Infinite Dimensional Cohomology Groups and Periodic Solutions of Asymptotically Linear Hamiltonian Systems

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In this paper we study the existence of nontrivial $2\pi$-periodic solutions of asymptotically linear Hamiltonian systems. We consider the case of resonance both at zero and at infinity, and we permit time-dependent asymptotic matrices. Our main tools are an infinite dimensional cohomology theory and a corresponding Morse theory recently constructed by W. Kryszewski and the first author. We develop a method to compute the new critical groups.

Key Words: Hamiltonian; filtration; $E$-cohomology; critical groups; $\delta$-Morse index; Morse inequalities.

1. INTRODUCTION

We consider the existence of nontrivial $2\pi$-periodic solutions of asymptotically linear Hamiltonian systems

\[ \dot{z} = JH'(z, t), \quad z \in \mathbb{R}^{2N}, \quad (S) \]

where

\[ J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \]

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is the standard symplectic matrix, $H \in C^2(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R})$ is $2\pi$-periodic in $t$, $H'$ denotes the gradient of $H$ with respect to the first $2N$ variables and there exist $s > 0$, $c > 0$ such that
\[(H) \quad |H_{zz}(z, t)| \leq c(1 + |z'|) \text{ for all } (z, t) \in \mathbb{R}^{2N} \times \mathbb{R}.
\]

In what follows we assume that there exist two symmetric $2N \times 2N$ matrices $A(t)$ and $A_0(t)$ with continuous and $2\pi$-periodic entries such that
\[H(z, t) = \frac{1}{2} A(t) z \cdot z + G(z, t),
\]
where $G'(z, t) = o(|z|)$ uniformly in $t$ as $|z| \to \infty$ and
\[H(z, t) = \frac{1}{2} A_0(t) z \cdot z + G_0(z, t),
\]
with $G_0'(z, t) = o(|z|)$ uniformly in $t$ as $|z| \to 0$. We denote by $\cdot$ and $|\cdot|$ the usual inner product and norm in $\mathbb{R}^{2N}$. The Hamiltonian system $(S)$ satisfying (1.1) and (1.2) is called asymptotically linear both at infinity and at zero. Moreover, it is called nonresonant at infinity if $1$ is not a Floquet multiplier of the linear system $\dot{z} = JA(t) z$; nonresonance at $0$ is defined in a similar way by replacing $A(t)$ with $A_0(t)$.

Before introducing our assumptions on $H(z, t)$ and stating the main results, let us recall some earlier work on asymptotically linear Hamiltonian systems. The case of $(S)$ nonresonant at infinity was considered in [2, 3] under the additional assumptions that $H_{zz}$ is bounded and $A, A_0$ are time-independent; in [4] $H_{zz}$ was bounded and $(S)$ was also nonresonant at zero. In [5] $A, A_0$ were time-independent and in [6] $H_{zz}$ was bounded. For $(S)$ resonant at infinity it was assumed in [7] that $A(t)$ is a constant matrix; [8, 9] considered the strongly resonant case and [14] studied $(S)$ under the assumption that $A(t), A_0(t)$ are so-called finitely degenerate, which is a strong condition. Moreover, no results on the existence of multiple solutions were obtained in [7–9, 14]. Recently Kryszewski and the first author [1] constructed an infinite dimensional cohomology theory and a Morse theory corresponding to it. These theories were applied to the study of Hamiltonian systems and wave equations. In particular, the case of $(S)$ resonant at infinity was studied in [1] under the hypotheses that $G'(z, t)$ is bounded and $G(z, t) \to \infty$ (or $-\infty$) uniformly in $t$ as $|z| \to \infty$. This was done by computing the new critical groups (the $\delta$-cohomology groups) at zero and at infinity. However, in the case of resonance at $0$, [1] contained no detailed computation of critical groups there; it was only shown that the groups at zero and at infinity were different under certain assumptions.

The purpose of the present paper is to develop a method to compute the $\delta$-cohomology groups both at infinity and at zero when resonance occurs at infinity and at zero simultaneously. We admit $H$ such that $G'(z, t)$ and
$G_0(z, t)$ are unbounded and $G(z, t), G_0(z, t)$ may change sign. Under rather weak conditions we obtain at least two nontrivial solutions for $(S)$.

In order to state our assumptions, we introduce a control function $h_\infty : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h_\infty(t)$ is increasing in $t$

$$1 \leq \frac{h_\infty(t)}{H_\infty(t)} \leq \alpha < 2, \quad h_\infty(s + t) \leq m(h_\infty(s) + h_\infty(t)) \quad \text{for any} \quad s, t \in \mathbb{R}^+,$$

where $H_\infty(t) = \int_0^t h_\infty(s) \, ds$ and $\alpha, m$ are constants. Evidently, $h_\infty(t) = t^\sigma$ with $0 < \sigma < 1$ is a simple example. Now we assume

$$(H_1) \quad |G(z, t)| \leq c(1 + h_\infty(|z|)) \quad \text{for all} \quad z \in \mathbb{R}^{2N} \quad \text{and} \quad t \in \mathbb{R};$$

$$(H_2) \quad \liminf_{|t| \to \infty} \frac{|G(z, t)|}{H_\infty(t)} := b^\pm(t) \geq 0 \quad \text{uniformly for} \quad t \in \mathbb{R}. $$

Here and in the following the letter $c$ will be repeatedly used to denote various positive constants whose exact value is irrelevant. For a function $a$ we write $a(t) = O(t)$ if $a(t) \leq c(t)$ for large $t$. Moreover, if $h_0$ is defined only for small $t > 0$, we may assume without loss of generality that it has been extended so that (1.3) holds for all $t \in \mathbb{R}^+$. We suppose that

Let $h_0 : \mathbb{R}^+ \to \mathbb{R}^+$ be a control function (for $G_0$) such that

$$2 < \beta \leq \frac{h_0(t)}{H_0(t)} \leq \gamma \quad \text{for} \quad t \text{ small,} \quad (1.3)$$

where $H_0(t) = \int_0^t h_0(s) \, ds$, and $\beta, \gamma$ are constants. Obviously, $h_0(t) = t^\delta$ with $\delta > 1$ satisfies (1.3). Moreover, although $h_0$ is defined only for small $t > 0$, we may assume without loss of generality that it has been extended so that (1.3) holds for all $t \in \mathbb{R}^+$. We suppose that

$$(H_3) \quad |G_0(z, t)| \leq ch_0(|z|) \quad \text{for} \quad |z| \text{ small;}$$

$$(H_3') \quad \liminf_{|t| \to \infty} \frac{|G_0(z, t)|}{H_0(t)} := b^0(t) \geq 0 \quad \text{uniformly for} \quad t \in \mathbb{R}.$$
and similarly, \( \int_0^t G(t, z(t)) \cdot w dt \leq c \|z\|^{p-1} \|w\|_p \) \((\| \cdot \|_p \) denotes the usual norm in \( L^p([0, 2\pi], \mathbb{R}^{2N})\)).

**Remark 1.2.** \((H_1)\) and \((H_4)\) imply that \(G'(z, t) = o(|z|)\) uniformly in \(t\) as \(|z| \to \infty\) and \(G''(z, t) = o(|z|)\) uniformly in \(t\) as \(|z| \to 0\). However, \((H_1)\) does not imply that \(G'(z, t)\) is bounded. Since \(a^2(t)\) and \(b^2(t)\) may be zero on a set of positive measure, \(G(z, t)\) and \(G''(z, t)\cdot z\) may not be of constant sign; moreover, \(G(z, t)\) may be bounded on a subset of positive measure. So our results will extend different conclusions contained in \([1]\) (and \([7, 10, 14, 18]\)). In \([1]\) it was assumed that \(G'(z, t)\) is bounded and \(G(z, t) \to \infty\) (or \(-\infty\)) uniformly in \(t\) as \(|z| \to \infty\).

In order to state our main result, we shall need the notion of \(d\)-Morse index which was introduced in \([1]\) and will be recalled in Section 2. It is a kind of relative Morse index for the quadratic form \(\sum_{k=1}^n (-Jz^2 - Az) \cdot zdh\), where \(A = A(t)\) is a symmetric \(2N \times 2N\) matrix. Denote this index by \(j^-(A)\) and the nullity of this quadratic form by \(j^0(A)\) and let \(j^+(A) = -j^-(A) - j^0(A)\). If we denote the Maslov-type index (cf. \([3, 4, 6]\)) of \(A\) by \((j, n)\), then \(j = j^-(A)\) and \(n = j^0(A)\) (cf. Remark 7.2 of \([1]\)). Now we state the main results.

**Theorem 1.1.** Suppose that \(H \in C^2(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R})\) satisfies \((H_0), (H_1)\) and one of the conditions \((H_2+)\). Then \((S)\) has a nontrivial \(2\pi\)-periodic solution in each of the following two cases:

(i) \((H_2^+)\) and \(j^-(A) \neq j^-(A_0) + j^0(A_0)\);

(ii) \((H_2^+)\) and \(j^+(A) \neq j^+(A_0) + j^0(A_0)\).

**Theorem 1.2.** Suppose that \(H \in C^4(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R})\) satisfies \((H_1)\) and \((H_3)\). Then \((S)\) has a nontrivial \(2\pi\)-periodic solution in each of the following four cases:

(i) \((H_2^+), (H_4^-), \) and \(j^-(A) + j^0(A) \neq j^-(A_0) + j^0(A_0)\);

(ii) \((H_2^+), (H_4^-), \) and \(j^+(A) \neq j^+(A_0)\);

(iii) \((H_2^-), (H_4^+), \) and \(j^-(A) \neq j^+(A_0) + j^0(A_0)\);

(iv) \((H_2^-), (H_4^+), \) and \(j^-(A) \neq j^-(A_0)\).

If the difference between the \(d\)-Morse indices at zero and at infinity is large enough, we obtain the following results on the existence of multiple solutions.
Theorem 1.3. Suppose that \( H \in C^2(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R}) \) satisfies \((H_0), (H_1)\) and \((H_3)\). Then \((S)\) has at least two nontrivial \(2\pi\)-periodic solutions in each of the following four cases:

(i) \((H_+^2), (H_+^4), \text{ and } |j^+(A) - j^+(A_0)| \geq 2N;\)

(ii) \((H_-^2), (H_-^4), \text{ and } |j^+(A) + j^-(A_0)| \geq 2N;\)

(iii) \((H_-^2), (H_+^4), \text{ and } |j^- - j^+(A_0)| \geq 2N;\)

(iv) \((H_-^2), (H_-^4), \text{ and } |j^-(A) - j^- - A_0)| \geq 2N.\)

Corollary 1.1. Suppose that \( H \in C^2(\mathbb{R}^{2N} \times \mathbb{R}, \mathbb{R}) \) satisfies \((H_0), (H_1)\), one of the conditions \((H_2^\pm)\), and \( A(t) = A_0(t) \equiv 0 \) (hence \( H(z, t) = G(z, t) = G_0(z, t) \)). Furthermore, let \( H'(z, t) = \alpha(|z|) \) uniformly in \( t \) for \( |z| \to 0 \). Then \((S)\) has at least two nontrivial \(2\pi\)-periodic solutions in each of the following two cases:

(i) \((H_+^2)\) and either there exists a \( \delta > 0 \) such that \( H(z, t) \leq 0 \) whenever \( |z| < \delta \) or \((H_3)\) are satisfied;

(ii) \((H_-^2)\) and either there exists a \( \delta > 0 \) such that \( H(z, t) \geq 0 \) whenever \( |z| < \delta \) or \((H_3)\) are satisfied.

Remark 1.3. Theorem 1.1 extends Theorem 7.5 in [1] where \( G' \) was assumed to be bounded and \( G(z, t) \to \infty \) (or \( -\infty \)) uniformly in \( t \) as \( |z| \to \infty \). Theorem 1.2 is a new result. Theorem 1.3 extends Theorem 7.8 in [1] where \( 0 \) was nondegenerate \( (j^0A_0) = 0 \), i.e., \((S)\) is nonresonant at zero), \( G' \) was bounded and \( G(z, t) \to \infty \) (or \( -\infty \)) uniformly in \( t \) as \( |z| \to \infty \). Corollary 1.1 is a generalization of Corollary 7.9 of [1].

2. PRELIMINARIES

In this section we recall some basic facts about the infinite dimensional cohomology theory and Morse theory of [1].

Assume that \( E \) is a real Hilbert space and there is a filtration \((E_n)_{n=1}^\infty\) of \( E \), i.e., an increasing sequence of closed subspaces of \( E \) such that \( E = \text{cl}(\bigcup_{n=1}^\infty E_n) \) (\( \text{cl} \) denotes the closure). Suppose that a sequence \((d_n)_{n=1}^\infty\) of nonnegative integers is given and let \( \mathcal{E} = \{E_n, d_n\}_{n=1}^\infty \). If \((X, A)\) is a closed pair of subsets of \( E \), then for any integer \( q \) we define the \( q \)-th \( \mathcal{E} \)-cohomology group of \((X, A)\) with coefficients in \( \mathcal{F} \) by the formula

\[
H^q_{\mathcal{E}}(X, A) := [(H^{q+d_n}(X \cap E_n, A \cap E_n))_{n=1}^\infty],
\]
where \( [(\xi_n)_n] \) is the equivalence class of sequences \((\xi_n)_n \) such that 
\( \xi_n = \delta_n \) for almost all \( n \) (cf. [1]). When \( \mathcal{F} \) is a field, \( H^*_E(\mathcal{F}, A) \) is a (graded) vector space over \( \mathcal{F} \). We shall use the symbol \([\mathcal{F}] \) to denote the group \([[(\mathcal{F})_n]_n] \) if \( \mathcal{F}_n = \mathcal{F} \) for almost all \( n \).

Let \( \Phi \in C^1(E, \mathbb{R}) \) be a functional satisfying the \((PS)^*\)-condition with respect to \( \delta \); that is, whenever a sequence \((y_j)_j \) is such that \( 8(y_j) \) is bounded, \( y_j \in E_n \) for some \( n_j \), \( n_j \to \infty \) and \( P_n \delta \Phi(y_j) \to 0 \) as \( j \to \infty \), then \((y_j)_j \) has a convergent subsequence. Here \( P_n \) denotes the orthogonal projector of \( E \) onto \( E_n \). If \( p \) is an isolated critical point of \( \Phi \), then there exists an admissible pair \((W, W^-)\) for \( \Phi \) and \( p \) (i.e., a kind of Gromoll–Meyer pair with filtration; see Definition 2.3 and Proposition 2.6 of [1]) and the \( q \)th critical group \((q \in \mathbb{Z})\) of \( \Phi \) at \( p \) with respect to \( \delta \) can be defined by

\[
C^*_q(\Phi, p) := H^*_q(W, W^-)
\]

It was proved in [1] that the critical groups \( C^*_q(\Phi, p) \) are well defined and have a certain continuity property (see Propositions 2.7 and 2.8 of [1]).

If the critical set \( K = K(\Phi) \) is compact, then there exists an admissible pair \((W, W^-)\) for \( \Phi \) and \( K \) (cf. Lemma 2.13 of [1]). The critical groups of \((\Phi, K)\) given by

\[
C^*_q(\Phi, K) := H^*_q(W, W^-)
\]

are well defined and have a continuity property (cf. Propositions 2.12 and 2.14 of [1]). Further properties of critical groups and \( \delta \)-cohomology groups, including the Morse inequalities, may be found in [1].

For an arbitrary linear self-adjoint operator \( L \), denote the Morse index of \( L \) by \( M(L) \). Suppose that \( L \) is a Fredholm operator of index 0 and \( Q_n : R(L) \to R(L) \cap E_n \) is the orthogonal projector of \( R(L) \) onto \( R(L) \cap E_n \).

Define the \( \delta \)-Morse index \( M^*_\delta(L) \) of \( L \) by the formula

\[
M^*_\delta(L) := \lim_{n \to \infty} (M(Q_n L)_{R(L) \cap E_n}) - d_n.
\]

Although this limit does not exist in general, it exists for operators \( L \) associated with \((S)\) provided the sequence \( (d_n) \) is chosen properly.

Now we turn to the asymptotically linear Hamiltonian system \((S)\). Let \( E := H^1(\mathcal{S}, \mathbb{R}^{2N}) \) be the Sobolev space of \( 2\pi \)-periodic \( \mathbb{R}^{2N} \)-valued functions

\[
z(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \quad a_0, a_k, b_k \in \mathbb{R}^{2N},
\]
such that \( \sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty \). Then \( E \) is a Hilbert space with a norm \( \| \cdot \| \) induced by the inner product \( \langle \cdot, \cdot \rangle \) given by
\[
\langle z, z' \rangle := 2\pi a_0 \cdot a_0' + \pi \sum_{k=1}^{\infty} k(a_k \cdot a_k' + b_k \cdot b_k').
\]

Set
\[
F_k := \{ a_k \cos kt + b_k \sin kt : a_k, b_k \in \mathbb{R}^{2N} \}, \quad k \geq 0,
\]
and
\[
E'_n := \bigoplus_{k=0}^{n} F_k \equiv \{ z \in E : z(t) = a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \}.
\]

Then \( (E_n')_{n=1}^{\infty} \) is a filtration of \( E \). Denote \( \mathcal{E} = \{ E_n, d_n \} \) with \( d_n := N(1 + 2n) = \frac{1}{4} \dim E_n \).

Suppose that \( B(t) \) is a symmetric \( 2N \times 2N \) matrix with continuous \( 2\pi \)-periodic entries. Then the operator \( B \) given by the formula
\[
\langle Bz, w \rangle := \int_{0}^{2\pi} B(t) z \cdot w \ dt
\]
is compact. According to Proposition 5.2 of [1] (see also the argument following Proposition 7.1 there), the operator \( L_B \) given by
\[
\langle L_B z, w \rangle := \int_{0}^{2\pi} (-Jz - B(t) z) \cdot w \ dt
\]
is \( A \)-proper and \( M_\mathcal{E}(L_B) \) is well defined and finite.

Denote
\[
\begin{align*}
\mathbf{j}^- (B) &:= M_\mathcal{E}(L_B), \\
\mathbf{j}^+ (B) &:= M_\mathcal{E}(L_B) := M_\mathcal{E}(-L_B), \\
\mathbf{j}^0 (B) &:= M_\mathcal{E}(L_B) := \dim \ker (L_B).
\end{align*}
\]

Then \( \mathbf{j}^- (B) + \mathbf{j}^+ (B) + \mathbf{j}^0 (B) = 0 \) (cf. p. 3214 of [1]). Since \( M_\mathcal{E}(L_B) \) is in fact the number of linearly independent \( 2\pi \)-periodic solutions of the linear system \( \dot{z} = JB(t) z \), \( 0 \leq M_\mathcal{E}(L_B) \leq 2N \).
It is well known (cf. [11]) that under condition \((H_1)\) \(z(t)\) is a \(2\pi\)-periodic solution of \((S)\) if and only if it is a critical point of the \(C^1\)-functional

\[
\Phi(z) = \frac{1}{2} \int_0^{2\pi} (-J_z - A(t)z) \cdot z \, dt - \int_0^{2\pi} G(z, t) \, dt := \frac{1}{2} \langle Lz, z \rangle - \varphi(z)
\]

Moreover, \(\Phi \in C^2(E, \mathbb{R})\) if \((H_0)\) is satisfied. By (1.1), (1.2) and [1, 5, 11] (or by Remark 1.1), \(\mathcal{V} \varphi(z) = o(\|z\|)\) as \(\|z\| \to \infty\) and \(\mathcal{V} \varphi(t) = o(\|z\|)\) as \(\|z\| \to 0\). In particular, \((S)\) has the trivial solution \(z = 0\).

\[3.\text{ COMPUTATION OF CRITICAL GROUPS}\]

Let \(L := L_{R} \) and \(L_0 := L_{B_0}\) (cf. (2.1)) and introduce a new filtration \(\mathcal{E}' := \{ E_n', d_n' \}_{n=1}^{\infty} \), where \(E_n' := (R(L) \cap E_n) \oplus \ker(L)\) and \(d_n = N(1 + 2n)\) as before. Then \(L, L_0\) are \(A\)-proper with respect to \(\mathcal{E}'\) (because they are with respect to \(\mathcal{E}\)) and

\[
M_{\mathcal{E}'}(L) = M_{\mathcal{E}'}(A) \quad \text{and} \quad M_{\mathcal{E}'}(L_0) = M_{\mathcal{E}'}(A_0)\]

(see the proof of Theorem 7.5 of [1]). In this section we will compute the critical groups \(C^\bullet_{\mathcal{E}'}(\Phi, 0)\) and \(C^\bullet_{\mathcal{E}'}(\Phi, K(\Phi))\). For this aim, we first show how conditions \((H_1)\) and \((H_\frac{3}{2})\) imply \((PS)^*\) with respect to \(\mathcal{E}'\).

\[\text{Lemma 3.1. Suppose that } (H_\frac{3}{2}) \text{ holds. Then}
\]

\[
\liminf_{\|z\| \to \infty} \frac{\int_0^{2\pi} G(z, t) \, dt}{H_{\infty}(\|z\|)} > 0.
\]

\[\text{Proof. Since } \dim \ker(L) < \infty, \text{ the norm } \|\cdot\| \text{ and the } L^\infty\text{-norm are equivalent on } \ker(L). \text{ Moreover, if } z \in \ker(L) \text{ and } z(t_0) = 0 \text{ for some } t_0, \text{ then } z = 0. \text{ Therefore } \delta (\|z\|) \leq |z(t)| \leq c \|z\| \text{ for some } \delta, c > 0 \text{ and all } t. \text{ Since } h_{\infty} \text{ is increasing and } h_{\infty}(s + t) \leq m(h_{\infty}(s) + h_{\infty}(t)), \text{ it is easy to see that } c_1 h_{\infty}(\|z\|) \leq h_{\infty}(\|z\|) \leq c_2 h_{\infty}(\|z\|) \text{ and therefore } c_3 H_{\infty}(\|z\|) \leq H_{\infty}(\|z\|) \leq c_4 H_{\infty}(\|z\|) \text{ for a suitable choice of constants. Hence it follows from } (H_\frac{3}{2}) \text{ that for any } \varepsilon > 0 \text{ and } \|z\| > R = R(\varepsilon),\]
\[ \pm \int_0^{2\pi} \frac{G(z, t)}{H_\infty(\|z\|)} H_\infty(|z|) \, dt \]
\[ \geq \int_0^{2\pi} (a^2(t - \varepsilon) - \varepsilon) \, dt \]
\[ \geq c_3 \int_0^{2\pi} a^2(t) \, dt - 2\pi c_4. \]

Since \( a^2(t) \geq 0 \) and \( \varepsilon \) is arbitrary, the conclusion follows.

**Lemma 3.2.** Assume \((H_1)\) and \((H_2^+\)). Then \( \Phi \) satisfies \((PS)^*\) with respect to \( \delta^* \). Moreover, under these hypotheses, \( \Phi \) satisfies the usual \((PS)\)-condition for each \( n \).

**Proof.** We only consider the case where \((H_2^-)\) holds, the other one is similar. Let \((z_j)\) be a \((PS)^*\)-sequence, i.e., \( z_j \in E_n \), \( \Phi(z_j) \) is bounded, \( P_n \nabla \Phi(z_j) \to 0 \) and \( n_j \to \infty \) as \( j \to \infty \) (\( P_n \) is the orthogonal projector onto \( E_n \)). By Theorem 4.5 in [1], we may find \( c > 0 \) and \( n_0 > 0 \) such that
\[ \|P_n Lz\| \geq c \|z\| \quad \text{for all} \quad z \in R(L) \cap E_n \quad \text{and} \quad n \geq n_0. \]

For \( z \in E_n \), write \( z = w + z_0 \in R(L) \cap E_n \otimes \ker(L) \). Then \( P_n \nabla \Phi(z_j) = P_n Lw_j - P_n \nabla \varphi(z_j) \to 0 \).

Since
\[ \int_0^{2\pi} h_\infty(\|z_0\|) \, |y| \, dt \leq c \int_0^{2\pi} h_\infty(\|z_0\|) \, |y| \, dt \leq ch_\infty(\|z_0\|) \|y\| \]
(cf. the proof of Lemma 3.1), we obtain by Remark 1.1 and the Sobolev embedding theorem that
\[ c \|w_j\| \leq \|P_n Lw_j\| \leq c(1 + \|w_j\|^{s-1} + h_\infty(\|z_0\|)). \]

Therefore \( \|w_j\| \leq c(1 + h_\infty(\|z_0\|)) \). Moreover, by Remark 1.1 again and by the mean value theorem,
\[ \Phi(z_j) \geq -c \|w_j\|^2 - \varphi(z_j) + \varphi(z_0) - \varphi(z_0) \]
\[ = -c \|w_j\|^2 - \int_0^{2\pi} (G(z_j, t) - G(z_0, t)) \, dt - \varphi(z_0) \]
\[ \geq -c \|w_j\|^2 - c(1 + \|w_j\|^{s-1} + h_\infty(\|z_0\|)) \|w_j\| - \varphi(z_0) \]
\[ \geq -c(1 + h_\infty(\|z_0\|)) \varphi(z_0). \]
If $|z_0^j| \to \infty$, then it follows from Lemma 3.1 that
\[
\Phi(z_j) \geq -c \frac{\varphi(z_j^0)}{h_\infty(\|z_j^0\|)}
\]
\[
= -c + \frac{\varphi(z_j^0)}{h_\infty(\|z_j^0\|)} H_\infty(\|z_j^0\|)
\]
\[
\to \infty
\]
as $j \to \infty$ because
\[
H_\infty(t) \frac{1}{H_\infty(t)} \geq ct^{2-n} \to \infty \quad \text{whenever} \quad t \to \infty.
\]
This contradicts the boundedness of $\Phi(z_j)$. It follows that $\|z_0^j\|$ and hence $\|z_j\|$ is bounded. Recalling the compactness of $\nabla \varphi$, we see that $(z_j)$ has a convergent subsequence. $lacksquare$

In order to compute $C^*_R(\Phi, 0)$, we first prove the following auxiliary results.

**Lemma 3.3.** Suppose that (H$_3$) and (H$_4^+$) hold. Then for any sequence $(z_n) \in E$ such that $z_n = z_0^i + w_n$, where $z_0^i \in \ker(L_0)$, $w_n \in (\ker(L_0))^\perp$, $\|z_n^i\| \to 0$ and $\|z_0^0^j/\|z_n\| \to 1$, we have
\[
\liminf_{n \to \infty} \frac{\int_0^{2\pi} G_0(z_n, t) \cdot z_n \ dt}{H_0(\|z_n\|)} > 0.
\]

**Proof.** First, by the definition of $h_0$, it is easy to check that
\[
\left( \frac{s}{t} \right)^\theta \leq H_0(s) / H_0(t) \leq \left( \frac{s}{t} \right)^7 \quad \text{for} \quad s \geq t > 0 \quad \text{and} \quad s, t \text{ small} \quad (3.1)
\]
Since $h_0$ may be extended in such a way that (1.3) holds for all $t > 0$, we may assume that also the above inequality holds for all $t > 0$.

Let $z = w + z^0 \in (\ker(L_0))^\perp \ker(L_0)$. Since $w \in L^2([0, 2\pi], \mathbb{R}^{2N})$, for each $\varepsilon_1 > 0$ there exists $R(\varepsilon_1) > 0$, independent of $w$ and such that
\[
\text{meas}\{ t \in [0, 2\pi] : |w(t)| > R(\varepsilon_1) \|w\| \} < \varepsilon_1.
\]
Set
\[
\Omega_n = \{ t \in [0, 2\pi] : |w_n(t)| \leq R(\varepsilon_1) \|w_n\| \};
\]

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then measurable $\{0, 2\pi\} \setminus \Omega_n < \varepsilon_1$. As $\int_0^{2\pi} h^\pm(t) \, dt > 0$, we may choose $\varepsilon_1$ so small that

$$\int_{\Omega_n} h^\pm(t) \, dt \geq \frac{1}{2} \int_0^{2\pi} h^\pm(t) \, dt > 0.$$  

Since $\ker L_0$ is finite dimensional, we may assume

$$|z_n(t)| \leq c(R(\varepsilon_1) + c) \|z_n\| \quad \text{whenever } t \in \Omega_n.$$  

For any $\varepsilon_2 > 0$, by $(H^\pm_4)$, we have that

$$\pm \frac{G_0(z_n, t)}{H_0(|z_n|)} \cdot z_n \geq b^\pm(t) - \varepsilon_2$$

whenever $t \in \Omega_n$ and $n$ is large enough. Since $H_0$ is increasing, $H_0(|z_n|) \geq H_0(\|z_n\|)$ for $|z_n| \geq \|z_n\|$. On the other hand, recalling that $\|z_n\| = \|z_0\| + c_1 \|w_n\| \to 1$, we obtain

$$\frac{|z_n(t)|}{\|z_n\|} \geq \frac{|z_0(t)| - |w_0(t)|}{\|z_n\|} \geq \frac{\delta \|z_n\|}{\|z_n\|} - R(\varepsilon_1) \|w_n\| \to \delta$$

as $t \in \Omega_n$ and $n \to \infty$, where $\delta$ is as in the proof of Lemma 3.1. This and (3.1) imply

$$\frac{H_0(|z_n|)}{H_0(\|z_n\|)} \geq \left( \frac{\delta}{2} \right)^{7} \quad \text{for } t \in \Omega_n, \quad |z_n(t)| \leq \|z_n\| \text{ and } n \text{ large enough}.$$  

Since it is easy to check by (3.1) that

$$\left| \int_0^{2\pi} \frac{H_0(|z_n|)}{H_0(\|z_n\|)} \, dt \right| \leq c_1$$

for some $c_1 > 0$, it follows, for $n$ large enough, that

$$\int_{\Omega_n} \frac{\pm G_0(z_n, t)}{H_0(\|z_n\|)} \cdot z_n \, dt \geq \int_{\Omega_n} (b^\pm(t) - \varepsilon_2) \frac{H_0(|z_n|)}{H_0(\|z_n\|)} \, dt \geq c_2 \int_{\Omega_n} b^\pm(t) \, dt - c_1 \varepsilon_2$$

$$\geq c_3 \int_0^{2\pi} b^\pm(t) \, dt - c_1 \varepsilon_2 = c_4 - c_1 \varepsilon_2.$$  

(3.2)
where the constants $c_i$ are independent of $\varepsilon_1, \varepsilon_2$. On the other hand, we may assume without loss of generality that $(H_3)$ holds for all $z$. Indeed, suppose that $(H_3)$ is satisfied whenever $|z| \leq \delta_0$. Since $h_0$ may be extended so that (1.3) holds for all $t$, then by (1.3) and (3.1) it is easy to check that

$$\frac{\beta (s)^{\beta - 1}}{\gamma (t)^{\gamma - 1}} \leq \frac{h_0(s)}{h_0(t)} \leq \frac{\gamma (s)^{\gamma - 1}}{\beta (t)^{\beta - 1}}$$

for all $s, t > 0$.

It follows that $h_0(t) \geq c t^{\beta - 1}$ for $t > \delta_0$. Hence by the asymptotic linearity of $H'(z, t)$,

$$|G_0(z, t)| \leq c |z| \leq \hat{c} h_0(|z|)$$

for some $\hat{c} > 0$ and all $|z| > \delta_0$. (3.3)

Using $(H_3)$, which now holds for all $z$, we see that

$$\frac{|\pm G_0(z_n, t) \cdot z_n|}{H_0(|z_n|)} \leq \frac{c h_0(|z_n|)}{H_0(|z_n|)} \leq c.$$ 

Since $\text{meas}([0, 2\pi] \setminus \Omega_n) < \varepsilon_1$, it follows that

\[
\left| \frac{1}{(0, 2\pi) \cap \Omega_n} \int_{(0, 2\pi) \cap \Omega_n} \frac{\pm G_0(z_n, t) \cdot z_n}{H_0(|z_n|)} \, dt \right| \\
\leq c \int_{(0, 2\pi) \cap \Omega_n} \frac{H_0(|z_n|)}{H_0(|z_n|)} \, dt \\
\leq c \varepsilon_1^{1/2} \left( \int_0^{2\pi} \frac{H_2(z_n)}{H_0(|z_n|)} \, dt \right)^{1/2}.
\]

If $|z_n| \leq \|z_n\|$, then $H_0(|z_n|)/H_0(|z_n|) \leq 1$. Otherwise, by (3.1),

$$\frac{H_0(|z_n|)}{H_0(|z_n|)} \leq \left( \frac{|z_n|}{\|z_n\|} \right)^\gamma.$$ 

Using this and the Sobolev embedding of $E$ into $L^{2\gamma}([0, 2\pi], \mathbb{R}^{2N})$, we obtain that

\[
\left| \frac{1}{(0, 2\pi) \cap \Omega_n} \int_{(0, 2\pi) \cap \Omega_n} \frac{\pm G_0(z_n, t) \cdot z_n}{H_0(|z_n|)} \, dt \right| \leq c \varepsilon_1^{1/2}
\]

for $n$ large enough. Combining (3.2) and (3.4) and letting $n$ be large enough, we have

$$\int_0^{2\pi} \frac{\pm G_0(z_n, t)}{H_0(|z_n|)} \, dt \geq c_4 - c_1 \varepsilon_2 - c_4^{1/2} > 0$$

since $c, c_1, c_4$ are independent of $\epsilon_1, \epsilon_2$ and $c_1, c_2$ may be chosen arbitrarily small.

**Lemma 3.4.** Assume $(H_3)$, $(H_4^\pm)$ and set

$$\mathcal{D}(p, \theta) := \{ z \in E : z = z^0 + w \in \ker(L_0) \oplus (\ker(L_0))^\perp, \quad 0 < \|z\| \leq p \text{ and } \|w\| \leq \theta \|z\| \}.$$  

Then there exist $p > 0$ and $\theta \in (0, 1)$ such that

$$\pm \langle \nabla \Phi(z), z^0 \rangle < 0 \quad \text{for all } z \in \mathcal{D}(p, \theta).$$

**Proof.** Assume by contradiction that for any $n$ there exists $z_n = z_n^0 + w_n \in \ker(L_0) \oplus (\ker(L_0))^\perp$ such that $0 < \|z_n\| < \frac{1}{p}$, $\|w_n\| \leq \frac{1}{2} \|z_n\|$ but

$$\pm \langle \nabla \Phi(z_n), z_n^0 \rangle \geq 0.$$  

This implies that $\|z_n\| \to 0$, $\|w_n\|/\|z_n\| \to 1$ as $n \to \infty$ and

$$-\int_0^{2n} \pm g_0(z_n, t) \cdot z_n^0 dt = -\langle \pm \varphi_0(z_n), z_n^0 \rangle = \pm \langle \nabla \Phi(z_n), z_n^0 \rangle \geq 0;$$

it follows that

$$\limsup_{n \to \infty} \frac{\int_0^{2n} \pm g_0(z_n, t) \cdot z_n^0 dt}{h_0(\|z_n\|/\|z_n\|)} \leq 0.$$  

By (3.1) and the definition of $h_0$,

$$\frac{h_0(\|z_n\|)}{h_0(\|z_n\|/\|z_n\|)} \leq c \max_{\beta, \gamma} \left( \left( \frac{\|z_n\|}{\|z_n\|/\|z_n\|} \right)^{\beta - 1}, \left( \frac{\|z_n\|}{\|z_n\|/\|z_n\|} \right)^{-\gamma} \right).$$

Therefore, using $(H_3)$ and (3.3), we obtain

$$\left| \int_0^{2n} \frac{\pm g_0(z_n, t) \cdot w_n dt}{h_0(\|z_n\|/\|z_n\|)} \right| \leq c \left( \left( \int_0^{2n} \frac{h^2_0(\|z_n\|)}{h_0(\|z_n\|/\|z_n\|)} \right)^{1/2} \left( \int_0^{2n} \frac{|w_n|^2 \|z_n\|^2 dt}{\|z_n\|^2} \right)^{1/2} \right) \leq c \frac{\|w_n\|}{\|z_n\|} \to 0.$$
as } n \to \infty. \text{ Finally, in view of Lemma 3.3, }
\liminf_{n \to \infty} \frac{1}{h_n(t)} \int_0^{s_n} G_0(z_n, t) \cdot z_n \, dt = \liminf_{n \to \infty} \frac{1}{h_n(t)} \int_0^{s_n} G_0(z_n, t) \cdot z_n \, dt > 0.\]

This contradicts the preceding estimate about the upper limit. \hfill \blacksquare

Using the above lemmas we can now compute the critical groups \( C^q_\omega(\Phi, 0) \) by making a perturbation and using the continuity property of \( C^q_\omega(\Phi, 0) \).

**Lemma 3.5.** Assume \((H_3)\) and \((H^-)\) (or \((H^-)\)). Then

(i) \((H^-)\) implies that \( C^q_\omega(\Phi, 0) = \{ \mathcal{F} \} \) for \( q = f^-(A_0) + f^0(A_0) \) and \([0]\) otherwise;

(ii) \((H^-)\) implies that \( C^q_\omega(\Phi, 0) = \{ \mathcal{F} \} \) for \( q = f^-(A_0) \) and \([0]\) otherwise.

**Proof.** (i) For any \( \lambda \in [0, 1] \) and \( z = z^0 + w \in \ker(L_0) \oplus (\ker(L_0))^\perp = E \) we consider the following perturbation of \( \Phi \):
\[
\Phi_\lambda(z) := \Phi(z) - \frac{1}{2} \lambda \| z^0 \|^2 = \frac{1}{2} \langle L_0 z - \lambda z^0, z \rangle - \varphi_0(z). \]

We claim that there exists a neighborhood \( \mathcal{N} \) of 0 such that 0 is the unique critical point of \( \Phi_\lambda \) in \( \mathcal{N} \) for any \( \lambda \in [0, 1] \). In fact, if \( z \in \mathcal{G}(\rho, \theta) \), then by Lemma 3.4 \( z^0 \neq 0 \) and
\[
\langle \nabla \Phi_\lambda(z), z^0 \rangle = \langle \nabla \Phi(z), z^0 \rangle - \lambda \langle z^0, z^0 \rangle < 0. \]

If \( z \in \{ z \in E : 0 < \| z \| \leq \rho \} \setminus \mathcal{G}(\rho, \theta) \), then \( \| w \| > \theta \| z \| \). Let \( w = w^+ + w^- \); then there exists a constant \( c \) such that
\[
\langle L_0 w, w^+ - w^- \rangle > c \| w^+ \|^2. \]
Therefore
\[
\langle \nabla \Phi_\lambda(z), w^+ - w^- \rangle = \langle L_0 w, w^+ - w^- \rangle - \langle \nabla \varphi_0(z), w^+ - w^- \rangle \]
\[
\geq \| w^+ + w^- \|^2 \left( c - \frac{\| \nabla \varphi_0(z) \|}{\| w^+ + w^- \|} \right) \]
\[
\geq \| w^+ + w^- \|^2 \left( c - \frac{\| \nabla \varphi_0(z) \|}{\theta \| z \|} \right) \]
\[
> 0 \]
for sufficiently small \( \rho \) and \( \| z \| \leq \rho \). The above arguments imply that 0 is the only critical point of \( \Phi_\lambda \) in \( \mathcal{N} := \{ z : \| z \| \leq \rho \} \) for all \( \lambda \in [0, 1] \). Since \( \| P_n L_0 w \| \geq c \| w \| \) whenever \( w \in \text{R}(L_0) \cap E^*_n \) and \( n \) is large enough, it is easy
to see that \( \Phi_i \) satisfies \((PS)^*\) in \( N \). Moreover, \( \operatorname{sup}_{x} |\Phi_i| < \infty \) and the mapping \( \lambda \mapsto \lambda \Phi_i \) is continuous uniformly in \( z \in N \). By Corollary 2.9 of [1], \( C^*_\Phi(0, 0) \) is independent of \( \lambda \in [0, 1] \). Therefore

\[
C^*_\Phi(0, 0) = C^*_\Phi(1, 0).
\]

On the other hand, since \( \ker L_0 \) is finite dimensional and \( L_0 \) is A-proper, it is easy to check that the operator \( L_0 \) defined by \( L_0 z = L_0 z - z^0 \) is invertible and A-proper.

Next we turn to the computation of the critical groups \( C^q_{\Phi}(N, 0) \) for almost all \( q \). By Lemma 4.1 of [1], \( E^q = R(L_0) \cap E^q_n \oplus P^q_n \ker(L_0); \) therefore \( z = w + z^0 = \tilde{w} + z^0 \in R(L_0) \cap E^q_n \oplus P^q_n \ker(L_0) \) and \( w - \tilde{w} = z^0 - z^0 \). Since \( P^q_n y \to y \) uniformly for \( y \) on bounded subsets of \( \ker(L_0) \) and \( w - \tilde{w} \in R(L_0) \), it follows that

\[
\sup_{n} \left\{ \|w - \tilde{w}\| : z = w + z^0 = \tilde{w} + z^0 \in E^q_n, \|z\| = 1 \right\} \to 0 \quad \text{as} \quad n \to \infty.
\]

So for \( n \) large, \( M^q_{\Phi}(E^q_n \cap \ker(L_0)) \) is the sum of the Morse indices of the form \( \langle L_0 z, z \rangle = \langle L_0 w, w \rangle - \langle 0, z^0 \rangle, \quad z \in E^q_n, \)

and according to Theorem 4.5 in [1], this form is nondegenerate for almost all \( n \). By Lemma 4.2 of [1], \( E^q = R(L_0) \cap E^q_n \oplus P^q_n \ker(L_0); \) therefore \( z = w + z^0 = \tilde{w} + z^0 \in R(L_0) \cap E^q_n \oplus P^q_n \ker(L_0) \) and \( w - \tilde{w} = z^0 - z^0 \). Since \( P^q_n y \to y \) uniformly for \( y \) on bounded subsets of \( \ker(L_0) \) and \( w - \tilde{w} \in R(L_0) \), it follows that

\[
M^q_{\Phi}(L_0) = M^q_{\Phi}(L_0) + \dim \ker(L_0) = j^q(A_0) + j^q(A_0),
\]

and by Theorem 5.3 of [1],

\[
C^q_{\Phi}(1, 0) = \begin{cases} \mathcal{F} & \text{for } q = j^q(A_0) + j^q(A_0) \text{ and } [0] \text{ otherwise;} \\ \end{cases}
\]

(iii) The proof is analogous with \( \Phi_i(z) := \frac{1}{2} \langle L_0 z + \lambda z^0, z \rangle - \varphi_i(z). \]

Next we turn to the computation of the critical groups \( C^q_{\Phi}(N, K) \).

**Lemma 3.6.** Suppose that \((H_1)\) and one of the conditions \((H_2^+)\) hold and \( K = K_{\Phi} \) is finite. Then

(i) \((H_2^+)\) implies that \( C^q_{\Phi}(N, K) = \begin{cases} \mathcal{F} & \text{for } q = j^q(A) + j^q(A) \text{ and } [0] \text{ otherwise;} \\ \end{cases} \)

(ii) \((H_2^+)\) implies that \( C^q_{\Phi}(N, K) = \begin{cases} \mathcal{F} & \text{for } q = j^q(A) \text{ and } [0] \text{ otherwise.} \\ \end{cases} \)

**Proof.** (i) Let \( E^+_n = (R(L) \cap E^+_n) \oplus \ker(L) = E^+_n \oplus E^-_n \oplus \ker(L) \) be the decomposition corresponding to the positive, the negative, and the zero
part of the operator $L$ on $E_n$. Then there exist $c^* > 0$ and $n_0 > 0$ such that 
\[ \pm \langle Lz^+, z^* \rangle \geq c^* \| z^+ \|^2 \] 
for all $z^* \in E_\infty^n$, $n \geq n_0$. Consider the set 
\[ \mathcal{U}_n := \{ z = z^+ + z^- + z^0 \in E_n^*: \| z^+ \|^2 - \frac{c^*}{8 \| L \|} \| z^- \|^2 - \frac{\lambda H^2_z(\| z^0 \|)}{1 + \| z^0 \|^2} \leq M \}, \]
where $z^* \in E_\infty^n$, $z^0 \in \ker(L)$; the constants $\lambda > 0, M > 0$ will be determined later. An outer normal vector to $\partial \mathcal{U}_n$ (the boundary of $\mathcal{U}_n$) is
\[ n = n(z) = z^+ - d z^- - \frac{\lambda}{2} p'(\| z^0 \|) \frac{z^0}{\| z^0 \|}, \]
where $d = \frac{8 \| L \|}{c^*}$ and $p(t) = H^2_z(t)/(1 + t^2)$. We claim that $\Phi|_{\mathcal{U}_n}$ has no critical point in $E_n \setminus \mathcal{U}_n$. In fact, by Remark 1.1, it is easy to check that
\[ \| \nabla \Phi(z) \| \leq c (1 + \| z^+ \|^s - 1 + \| z^- \|^s - 1 + h_{aw}(\| z^0 \|)) \] 
for $z \in E$.

Therefore, for $\varepsilon$ small enough and $n \geq n_0$,
\[ \langle \nabla \Phi(z), v_n \rangle = \langle Lz^+, z^* \rangle - d \langle Lz^-, z^- \rangle - \langle \nabla \Phi(z), v_n \rangle \]
\[ \geq c^* \| z^+ \|^2 + d c^* \| z^- \|^2 - c_1 (1 + h_{aw}(\| z^0 \|)) + \| z^+ \|^s - 1 + \| z^- \|^s - 1(\| z^+ \|^2 + d \| z^- \|^2 + \lambda \| p'(\| z^0 \|) \|) \]
\[ \geq \frac{1}{2} c^* \| z^+ \|^2 - \frac{d}{2} c^* \| z^- \|^2 - c_1 e \lambda^2 | p'(\| z^0 \|) |^2 - c_1 e^{-1} h_{aw}^2(\| z^0 \|) - c_2. \]

Here we have used the inequalities $xy \leq e^{-1} x^2 + e y^2$ and $xy^{s-1} \leq x^2 + e y^2 + c$ which hold for all $x, y \geq 0, c > 0$ and an appropriate $c = c(e)$. By the definition of $h_{aw}$, we see that
\[ | p'(\| z^0 \|) |^2 \leq \frac{4H^2_z(t)}{(1 + t^2)^2} \left( \frac{z}{t} (1 + t^2) + t \right)^2, \]
\[ h_{aw}^2(t) \leq \frac{4H^2_z(t)}{1 + t^2} + c \]
for $t > 0$. Let $\lambda > 10 c_1/(e c^*)$. Since $H_{aw}(t)/(1 + t^2) \to 0$ as $t \to \infty$, it is easy to verify that
\[ c_1 e \lambda^2 | p'(\| z^0 \|) |^2 + c_1 e^{-1} h_{aw}^2(\| z^0 \|) \leq \frac{\lambda c^* H^2_z(\| z^0 \|)}{2 (1 + \| z^0 \|)^2} + c. \]
Therefore
\[
\langle \nabla \Phi(z), v_n \rangle \geq \frac{c^+}{2} (|z^+| - d |z^-|^2 - \lambda p(|z^0|)) - c
\]
\[
\geq \frac{c^+}{2} M - c
\]
\[
> 0
\]
for an appropriate \( M \). So \( \Phi|_{\mathcal{E}_n} \) has no critical point outside \( \mathcal{U}_n \) and on \( \partial \mathcal{U}_n \).

It is easy to construct a pseudogradient vector field \( V \) on \( \mathcal{E}_n \) such that \( \langle V(z), v_n(z) \rangle > 0 \) on \( \partial \mathcal{U}_n \). This implies that the flow of \(-V\) points into \( \mathcal{U}_n \) on \( \partial \mathcal{U}_n \).

Next we show that on \( \mathcal{U}_n \)
\[
\Phi(z) \to -\infty \quad \text{if and only if} \quad |z^0 + z^-| \to \infty \quad (3.5)
\]
and the convergence is uniform with respect to the choice of \( n \geq n_0 \). Indeed, if \( z \in \mathcal{U}_n \), then \( |z^+|^2 \leq M + d |z^-|^2 + \lambda p(|z^0|) \), and since \( p(t) \leq c(1 + h^2_{\alpha}(t)) \), it follows using the mean value theorem as in the proof of Lemma 3.2 that
\[
\Phi(z) = \frac{1}{2} (\langle Lz^+, z^+ \rangle + \langle Lz^-, z^- \rangle) - \varphi(z)
\]
\[
\leq \frac{1}{2} \|L\| |z^+|^2 - \frac{1}{2} c^* |z^-|^2 - \varphi(z^0) + \varphi(z^0) - \varphi(z)
\]
\[
\leq \frac{1}{2} \|L\| |z^+|^2 - \frac{1}{2} c^* |z^-|^2 - \varphi(z^0)
\]
\[
+ c(1 + h_{\alpha}(|z^0|)) + |z^+|^s - 1 + |z^-|^s - 1 |z^+ + z^-|
\]
\[
\leq \|L\| |z^+|^2 - \frac{1}{4} c^* |z^-|^2 + ch^2_{\alpha}(|z^0|) - \varphi(z^0) + c
\]
\[
\leq (\frac{1}{4} c^* + d \|L\|) |z^-|^2 + \|L\| \lambda p(|z^0|)
\]
\[
+ ch^2_{\alpha}(|z^0|) + \|L\| M - \varphi(z^0) + c
\]
\[
\leq -\frac{c^*}{8} |z^-|^2 + ch^2_{\alpha}(|z^0|) - \varphi(z^0) + c.
\]
In view of the definition of $h_\infty$ and Lemma 3.1, we have that
\[
\lim_{t \to \infty} \frac{h_\infty^2(t)}{H_\infty(t)} \leq c t^{-2} = 0 \quad \text{and} \quad \liminf_{|z| \to \infty} \frac{\varphi(z)}{H_\infty(|z|)} > 0;
\]
consequently,
\[
\lim_{|z| \to \infty} \frac{\varphi(z)}{h_\infty^2(|z|)} = \infty,
\]
and $\Phi(z) \to -\infty$ uniformly in $n$ as $|z^0| \to \infty$.

On the other hand, if $z \in U_n$ and $|z^0 + z^-| \leq c$, then $|z^+| \leq \tilde{c}$ for an appropriate $\tilde{c} > 0$; hence $\Phi(z) \to -\infty$ implies that $|z^0 + z^-| \to \infty$.

Now we adapt an argument of Lemma 7.6 in [1]. Choose $a > 0$ such that $K = K(\Phi) \subset \{ z \in E : |\Phi(z)| < a \}$. By (3.5), there exists $R_2 = R_2(a)$ ($R_2$ independent of $n$) such that
\[
D_2 := \{ z \in U_n : |z^- + z^0| \geq R_2 \} \subset U_n \cap \Phi^{-n}.
\]
Using (3.5) again, we first find $b > a$ with the property that $\Phi^{-b} \cap U_n \subset D_2$, and then $R_1 > R_2$ such that
\[
D_1 := \{ z \in U_n : |z^0 + z^-| \geq R_1 \} \subset \Phi^{-b} \cap U_n.
\]
Define $\xi : [0, 1] \times D_2 \to D_1$ as follows:
\[
\xi(t, z) = \begin{cases} 
    z & \text{if } |z^- + z^0| \geq R_1, \\
    z + \frac{z^- + z^0}{|z^- + z^0|} \{ tR_1 + (1 - t) |z^- + z^0| \} & \text{if } |z^- + z^0| < R_1.
\end{cases}
\]

It is easy to see that $\xi$ is a strong deformation retraction of $D_2$ onto $D_1$ (since $p^* > 0$, $\xi$ does not leave $U_n$). By $(PS)^*$, $K(\Phi|_{E_n}) \subset U_n \backslash \Phi^{-1}(\{ -b, -a \})$ for $n \geq n_0$ (possibly after choosing a larger $n_0$). Therefore, using the flow of $-V$, it is easy to construct a strong deformation retraction $\eta$ of $\Phi^{-n} \cap U_n$ onto $\Phi^{-b} \cap U_n$. Let $\xi * \eta$ denote the deformation $\eta$ followed by $\xi$. Then $\xi * \eta$ is a strong deformation retraction of $\Phi^{-n} \cap U_n$ onto $D_1$.

Applying the flow of $-V$ again, we obtain a strong deformation retraction of $\Phi^{-n} \cap U_n$ onto $\Phi^{-(n+1)} \cap U_n$. Finally, by the above-mentioned properties and the strong excision (cf. Property 1.2 of [1]), we have that for $n \geq n_0$,
\[ H^q(\Phi^a \cap E^a_n, \Phi^{-a} \cap E^{-a}_n) \cong H^q((\Phi^{-a} \cap E^a_n) \cup \varnothing_n, \Phi^{-a} \cap E^{-a}_n) \]
\[ \cong H^q(\varnothing_n, \Phi^{-a} \cap \varnothing_n) \quad \text{(excision)} \]
\[ \cong H^q(\varnothing_n, D_1) \]
\[ \cong \begin{cases} \mathcal{F} & \text{if } q = j^-(A) + j^0(A) + d_n, \\ 0 & \text{otherwise.} \end{cases} \]

Since the excision property implies that
\[ H^q_e(\Phi^a, \Phi^{-a}) \cong H^q_e(\Phi^{-1}([-a, a]), \Phi^{-1}(-a)) \]
and \((\Phi^{-1}([-a, a]), \Phi^{-1}(-a))\) is an admissible pair for \(\Phi\) and \(K\) (cf. Proposition 2.5 of [1]), the conclusion of case (i) follows from the definition of \(C^*_e(\Phi, K(\Phi))\).

(ii) Set
\[ V_n := \left\{ \begin{array}{l} z \in E_n : \|z^-\|^2 - \frac{c^*}{8\|L\|} \|z^+\|^2 - \frac{jH^2(\|z^0\|)}{1 + \|z^0\|^2} \leq M, \end{array} \right\}. \]

Then an outer normal vector to \(\partial V_n\) is
\[ v_n = v_n(z) = z^- - \frac{c^*}{8\|L\|} z^+ - \frac{\lambda}{2} p'(\|z^0\|) \frac{z^0}{\|z^0\|}, \quad \text{where } p(t) = \frac{H^2_n(\lambda t)}{1 + t^2}. \]

By an argument similar to that in case (i), there exist \(\lambda\) and \(M\) such that
\[ \langle \nabla \Phi(z), v_n \rangle \leq -\frac{c^*}{2} \left( \|z^-\|^2 - \frac{c^*}{8\|L\|} \|z^+\|^2 - \frac{jH^2(\|z^0\|)}{1 + \|z^0\|^2} \right) + c \]
\[ \leq -\frac{c^*}{2} M + c \]
\[ < 0, \]
where \(c\) is independent of \(n \geq n_0\). It follows that \(\Phi|_{\varnothing_n}\) has no critical point in \(E_n \setminus V_n\) and there exists a pseudogradient vector field \(V\) such that the flow of \(-V\) points outwards on \(\partial V_n\). Furthermore,
\[ \|z^-\|^2 \leq \frac{c^*}{8\|L\|} \|z^+\|^2 + \frac{jH^2(\|z^0\|)}{1 + \|z^0\|^2} + M \quad \text{for } z \in \varnothing_n; \]
consequently, \( \Phi(z) = \frac{1}{2} \langle Lz^+, z^+ \rangle + \frac{1}{2} \langle Lz^-, z^- \rangle - \varphi(z) \)
\[ \geq \frac{1}{2} \epsilon^* \|z^+\|^2 - \frac{1}{2} L \|z^+\|^2 - \varphi(z^0) \]
\[ - c(1 + h_n(\|z^0\|) + \|z^+\|^s - 1 + \|z^-\|^s - 1) \|z^+ + z^-\| \]
\[ \geq \frac{c^*}{8} \|z^+\|^2 - ch_n^2(\|z^0\|) - \varphi(z^0) - c. \]

Since by Lemma 3.1,
\[ \lim_{\|z^0\| \to \infty} \frac{- \varphi(z^0)}{h_n^2(\|z^0\|)} = \infty, \]
it follows that \( \Phi(z) \to \infty \) uniformly in \( n \) as \( \|z^+ + z^0\| \to \infty \). As in case (i) we also see that the reverse implication is true.

It follows that we can find \( a > 0 \) such that \( K = K(\Phi) \subset \{ z \in E : |\Phi(z)| < a \} \) and \( \Phi^{-a} \cap E_n^+ \subset E_n^+ \setminus \mathcal{V}_n^0 \). Since \( \Phi^a \cap \mathcal{V}_n \) is a bounded set, we find \( R_0 > 0 \) such that
\[ \Phi^a \cap \mathcal{V}_n \subset D := \{ z \in \mathcal{V}_n : \|z^+ + z^0\| \leq R_0 \}. \]

Since \( D \) is also bounded, there exists \( b > a \) such that \( D \subset \Phi^b \cap \mathcal{V}_n \). Similar to the proof of Lemma 7.6 in [1], we find a strong deformation retraction \( \xi \) of \( E_n^+ \) onto \( D \cap \mathcal{V}_n \) (we can e.g. use the flow of \( -v_n \) to deform \( E_n^+ \) onto \( \mathcal{V}_n \) and that of \( v_n \) to deform \( \mathcal{V}_n \) onto \( D \cup \partial \mathcal{V}_n \)). By \( (PS)^* \), we may assume that \( K(\Phi|_{E_n^+}) \subset \mathcal{V}_n \setminus \Phi^{-1} [a, b] \) for \( n \geq n_0 \), so the flow of \( -V \) provides a strong deformation retraction of \( E_n^+ \setminus \mathcal{V}_n \) onto \( \Phi^{-a} \cap E_n^+ \). Moreover, the flow of \( -V \) induces a strong deformation retraction \( \eta \) of \( (E_n^+ \setminus \mathcal{V}_n) \cup D \) onto \( \Phi^a \cap E_n^+ \). Now it is easy to see that the mapping \( \eta \ast \xi \) is a strong deformation retraction of \( E_n^+ \) onto \( \Phi^a \cap E_n^+ \). Therefore
\[ H^q(\Phi^a \cap E_n^+, \Phi^{-a} \cap E_n^+) \cong H^q(E_n^+, \Phi^{-a} \cap E_n^+) \]
\[ \cong H^q(E_n^+, E_n^+ \setminus \mathcal{V}_n) \]
\[ \cong \begin{cases} \mathcal{F} & \text{if } q = j^*(A) + d_n, \\ 0 & \text{otherwise.} \end{cases} \]

Now by the same argument as in case (i) we get the conclusion. \( \blacksquare \)
Remark 3.1. For the computation of the usual relative homology groups, see [12, 13, 15]. We emphasize that the results of [12, 13, 15] cannot be used directly to deal with strongly indefinite functionals.

4. PROOFS OF THE MAIN RESULTS

Based on the computations of the critical groups $C^*_q(\Phi, 0)$ and $C^*_q(\Phi, K)$, we can prove the main results of Section 1.

Proof of Theorem 1.1. (i) By Lemma 3.6, $(H^-_2)$ implies that $C^*_q(\Phi, K) = \begin{bmatrix} \mathcal{F} \\ 0 \end{bmatrix}$ for $q = j^-(A)$ and $[0]$ otherwise. On the other hand, if 0 is the only critical point of $\Phi$, then $C^*_q(\Phi, K) = C^*_q(\Phi, 0)$. It follows from the shifting theorem (cf. Theorem 5.4 of [1]) that $C^*_q(\Phi, 0) = \begin{bmatrix} C^{q - j^{-}(A_0)}(\tilde{\Phi}_0, 0) \end{bmatrix}$, where $\tilde{\Phi}_0$ is defined on a subset of $\ker(L_0)$. Since $\dim \ker(L_0) = j^-(A_0)$, $C^*_q(\Phi, 0) = [0]$ whenever $q \notin \{ j^{-}(A_0), j^{-}(A_0) + j^0(A_0) \}$. So by our assumption, $C^*_q(\Phi, 0) = \{ 0 \} \neq C^*_q(\Phi, K)$, a contradiction.

(ii) Since $j^-(A) + j^0(A) + j^+(A) = 0$, the conclusion follows from Lemma 3.6(i) and a similar argument.

Proof of Theorem 1.2. It follows from Lemmas 3.5 and 3.6 that $C^*_q(\Phi, 0) \neq C^*_q(\Phi, K)$ for some $q$; hence $K \neq \{ 0 \}$.

Proof of Theorem 1.3. We only prove the case (i) as an example. The other cases are similar. Since

$$(H^-_2)\text{ implies that } C^*_q(\Phi, K) = \begin{bmatrix} \mathcal{F} \\ 0 \end{bmatrix} \quad \text{for } q = j^-(A) + j^0(A),$$

and

$$(H^-_4)\text{ implies that } C^*_q(\Phi, 0) = \begin{bmatrix} \mathcal{F} \\ 0 \end{bmatrix} \quad \text{for } q = j^{-}(A_0) + j^0(A_0),$$

there exists a nonzero critical point $z_0$. Suppose there are no other ones, then by Theorem 5.4 of [1], $C^*_q(\Phi, z_0) = \begin{bmatrix} C^{q - q}(\tilde{\Phi}_0, 0) \end{bmatrix}$ for some $r_0 \in \mathbb{Z}$ and some functional $\tilde{\Phi}_0$ defined on a space $Z$ with $\dim Z \leq 2N$. In this case the Morse inequalities read

$$tf^-(A_0) + j^0(A_0) + \sum_{i=0}^{2N-2} b_i t^{r_i + i} = tf^-(A) + j^0(A) + (1 + t) Q(t),$$
where $b_i \in [Z]$ and $x \in Z$. That the sum on the left-hand side above contains at most $2N - 1$ nonzero terms follows from the fact that if $C^0(\tilde{\phi}_0, 0) \neq 0$, then $\tilde{\phi}_0$ has a local minimum at 0 and $C^0(\tilde{\phi}_0, 0) = 0$ for $p \neq 0$, and if $C^{2N}(\tilde{\phi}_0, 0) \neq 0$, then $\tilde{\phi}_0$ has a local minimum there and $C^p(\tilde{\phi}_0, 0) = 0$ for $p \neq 2N$. By comparing the exponents, we can find $i$ and $j$ such that $\alpha + i = j^\ast(A) + j^\ast(A_0)$ and $\alpha + j = j^\ast(A_0) + j^\ast(A_0) \pm 1$, where $i, j \in \{0, 1, \ldots, 2N - 2\}$. So $|j^\ast(A) - j^\ast(A_0)| = |j^\ast(A) + j^\ast(A_0) - j^\ast(A_0) - j^\ast(A_0)| = |i - j \pm 1| \leq 2N - 1$, a contradiction.

Proof of Corollary 1.1. We only prove case (ii). Since $A = A_0 \equiv 0$, $j^\ast(0) = -N$ and $j^0(0) = 2N$ (cf. Proposition 7.1 of [1]). Consequently, by Lemma 3.6, $C^q_\ast(\Phi, K) = [\mathcal{F}]$ if $q = N$ and $[0]$ otherwise. On the other hand, by Corollary 5.5 of [1] and Lemma 3.5, $C^q_\ast(\Phi, 0) = [\mathcal{F}]$ if $q = -N$ and $[0]$ otherwise. If $\Phi$ has only one nontrivial critical point, then by the Morse inequalities,

$$t^{-N} + \sum_{i=0}^{2N-2} b_i t^{\ast+i} = t^N + (1 + t) \mathcal{Q}(t),$$

and similar to the proof of Theorem 1.3, we get a contradiction.

Note added in proof. After submission of this paper, a related work by Abbondandolo [16] appeared. In [16] a different infinite dimensional Morse theory (which goes back to [17]) was introduced and it was shown that (S) has a nontrivial periodic solution under certain conditions of asymptotic linearity. The asymptotic conditions in [16] and here are rather different, and when they coincide, our conclusions are stronger.

REFERENCES

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