Factorially graded rings and Cox rings

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ARTICLE INFO

Article history:
Received 19 December 2011
Available online 9 August 2012
Communicated by Alberto Elduque

MSC:
13A02
13F15
14L30

Keywords:
Commutative algebra
Algebraic geometry
Graded rings
Unique factorization
Cox rings

ABSTRACT

Cox rings of normal prevarieties are factorially graded, i.e. homogeneous elements allow a unique decomposition into homogeneous factors. We study this property from an algebraic point of view and give a criterion which in a sense reduces it to factoriality. This will allow us to detect and construct Cox rings in a purely algebraic manner.

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1. Introduction

Throughout this article, $K$ is an abelian group and $R$ is a $K$-graded, associative, commutative algebra with unit over a Noetherian ring $A$. We assume that $R$ has no $K$-homogeneous zero divisors and denote by $R^+$ the multiplicative monoid of $K$-homogeneous elements of $R$. Following [2,9], we say that a non-zero $f \in R^+ \setminus R^*$ is $K$-prime if $f|gh$ with $g, h \in R^+$ implies $f|g$ or $f|h$. The ring $R$ is said to be factorially $K$-graded if every non-zero $f \in R^+ \setminus R^*$ is a product of $K$-primes. In general, factorially graded rings need not be UFDs [2,10] and it is only in the case of free grading groups that these notions are equivalent [1].

Factorially graded rings naturally occur in algebraic geometry: Let $K$ be an algebraically closed field of characteristic zero and let $X$ be a normal prevariety over $K$ with $\mathcal{O}(X)^* = K^*$ and finitely...
generated divisor class group $\text{Cl}(X)$, i.e. the group of Weil divisors modulo principal divisors. Then the Cox ring of $X$, defined as

$$\mathcal{R}(X) := \bigoplus_{\text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)),$$

is factorially $\text{Cl}(X)$-graded. For details on the precise construction and a proof of well-definedness, see [2,9]. Since the initial paper by Cox [5], the Cox ring or homogeneous coordinate ring has become a much studied invariant of normal prevarieties. The case of finitely generated Cox rings is of particular interest because the prevariety $X$ may then be realized as a good quotient of an open subvariety of $\text{Spec}(\mathcal{R}(X))$. The $\mathbb{Q}$-factorial projective varieties with finitely generated Cox ring are also known as Mori Dream Spaces.

We say that a finitely generated factorially $K$-graded $K$-algebra is almost freely graded if there is a system $f_1, \ldots, f_m \in R$ of pairwise non-associated $K$-prime generators such that each $m-1$ of their degrees together generate $K$. Here, the existence of a system of pairwise non-associated $K$-prime generators follows from finite generation of $R$ and graded factoriality, and whether or not all but one of their degrees generate $K$ only needs to be checked for one such system, because it either holds for all or for none of them [3].

Finitely generated Cox rings of normal prevarieties are characterized in algebraic terms as follows:

**Definition 1.1 (Finitely generated Cox ring, algebraic definition).** A finitely generated $K$-algebra $R$ graded by a finitely generated abelian group $K$ is an algebraic Cox ring if and only if the following hold:

(i) $R$ is normal,
(ii) $R$ is factorially $K$-graded,
(iii) $R^+ \cap R^* = K^*$,
(iv) $R$ is almost freely $K$-graded.

Indeed, starting from an algebraic Cox ring $R$ and some combinatorial data, one may explicitly construct a prevariety $X$ with Cox ring $\mathcal{R}(X) = R$ [3,9]. The geometry of the prevarieties thus obtained is largely described in combinatorial terms, generalizing the combinatorial description of toric varieties. This makes it very desirable to have a large pool of algebraic Cox rings at ones disposal.

Our aim is then to provide algebraic criteria for the detection and construction of Cox rings. The main difficulty appears to be ascertaining property (ii), graded factoriality. The first result states that graded factoriality may be reduced to factoriality in the following sense:

**Theorem 1.2.** Let $R = A[f_1, \ldots, f_m]$ be a $K$-graded ring with $K$-prime generators $f_1, \ldots, f_m$ and let $S$ be the multiplicative system generated by the $f_i$. Then the following are equivalent:

(i) $R$ is factorially $K$-graded,
(ii) $(S^{-1}R)_0$ is factorial.

This allows generalization in the following way. Let $K'$ be a subgroup of $K$. The Veronese subalgebra of $R$ corresponding to $K'$ is denoted by $R_{K'}$. If any $K$-homogeneous element in $R$ is associated to a homogeneous element in $R_{K'}$, then $R$ is called $K'$-associated. Moreover, we say that $R$ is $K$-Noetherian if any $K$-homogeneous ideal is generated by finitely many $K$-homogeneous elements. Now our more general result is the following theorem (see Section 2 for the proof):

**Theorem 1.3.** Let $R$ be a $K$-graded $K$-Noetherian ring. Then for every subgroup $K'$ of $K$ the following are equivalent:

(i) $R$ is factorially $K$-graded,
(ii) there is a multiplicative system $S$ generated by $K$-primes such that
Theorem 1.4. Let R be a K-graded ring without homogeneous zero divisors and let R[T] be K-graded through a choice of a K-degree for T. Then R is factorially K-graded if and only if R[T] is so.

Another way to ascertain that R is almost freely graded is to coarsen the grading. By the following theorem (proven in Section 4), graded factoriality is preserved if the coarsening consists in dropping a free direct summand of the grading group:

Theorem 1.5. Let R be a K ⊕ ℤm-graded K-Noetherian ring without K ⊕ ℤm-homogeneous zero divisors. Then R is factorially K ⊕ ℤm-graded if and only if it is factorially K-graded. Furthermore, if R is factorially graded, then a K ⊕ ℤm-homogeneous element has one and the same decomposition with respect to both gradings.

In Section 5 we give examples of Cox rings constructed via the tools presented above. Among others, we treat the rings C[T1, . . . , T4]/⟨T1m1 + · · · + T4m4⟩ whose divisor class groups are calculated in [13]. It is an immediate consequence of our results that all those rings, UFDs and non-UFDs, are factorially graded and become Cox rings after the adjunction of a further homogeneous variable. The class of rings that our results allow us to verify algebraically as Cox rings includes all Cox rings calculated geometrically in [6,8,11].

2. Proof of Theorem 1.3

A non-zero f ∈ R+ \ R* is K-irreducible if f = gh with g, h ∈ R+ implies g ∈ R* or h ∈ R*. We now study the behaviour of factorially graded rings with respect to localizations. Let R = ⊕w∈K Rw be a K-graded ring and S ⊂ R+ a multiplicative system. Then the localization S−1R is K-graded via

\[ (S^{-1}R)_w := \left\{ \frac{r}{s} \in S^{-1}R : r \in R_{\text{deg}(s)+w} \right\} \quad \text{for } w \in K. \]

Note that the localization S−1a of a K-homogeneous ideal is again K-homogeneous.

Proposition 2.1. Let R be a K-graded ring and S ⊂ R+ a multiplicative system. If R is factorially K-graded then so is S−1R.

Proof. In general, for a K-prime f ∈ R the fraction f/1 ∈ S−1R is either K-prime or a unit, depending on whether or not ⟨f⟩ ∩ S is empty: If ⟨f⟩ ∩ S = ∅ and f/1 divides a product of homogeneous elements g/s, h/t ∈ S−1R, then there are r ∈ R+ and q ∈ S with ghq = fstr ∈ ⟨f⟩. Since q cannot lie in ⟨f⟩ either g or h must, in particular, f/1 divides either g/1 or h/1 in S−1R. If otherwise ⟨f⟩ ∩ S is non-empty then there is an r ∈ R with rf = s ∈ S, so (f/1)(r/s) = 1.

Now, if R is factorially K-graded, we may write an arbitrary f/s as (1/s)(f/1) and decompose f into K-prime factors f1 · · · fm. Grouping the fractions f1/1 that are units together with 1/s we obtain the desired decomposition. □
The converse of the above is true if \( R \) is \( K \)-Noetherian and \( S \) is generated by \( K \)-primes, as the following two statements show, the latter of which was formulated by Nagata [12] with respect to localizations of UFDs.

**Lemma 2.2.** Let \( R \) be a \( K \)-graded \( K \)-Noetherian ring. Then every homogeneous non-zero non-unit is a product of \( K \)-irreducible elements.

**Proof.** Suppose that the set \( M \) of principal ideals \( \langle r \rangle \) generated by elements \( r \in R^+ \setminus \{0\} \) that are not products of \( K \)-irreducible elements is non-empty. Then, by \( K \)-Noetherianity it has a maximal element \( \langle r' \rangle \) whose generator \( r' \) is in particular not \( K \)-irreducible. So there are \( s, t \in R^+ \setminus R^* \) with \( r' = st \), and \( \langle r' \rangle \subseteq \langle s \rangle, \langle t \rangle \) are proper inclusions. Thus, by maximality of \( \langle r' \rangle \) the elements \( s \) and \( t \) are products of \( K \)-irreducible elements. But then, so is \( r' \) – a contradiction. \( \square \)

**Lemma 2.3.** Let \( R \) be a \( K \)-graded ring such that every homogeneous non-zero non-unit is a product of \( K \)-irreducible elements. Let \( S \) be a multiplicative system generated by \( K \)-prime elements of \( R \). If \( S^{-1}R \) is factorially \( K \)-graded then so is \( R \).

**Proof.** By Lemma 2.2, we have to show that every \( K \)-irreducible \( f \in R \) is \( K \)-prime. Let \( G \) be a set of \( K \)-prime generators of \( S \). If \( f \) is divided by an element of \( G \), then it is already associated to that element and hence also \( K \)-prime. So assume that no element of \( G \) divides \( f \). Then \( f/1 \in S^{-1}R \) is not a unit and is in fact \( K \)-irreducible: For if \( fs_1s_2 = r_1r_2 \) with \( r_1 \in R^+ \) and \( s_1, s_2 \in S \), the \( K \)-prime factors of \( s_1s_2 \) can be factored out of \( r_1r_2 \), leaving a decomposition \( f = r'_1r'_2 \). But then \( r'_1 \in R^* \) or \( r'_2 \in R^* \), so \( r_1 \in (S^{-1}R)^* \) or \( r_2 \in (S^{-1}R)^* \). Since \( S^{-1}R \) is factorially \( K \)-graded, we can conclude that \( f/1 \) is \( K \)-prime in \( S^{-1}R \).

Now consider \( g, h \in R^+ \) with \( f|gh \). By \( K \)-primality of \( f/1 \) we may assume \( f|g \) in \( S^{-1}R \), so there exist \( s \in S \) and \( r \in R^+ \) with \( rf = sg \). Since the \( K \)-prime factors of \( s \) have to occur in \( r \), we obtain \( r'f = g \) for some \( r' \in R \), so \( f|g \). \( \square \)

The above preliminaries show that factorially graded rings behave with respect to graded localizations analogously to UFDs. However, the following lemma, which is essential for our proof of Theorem 1.3, is a priori unique to the setting of graded rings. For it relies on the fact that if a product \( fg \) of homogeneous elements and one of its factors \( f \) lie in the Veronese subalgebra \( R_{K'} \), then so does the other factor \( g \). An analogous statement on subrings of a non-graded ring is generally false.

**Lemma 2.4.** Let \( R \) be a \( K \)-graded ring and \( K' \) a subgroup of \( K \) such that \( R \) is \( K' \)-associated. Then the following hold:

(i) An element \( f \in R_{K'}^+ \) is \( K' \)-prime in \( R_{K'} \) if and only if it is \( K \)-prime in \( R \).

(ii) The ring \( R \) is factorially \( K \)-graded if and only if \( R_{K'} \) is factorially \( K' \)-graded.

**Proof.** First, note that for homogeneous \( f, g \) with \( f \in R_{K'} \) and \( fg \in R_{K'} \) we indeed have \( \deg_{K'}(g) + \deg_{K'}(f) \in K' \) and \( \deg_K(g) \in K' \), so \( \deg_K(g) \in K' \), i.e. \( g \in R_{K'} \).

For (i), consider a \( K' \)-prime \( f \in R_{K'} \). Given \( g, h, t \in R^+ \) with \( ft = gh \), we fix \( s_g, s_h \in R^* \cap R^+ \) with \( gs_g, hs_h \in R_{K'} \) and get \( f(tgsh) = (gs_g)(hs_h) \). Then \( ts_gsh \) already lies in \( R_{K'} \), so \( f|gs_g \) or \( f|h \) in \( R_{K'} \), which implies \( f|g \) or \( f|h \) in \( R \). Thus, \( f \) is \( K \)-prime. Conversely, a \( K \)-prime \( f \in R_{K'} \) clearly is \( K' \)-prime in \( R_{K'} \).

For (ii), first suppose that \( R \) is factorially \( K \)-graded and let \( f \in R_{K'}^+ \setminus R^+ \). Then \( f \) has a decomposition \( f = f_1 \cdots f_m \) in \( R \). Taking \( s_i \in R^* \cap R^+ \) with \( f_is_i \in R_{K'} \), we get \( f = (s_1^{-1} \cdots s_m^{-1}f_1)(s_2s_2) \cdots (f_ms_m) \) with all factors in brackets lying in \( R_{K'} \). By (i), this is a decomposition into \( K' \)-primes as needed. For the converse, let \( R_{K'} \) be factorially \( K' \)-graded and \( f \in R^+ \setminus R^* \). Again, we fix \( s_f \in R^* \cap R^+ \) with \( fs_f \in R_{K'} \). Then \( fs_f \) has a decomposition into \( K' \)-primes which are also \( K \)-primes in \( R \) by (i), and multiplication by \( s_f^{-1} \) yields a decomposition of \( f \) into \( K \)-primes. \( \square \)
Proof of Theorem 1.3. Let \( R \) be factorially \( K \)-graded and let \( K' \) be a subgroup of \( K \). The existence of an \( S \) such that \( S^{-1}R \) is \( K' \)-associated is obvious: Take \( S := R^+ \setminus \{0\} \). Now let \( S \) be any such multiplicative system. By Proposition 2.1 the localization \( S^{-1}R \) is factorially \( K \)-graded and by definition of \( S \), we may apply Lemma 2.4 and conclude that \( (S^{-1}R)_{K'} \) is factorially \( K' \)-graded.

Conversely, suppose there is an \( S \) generated by \( K \)-primes and a subgroup \( K' \) such that \( S^{-1}R \) is \( K' \)-associated and \( (S^{-1}R)_{K'} \) is factorially \( K' \)-graded. Then Lemma 2.4 yields that \( S^{-1}R \) is factorially \( K \)-graded. And since \( R \) is \( K \)-Noetherian, Lemmas 2.2 and 2.3 imply that \( R \) is factorially \( K \)-graded. \( \square \)

Theorem 1.2 is now obtained from Theorem 1.3 by setting \( K' = 0 \) and taking \( S \) as the multiplicative system generated by \( f_1, \ldots, f_m \). Note that \( R \) is \( K \)-Noetherian because \( A \) is Noetherian and \( R = A[f_1, \ldots, f_m] \).

3. Proof of Theorem 1.4

In this section, let \( R \) be a \( K \)-graded ring (without \( K \)-homogeneous zero divisors) and let \( R[T] \) have the \( K \)-grading extending that of \( R \) which is obtained through the choice of an arbitrary \( w \in K \) as the \( K \)-degree of \( T \). As before, the \( K \)-degree of a \( K \)-homogeneous polynomial \( f \in R[T] \) is denoted by \( \deg_K(f) \), whereas \( \deg(f) \in \mathbb{Z}_{\geq 0} \) is the maximal exponent occurring in \( f \) with a non-vanishing coefficient, as usual.

Since \( T \) is \( K \)-homogeneous, a \( K \)-homogeneous polynomial \( f \in R[T] \) has only \( K \)-homogeneous coefficients. So, as \( R \) has no \( K \)-homogeneous zero divisors, the leading term of a product \( gh \) with \( g, h \in R[T]^+ \) is just the product of the leading terms of \( g \) and \( h \). In particular, \( R[T] \) has no \( K \)-homogeneous zero divisors either.

Lemma 3.1. Let \( R \) and \( R[T] \) be \( K \)-graded as above. Then a non-zero \( p \in R^+ \setminus R^* \) is \( K \)-prime if and only if it is \( K \)-prime in \( R[T] \). Thus, if \( R \) is factorially graded then the decomposition of \( r \in R^+ \) into \( K \)-prime factors in \( R \) is also a decomposition into \( K \)-prime factors in \( R[T] \).

Proof. By the above, \( R/(p) \) has no homogeneous zero divisors precisely if \( (R/(p))[T] \), which is isomorphic to \( R[T]/(p) \), has none. \( \square \)

Proof of Theorem 1.4 ("if"-direction). Let \( R[T] \) be factorially \( K \)-graded and let \( r \in R^+ \). Then all \( K \)-prime factors of the decomposition of \( r \) in \( R[T] \) are constant and are therefore \( K \)-prime in \( R \). So \( R \) is factorially \( K \)-graded. \( \square \)

Next, we prepare the proof of the converse.

Proposition 3.2. For \( R \) and \( R[T] \) as above the following are equivalent:

(i) \( R^+ \setminus \{0\} \subseteq R^* \),

(ii) every \( K \)-homogeneous ideal \( a \) of \( R[T] \) is generated by a \( K \)-homogeneous element \( f \in a \) of minimal degree \( \deg(f) \in \mathbb{Z}_{>0} \).

Proof. Let \( R^+ \setminus \{0\} \subseteq R^* \). First, we observe that \( R[T] \) allows division with remainder for \( K \)-homogeneous elements in the following sense: For non-zero polynomials \( f, g \in R[T]^+ \) there are \( q, r \in R[T]^+ \) with \( f = qg + r \) such that \( r = 0 \) or \( \deg(r) < \deg(g) \). So let

\[
f = a_0 T^0 + \cdots + a_m T^m, \quad g = b_0 T^0 + \cdots + b_n T^n \in R[T]^+.
\]

If \( m = 0 \), then \( f = g \), as required. Now let \( m > 0 \). We only need to consider the case \( m \geq n \). Then \( f' := f - b_n^{-1} a_m T^{m-n} g \) is \( K \)-homogeneous of degree \( \deg_K(f) \) and by induction we find \( q', r \in R[T]^+ \) with \( f' = q'g + r \) and \( r = 0 \) or \( \deg(r) < \deg(g) \). Thus, we obtain \( f = qg + r \) where \( q := q' + b_n^{-1} a_m T^{m-n} \).
Now consider a $K$-homogeneous ideal $a \trianglelefteq R[T]$. Let $f \in a \cap R[T]^+$ have minimal degree $\deg(f) \in \mathbb{Z}_{\geq 0}$. For $g \in a \cap R[T]^+$ there are $q, r \in R[T]^+$ with $g = qf + r$ and $\deg(r) < \deg(f)$ or $r = 0$. Minimality of $\deg(f)$ implies $r = 0$.

For the proof of the converse, we first note that since $R$ has no $K$-homogeneous zero divisors, $T$ is $K$-irreducible. Thus, $\langle T \rangle$ is maximal among principal ideals of $K$-homogeneous elements, i.e. among all homogeneous ideals by (ii). Equivalently, all non-zero $K$-homogeneous elements of $R \cong R[T]/(R)$ are units.

**Proposition 3.3.** Let $R$ be a $K$-graded ring such that every $K$-homogeneous ideal is generated by a single homogeneous element. Then $R$ is factorially graded.

**Proof.** Since $R$ is $K$-Noetherian, Lemma 2.2 reduces the problem to showing that every $K$-irreducible $p \in R$ is $K$-prime. By definition, $\langle p \rangle$ is maximal among all the principal ideals of $K$-homogeneous elements − so in our case among all homogeneous ideals. Thus, in $R/\langle p \rangle$ every non-zero $K$-homogeneous element is a unit, in particular, $R/\langle p \rangle$ has no $K$-homogeneous zero divisors. Hence, $p$ is $K$-prime. \qed

**Corollary 3.4.** Let $R$ be a $K$-graded ring. Then $(R^+ \setminus \{0\})^{-1}R[T]$ is factorially $K$-graded.

Now let $R$ be factorially $K$-graded and let $P$ be a system of $K$-prime elements such that every $K$-prime element of $R$ is associated to exactly one $p \in P$. For brevity set $R' := (R^+ \setminus \{0\})^{-1}R$. We write $v_p(f)$ for the minimal multiplicity with which $p \in P$ occurs in the decompositions of the coefficients of $f \in R'[T]^+$.

**Lemma 3.5.** In the above notation, for any $p \in P$ the multiplicities of two $K$-homogeneous polynomials $f, g \in R'[T]$ satisfy

$$v_p(fg) = v_p(f) + v_p(g).$$

Furthermore, a polynomial $f \in R[T]^+$ with $v_p(f) = 0$ for all $p \in P$ is $K$-prime in $R[T]$ if and only if it is $K$-prime in $R'[T]$.

**Proof.** Let $f, g \in R'[T]^+$. The above equation clearly holds if $f$ or $g$ is constant. So we only consider the case that $v_p(f) = v_p(g) = 0$ for all $p \in P$. In particular, $f$ and $g$ are polynomials over $R$. Now for all $p \in P$, the classes of $f$ and $g$ in $(R/\langle p \rangle)[T]$ are non-zero and therefore their product $fg$ is non-zero as well, because $R/\langle p \rangle$ and thereby $(R/\langle p \rangle)[T]$ has no $K$-homogeneous zero divisors. But this is equivalent to $v_p(fg) = 0$ for all $p \in P$, which we had to show.

Now, let $f \in R[T]$ satisfy $v_f(p) = 0$ for all $p \in P$ and let $f$ be $K$-prime in $R'[T]$. If $f$ divides in $R[T]$ the product of two $K$-homogeneous $g, h \in R[T]$, it must divide either $g$ or $h$ in $R'[T]$. So we may assume that there is a $q \in R'[T]$ with $fq = g$. Applying the above, we obtain $v_p(q) = v_p(fq) = v_p(g) \geq 0$ for all $p \in P$, so $q \in R[T]$ and $f|g$ in $R[T]$. Thus, $f$ is $K$-prime in $R[T]$. We omit the proof of the converse, as it is not needed in the following. \qed

**Proof of Theorem 1.4 (“only if”-direction).** Let $R$ be factorially $K$-graded and $f \in R[T]^+ \setminus R[T]^*$. By Corollary 3.4, $R[T]$ is factorially $K$-graded. Thus, $f$ has a decomposition $f = f_1 \cdots f_m$ with $K$-prime elements $f_i \in R'[T]$. Multiplication by a suitable fraction of $K$-homogeneous elements of $R$ yields $f = cf_1^{\prime} \cdots f_m^{\prime}$ where the $f_i^{\prime}$ lie in $R[T]$ and satisfy $v_p(f_i^{\prime}) = 0$ for all $p \in P$. By Lemma 3.5, the $f_i^{\prime}$ are $K$-prime in $R[T]$ and $c \in R$ because $v_p(c) = v_p(f) \geq 0$ for all $p \in P$. Since the decomposition of $c$ in $R$ is also a decomposition in $R[T]$, the proof is complete. \qed

4. **Proof of Theorem 1.5**

**Proposition 4.1.** Let $R$ be a $K \oplus \mathbb{Z}^m$-graded ring without $K \oplus \mathbb{Z}^m$-homogeneous zero divisors and $f \in R$ a $K \oplus \mathbb{Z}^m$-homogeneous element. Then $f$ is $K \oplus \mathbb{Z}^m$-prime if and only if it is $K$-prime.
Proof. We only show the “only if”-direction for $m = 1$. Let $f$ be $K \oplus \mathbb{Z}$-prime. Let $g$ and $h$ be $K$-homogeneous with $f|gh$. Let $g = g_{n_0} + \cdots + g_{n_2}$ and $h = h_{m_1} + \cdots + h_{m_2}$ be the decompositions into $\mathbb{Z}$-homogeneous parts. Since $R$ has no $K \oplus \mathbb{Z}$-homogeneous zero divisors, $g_{n_2}h_{m_2}$ is the leading term of $gh$, so $\mathbb{Z}$-homogeneity of $f$ yields $g_{n_2}h_{m_2} \in (f)$, so $g_{n_2} \in (f)$ or $h_{m_2} \in (f)$ by $K \oplus \mathbb{Z}$-primality of $f$. Now we have $(g - g_{n_2})h \in (f)$ or $g(h - h_{m_2}) \in (f)$ and by induction on the added number of $\mathbb{Z}$-homogeneous parts of $g$ and $h$, we obtain $f|g$ or $f|h$. □

Proof of Theorem 5.5. Let $R$ be factorially $K \oplus \mathbb{Z}^m$-graded. By Theorem 1.3, there is a multiplicative system $S$ generated by $K \oplus \mathbb{Z}^m$-primes such that $S^{-1}R$ is $(0_K) \oplus \mathbb{Z}^m$-associated and $(S^{-1}R)_{(0_K) \oplus \mathbb{Z}^m}$ is factorially graded, i.e. factorial. With respect to the coarsened $K$-grading, this is just the Veronese subalgebra $(S^{-1}R)_{0_K}$. Since the generators of $S$ are $K$-prime by the above proposition and $S^{-1}R$ is clearly $(0_K) \oplus \mathbb{Z}^m$-associated, Theorem 1.3 implies that $R$ is factorially $K$-graded.

Conversely, let $R$ be factorially $K$-graded and let $f$ be a non-zero $K \oplus \mathbb{Z}^m$-homogeneous non-unit. Then all $K$-prime factors in the decomposition of $f$ are $K \oplus \mathbb{Z}$-homogeneous and therefore $K \oplus \mathbb{Z}$-prime by the above proposition. □

5. Examples

In the following, let $K$ be algebraically closed of characteristic zero and $K$ finitely generated. In order to give examples of factorially graded rings and Cox rings, we first fix the notation for a more explicit version of our criterion for graded factoriality. Let $F := K[T_1, \ldots, T_m]/(a \mid b)$ be an integral $K$-graded $K$-algebra with $K$-prime generators $T_i$. Let $Q : \mathbb{Z}^m \to K$ denote the map given by $e_i \mapsto \deg_K(T_i)$. For a free subgroup $K'$ of $K$, we fix a monomorphism $B : \mathbb{Z}^m \to \mathbb{Z}^m$ mapping onto $Q^{-1}(K')$ and denote by $\beta : K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m] \to K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]$ the corresponding homomorphism of group algebras which is an isomorphism onto the Veronese subalgebra $K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]_{K'}$. Let $g_1, \ldots, g_d$ be $K$-homogeneous generators of $a$ and $\mu_1, \ldots, \mu_d \in \mathbb{Z}^m$ with $Q(\mu_j) - \deg_K(g_j) \in K'$. We denote by $h_j$ the unique preimage of $T^{-\mu_j}g_j \in K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]_{K'}$. Here, we write the Laurent monomial $T^{v_1} \cdots T^{v_m}$ as $T^v$ where $v := (v_1, \ldots, v_m) \in \mathbb{Z}^m$.

Theorem 5.1. In the above setting, $R$ is factorially $K$-graded if and only if $K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]/(h_1, \ldots, h_d)$ is factorial. Moreover, if one of the above holds and $R$ is normal, almost freely graded and $R^+ \cap R^* = \mathbb{K}^*$ holds, then $R$ is a Cox ring.

Proof. Let $S$ be the multiplicative system given by products of the $T_i$. Then $S^{-1}R$ is $K'$-associated and by Theorem 1.3 we only have to show that $(S^{-1}R)_{K'}$ is isomorphic to $K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]/(h_1, \ldots, h_d)$ which we see as follows:

$$(S^{-1}R)_{K'} \cong \left(\frac{K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]}{\langle a_T \rangle}_{K'}\right)_{K'} \cong K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]/(a_T \cap \mathbb{K}[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]_{K'}) \cong K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]/(a_T \cap \mathbb{K}[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]_{K'}).$$

Here, $a_T$ denotes the localization of $a$ by $(T_1 \cdots T_m)$. By construction, the $h_j$ generate $\beta^{-1}(a_T \cap \mathbb{K}[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]_{K'})$, which concludes the proof. □

Corollary 5.2. If in the above setting the elements of $(\text{Supp}(T^{-\mu_1}g_1) \cup \cdots \cup \text{Supp}(T^{-\mu_d}g_d)) \setminus \{0\}$ form a basis of a primitive sublattice of $Q^{-1}(K')$, then $R$ is factorially $K'$-graded.

Proof. After a unimodular transformation of the variables, the $h_j$ are affine linear polynomials. So $K[T^{\pm 1}_1, \ldots, T^{\pm 1}_m]/(h_1, \ldots, h_d)$ is as a graded ring isomorphic to the localization of some Laurent algebra $K[T^{\pm 1}_1, \ldots, T^{\pm 1}_s]$ by a Laurent polynomial. In particular, it is factorially $K'$-graded, and we can apply the above theorem. □
Remark 5.3. Note that the Cox rings calculated in [8], the complexity one Cox rings of [11] and the hypersurface Cox rings of [6] may all be obtained through Corollary 5.2 in the same way as the examples below.

For the discussion of examples, the following well-known statement is useful; see [7, §18]:

Lemma 5.4. Let \( g \in \mathbb{K}[T_1, \ldots, T_m] \). If the hypersurface \( V(g) \) is connected and the zero set \( V(\partial g/\partial T_1, \ldots, \partial g/\partial T_m) \) is of codimension at least two in \( V(g) \), then \( \mathbb{K}[T_1, \ldots, T_m]/\langle g \rangle \) is normal and in particular, \( g \) is prime.

In the following, we construct Cox rings of the form \( R = \mathbb{C}[T_1, \ldots, T_m]/\langle g \rangle \) algebraically, applying Corollary 5.2 with \( K^0 = 0 \), i.e. \( Q^{-1}(K^0) = \ker(Q) \).

Example 5.5. Consider the ring \( R = \mathbb{C}[T_1, \ldots, T_4]/\langle g \rangle \) with \( g = T_1^{m_1} + \cdots + T_4^{m_4} \). In [13], Storch showed that \( R \) is a UFD if and only if one of the following conditions holds:

(i) One of the numbers \( m_i \) is coprime to the others;
(ii) After renumbering, \( m_1, m_2, m_3 \) are pairwise coprime, \( d_i := \gcd(m_i, m_4) \neq 1 \) holds for \( i = 1, 2, 3 \), and there is no permutation \( \pi \in S_4 \) with \( 2|d_{\pi(1)}, 3|d_{\pi(2)}, 5|d_{\pi(3)} \).

While only some of the rings \( R \) are UFDs, all of them can be factorially graded as we show now.

First note that \( R \) is \( \mathbb{Z}_{\geq 0} \)-graded via \( \deg(T_i) = m_i^{-1}m_1 \cdots m_4 \). For this grading, we have \( R_0 = \mathbb{C} \), which implies \( R^* = \mathbb{C}^* \). Moreover, \( V(g) \) is invariant under the \( \mathbb{C}^* \)-action on \( \mathbb{C}^4 \) associated to the \( \mathbb{Z}_{\geq 0} \)-grading. Since the origin lies in any orbit closure, \( V(g) \) is connected. Thus, Lemma 5.4 applies and we obtain that \( R \) is normal. In the same way we see that the \( \overline{T_i} \) are prime. For example, for \( \overline{T_1} \), it suffices to show that \( R/(\overline{T_1}) = \mathbb{C}[T_2, T_3, T_4]/(T_2^{m_2} + T_3^{m_3} + T_4^{m_4}) \) is normal which follows exactly as before.

Now let \( K := \mathbb{Z}^4/M \) with the sublattice \( M \subseteq \mathbb{Z}^4 \) spanned by \((-m_1, m_2, 0, 0), (-m_1, 0, m_3, 0), (-m_1, 0, 0, m_4)\) and let \( T_i \) have the degree \( \deg_k(T_i) := Q(e_i) := e_i + M \). Then \( g \) is \( K \)-homogeneous, so \( R \) is \( K \)-graded. Since the elements of \( \text{Supp}(T_i^{m_i}, g) \setminus \{0\} \) by definition form a basis for \( \ker(Q) = M \), the ring \( R \) is factorially \( K \)-graded by Corollary 5.2.

Not all of the above rings are Cox rings yet, but using Theorem 1.4 we do get Cox rings:

Example 5.6. In the above notation, let \( m_1 = m_2 = 2 \) and \( m_3 = m_4 = 3 \). According to Storch’s criterion, \( R \) is not a UFD. The grading by \( K = \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \) is given via \( \deg_k(T_1) = (3, 0), \deg_k(T_2) = (3, 3), \deg_k(T_3) = (2, 0) \) and \( \deg_k(T_4) = (2, 2) \). The \( \overline{T_i} \) are pairwise non-associated because they have different \( K \)-degrees and \( R^* = \mathbb{C}^* \). But no three of their degrees generate \( K \), so \( R \) is not a Cox ring. However, by adding another variable \( T_5 \) (which automatically is prime because \( R \) is integral) with degree e.g. \( (2, 1) \) we do get a Cox ring \( R[T_5] \) since Theorem 1.4, respectively Lemma 3.1, ensure graded factoriality of \( R[T_5] \) and \( K \)-primality of \( \overline{T_1}, \ldots, \overline{T_4} \).

Next, we consider an example of an application of Theorem 1.5:

Example 5.7. Let \( R := \mathbb{K}[T_1, \ldots, T_5]/\langle g \rangle \) with \( g := T_1^2T_2^3T_3 + T_4^2 \). As before, \( R \) is normal, the \( \overline{T_i} \) are prime and pairwise non-associated, and \( R^* = \mathbb{K}^* \). \( R \) is graded by \( \tilde{K} := \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \), respectively \( K := \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \), via the matrices

\[
\tilde{Q} := \begin{pmatrix}
0 & -3 & 2 & 0 & 0 \\
6 & 6 & 0 & 4 & 3 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}, \quad Q := \begin{pmatrix}
6 & 3 & 2 & 4 & 3 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix},
\]
where the $i$-th column denotes $\deg_K(T_i)$, respectively $\deg_\tilde{K}(T_i)$. Again, $R$ is factorially $\tilde{K}$-graded by Corollary 5.2. However, the $\tilde{K}$-grading is not almost free, so as a $\tilde{K}$-graded ring, $R$ is not a Cox ring. But by Theorem 1.5, $R$ is also factorially $K$-graded, and this grading clearly is almost free. So as a $K$-graded ring, $R$ is a Cox ring.

**Remark 5.8.** In general, we may use Corollary 5.2 to show that $R$ is factorially graded, if $R$ is integral, the $T_i$ are $K$-prime and the monomials $q_{i,j}$ occurring in the defining relations $g_i$ each comprise a variable that occurs in no other $q_{i,k}$. The example below shows that our method is not restricted to these cases.

**Example 5.9.** Let the ring $R = \mathbb{K}[T_1, \ldots, T_4]/\langle g \rangle$ with $g = T_1^2 + T_2^2T_4 + T_2^3T_3 + T_4^2$ be graded by $K := \mathbb{Z}$ via $\deg_K(T_1) = 2, \deg_K(T_2) = 3, \deg_K(T_3) = 5$ and $\deg_K(T_4) = 7$, and let $Q : \mathbb{Z}^4 \to K$ denote the corresponding linear map. As before, we see that $R$ is normal and $R^* = K^*$ holds and the $T_i$ are pairwise non-associated. Also, for each $i$ direct computations show that the polynomial $g_i$ obtained by cancelling all summands who feature $T_i$ is $K$-irreducible, and by graded factoriality of the polynomial ring, $g_i$ is also $K$-prime which implies that $\overline{T_i} \in R$ is $K$-prime. The grading is almost free and since the elements of $\text{Supp}(T_2^{-3}T_3^{-1}g) \setminus \{0\}$ form a basis of $\ker(Q)$, $R$ is factorially graded by Corollary 5.2 (and thus even a UFD, because $K = \mathbb{Z}$). Hence, $R$ is a Cox ring.

**Acknowledgments**

The author wishes to thank Jürgen Hausen for many fruitful discussions and the referee for a thorough reading of the preliminary draft for this article and helpful comments and suggestions.

**References**