and so necessarily \( y_k = -\tilde{\sigma}_{2k} \). Also (4) gives \( \sigma_{2k} \geq -1 \). So by (5), \( \tilde{\sigma}_{22} \leq 1 \). This contradicts (7) and so the claim is shown. Next we claim \( \sigma_{11} = 1 \). Since \( s_1 = 0 \), this is true unless \( a_1 = 3 \), \( b_1 = 1 \), and \( \sigma_{11} = 2 \). Since \( \sigma_{11} \) is even and \( a_1 \) is odd, (3) and (4) yield \( \sigma_{21} \) is even if and only if \( \tilde{\sigma}_{21} \) is even if and only if \( y_1 \) is even. Clearly \( \sigma_{21} \geq -1 \). If \( \sigma_{21} \geq -1 \), the requirement \( \tilde{\sigma}_{22} > 1 \) forces \( y_1 \) to be a proper divisor of \( \tilde{\sigma}_{21} \). Since \( y_1 \leq -3 \), we must have \( y_1 = -3 \) and \( \tilde{\sigma}_{21} = 6 \). Since one is odd and the other is even, this is impossible. If \( \sigma_{21} = -2 \), we must have \( y_1 = -6 = -\tilde{\sigma}_{21} \). Here \( \tilde{\sigma}_{22} \leq 2 \). To obtain equality, we need \( D_2 = M_1 = 4 \) and \( E_2 = 10 \). We cannot stop since \( 2 < 4 < 0 \). Since 4, 10 are even, we must have \( y_2 = -2 = -\tilde{\sigma}_{22} \). To obtain \( \tilde{\sigma}_{22} = 2 \), we need \( \sigma_{22} = -2 \), \( a_2 = 1 \), and so \( b_2 = 2 \). Thus \( s_2 = \sigma_{12} = 1 \) and we are done by (9).

So we are reduced to the situation \( \sigma_{11} = s_2 = 1 \). So we are done by (9) provided \( \tilde{\sigma}_{22} \leq 5 \). As \( a_1 \geq 2 \), \( \sigma_{21} \geq -3 \). Thus, if \( y_k = -\tilde{\sigma}_{21} \), (5) gives \( \tilde{\sigma}_{22} \leq 3 \) as desired. So \( y_1 \) is a proper divisor of \( \tilde{\sigma}_{21} \). Hence \( y_1 \geq -3 \) and so \( \sigma_{21} \geq -1 \). Also \( y_1 \leq -2 \), \( \tilde{\sigma}_{21} \leq 6 \), and so \( \tilde{\sigma}_{22} \leq 6 + (1 - 2) = 5 \).

\[ \text{(9)} \]

\[ \text{Proposition 2.22.} \quad F_1 \geq 4. \]

\[ \text{Proof.} \] Assume \( E_1 \leq 3 \). First consider the case \( d_1 = 0 \). Then \( a_1 \mid y_1 \) and \( a_1 \mid D_1 \). Since \( y_1 \) divides \( \tilde{\sigma}_{21} \), \( a_1 \mid \tilde{\sigma}_{21} = D_1 - E_1 \). Thus \( a_1 \) divides \( E_1 \). Since \( a_1 > 1 \), we must have \( a_1 = E_1 \). Since \( y_1 < 0 \), \( e_1 > 0 \) and so \( E_1 - e_1 \leq 2 \). So \( M_1 \leq 2 \). Now \( \tilde{\sigma}_{21} > 0 \); so \( 0 < M_1 a_1 - E_1 = (M_1 - 1) a_1 \) and \( M_1 = 2 \). Thus \( (D_1, E_1) = (6, 3) \) and \( (d_1, e_1) = (0, 1) \). We obtain \( \tilde{\sigma}_{22} = 1 \), a contradiction.

So \( d_1 \neq 0 \). Since a side of \( P_1 \) lies along the \( y \)-axis, this gives \( c_1 > 1 \); hence \( c_1 = 2 \) and \( E_1 = 3 \). Then \( b_1 = 1 \), \( a_1 = D_1 - d_1 \), and \( y_1 = d_1 - 2(D_1 - d_1) = 3d_1 - 2D_1 \). Since \( y_1 \mid \tilde{\sigma}_{21} \), we have \( 0 < \gamma_1 < \tilde{\sigma}_{21} \), i.e., \( 0 < 3D_1 - 3d_1 < D_1 - 3 \). So \( d_1 > 2 \) and \( \sigma_{21} > 0 \). If \( F_1 \) is the leading form of \( f \) corresponding to \( \tilde{\varepsilon}_1 \), \( F_1 \) is a \( p \)th power and no higher power. If we let \( v \) be the valuation corresponding to the edge to the left of \( \tilde{\varepsilon}_1 \), then the \( v \)-leading form \( F \) of \( f \) is also a \( p \)th power. Let \( v' \) be a valuation corresponding to the vertex \( (d_1, 2) \). Then we may apply Proposition 1.10(3) to the pair \{\( v, v' \)\} and note that \( \tau(v') \) must have strictly more terms than \( \tau(v) \) while Proposition 1.10(2) says \( \tau(v) = \tau(v') \). The divisibility remark in Proposition 1.10(3) now contradicts the observation that \( F_1 \) and \( F \) are both precisely \( p \)th powers.

\[ \text{Notation.} \] There are three vertices of \( P \) of primary interest: the vertex \( (D_1, E_1) \) of the last several results, the vertex to the immediate right of the ascending portion
of the left side, and the upper right vertex. We actually have no further use of the first and it is most convenient to designate the second as \((D_1, E_1)\) hereafter. In most of the low degree examples, they coincide and the previous results are always true. We designate the third as \((D, E)\). Earlier, we used this notation for the corresponding vertex of \(N(f)\) but our primary concern is \(P\).

**Theorem 2.23.** \(\gcd(\deg(f), \deg(g)) \geq 16\).

**Proof.** We know \(E_1 \geq 4\) and \(D_1 \geq E_1 + 8 = 12\). Also \(E = E_1\) and \(D \geq D_1\). (This is an inequality if there is a horizontal edge.) So \(\gcd(\deg(f), \deg(g)) = D + E \geq 16\). □

With techniques such as those used to obtain Theorem 2.23 based on the results of Sections 1 and 2, we employ a computer search to find which pairs \((D_1, E_1)\) are compatible with the results. There are presumably infinitely many so we bound the search. We actually want to know which pairs \((D, E)\) are allowed. So suppose we have a horizontal edge \((D_1, E_1)\)-(\(D, E)\) with \(E - E_1\). Let \(F'\) be the leading form corresponding to this edge. Use the notation above and let \(c = \gcd(y_1, \sigma_{11})\). Then, calculating as in Proposition 1.10, we actually see that \(F\) is a \((p \gamma_1/c)\)th power. Then the arguments leading up to Theorem 1.16 say \(x^{pD}y^{pE}\) is a \((p \gamma_1 A/c)\)th power with \(A \geq 1\). Thus \(\gamma_1/c\) is a proper divisor of \(\gcd(D, E)\). In particular, if \(\gamma_1/c = E_1\), there cannot be a horizontal edge. This always happens if \(E_1\) is prime or 4.

Suppose \(D_1 + E_1 \leq 100\). The computer search allows 132 possible pairs \((D_1, E_1)\). While this is a comparatively small number, it is far too large to allow us to seriously consider the cases. So we restrict to \(D_1 + E_1 \leq 50\).

**Theorem 2.24.** Suppose we have a counterexample with \(D + E \leq 50\). Then

1. \((D, E) \in \{(12, 4), (15, 6), (20, 5), (21, 7), (21, 9), (24, 8), (24, 9), (25, 10), (28, 8), (28, 12), (30, 6), (30, 10), (30, 12), (30, 18), (32, 8), (33, 11), (33, 12), (35, 7), (35, 14), (35, 15), (36, 9), (36, 12), (40, 8), (40, 10), (42, 7)\}.

2. Except for \((32, 8)\), each of the above can occur as \((D_1, E_1)\). \((32, 8)\) can only occur with \((D_1, E_1) = (28, 8)\).

3. If we have a horizontal edge \((D_1, E_1)-(D, E)\), there is a number \(\alpha\) such that \(\gcd(D, E)\) must be a proper multiple of \(\alpha\). With three exceptions, \(\alpha = \gcd(D_1, E_1)\). For \((30, 12)\), \(\alpha = 3\); for \((36, 12)\), \(\alpha\) can equal 4 or 6; and for \((40, 10)\), \(\alpha = 5\).

**Proof.** The \((D_1, E_1)\) and \(\alpha\) values were found by computer search. Then it is a trivial exercise to find which additional \((D, E)\) values are compatible with each \((D_1, E_1)\). Of course, if \(\alpha = E_1\), there can be no horizontal edge. □

Lastly, we want to consider \(\{\deg(f), \deg(g)\}\). In addition to knowing \(D + E\), which gives the gcd of the degrees, we need to know the pair \((p, q)\). For a given \((D_1, E_1)\), some pairs are permissible and others are not. To find allowable pairs, we take a resolution to reach the edge considered in Theorem 2.19. Then we have two
homogeneous polynomials such that \( J(F, G) = w^5 \). Necessarily, there must be monomials \( \tilde{F}, \tilde{G} \) in the respective supports with \( \tilde{F}\tilde{G} = z w^{p+q} \). The product of the highest degree monomials is \( (z^j w^i)^{p+q} \), where \( i, j \) are determined by the process.

If \( a = v_k(w) \) and \( b = v_k(z) \), there is an integer \( B \) such that \( i(p + q) - 1 = bB \) and \( j(p + q) - (s+1) = aB \). Since \( a, b \) are relatively prime, there is exactly one solution for each \( (p + q) \). However, since \( a \mid p \) or \( a \mid q \), we are only interested in those solutions for which \( a < p + q - 1 \). Moreover, if \( a \geq (p + q - 1)/2 \), the pair \( (p, q) \) will be determined by this divisibility. We submit without proof the possibilities for some of the smaller cases.

**Theorem 2.25.** If \( (D_1, E_1) = (12,4) \), \( (p, q) = (3k + 1, 2k + 1) \).

If \( (D_1, E_1) = (15,6) \), \( (p, q) = (5k + 2, 2k + 1) \) or \( (4k + 3, k + 1) \).

If \( (D_1, E_1) = (20,5) \), \( (p, q) = (4k + 1, 7k + 2) \), \( (8k + 2, 3k + 1) \), \( (3k + 2, 4k + 3) \), \( (6k + 4, k + 1) \), or \( (2k + 1, k + 1) \).

If \( (D_1, E_1) = (21,7) \), \( (p, q) = (3k + 1, 5k + 2) \), \( (6k + 2, 2k + 1) \), \( (2k + 1, k + 1) \), \( (5k + 3, 3k + 2) \), or \( (4k + 3, k + 1) \).

If \( (D_1, E_1) = (21,9) \), \( (p, q) = (7k + 6, k + 1) \).

If \( (D_1, E_1) = (24,8) \), \( (p, q) = (5k + 2, 2k + 1) \).

If \( (D_1, E_1) = (24,9) \), \( (p, q) = (8k + 3, 5k + 2) \), \( (7k + 5, 4k + 3) \), \( (5k + 2, 2k + 1) \), or \( (4k + 3, k + 1) \).

\[ \square \]

**Corollary 2.26.** If \( \deg(f), \deg(g) \) are both less than 100, then we must have \( (\deg(f), \deg(g)) \in \{64, 48\}, \{75, 50\}, \{84, 56\}, \{99, 66\} \). \[ \square \]

These are precisely the four exceptional cases obtained by Moh [3]. He disposes of them by hand, using a reduction of degree technique. We have no nice way to handle these cases.

### 3. An example

The purpose of this section is to consider the nature of possible counterexamples permitted by Sections 1 and 2. The pairs \( (D_1, E_1) \) which I studied were \( (12,4) \), \( (36,9) \), and \( (42,7) \). While much can be said about the original polynomials \( f, g \), there are simply too many parameters to allow us to really see what is going on.

First consider \( (12,4) \). The polygon \( P \) has vertices \( (0,0), (0,1), (12,4), (4k,0) \) where \( k=1 \) or \( 2 \). I assumed the \( k=1 \) case (the easier one) and also took \( \deg(f) = 48, \deg(g) = 64 \). The leading form corresponding to the edge \( (0,3)-(36,12) \) must be \( y^3(x^4y - \beta)^5 \) with \( \beta \neq 0 \). Actually, by change of variables, we can assume \( \beta = 1 \). Set \( z = (x^4y - 1) \). When we resolve the edge, we obtain new variables \( z, w \) so that \( y = w^4 + \text{terms of higher value} \) and the relevant part of our new polygon is \( (0,1)-(1,2)-(9,12) \). (The corresponding path of \( N(g^*) \) is \( (0,1)-(12,16) \).) The edge \( (1,2)-(9,12) \) represents two points at infinity with \( v(z) = 5, v(w) = -4, v(y) = -16, v(x) = 4 \), and
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$\nu(g) = -4$. The edge $(0, 1)-(1, 2)$ represents one point at infinity with $\nu(z) = 1$, $\nu(w) = -1$, $\nu(y) = -4$, $\nu(x) = 1$, and $\nu(g) = -1$. The edge $(36, 12)-(12k, 0)$ of $N(f)$ represents one point at infinity. The corresponding residue field has degree 12 over $C$. We can compute $\nu(\text{d}Y/\text{f}_x) = 1$. So we may remark that a divisor in the canonical class has degree $2(-5) + (-2) + 12 = 0$ and the curve $C$ has genus one.

In order to get the proper resolution of the edge $(0, 3)-(36, 12)$, we have

$$f = y^3(\alpha_{12,9}z^9 + \alpha_{12,10}z^{10} + \cdots) + y^3x(\alpha_{11,9}z^9 + \cdots) + y^3x^2(\alpha_{10,8}z^8 + \cdots)$$

$$+ y^3x^3(\alpha_{9,7}z^7 + \cdots) + y^2(\alpha_{8,6}z^6 + \cdots) + y^2x(\alpha_{7,5}z^5 + \cdots)$$

$$+ y^2x^2(\alpha_{6,4}z^4 + \cdots) + y^2x^3(\alpha_{5,4}z^4 + \cdots) + y(\alpha_{4,3}z^3 + \cdots)$$

$$+ yx(\alpha_{3,2}z^2 + \cdots) + yx^2(\alpha_{2,1}z + \cdots) + yx^3(\alpha_{1,1}z + \cdots) + (\alpha_{0,0} + \cdots)$$

$$+ x(\alpha_{-1,0} + \cdots) + x^2(\alpha_{-2,0} + \cdots) + \cdots.$$ 

On the other hand, the edge $(36, 12)-(12, 0)$ forces

$$f = x^{12}(\gamma_{12,12}u^{12} + \gamma_{12,13}u^{13} + \cdots) + x^{11}(\gamma_{11,11}u^{11} + \cdots)$$

$$+ x^{10}(\gamma_{10,10}u^{10} + \cdots) + \cdots + x(\gamma_{1,1}u + \cdots) + (\gamma_{0,0} + \cdots)$$

$$+ y(\gamma_{-1,0} + \cdots) + y(\gamma_{-2,0} + \cdots) + y^2x(\gamma_{-3,0} + \cdots) + \cdots,$$

where $u = x^2y - \delta$, $\delta \neq 0$, and as before we can assume $\delta = 1$. (Since we have two variables, we are allowed two such assumptions.) These two representations determine each other and in fact, it can be shown that $f$ is a linear combination of six polynomials, which we may designate $\theta, \theta^2, \theta^3, \varphi, \theta\varphi, \eta$. Similarly, $g$ is a linear combination of 10 polynomials – these six plus $\theta^4, \theta^2\varphi, \theta\eta, \nu$.

There are some constraints on our choice of linear combinations for $f$. To actually get the correct Newton polygon and not one inside it, the coefficient of $\theta^3$ must be nonzero. Also, to insure that we get the desired three poles of $g$, it is necessary that $\alpha_{12,9}X^2 + \alpha_{7,5}X + \alpha_{2,1} = 0$ have distinct nonzero roots. Finally, multiplying $f$ by a nonzero constant will not affect $[dg]$. So we are really looking for potential counterexamples in a collection of pairs $(f, g)$ parametrized by an open subset of $\mathbb{P}^5 \times \mathbb{P}^9$.

It is relatively easy to insure $\nu^*(dg) = \nu^*(\text{d}Y/\text{f}_x)$ for each pole $\nu^*$ of $\text{d}Y/\text{f}_x$. The difficulty is obtaining $\nu(dg) = 1$ for the lone zero $\nu$ of $\text{d}Y/\text{f}_x$. In fact, for this specific example, it proved to be impossible. Unfortunately, the reason for the failure, more equations than unknowns, does not convincingly rule out other potential counterexamples.

Alternatively, we take a more geometric approach and employ Riemann-Roch. Let $v_1, v_2, v_3$ denote the three poles and let $A$ denote the space of differentials $D$ such that $v_1(D)$, $v_2(D) \geq -5$ and $v_3(D) \geq -2$. Since the genus equals one, $\dim(A) = 12$. Also, if $A^*$ is the subspace of exact differentials, $\dim(A^*) = 8$. It then would appear that imposing four conditions on a differential $D$ in $A$ would force it to be exact. But it is not clear how to obtain the four conditions. In general, the codimension of $A^* = 2(\text{genus}) + (# \text{ of poles} - 1)$. The pole conditions are simply that $\text{d}Y/\text{f}_x$ can have no residues. These can be converted into equations, though not linear ones. The genus conditions seem harder to quantify.
What is particularly intriguing however is the observation that we can choose $f$ from an open subset of a 5-parameter family and need only meet four conditions. On the other hand, when we try to select $(f, g)$ so that $u(dg) = 1$, we must solve 15 quadratic equations in 13 variables. ($13 = 9 + 5 - 1$; we lose one variable as replacing $g$ by $g - \alpha f$ does not affect $dg$. The number 15 is not so easily explained.) Yet Theorem 2.5 tells us that the two problems are equivalent. This seems to suggest that the inability to find a counterexample here is not simply a case of more equations than unknowns but rather due to some obstruction related to the genus conditions.

The study of $(36, 9)$ and $(42, 7)$ was less rewarding and there is no need to detail that here. The following appraisal seems justified. If we can see how $dy/f_x$ varies on a family of curves, we might be able to make further progress. However, if we cannot do this, or develop some new trick, the methods of this paper should lead no further.

References