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Classification of irreducible weight modules over higher rank Virasoro algebras ☆

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Abstract

Let G be a rank n additive subgroup of $\mathbb{C}$ and $\mathrm{Vir}[G]$ the corresponding Virasoro algebra of rank n. In the present paper, irreducible weight modules with finite dimensional weight spaces over $\mathrm{Vir}[G]$ are completely determined. There are two different classes of them. One class consists of simple modules of intermediate series whose weight spaces are all 1-dimensional. The other is constructed by using intermediate series modules over a Virasoro subalgebra of rank $n-1$. The classification of such modules over the classical Virasoro algebra was obtained by O. Mathieu in 1992 using a completely different approach.
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1. Introduction

Let $\mathbb{C}$ be the field of complex numbers. The *Virasoro algebra* $\mathrm{Vir} := \mathrm{Vir}[\mathbb{Z}]$ (over $\mathbb{C}$) is the Lie algebra with the basis $\{c, d_i \mid i \in \mathbb{Z}\}$ and the Lie bracket defined by

$$[c, d_i] = 0,$$
$$[d_i, d_j] = (j - i)d_{i+j} + \delta_{i,-j}\frac{i^3 - i}{12}c, \quad \forall i, j \in \mathbb{Z}.$$

The structure theory of the Virasoro algebra weight modules with finite dimensional weight spaces is fairly well developed. For details, we refer the readers to [6], the book [4] and the references therein.

The centerless Virasoro algebra is actually a Witt algebra, and generalized Witt algebras in positive characteristic and characteristic 0 were studied by many authors, for instance, Zassenhaus [19], Kaplansky [5], Ree [12], Wilson [17], Strade [13]; and Osborn [9], Djokovic and Zhao [2], Passman [10], Xu [18].

Patera and Zassenhaus [11] introduced the *generalized Virasoro algebra* $\mathrm{Vir}[G]$ for any additive subgroup G of $\mathbb{C}$. This Lie algebra can be obtained from Vir by replacing the index group $\mathbb{Z}$ with G (see Definition 2.1). If $G \simeq \mathbb{Z}^n$, then $\mathrm{Vir}[G]$ is called a *rank n Virasoro algebra* (or a *higher rank Virasoro algebra* if $n \geqslant 2$).

Representations for generalized Virasoro algebras $\mathrm{Vir}[G]$ have been studied by several authors. Mazorchuk [8] proved that all irreducible weight modules with finite dimensional weight spaces over $\mathrm{Vir}[\mathbb{Q}]$ are intermediate series modules (where $\mathbb{Q}$ is the field of rational numbers). In [7], Mazorchuk determined the irreducibility of Verma modules with zero central charge over higher rank Virasoro algebras. In [3], Hu, Wang and Zhao obtained a criterion for the irreducibility of Verma modules over the generalized Virasoro algebra $\mathrm{Vir}[G]$ over an arbitrary field F of characteristic 0 (G is an additive subgroup of F). Su and Zhao [16] proved that weight modules with all weight spaces 1-dimensional are some so-called intermediate series of modules. In [14,15], Su proved that the irreducible weight modules over higher rank Virasoro algebras are divided into two classes: intermediate series modules, and GHW modules. In [1], Billig and Zhao constructed a new class of irreducible weight modules with finite dimensional weight spaces over some generalized Virasoro algebras.

The aim of this paper is to complete the classification of irreducible weight modules with finite dimensional weight spaces over higher rank Virasoro algebras $\mathrm{Vir}[G]$. The result for $n = 1$ was obtained by Mathieu [6] by using a completely different method.

This paper is arranged as follows.

In Section 2, we collect some known results. For any total order "$\succ$" on G, which is compatible with the group addition, and for any $\dot{c}, h \in \mathbb{C}$, we recall the definition of the Verma module $M(\dot{c}, h, \succ)$ over $\mathrm{Vir}[G]$ and some known facts about such modules (see [3]). We recall from [1] the construction of a class of irreducible weight modules with finite dimensional weight spaces over some generalized (including higher rank) Virasoro algebras. These modules are denoted by $V(\alpha, \beta, b, G_0)$ (see (2.5) for definition) for some $\alpha, \beta \in \mathbb{C}$, $b \in G \setminus \{0\}$, and a subgroup G_0 of G with $G = \mathbb{Z}b \oplus G_0$. We also recall in Theorem 2.5 a useful result from [14].

In Section 3, we give a classification of irreducible weight modules with finite dimensional weight spaces over Vir$[G]$ for $G \simeq \mathbb{Z}^n$, i.e., any such module is either an intermediate series module $V'(\alpha, \beta, G)$ or $V(\alpha, \beta, b, G_0)$ for suitable parameters (Theorem 3.9). We show that all GHW modules (see the definition preceding Theorem 2.5) over Vir$[G]$ are isomorphic to modules $V(\alpha, \beta, b, G_0)$. The main technique we employ in this paper is to thoroughly study the weight set supp(V) (sometimes also called the support) of nontrivial irreducible weight modules V with finite dimensional weight spaces. We first spend a lot of effort to handle the case $n = 2$ (Lemma 3.3–Theorem 3.7), and then use induction on n to deal with all other cases. The induction turns out to be rather difficult.

We hope that our results will have some applications in physics since the Lie algebras studied in the present paper have similar properties as the classical Virasoro algebra which is widely used in physics.

2. Weight modules over generalized Virasoro algebras

In this section we recall the construction of various modules and collect some known results for later use.

Definition 2.1. Let G be a nonzero additive subgroup of $\mathbb{C}$. The *generalized Virasoro algebra* Vir$[G]$ (over $\mathbb{C}$) is the Lie algebra with the basis $\{c, d_x \mid x \in G\}$ and the Lie bracket defined by

$$[c, d_x] = 0,$$
$$[d_x, d_y] = (x - y)d_{x+y} + \delta_{x,-y}\frac{x^3 - x}{12}c, \quad \forall x, y \in G.$$

It is clear that $\mathrm{Vir}[G] \simeq \mathrm{Vir}[aG]$ for any $a \in \mathbb{C}^*$. For any $x \in G^* := G \setminus \{0\}$, Vir$[x\mathbb{Z}]$ is a Lie subalgebra of Vir$[G]$ isomorphic to Vir.

Fix a total order "$\succ$" on G which is compatible with the addition, i.e., $x \succ y$ implies $x + z \succ y + z$ for any $z \in G$. Let

$$G_+ := \{x \in G \mid x \succ 0\}, \qquad G_- := \{x \in G \mid x \prec 0\}.$$

Then $G = G_+ \cup \{0\} \cup G_-$ and we have the triangular decomposition

$$\mathrm{Vir}[G] = \mathrm{Vir}[G]_+ \oplus \mathrm{Vir}[G]_- \oplus \mathrm{Vir}[G]_0,$$

where $\mathrm{Vir}[G]_+ = \bigoplus_{x\in G_+} \mathbb{C}d_x$, $\mathrm{Vir}[G]_- = \bigoplus_{x\in G_-} \mathbb{C}d_x$, $\mathrm{Vir}[G]_0 = \mathbb{C}d_0 + \mathbb{C}c$.

It is clear that either

$$\#\{y \in G \mid 0 \prec y \prec x\} = \infty, \quad \forall x \in G_+, \tag{2.1}$$

or

$$\exists a \in G_+, \quad \#\{y \in G \mid 0 \prec y \prec a\} = 0. \tag{2.2}$$

We say that the order is *dense* respectively *discrete* if (2.1) respectively (2.2) holds.

A Vir[G]-module V is called *trivial* if $\mathrm{Vir}[G]V = 0$. For any Vir[G]-module V and $\dot{c}, \lambda \in \mathbb{C}$, let $V_{\dot{c},\lambda} := \{v \in V \mid d_0 v = \lambda v, cv = \dot{c}v\}$ denote the *weight space* of V corresponding to a weight $(\dot{c}, \lambda)$. When c acts as the scalar $\dot{c}$ on the whole space V, we shall simply write V_λ instead of $V_{\dot{c},\lambda}$.

A Vir[G]-module V is called a *weight module* if V is the sum of its weight spaces. For a weight module V we define $\mathrm{supp}(V) := \{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\}$, which is generally called the *weight set* (or the *support*) of V.

For any Lie algebra L, we shall use $U(L)$ to denote its universal enveloping algebra. For any $\dot{c}, h \in \mathbb{C}$, let $I(\dot{c}, h, \succ)$ be the left ideal of $U := U(\mathrm{Vir}[G])$ generated by the elements

$$\{d_i \mid i \in G_+\} \cup \{d_0 - h \cdot 1, c - \dot{c} \cdot 1\}.$$

Then the *Verma module* with the highest weight $(\dot{c}, h)$ for Vir[G] is defined as

$$M(\dot{c}, h, \succ) := U/I(\dot{c}, h, \succ).$$

This module has a basis consisting of the following vectors

$$d_{-i_1} d_{-i_2} \cdots d_{-i_k} v_h, \quad k \in \mathbb{N} \cup \{0\}, \quad i_j \in G_+, \quad \forall j \text{ and } i_k \geqslant \cdots \geqslant i_2 \geqslant i_1 > 0,$$

where $v_h = 1 + I(\dot{c}, h, \succ)$ is the highest weight vector. Let $V(\dot{c}, h, \succ)$ be the unique irreducible quotient of $M(\dot{c}, h, \succ)$. Let us recall

Theorem 2.2. [3, Theorem 3.1] *Let* $\dot{c}, h \in \mathbb{C}$.

(1) *Assume that the order "$\succ$" is dense. Then the Verma module $M(\dot{c}, h, \succ)$ is an irreducible* Vir[G]*-module if and only if* $(\dot{c}, h) \neq (0, 0)$. *Moreover,*

$$M'(0, 0, \succ) := \sum_{i_1, \ldots, i_k \in G_+,\, k>0} \mathbb{C} d_{-i_1} \cdots d_{-i_k} v_0$$

is an irreducible submodule of $M(0, 0, \succ)$.

(2) *Assume that the order "$\succ$" is discrete. Then the Verma module $M(\dot{c}, h, \succ)$ is an irreducible* Vir[G]*-module if and only if for the minimal positive element $a \in G$ with respect to "$\succ$", the* Vir[$a\mathbb{Z}$]*-module $M_a(\dot{c}, h, \succ) = U(\mathrm{Vir}[\mathbb{Z}a])v_h$ is irreducible.*

Now we give another class of Vir[G]-modules $V(\alpha, \beta, G)$ for any $\alpha, \beta \in \mathbb{C}$ (see [16]). These Vir[G]-modules all have basis $\{v_x \mid x \in G\}$ with actions given by the following formula

$$cv_y = 0, \qquad d_x v_y = (\alpha + y + x\beta) v_{x+y}, \quad \forall x, y \in G.$$

One knows from [16] that $V(\alpha, \beta, G)$ is reducible if and only if $\alpha \in G$ and $\beta \in \{0, 1\}$. By $V'(\alpha, \beta, G)$ we denote the unique nontrivial irreducible sub-quotient of $V(\alpha, \beta, G)$. Then $\mathrm{supp}(V'(\alpha, \beta, G)) = \alpha + G$ or $\mathrm{supp}(V'(\alpha, \beta, G)) = G \setminus \{0\}$. We now recall

Theorem 2.3. [16, Theorem 4.6] *Let V be a nontrivial irreducible weight module over* $\mathrm{Vir}[G]$ *with all weight spaces* 1*-dimensional. Then $V \simeq V'(a,b,G)$ for some $a,b \in \mathbb{C}$.*

Now we assume that $G = \mathbb{Z}b \oplus G_0 \subset \mathbb{C}$ where $0 \neq b \in \mathbb{C}$ and G_0 is a nonzero subgroup of $\mathbb{C}$. (Note that some G lack this property.) We temporarily set $L = \mathrm{Vir}[G]$. For any $i \in \mathbb{Z}$, we set

$$L_{ib} = \bigoplus_{a \in G_0} \mathbb{C} d_{ib+a},$$

$$L_+ = \bigoplus_{i>0} L_{ib}, \qquad L_- = \bigoplus_{i<0} L_{ib}, \qquad L_0 \simeq \mathrm{Vir}[G_0].$$

For any $\alpha, \beta \in \mathbb{C}$, we have the irreducible L_0-module $V'(\alpha,\beta,G_0)$. We extend the L_0-module structure on $V'(\alpha,\beta,G_0)$ to an $(L_+ + L_0)$-module structure by defining $L_+ V'(\alpha,\beta,G_0) = 0$. Then we obtain the induced L-module

$$\begin{aligned}\bar{M}\big(b, G_0, V'(\alpha,\beta,G_0)\big) &= \mathrm{Ind}_{L_+ + L_0}^{L} V'(\alpha,\beta,G_0) \\ &= U(L) \otimes_{U(L_+ + L_0)} V'(\alpha,\beta,G_0).\end{aligned} \tag{2.3}$$

As vector spaces, $\bar{M}(b, G_0, V'(\alpha,\beta,G_0)) \simeq U(L_-) \otimes_{\mathbb{C}} V'(\alpha,\beta,G_0)$. The L-module $\bar{M}(b, G_0, V'(\alpha,\beta,G_0))$ has a unique maximal proper submodule J. Then we obtain the irreducible quotient module

$$M\big(b, G_0, V'(\alpha,\beta,G_0)\big) = \bar{M}\big(b, G_0, V'(\alpha,\beta,G_0)\big)/J. \tag{2.4}$$

It is clear that this module is uniquely determined by α, β, b and G_0; and that

$$\mathrm{supp}\big(M\big(b, G_0, V'(\alpha,\beta,G_0)\big)\big) = \mathbb{Z}^+ b + G_0 \quad \text{or} \quad \big(\mathbb{Z}^+ b + G_0\big) \setminus \{0\}. \tag{2.5}$$

Note that b can be replaced by any element in $b + G_0$.

To simplify notation, set

$$V = V(\alpha,\beta,b,G_0) = M\big(b, G_0, V'(\alpha,\beta,G_0)\big). \tag{2.6}$$

It is clear that $V = \bigoplus_{i \in \mathbb{Z}_+} V_{-ib+\alpha+G_0}$, where

$$V_{-ib+\alpha+G_0} = \bigoplus_{a \in G_0} V_{-ib+\alpha+a}, \qquad V_{-ib+\alpha+a} = \big\{v \in V \mid d_0 v = (-ib+\alpha+a)v\big\}.$$

Now we recall

Theorem 2.4. [1, Theorem 3.1] *All weight spaces of the* $\mathrm{Vir}[G]$*-module $V(\alpha,\beta,b,G_0)$, defined above, are finite dimensional. More precisely,* $\dim V_{-ib+\alpha+a} \leqslant (2i+1)!!$ *for all $i \in \mathbb{N}$, $a \in G_0$.*

From now on in this paper we assume that $G \simeq \mathbb{Z}^n$ for some integer $n > 1$, $V = \bigoplus_{x\in G} V_{a+x}$ is an irreducible weight module over $\mathrm{Vir}[G]$ with finite dimensional weight spaces (i.e., $\dim V_{a+x} < \infty$ for all $x \in \mathbb{C}$) where $a \in \mathbb{C}$. If there exists $N \in \mathbb{N}$ such that $\dim V_{a+x} < N$ for all $x \in \mathbb{C}$, we say that V is *uniformly bounded*. If there exists a $\mathbb{Z}$-basis $B = \{b_1, \dots, b_n\}$ of G and $v_{\Lambda_0} \in V_{\Lambda_0}$ such that

$$d_x v_{\Lambda_0} = 0, \quad \forall 0 \neq x \in \mathbb{Z}^+ b_1 + \cdots + \mathbb{Z}^+ b_n,$$

we say that V is a *generalized highest weight module* (GHW module for short) with GHW Λ_0 w.r.t. B (see [14]). The vector v_{Λ_0} is called a GHW vector with respect to B, or simply a GHW vector. Finally we recall

Theorem 2.5. [14, Theorem 1.2] *Suppose that $G \simeq \mathbb{Z}^n$, $n > 1$ and V is a nontrivial irreducible weight* $\mathrm{Vir}[G]$*-module with finite dimensional weight spaces.*

(a) *If V is uniformly bounded, then $V \simeq V'(\alpha, \beta, G)$ for suitable $\alpha, \beta \in \mathbb{C}$.*
(b) *If V is not uniformly bounded, then V is a GHW module.*

3. Classification of weight modules

In this section we give a classification of all irreducible weight modules with finite dimensional weight spaces over higher rank Virasoro algebras. More precisely, we prove that any such module is either $V'(\alpha, \beta, G)$ or $V(\alpha, \beta, b, G_0)$ (Theorem 3.9). To this end, by Theorem 2.5, we need only study GHW modules.

Recall that G is an additive subgroup of $\mathbb{C}$ with $G \simeq \mathbb{Z}^n$ and $n > 1$, and that $V = \bigoplus_{x\in G} V_{a+x}$ is an irreducible weight module over $\mathrm{Vir}[G]$ with finite dimensional weight spaces.

By "$\succ$" we denote the lexicographic order on $\mathbb{Z}^n$, i.e., $(x_1, \dots, x_n) \succ (y_1, \dots, y_n)$ if and only if there exists s: $1 \leqslant s \leqslant n$ such that $x_i = y_i$ for $1 \leqslant i \leqslant s-1$ and $x_s > y_s$.

We write $(x_1, \dots, x_n) > (y_1, \dots, y_n)$ if $x_i > y_i$ for $1 \leqslant i \leqslant n$; and $(x_1, \dots, x_n) \geqslant (y_1, \dots, y_n)$ if $x_i \geqslant y_i$ for $1 \leqslant i \leqslant n$.

In this section, the letters $i, j, k, l, m, n, p, q, r, s, t, x, y$ denote integers. For convenience, we set $[p, q] = \{x \mid x \in \mathbb{Z},\ p \leqslant x \leqslant q\}$ and define similarly the infinite intervals $(-\infty, p]$, $[q, \infty)$ and $(-\infty, +\infty)$. For $a \in G$ or $S \subset G$, we denote by $\mathrm{Vir}[a]$ or $\mathrm{Vir}[S]$ the subalgebra of $\mathrm{Vir}[G]$ generated by $\{d_{\pm a}, d_{\pm 2a}\}$ or $\{d_{\pm a}, d_{\pm 2a} \mid a \in S\}$, respectively.

Lemma 3.1. *Suppose that $B = (b_1, b_2, \dots, b_n)$ is a $\mathbb{Z}$-basis of G and $n \geqslant 2$. Let V be a nontrivial irreducible GHW* $\mathrm{Vir}[G]$*-module with GHW Λ_0 w.r.t. B.*

(a) *For any $v \in V$, there exists $p > 0$ such that $d_{i_1 b_1 + i_2 b_2 + \cdots + i_n b_n} v = 0$ for all $(i_1, i_2, \dots, i_n) \geqslant (p, p, \dots, p)$.*
(b) *If $\Lambda_0 + i_1 b_1 + i_2 b_2 + \cdots + i_n b_n \in \mathrm{supp}(V)$, then for any positive integers $k_1, k_2, \dots, k_n$, there exists $m \geqslant 0$ such that $\{x \in \mathbb{Z} \mid \Lambda_0 + i_1 b_1 + i_2 b_2 + \cdots + i_n b_n + x(k_1 b_1 + k_2 b_2 + \cdots + k_n b_n) \in \mathrm{supp}(V)\} = (-\infty, m]$.*

(c) *Let S be any subgroup of G of rank n, then any nonzero* $\mathrm{Vir}[S]$*-submodule of V is nontrivial.*

(d) *There exists a $\mathbb{Z}$-basis $B' = \{b'_1, b'_2, \ldots, b'_n\}$ of G such that*

(d1) *V is a GHW module with GHW Λ_0 w.r.t. B'*;

(d2) $(\Lambda_0 + \mathbb{Z}^+ b'_1 + \mathbb{Z}^+ b'_2 + \cdots + \mathbb{Z}^+ b'_n) \cap \mathrm{supp}(V) = \{\Lambda_0\}$;

(d3) $(\Lambda_0 - \mathbb{Z}^+ b'_1 - \mathbb{Z}^+ b'_2 - \cdots - \mathbb{Z}^+ b'_n) \cap \mathrm{supp}(V) = \Lambda_0 - \mathbb{Z}^+ b'_1 - \mathbb{Z}^+ b'_2 - \cdots - \mathbb{Z}^+ b'_n$;

(d4) $\Lambda_0 + k_1 b'_1 + k_2 b'_2 + \cdots + k_n b'_n \notin \mathrm{supp}(V),\ \forall (k_1, k_2, \ldots, k_n) \geqslant (i_1, i_2, \ldots, i_n)$ *if* $\Lambda_0 + i_1 b'_1 + i_2 b'_2 + \cdots + i_n b'_n \notin \mathrm{supp}(V)$;

(d5) $\Lambda_0 + k_1 b'_1 + k_2 b'_2 + \cdots + k_n b'_n \in \mathrm{supp}(V),\ \forall (k_1, k_2, \ldots, k_n) \leqslant (i_1, i_2, \ldots, i_n)$ *if* $\Lambda_0 + i_1 b'_1 + i_2 b'_2 + \cdots + i_n b'_n \in \mathrm{supp}(V)$;

(d6) *For any* $0 \neq (k_1, k_2, \ldots, k_n) \geqslant 0$ *and* $(i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$, *we have* $\{x \in \mathbb{Z} \mid \Lambda_0 + \sum_{l=1}^n i_l b'_l + x(\sum_{l=1}^n k_l b'_l) \in \mathrm{supp}(V)\} = (-\infty, m]$ *for some* $m \in \mathbb{Z}$.

Proof. For $n = 2$ a slightly weaker form of this lemma is a combination of several lemmas in [15].

(a) Without loss of generality, we may assume that $v = u v_{\Lambda_0}$, where

$$u = d_{i_1^{(1)} b_1 + i_2^{(1)} b_2 + \cdots + i_n^{(1)} b_n} d_{i_1^{(2)} b_1 + i_2^{(2)} b_2 + \cdots + i_n^{(2)} b_n} \cdots d_{i_1^{(m)} b_1 + i_2^{(m)} b_2 + \cdots + i_n^{(m)} b_n} \in U\big(\mathrm{Vir}[G]\big).$$

Take $p = \max\{-\sum_{i_1^{(s)} < 0} i_1^{(s)}, -\sum_{i_2^{(s)} < 0} i_2^{(s)}, \ldots, -\sum_{i_m^{(s)} < 0} i_m^{(s)}\} + 1$. By induction on m, and using the Lie bracket in $\mathrm{Vir}[G]$, we easily obtain

$$d_{i_1 b_1 + i_2 b_2 + \cdots + i_n b_n} v = 0, \quad \forall (i_1, i_2, \ldots, i_n) \geqslant (p, p, \ldots, p).$$

(b) Let $J = \{x \in \mathbb{Z} \mid \Lambda_0 + \sum_{l=1}^n i_l b_l + x(\sum_{l=1}^n k_l b_l) \in \mathrm{supp}(V)\}$.

Claim 1. *For any nonzero $v \in V$, we have $d_{-(k_1 b_1 + k_2 b_2 + \cdots + k_n b_n)} v \neq 0$.*

Proof. Suppose that $d_{-(k_1 b_1 + k_2 b_2 + \cdots + k_n b_n)} v = 0$ for some nonzero $v \in V$. Let p be as in (a). Then $d_{-(k_1 b_1 + k_2 b_2 + \cdots + k_n b_n)}$ and $d_{b_i + p(k_1 b_1 + k_2 b_2 + \cdots + k_n b_n)}$ for $i \in [1, n]$ act trivially on v. Since $\mathrm{Vir}[G]$ is generated by these elements, we see that $\mathrm{Vir}[G]v = 0$, contradicting the fact that V is a nontrivial irreducible module. Claim 1 follows. □

It follows from this claim that $J = (-\infty, m]$ for some $m \geqslant 0$ or $J = \mathbb{Z}$.

Suppose that $J = \mathbb{Z}$. For any $x \in \mathbb{Z}$, let

$$\lambda_x = \Lambda_0 + i_1 b_1 + i_2 b_2 + \cdots + i_n b_n + x(k_1 b_1 + k_2 b_2 + \cdots + k_n b_n).$$

We know that $\mathrm{Vir}^{[k]} := \mathrm{Vir}[k_1 b_1 + k_2 b_2 + \cdots + k_n b_n]$ is a rank one Virasoro subalgebra, and $W = \bigoplus_{x \in \mathbb{Z}} V_{\lambda_x}$ is a $\mathrm{Vir}^{[k]}$-module. From (a) and a well-known result in [6, Lemma 1.6] for any $x \in \mathbb{Z}$ there exists $y \geqslant x$ such that V_{λ_y} contains a $\mathrm{Vir}^{[k]}$ primitive vector (a nonzero weight vector v such that $d_{l(k_1b_1+k_2b_2+\cdots+k_nb_n)}v=0$ for all $l\in\mathbb{N}$). So there are infinitely many nontrivial highest weight $\mathrm{Vir}^{[k]}$-modules having the same weight λ_0, which implies $\dim V_{\lambda_0}=\infty$. This contradiction yields that $J\neq\mathbb{Z}$. Hence, (b) is proved.

(c) For any p, let $I_1=(p+1,p,\dots,p)$, $I_2=(p+2,p+1,p,p,\dots,p)$, $I_k=I_1+(0,0,\delta_{3,k},\dots,\delta_{n,k})\in\mathbb{Z}^n$, $k=3,\dots,n$. Let

$$A=\begin{pmatrix} I_1\\ I_2\\ I_3\\ \vdots\\ I_n\end{pmatrix}. \tag{3.1}$$

Then $\det(A)=1$. Suppose that there exists a rank n subgroup S of G and a nonzero $v_0\in V$ such that $\mathrm{Vir}[S]v_0=0$. Now take p as in (a), that is, $d_{i_1b_1+i_2b_2+\cdots+i_nb_n}v_0=0$ for all $(i_1,i_2,\dots,i_n)\geqslant(p,p,\dots,p)$. Let $(b_1^*,b_2^*,\dots,b_n^*)=(b_1,b_2,\dots,b_n)A$. Then $d_{b_i^*}v_0=0$ for all $i=1,2,\dots,n$. Since G/S is a finite group, there exists some $i>0$, such that $-i(b_1^*+b_2^*+\cdots+b_n^*)\in S$. Clearly $d_{-b_1^*},d_{-b_2^*},\dots,d_{-b_n^*}$ belongs to the subalgebra generated by the elements:

$$d_{-i(b_1^*+b_2^*+\cdots+b_n^*)},\ d_{b_i^*},\quad i=1,2,\dots,n;$$

and $\mathrm{Vir}[G]$ is generated by $d_{\pm b_i^*}$, $i=1,2,\dots,n$. Hence we have $\mathrm{Vir}[G]v_0=0$, a contradiction to the fact that V is nontrivial.

(d) By (b) we can suppose that $\{x\in\mathbb{Z}\mid \Lambda_0+x(b_1+b_2+\cdots+b_n)\in\mathrm{supp}(V)\}=(\infty,p-2]$ for some $p\geqslant 2$. Take A as in (3.1), and $(b_1',b_2',\dots,b_n')=(b_1,b_2,\dots,b_n)A$. One can easily check (d1)–(d6) by using (b) and Claim 1. We omit the details. □

To better understand the proof of Lemma 3.1(d) and the lemmas that follow it might help if one draws a diagram in the Ob_1b_2-plane for $n=2$ to describe those sets. For instance, if $\lambda=x_1b_1+x_2b_2$ in the first quadrant, i.e., $x_1>0,x_2>0$, then $\Lambda_0+\lambda\notin\mathrm{supp}(V)$ and $\Lambda_0-\lambda\in\mathrm{supp}(V)$.

In the next lemma we do not assume the irreducibility of V.

Lemma 3.2. *If V is a nonzero uniformly bounded weight module over* Vir[G]*, then V has an irreducible submodule.*

Proof. Fix $a\in\mathrm{supp}(V)$. Then $\bigoplus_{g\in G}V_{g+a}$ is a $\mathrm{Vir}[G]$-submodule. Thus it is enough to prove the lemma for $V=\bigoplus_{g\in G}V_{g+a}$. We may assume that V does not have any nonzero trivial submodules. So we can further assume that $a\neq 0$.

We shall prove the lemma by induction on $\dim V_a$.

If $\dim V_a=1$, let W be the submodule generated by V_a. We know that there exists a maximal proper submodule W' of W not containing V_a. So W/W' is irreducible. By Theorem 2.5, we know that W/W' is a $V'(\alpha,\beta,G)$. So $W'_a=0$. If $W'_{a'}\neq 0$ for $a'\neq a$, then consider the $\mathrm{Vir}[a-a']$-module generated by $W'_{a'}$. From the Virasoro algebra theory, we see that $a' = 0$, hence $\mathrm{Vir}[G]W' = 0$, a contradiction. So $W' = 0$ and W itself is an irreducible $\mathrm{Vir}[G]$-submodule. The lemma follows in this case.

In general, for any nonzero $v \in V_a$, let W be the submodule generated by v. By Zorn's Lemma, there exists a maximal proper submodule W' of W not containing v. By Theorem 2.5, $W/W' \simeq V'(\alpha, \beta, G)$ for some $\alpha, \beta \in \mathbb{C}$. If $W' = 0$ we are done. If $W' \neq 0$, then, applying the inductive hypothesis to W', we have an irreducible submodule of W'. The lemma is proved. □

In the rest of this section we further assume that $V = \bigoplus_{g\in G} V_{\Lambda_0+g}$ is a nontrivial irreducible GHW $\mathrm{Vir}[G]$-module with GHW Λ_0 w.r.t. $B = \{b_1, b_2, \ldots, b_n\}$, where $\Lambda_0 \in \mathbb{C}$, and B satisfies the properties of Lemma 3.1(d).

Lemma 3.3. *If there exist $(i_1, i_2, \ldots, i_n), (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1, \ldots, k_n$ relatively prime, and $(s_1, \ldots, s_n) > 0$ satisfying*

$$\left\{\Lambda_0 + \sum_{t=1}^{n} i_t b_t + \sum_{t=1}^{n} x_t s_t b_t \;\middle|\; (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n,\ \sum_{t=1}^{n} k_t s_t x_t = 0\right\} \cap \mathrm{supp}(V) = \emptyset,$$

then $V \simeq M(b', G_0, V'(\alpha, \beta, G_0))$ for some $\alpha, \beta \in \mathbb{C}$, and $G = \mathbb{Z}b' \oplus G_0$, where $0 \neq b' \in \mathbb{C}$, G_0 is a subgroup of G.

Remark. The above condition means that a lattice in some affine hyperplane of $\mathbb{Z}^n$ orthogonal to $(k_1, k_2, \ldots, k_n)$ contains no weights of V.

Proof. As mentioned earlier, to understand the proof of this lemma better it may be helpful to sketch in the Ob_1b_2-plane for $n = 2$ the sets used in the proof.

By Lemma 3.1(d6), we have $k_i > 0$ for all $i = 1, 2, \ldots, n$ or $k_i < 0$ for all $i = 1, 2, \ldots, n$. We may assume that $(k_1, k_2, \ldots, k_n) > 0$. Let

$$G_0 = \left\{\sum_{t=1}^{n} x_t b_t \in G \;\middle|\; \sum_{i=1}^{n} k_i x_i = 0\right\}. \tag{3.2}$$

Claim 1. *There exists $m_0 \in \mathbb{Z}$ such that*

$$\left\{\Lambda_0 + \sum_{t=1}^{n} x_t b_t \;\middle|\; (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n,\ \sum_{i=1}^{n} k_i x_i \geqslant m_0\right\} \cap \mathrm{supp}(V) = \emptyset. \tag{3.3}$$

Proof. Let $A_t = s_t s_1(-\delta_{1,t}k_1 + k_t, -\delta_{2,t}k_1, \ldots, -\delta_{n,t}k_1)$ whose corresponding element in G is $s_t s_1(-k_1 b_t + k_t b_1) \in G_0$. Note that $k_1 \neq 0$. One may easily check that for any $(z_1, z_2, \ldots, z_n) \in \mathbb{Z}^n$ with $\sum_{t=1}^{n} z_t k_t \geqslant 0$, there exist suitable $l_t \in \mathbb{Z}$, $t = 1, 2, \ldots, n$, such that

$$(z_1, z_2, \ldots, z_n) = \left(z_1', z_2', \ldots, z_n'\right) + \sum_{t=1}^{n} l_t A_t, \tag{3.4}$$

where $0 \geqslant z'_t > -k_1 s_1 s_t$ for all $t \in \{2, 3, \dots, n\}$. Hence $z'_1 \geqslant 0$. Now let $N = \max\{k_1 s_1 s_1, k_1 s_1 s_2, \dots, k_1 s_1 s_n\}$, and $m_0 = \sum_{t=1}^n k_t(N + i_t)$. Then using (3.4), for any $(x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$ with $\sum_{i=1}^n k_i x_i \geqslant m_0$ we have

$$(x_1, x_2, \dots, x_n) - (i_1 + N, i_2 + N, \dots, i_n + N) = \left(x'_1, x'_2, \dots, x'_n\right) + \sum_{t=1}^n l_t A_t,$$

where $0 \geqslant x'_t > -k_1 s_1 s_t$, i.e.,

$$(x_1, x_2, \dots, x_n) = (i_1, i_2, \dots, i_n) + \left(N + x'_1, N + x'_2, \dots, N + x'_n\right) + \sum_{t=1}^n l_t A_t. \quad (3.5)$$

Let $(y_1, y_2, \dots, y_n) = \sum_{t=1}^n l_t A_t$. We have

$$\Lambda_0 + \sum_{t=1}^n x_t b_t = \Lambda_0 + \sum_{t=1}^n i_t b_t + \sum_{t=1}^n y_t b_t + \sum_{t=1}^n \mathbb{Z}^+ b_t.$$

Note that $\sum_{t=1}^n y_t b_t = \sum_{t=1}^n y'_t s_t b_t$ with $\sum_{t=1}^n y'_t s_t k_t = 0$. From the assumption we know that

$$\Lambda_0 + \sum_{t=1}^n i_t b_t + \sum_{t=1}^n y_t b_t \notin \operatorname{supp}(V). \quad (3.6)$$

By applying Lemma 3.1(d4) we obtain $\Lambda_0 + \sum_{t=1}^n x_t b_t \notin \operatorname{supp}(V)$. The claim follows. □

From Claim 1 we have a unique integer m with the following two properties:

(1) $\{\Lambda_0 + \sum_{t=1}^n x_t b_t \in \operatorname{supp}(V) \mid x_1, x_2, \dots, x_n \in \mathbb{Z},\ \sum_{i=1}^n k_i x_i \geqslant m\} = \emptyset$, and
(2) $P := \{\Lambda_0 + \sum_{t=1}^n x_t b_t \in \operatorname{supp}(V) \mid x_t \in \mathbb{Z},\ \sum_{i=1}^n k_i x_i = m - 1\} \neq \emptyset$.

Fix some $b'_1 = t_1 b_1 + t_2 b_2 + \cdots + t_n b_n$ with $\sum_{i=1}^n k_i t_i = 1$. Since for any $g = \sum_{i=1}^n g_i b_i \in G$, we see that $g - (\sum_{i=1}^n k_i g_i) b'_1 \in G_0$, then $G = \mathbb{Z} b'_1 \oplus G_0$. Fix $\lambda_0 \in P$. We have $P = (\lambda_0 + G_0) \cap \operatorname{supp}(V)$. Let $W = \bigoplus_{\lambda \in \lambda_0 + G_0} V_\lambda$, which is a $\mathrm{Vir}[G_0]$-submodule of V.

Claim 2. *W is a uniformly bounded* $\mathrm{Vir}[G_0]$*-module.*

Proof. Let $0 \neq w \in V_\lambda$ for some $\lambda \in P$. Noting that $(P + G_0 + b'_1) \cap \operatorname{supp}(V) = \emptyset$, and that for any $a_0 \in G_0$, the set $\{d_{a+b'_1}, d_{-a_0-b'_1} \mid a \in G_0\}$ generates the Lie algebra $\mathrm{Vir}[b'_1, G_0] = \mathrm{Vir}[G]$, we deduce

$$d_{-a_0-b'_1} w \neq 0 \quad \text{for any } a_0 \in G_0.$$

Thus we obtain a linear injection $d_{-a_0-b_1'}: V_{\lambda+a_0} \to V_{\lambda-b_1'}$. Thus $\dim V_{\lambda+a_0} \leqslant \dim V_{\lambda-b_1'}$ for all $a_0 \in G_0$, i.e., W is uniformly bounded. Claim 2 follows. □

By Lemma 3.2, W has an irreducible Vir$[G_0]$-submodule W'. By Theorem 2.5, any irreducible uniformly bounded module is either trivial or isomorphic to $V'(\alpha, \beta, G_0)$ for some $(\alpha, \beta) \in \mathbb{C}^2$. Now the center c acts as zero on W'. The $\mathrm{Vir}[G] = \mathrm{Vir}[b_1', G_0]$-module V is generated by W' and $d_{kb_1'+a_0} W' = 0$ for any $k \in \mathbb{N}, a_0 \in G_0$. So V is the unique irreducible quotient of $M(b_1', G_0, W')$. If $W' = \mathbb{C}v_0$ then $V = \mathbb{C}v_0$. Since V is nontrivial, we have $W' \simeq V'(\alpha, \beta, G_0)$ for some $(\alpha, \beta) \in \mathbb{C}^2$ and $V \simeq M(b_1', G_0, V'(\alpha, \beta, G_0))$. □

For any $\mathbb{Z}$-basis $B' = \{b_1', b_2', \ldots, b_n'\}$ of G, we define the total order "$\succ_{B'}$" on G as follows: $x_1 b_1' + x_1 b_2' + \cdots + x_n b_n' \succ_{B'} y_1 b_1' + y_1 b_2' + \cdots + y_n b_n'$ if $(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n)$.

Corollary 3.4. *Suppose that $G \simeq \mathbb{Z}^2$. For any $(0,0) \neq (\dot{c}, h) \in \mathbb{C}^2$, and any $\mathbb{Z}$-basis $B' = \{b_1', b_2'\}$ of G, there exists $\lambda \in \mathrm{supp}(V(\dot{c}, h, \succ_{B'}))$ such that $\dim(V(\dot{c}, h, \succ_{B'}))_\lambda = \infty$.*

Proof. Suppose that for any $\lambda \in \mathrm{supp}(V(\dot{c}, h, \succ_{B'}))$ we have $\dim(V(\dot{c}, h, \succ_{B'}))_\lambda < \infty$. It is easy to see that $\mathrm{supp}(V(\dot{c}, h, \succ_{B'})) \subset (h - \mathbb{N}b_1' + \mathbb{Z}b_2') \cup (h - \mathbb{Z}^+ b_2')$, hence $V(\dot{c}, h, \succ_{B'})$ is a GHW module with GHW h w.r.t. B'. Note that

$$\big(h + \mathbb{N}b_1' + \mathbb{Z}b_2'\big) \cap \mathrm{supp}\big(V(\dot{c}, h, \succ_{B'})\big) = \emptyset, \quad \text{and}$$
$$\big(h + \mathbb{Z}b_2'\big) \cap \mathrm{supp}\big(V(\dot{c}, h, \succ_{B'})\big) \neq \emptyset.$$

Using the same argument as in the proof of Claim 2 of Lemma 3.2, we see that

$$W = \bigoplus_{\lambda \in \mathbb{Z}b_2'} V(\dot{c}, h, \succ_{B'})_{h+\lambda}$$

is a uniformly bounded $\mathrm{Vir}[b_2']$ module. Since W contains the submodule $W' = U(\mathrm{Vir}[b_2'])(v_h)$ which is a highest weight module with highest weight $(\dot{c}, h)$, W' (and W) is not uniformly bounded. A contradiction. Hence $(\dot{c}, h) = (0, 0)$. The corollary follows. □

Lemma 3.5. *Suppose that $G \simeq \mathbb{Z}^2$. If there exist $(k, l) \neq 0$, $(i, j) \in \mathbb{Z}^2$, $p, q \in \mathbb{Z}$ such that*

$$\big\{x \in \mathbb{Z} \mid \Lambda_0 + ib_1 + jb_2 + x(kb_1 + lb_2) \in \mathrm{supp}(V)\big\} \supset (-\infty, p] \cup [q, \infty),$$

then $V \simeq M(b_1', \mathbb{Z}b_2', V'(\alpha, \beta, \mathbb{Z}b_2'))$ for some $\alpha, \beta \in \mathbb{C}$, and a $\mathbb{Z}$-basis $B' = (b_1', b_2')$ of G.

Proof. From Lemma 3.1(d6), we see $kl < 0$. We may assume that $l > 0$. Let

$$(i_0, j_0) = \begin{cases} (i + qk, j + pl), & \text{if } p < q - 1, \\ (i, j), & \text{if } p \geqslant q - 1. \end{cases}$$

Denote $L := \{i_0b_1 + j_0b_2 + x(kb_1 + lb_2) \mid x \in \mathbb{Z}\}$. If $p < q - 1$, write

$$i_0b_1 + j_0b_2 + x(kb_1 + lb_2) = ib_1 + jb_2 + (x + q)(kb_1 + lb_2) + (p - q)l_0b_2 \quad \text{or}$$
$$ib_1 + jb_2 + (x + p)(kb_1 + lb_2) + (q - p)k_0b_1$$

according to $x \geqslant 0$ or $x < 0$. From Lemma 3.1(d5) we see that all points in the set $\Lambda_0 + L$ are weights of V.

Write $(k, l) = s(k_0, l_0)$ with k_0, l_0 relatively prime, $s \geqslant 1$. By replacing (i, j) with $(i_0, j_0 - (s - 1)l_0)$, we may assume that $p = q$. Then similarly we have $L_0 := ib_1 + jb_2 + \mathbb{Z}(k_0b_1 + l_0b_2) \subset \mathrm{supp}(V)$. Using Lemma 3.1(d5) we see that

$$\bigl\{\Lambda_0 + xb_1 + yb_2 \mid l_0x - k_0y \leqslant l_0i - k_0\bigl(j_0 - (s - 1)l_0\bigr),\ (x, y) \in \mathbb{Z}^2\bigr\} \subset \mathrm{supp}(V),$$

i.e., all points under the line $\Lambda_0 - (s - 1)l_0b_2 + L_0$ are weights of V (It might help if one draws a diagram on the Ob_1b_2-plane.)

So we may assume that k, l are relatively prime, $k < 0$, $l > 0$, and there exists an integer m_0 such that

$$\bigl\{\Lambda_0 + xb_1 + yb_2 \mid lx - ky \leqslant m_0, (x, y) \in \mathbb{Z}^2\bigr\} \subset \mathrm{supp}(V). \tag{3.7}$$

Fix $(k', l') \in \mathbb{Z}^2$ with $lk' - kl' = 1$. Denote $b_1' = kb_1 + lb_2$ and $b_2' = k'b_1 + l'b_2$. If

$$\bigl\{\Lambda_0 - kb_1 + b_2' + tb_1' \mid t \in \mathbb{Z}\bigr\} \cap \mathrm{supp}(V) = \emptyset,$$

then the lemma follows from Lemma 3.3. Hence we may assume that

$$\bigl\{\Lambda_0 - kb_1 + b_2' + tb_1' \mid t \in \mathbb{Z}\bigr\} \cap \mathrm{supp}(V) \neq \emptyset. \tag{3.8}$$

Choose $\Lambda_0 - kb_1 + b_2' - sb_1' \in \mathrm{supp}(V)$, and a nonzero weight vector $v \in V_{\Lambda_0 - kb_1 + b_2' - sb_1'}$. Let

$$b_1'' = sb_1' - b_2', \qquad b_2'' = (s + 1)b_1' - b_2'.$$

Since $\Lambda_0 - kb_1, \Lambda_0 - kb_1 + b_1' \in \Lambda_0 + \mathbb{Z}^+b_1 + \mathbb{Z}^+b_2$, we obtain

$$d_{b_1''}v = 0, \qquad d_{b_2''}v = 0.$$

Thus

$$d_{mb_1'' + nb_2''}v = 0, \quad \forall m > 0,\ n > 0.$$

Using this, one sees that v is a GHW vector with respect to the $\mathbb{Z}$-basis $\{b_1'' + b_2'', b_1'' + 2b_2''\}$ of G. Now by Lemma 3.1(b) there exists some x_0 such that

$$\lambda_0 + b_2' + x\bigl(\bigl(b_1'' + b_2''\bigr) + \bigl(b_1'' + 2b_2''\bigr)\bigr) \notin \mathrm{supp}(V), \quad \forall x > x_0. \tag{3.9}$$

But

$$\begin{aligned}(b_1''+b_2'')+(b_1''+2b_2'')=2b_1''+3b_2''&=\big(2s+3(s+1)\big)b_1'-5b_2'\\&=\big((5s+3)k-5k'\big)b_1+\big((5s+3)l-5l'\big)b_2,\\ l\big((5s+3)k-5k'\big)-k\big((5s+3)l-5l'\big)&=-5(lk'-kl')=-5<0.\end{aligned}$$

Hence for x sufficiently large we have

$$\lambda_0+b_2'+x\big((b_1''+b_2'')+(b_1''+2b_2'')\big)\in\big\{\Lambda_0+xb_1+yb_2\mid lx-ky\leqslant m_0,\ (x,y)\in\mathbb{Z}^2\big\},$$

which is a contradiction to (3.7) and (3.9), hence (3.8) cannot occur. The lemma follows. □

Lemma 3.6. *Suppose that $G\simeq\mathbb{Z}^2$. If there exist $(i,j),(k,l)\in\mathbb{Z}^2$ and $x_1,x_2,x_3\in\mathbb{Z}$ with $x_1<x_2<x_3$, such that*

$$\Lambda_0+ib_1+jb_2+x_1(kb_1+lb_2)\notin\operatorname{supp}(V),$$
$$\Lambda_0+ib_1+jb_2+x_2(kb_1+lb_2)\in\operatorname{supp}(V),\quad \textit{and}$$
$$\Lambda_0+ib_1+jb_2+x_3(kb_1+lb_2)\notin\operatorname{supp}(V),$$

then

(a) *there exists $x\in\mathbb{Z}$ with $x_1<x<x_3$ such that*

$$\Lambda_0+ib_1+jb_2+x(kb_1+lb_2)=0,$$

and further,

(b) *such a module V does not exist.*

Proof. We may assume that k,l are relatively prime, and by Lemma 3.1(d6) we see $kl<0$. So we may assume that $k<0$ and $l>0$. Replacing x_2 by the largest $x<x_3$ with $\Lambda_0+ib_1+jb_2+x(kb_1+lb_2)\in\operatorname{supp}(V)$, and then replacing x_3 by x_2+1 and (i,j) by $(i,j)+x_2(k,l)$ we can assume that

$$x_1<x_2=0,\qquad x_3=1. \tag{3.10}$$

Fix a nonzero weight vector $v\in V_{\Lambda_0+ib_1+jb_2}$. Then (3.10) means

$$d_{kb_1+lb_2}v=0=d_{x_1(kb_1+lb_2)}v,$$

which yields $d_{\pm(kb_1+lb_2)}v=0$. By Lemma 3.1(b) we can choose $p,q>0$ such that $d_{pb_1+qb_2}v=0$. Since $kq-lp<0$, then $S=\{b_1'=kb_1+lb_2, b_2'=pb_1+qb_2\}$ is a $\mathbb{Z}$-linear independent subset of G. Note that $d_{mb_1'+nb_2'}$ for $n>0$ belong to the subalgebra generated by $d_{\pm b_1'}, d_{b_2'}$. Thus

$$d_{mb_1'+nb_2'}v=0, \quad \forall n>0, m\in\mathbb{Z}.$$

Consider the $\mathrm{Vir}[b_1']$-module $W=U(\mathrm{Vir}[b_1'])v$. By using the PBW basis of $U(\mathrm{Vir}[S])$ we have

$$\big(\Lambda_0+ib_1+jb_2+\mathbb{Z}b_1'+\mathbb{N}b_2'\big)\cap \mathrm{supp}\big(U\big(\mathrm{Vir}[S]\big)v\big)=\emptyset. \tag{3.11}$$

Case 1. W is not uniformly bounded.

From Virasoro algebra theory we see that W has a nontrivial irreducible sub-quotient $\mathrm{Vir}[b_1']$-module W_1/W_2 which is a highest (or lowest) weight $\mathrm{Vir}[b_1']$-module. Using (3.11) and PBW Theorem, we know that $W'=U(\mathrm{Vir}[S])W_1/U(\mathrm{Vir}[S])W_2$ is a highest weight $\mathrm{Vir}[S]$-module w.r.t. the lexicographic order determined by $\{b_1', b_2'\}$ with highest weight not equal to $(0,0)$. Now by Corollary 3.4, W' has a weight space of infinite dimension. So does S. This case does not occur.

Case 2. W is uniformly bounded.

First we can easily see that the center c acts as zero on V. From the fact that $\mathrm{supp}(V'(\alpha,\beta,\mathbb{Z}b_1'))=\alpha+\mathbb{Z}b_1'$ or $\mathbb{Z}b_1'\setminus\{0\}$ and the assumption (3.10), we know that $W\subset V_0$, the weight space with 0 weight. We deduce (a).

It is clear that $W=V_0=\mathbb{C}v_0$. Denote by W'' the $\mathrm{Vir}[S]$-module generated by W, which is a $\mathrm{Vir}[S]$-submodule of V. Now by (3.11), $\mathrm{Vir}[S]$-module W'' is a quotient module of $M(0,0,\succ_{B'})$, and W'' is nontrivial (from Lemma 3.1(c)), so W'' is reducible. If $d_{-b_2'+s_0b_1'}v_0=0$ for some s_0, from

$$d_{-b_2'+sb_1'}v_0=\big(-b_2'+(2s_0-s)b_1'\big)^{-1}[d_{-b_2'+s_0b_1'}, d_{sb_1'-s_0b_1'}]v_0=0,$$

and the fact that $\{d_{-b_2'+sb_1'}\mid s\in\mathbb{Z}\}$ generates $\{d_{-tb_2'+sb_1'}\mid s\in\mathbb{Z},\ t\in\mathbb{N}\}$, combining with (3.11) we deduce that W'' is a trivial $\mathrm{Vir}[S]$-submodule, a contradiction to Lemma 3.1(c). So we have

$$d_{-b_2'+sb_1'}v_0\neq 0 \quad \text{for any } s\in\mathbb{Z}.$$

Thus $\{-b_2'+sb_1'\mid s\in\mathbb{Z}\}\subset \mathrm{supp}(V)$. Now by Lemma 3.5, we have $V\simeq M(b_2',\mathbb{Z}b_1', V'(\alpha,\beta,\mathbb{Z}b_1'))$ for some $\alpha,\beta\in\mathbb{C}$, and a $\mathbb{Z}$-basis $B'=(b_1',b_2')$ of G. It is easy to see that $M(b_2',\mathbb{Z}b_1',V'(\alpha,\beta,\mathbb{Z}b_1'))$ does not satisfy condition (a). Thus such a module V does not exist.

This completes the proof. □

The idea of Claims 1 and 2 in the proof of the next theorem comes from the proof of [15, Theorem 1.1] for $n=2$.

Theorem 3.7. *Suppose that $B = (b_1, b_2)$ is a $\mathbb{Z}$-basis of the additive subgroup $G \subset \mathbb{C}$. If V is a nontrivial irreducible weight module with finite dimensional weight spaces over the higher rank Virasoro algebra* $\mathrm{Vir}[G]$*, then $V \cong V'(\alpha, \beta, G)$ or $V \cong M(b_1', \mathbb{Z}b_2', V'(\alpha, \beta, \mathbb{Z}b_2'))$ for some $\alpha, \beta \in \mathbb{C}$, and a $\mathbb{Z}$-basis $B' = (b_1', b_2')$ of G.*

Proof. To the contrary, we suppose that $V \not\cong V'(\alpha, \beta, G)$ or $M(b_1', \mathbb{Z}b_2', V'(\alpha, \beta, \mathbb{Z}b_2'))$ for any $\alpha, \beta \in \mathbb{C}$, and any $\mathbb{Z}$-basis of $B' = (b_1', b_2')$ of G. From Theorem 2.5 we may assume that V is a GHW module with GHW Λ_0 w.r.t. the basis $B = \{b_1, b_2\}$ for G. We need to prove that $V \cong M(b_1', \mathbb{Z}b_2', V'(\alpha, \beta, \mathbb{Z}b_2'))$ for proper parameters. We still assume that B satisfies Lemma 3.1(d). By Lemmas 3.3, 3.5 and 3.6, for any $(i, j), 0 \neq (k, l) \in \mathbb{Z}^2$, there exists $p \in \mathbb{Z}$ such that

$$\{x \in \mathbb{Z} \mid \Lambda_0 + ib_1 + jb_2 + x(kb_1 + lb_2) \in \mathrm{supp}(V)\} = (-\infty, p] \quad \text{or} \quad [p, \infty). \tag{3.12}$$

Then for any $i \in \mathbb{N}$, there exist $x_i, y_i \in \mathbb{Z}^+$ such that

$$(-\infty, y_i] = \max\{y \in \mathbb{Z} \mid \Lambda_0 - ib_1 + yb_2 \in \mathrm{supp}(V)\},$$
$$(-\infty, x_i] = \max\{x \in \mathbb{Z} \mid \Lambda_0 + xb_1 - ib_2 \in \mathrm{supp}(V)\}.$$

By Lemma 3.1(d5) we know that $y_{i+1} \geqslant y_i \geqslant 0, x_{i+1} \geqslant x_i \geqslant 0$. Let $j, t \in \mathbb{N}$, if $y_{jt} \geqslant t(y_j + 1)$, then $t > 1$ and $\Lambda_0, \Lambda_0 + t(-jb_1 + (y_j + 1)b_2) \in \mathrm{supp}(V)$, and by (3.12), $\Lambda_0 + (-jb_1 + (y_j + 1)b_2) \in \mathrm{supp}(V)$, contrary to the definition of y_j. So

$$y_{tj} < t(y_j + 1), \quad \forall t, j \in \mathbb{N}. \tag{3.13}$$

Since $\Lambda_0 + b_2 \notin \mathrm{supp}(V)$ and $\Lambda_0 - jb_1 + y_j b_2 = \Lambda_0 + b_2 + (-jb_1 + (y_j - 1)b_2) \in \mathrm{supp}(V)$, then (3.12) yields $\Lambda_0 + b_2 + t(-jb_1 + (y_j - 1)b_2) \in \mathrm{supp}(V)$ for all $t > 0$. Hence

$$y_{tj} \geqslant t(y_j - 1) + 1, \quad \forall t, j \in \mathbb{N}. \tag{3.14}$$

Using (3.13), (3.14) we obtain

$$j(y_i - 1) + 1 \leqslant y_{ij} < i(y_j + 1), \quad \forall i, j \in \mathbb{N}.$$

From $j(y_i - 1) + 1 < i(y_j + 1)$ and the one with i, j interchanged, we deduce

$$\frac{y_j}{j} - \frac{i + j - 1}{ij} < \frac{y_i}{i} < \frac{y_j}{j} + \frac{i + j - 1}{ij}, \quad \forall i, j \in \mathbb{N}. \tag{3.15}$$

This shows that the following limits exist:

$$\alpha = \lim_{i \to \infty} \frac{y_i}{i}, \qquad \beta = \lim_{i \to \infty} \frac{x_i}{i}, \tag{3.16}$$

where the second equation is obtained by symmetry. Note that (3.12) implies that there exists some $j_0 \in \mathbb{N}$ such that $y_{j_0} > 1$ (otherwise $(\Lambda_0 + 2b_2 + \mathbb{Z}b_1) \cap \operatorname{supp}(V) = \emptyset$). Hence by (3.14) we deduce

$$\frac{y_{tj_0}}{tj_0} \geqslant \frac{y_{j_0} - 1}{j_0} + \frac{1}{tj_0} > \frac{y_{j_0} - 1}{j_0} > 0,$$

thus $\alpha > 0$. Similarly $\beta > 0$.

Claim 1. $\alpha = \beta^{-1}$ *is an irrational number.*

Proof. Suppose that $\alpha > \beta^{-1}$. Choose $s, q \in \mathbb{N}$ with s, q relatively prime and $\alpha > s/q > \beta^{-1}$. Applying (3.12) to $\Lambda_0 + t(-qb_1 + sb_2)$, by the definition of α, we have $\Lambda_0 + t(-qb_1 + sb_2) \in \operatorname{supp}(V)$ for all sufficiently large t. From (3.12), hence, for all sufficiently large t we have $\Lambda_0 - t(-qb_1 + sb_2) \notin \operatorname{supp}(V)$, which implies that

$$\beta = \lim_{t\to\infty} \frac{x_{st}}{st} \leqslant \frac{q}{s},$$

i.e., $\beta^{-1} \geqslant s/q$, a contradiction. So we have $\alpha \leqslant \beta^{-1}$, and similarly we have $\alpha \geqslant \beta^{-1}$. Thus $\alpha = \beta^{-1}$.

Assume $\alpha = q/s$ is a rational number, where $s, q \in \mathbb{N}$ are relatively prime. By (3.12), there exists some $m_0 \in \mathbb{Z}$ such that $\Lambda_0 - b_2 + m_0(sb_1 - qb_2) \notin \operatorname{supp}(V)$. Say $m_0 > 0$. Since $\Lambda_0 \in \operatorname{supp}(V)$, by (3.12) again, we deduce

$$\Lambda_0 + i\big(-m_0 sb_1 + (m_0 q + 1)b_2\big) \in \operatorname{supp}(V), \quad \forall i \in [0, \infty].$$

However

$$\alpha = \frac{q}{s} = \lim_{i\to\infty} \frac{y_{im_0 s}}{im_0 s} \geqslant \frac{m_0 q+1}{m_0 s} > \frac{q}{s},$$

a contradiction. Hence α is an irrational number, and Claim 1 follows. □

We define a total order $>_\alpha$ on G as follows:

$$ib_1 + jb_2 >_\alpha kb_1 + lb_2 \quad \Leftrightarrow \quad i\alpha + j > k\alpha + l.$$

Let $G^+ = \{ib_1 + jb_2 \in G \mid ib_1 + jb_2 >_\alpha 0\}$. If $\lambda \in \operatorname{supp}(V)$ satisfies $(\lambda + G^+) \cap \operatorname{supp}(V) = \emptyset$, then V is a nontrivial highest weight module w.r.t. "$<_\alpha$". Since the order "$<_\alpha$" is dense, from Theorem 2.2 we see that V is a Verma module, which contradicts the fact that all weight spaces of V are finite dimensional. So for any $\lambda \in \operatorname{supp}(V)$ we have

$$\big(\lambda + G^+\big) \cap \operatorname{supp}(V) \neq \emptyset. \tag{3.17}$$

Claim 2. *If $\Lambda_0 + g \in \mathrm{supp}(V)$ for some $g = ib_1 + jb_2 \in G^+$ then*

$$\Lambda_0 + kb_1 + lb_2 \in \mathrm{supp}(V), \quad \forall kb_1 + lb_2 <_\alpha ib_1 + jb_2.$$

Proof. If there exists some $kb_1 + lb_2 <_\alpha ib_1 + jb_2$ such that $\Lambda_0 + kb_1 + lb_2 \notin \mathrm{supp}(V)$, (3.12) implies

$$\Lambda_0 + ib_1 + jb_2 + t\big((k-i)b_1 + (l-j)b_2\big) \notin \mathrm{supp}(V), \quad \forall t \in \mathbb{N}.$$

If $k - i < 0$ (then $l - j \geqslant 0$), from $(k-i)b_1 + (l-j)b_2 <_\alpha 0$ we see that $-(l-j)/(k-i) < \alpha$. On the other hand,

$$\alpha = \lim_{t\to\infty} \frac{y_{t(i-k)-i}}{t(i-k)-i} \leqslant \lim_{t\to\infty} \frac{j + t(l-j)}{t(i-k)-i} = \frac{l-j}{i-k},$$

a contradiction. If $k - i > 0$, from $(k-i)b_1 + (l-j)b_2 <_\alpha 0$ we know that $(l-j) < 0$, and $-(k-i)/(l-j) < \alpha^{-1}$. Similarly,

$$\alpha^{-1} = \lim_{t\to\infty} \frac{x_t}{t} \leqslant -\frac{k-i}{l-j},$$

again a contradiction. If $k - i = 0$, by Lemma 3.1(d5) we have $(l-j) > 0$, but by $(k-i)b_1 + (l-j)b_2 <_\alpha 0$, we have $l - j < 0$, which is also a contradiction. So we have Claim 2. □

Claim 2 implies that for any $\Lambda \in \mathrm{supp}(V)$, we have

$$\Lambda - G^+ \subset \mathrm{supp}(V). \tag{3.18}$$

Claim 3. *$d_{-g}v_\lambda \neq 0$ for any $g = ib_1 + jb_2 \in G^+$ and any nonzero weight vector $v_\lambda \in V_\lambda$.*

Proof. Suppose that $d_{-g}v_\lambda = 0$ for some $g = ib_1 + jb_2 \in G^+$ and $0 \neq v_\lambda \in V_\lambda$. By (3.12) and (3.18), we see that $d_{sg}v_\lambda = 0$ for all sufficiently large $s > 0$. Hence $d_g v_\lambda = 0$. By Lemma 3.1(b) we can choose $g_1 = pb_1 + qb_2$ such that $d_{g_1}v_\lambda = 0$ and $S = \{g, g_1\}$ is a $\mathbb{Z}$-linearly independent subset of G. Consider the Vir$[g]$-module $W = U(\mathrm{Vir}[g])v_\lambda$. Using the PBW basis of $U(\mathrm{Vir}[S])$ we have

$$(\lambda + \mathbb{Z}g + \mathbb{N}g_1) \cap \mathrm{supp}\big(U\big(\mathrm{Vir}[S]\big)v_\lambda\big) = \emptyset.$$

By (3.12) and (3.18) there exists some s_0 such that $\lambda + sg \notin \mathrm{supp}(V)$ for all $s > s_0$. Hence any irreducible Vir$[g]$-subquotient of W is a highest weight module. If W has a nontrivial irreducible Vir$[g]$-subquotient, using the arguments, analogous to those used in Case 1 in the proof of Lemma 3.6, we get a contradiction. So we deduce that $W = \mathbb{C}v_\lambda$ with $\lambda = 0$. With a similar discussion as in Case 2 in the proof of Lemma 3.6 we obtain that $\lambda + \mathbb{Z}g - g_1 \subset \mathrm{supp}(U(\mathrm{Vir}[S])v_\lambda)$, which contradicts (3.12). Hence Claim 3 follows. □

Fix $\Lambda_0 + ib_1 + jb_2 \in \mathrm{supp}(V)$, where $ib_1 + jb_2 \in G^+$. We are going to show that $\dim V_{\Lambda_0+ib_1+jb_2} = \infty$. For a given $n > 0$, let $\varepsilon = \frac{1}{n}(j + i\alpha) > 0$. Since the order "$<_\alpha$" is dense, we can choose $p, q \in \mathbb{Z}$ with $0 < q + p\alpha < \varepsilon$. Hence we obtain $0 <_\alpha pb_1 + qb_2$ and $npb_1 + nqb_2 <_\alpha (ib_1 + jb_2)$. Then from Claim 2 we deduce that

$$\Lambda_0 + m(pb_1 + qb_2) \in \mathrm{supp}(V), \quad \forall m \leqslant 0.$$

By (3.12) we assume that m_0 is the maximal integer such that $\Lambda_0 + m_0(pb_1 + qb_2) \in \mathrm{supp}(V)$, so $m_0 \geqslant n$. Let

$$M = \left\{g \in G^+ \mid 0 \neq \Lambda_0 + m_0(pb_1 + qb_2) + g \in \mathrm{supp}(V)\right\}.$$

By (3.17) M is an infinite set. Denote $\bar{g} = pb_1 + qb_2$.

Claim 4. *There exist $g_0 \in M$ such that for any k: $1 \leqslant k \leqslant n$, the k vectors*

$$d_{-\bar{g}}^{k-1} d_{-\bar{g}} v,\ d_{-\bar{g}}^{k-2} d_{-2\bar{g}} v,\ \ldots,\ d_{-\bar{g}} d_{-(k-1)\bar{g}} v,\ d_{-k\bar{g}} v$$

are linearly independent, where $v \in V_{\Lambda_0+g_0+m_0\bar{g}} \setminus \{0\}$.

Proof. We will prove the claim by induction on k.

Suppose that $v \in V_{\Lambda_0+g+m_0\bar{g}} \setminus \{0\}$ for $g \in M$.

If $k = 1$, from $d_{\bar{g}} v_{\Lambda_0+g+m_0\bar{g}} = 0$, we deduce that

$$d_{\bar{g}} d_{-\bar{g}} v = \left(-2\bar{g}\big(\Lambda_0 + g + m_0\bar{g}\big) + \frac{\bar{g}^3 - \bar{g}}{12} c\right) v.$$

Let $h_1(g) := -2\bar{g}(\Lambda_0 + g + m_0\bar{g}) + \frac{\bar{g}^3 - \bar{g}}{12} c$. Then the set $M_1 = \{g \in M \mid h_1(g) \neq 0\}$ is infinite and $d_{-\bar{g}} v \neq 0$ for any $g \in M_1$.

Suppose that $k > 1$ and there exist an infinite set $M_{k-1} \subset M$ such that

$$d_{-\bar{g}}^{k-2} d_{-\bar{g}} v,\ d_{-\bar{g}}^{k-3} d_{-2\bar{g}} v,\ \ldots,\ d_{-\bar{g}} d_{-(k-2)\bar{g}} v,\ d_{-(k-1)\bar{g}} v$$

are linearly independent for any $v \in V_{\Lambda_0+g+m_0\bar{g}} \setminus \{0\}$ and $g \in M_{k-1}$.

Now we consider k. If the vectors

$$d_{-\bar{g}}^{k} v,\ d_{-\bar{g}}^{k-2} d_{-2\bar{g}} v,\ \ldots,\ d_{-\bar{g}} d_{-(k-1)\bar{g}} v,\ d_{-k\bar{g}} v$$

are linearly dependent for some $g \in M_{k-1}$. Then there exist $a_1, \ldots, a_k \in \mathbb{C}$, not all zero, such that

$$w_k = a_1 d_{-\bar{g}}^{k} v + a_2 d_{-\bar{g}}^{k-2} d_{-2\bar{g}} v + \cdots + a_k d_{-(k)\bar{g}} v = 0.$$

Using $[d_{\bar{g}}, d_{-\bar{g}}^k] = -k\bar{g}(2d_0 + (k-1)\bar{g} - \frac{\bar{g}^2-1}{12} c) d_{-\bar{g}}^{k-1}$, we deduce that

$$
\begin{aligned}
0 &= d_{\bar g} w_k \\
&= -a_1 k\bar g\left(2\big(\Lambda_0 + g + (m_0 - k + 1)\bar g\big) + (k-1)\bar g - \frac{\bar g^2 - 1}{12}c\right) d_{-\bar g}^{k-1} v \\
&\quad + a_2(-k+2)\bar g\left(2\big(\Lambda_0 + g + (m_0 - k + 1)\bar g\big) + (k-3)\bar g - \frac{\bar g^2 - 1}{12}c\right) d_{-\bar g}^{k-3} d_{-2\bar g} v \\
&\quad - 3a_2 \bar g d_{-\bar g}^{k-1} v + \cdots \\
&\quad + a_{k-1}(-1)\bar g\left(2\big(\Lambda_0 + g + (m_0 - k + 1)\bar g\big) - \frac{\bar g^2 - 1}{12}c\right) d_{-(k-1)\bar g} v \\
&\quad + (-k)a_{k-1}\bar g d_{-\bar g} d_{-(n-2)\bar g} v_{\Lambda_0 + g + (m_0 - k + 1)\bar g} + a_k(-k-1)\bar g d_{-(k-1)\bar g} v.
\end{aligned}
$$

This together with the inductive hypothesis yields that

$$
a_i = a_1 f_i(g), \quad \forall i = 1, 2, \ldots, k, \tag{3.19}
$$

where $f_i(X)$ is a polynomial of degree $i-1$ in X. Using (3.19) and the following computations

$$
\begin{aligned}
0 &= d_{k\bar g} w_k \\
&= a_1(-k-1)\bar g(-k)\bar g \cdots (-3)\bar g\left(-2\bar g\big(\Lambda_0 + g + m_0\bar g\big) + \frac{\bar g^3 - \bar g}{12}c\right) v \\
&\quad + a_2(-k-1)\bar g(-k)\bar g \cdots (-4)\bar g\left(-4\bar g\big(\Lambda_0 + g + m_0\bar g\big) + \frac{(2\bar g)^3 - 2\bar g}{12}c\right) v + \cdots \\
&\quad + a_{k-1}(-k-1)\bar g\left(-2(k-1)\bar g\big(\Lambda_0 + g + m_0\bar g\big) + \frac{((k-1)\bar g)^3 - (k-1)\bar g}{12}c\right) v \\
&\quad + a_k\left(-2k\bar g\big(\Lambda_0 + g + m_0\bar g\big) + \frac{(k\bar g)^3 - k\bar g}{12}c\right) v \\
&= a_1 h_k(g) v,
\end{aligned}
$$

where $h_k(X)$ is a polynomial of degree k in X. Then $M_k = \{g \in M_{k-1} \mid h_k(g) \neq 0\}$ is infinite and the vectors in Claim 4 are linearly independent for $g_0 \in M_k$. Hence Claim 4 follows. $\square$

From Claim 4 we know that $\dim V_{\Lambda_0 + g_0 + (m_0 - n)\bar g} \geqslant n$ for some $g_0 \in M$ and for all $n \in \mathbb{N}$. Noting that $g_0, \bar g \in G^+$, by Claim 3 we deduce that $\dim V_{\Lambda_0} \geqslant n$ for all $n \in \mathbb{N}$. Hence $\dim V_{\Lambda_0} = \infty$. This proves that (3.12) cannot occur, and the theorem follows. $\square$

Lemma 3.8. *Suppose that $G = \mathbb{Z}b_1' \oplus G_0$. Then* $\mathrm{supp}(M(b_1', G_0, V'(\alpha, \beta, G_0)) = \mathrm{supp}(V'(\alpha, \beta, G_0)) \cup (\alpha + G_0 - \mathbb{N}b_1')$.

Proof. It is clear that

$$\operatorname{supp}\big(M\big(b_1', G_0, V'(\alpha,\beta,G_0)\big)\big) \subset \operatorname{supp}\big(V'(\alpha,\beta,G_0)\big) \cup \big(\alpha + G_0 - \mathbb{N}b_1'\big),$$
$$\operatorname{supp}\big(V'(\alpha,\beta,G_0)\big) \subset \operatorname{supp}\big(M\big(b_1', G_0, V'(\alpha,\beta,G_0)\big)\big).$$

Suppose that there exists $d = \alpha + g_0 - sb_1' \notin \operatorname{supp}(M(b_1', G_0, V'(\alpha,\beta,G_0))$, where $s > 0$ and $g_0 \in G_0$.

Choose $\alpha + g_1 \in \operatorname{supp}(V'(\alpha,\beta,G_0)) \setminus \{0\}$. Let $d' = g_1 - g_0 + sb'$. We see that $\alpha + g_1 \in \operatorname{supp}(V)$. Fix $v \in V'(\alpha,\beta,G_0)_{\alpha+g_1}$. Let W be the $\mathrm{Vir}[d', b_1']$-submodule generated by v. Then we have an irreducible sub-quotient module W' of W with $\alpha + g_1 \in \operatorname{supp}(W')$, and $\alpha + g_1 \pm d' \notin \operatorname{supp}(W')$. We get a contradiction to Lemma 3.6. This completes the proof of the lemma. □

From the lemma above we see that $\operatorname{supp}(M(b_1', G_0, V'(\alpha,\beta,G_0))$ equals either $\alpha - \mathbb{Z}^+ b_1' + G_0$ or $(-\mathbb{Z}^+ b_1' + G_0) \setminus \{0\}$. Finally we can handle the general case.

Theorem 3.9. *If V is a nontrivial irreducible weight module with finite dimensional weight spaces over the higher rank Virasoro algebra* $\mathrm{Vir}[G]$ *for $G \simeq \mathbb{Z}^n$ $(n \geqslant 2)$, then $V \simeq V'(\alpha,\beta,G)$ or $V \simeq M(b_1', G_0, V'(\alpha,\beta,G_0))$ for some $\alpha, \beta \in \mathbb{C}$, $b_1' \in G \setminus \{0\}$, and a subgroup G_0 of G with $G = \mathbb{Z}b_1' \oplus G_0$.*

Proof. From Theorem 2.5 we may assume that V is a nontrivial irreducible GHW module with GHW Λ_0 w.r.t. $B = \{b_1, b_2, \ldots, b_n\}$ over $\mathrm{Vir}[G]$, where B is a $\mathbb{Z}$-basis of the additive subgroup G of $\mathbb{C}$, then we need to prove that $V \simeq M(b_1', G_0, V'(\alpha,\beta,G_0))$. We still assume that B satisfies Lemma 3.1(d).

We shall prove this by induction on n. For $n = 2$ this is Theorem 3.7. Now suppose that the theorem holds for any $n \leqslant N-1$ where $N \geqslant 3$. We shall prove $V \simeq M(b_1', G_0, V'(\alpha,\beta,G_0))$ for $n = N$.

If there exist $g \in G$ and a corank 1 subgroup G_0 of G such that $(\Lambda_0 + g + G_0) \cap \operatorname{supp}(V) \subset \{0\}$, then the theorem follows from Lemma 3.3. (Indeed, If $(\Lambda_0 + g + G_0) \cap \operatorname{supp}(V) = \{0\}$, suppose that $G_0 = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_{N-1}$. We may assume that $\Lambda_0 + g = 0$. Then $(a_1 + \mathbb{Z}2a_1 + \cdots + \mathbb{Z}2a_{N-1}) \cap \operatorname{supp}(V) = \emptyset$. Using Lemma 3.3 we have the theorem). So we may assume that for any $g \in G$ and any corank 1 subgroup G_0,

$$(\Lambda_0 + g + G_0) \cap \operatorname{supp}(V) \not\subseteq \{0\}. \tag{3.20}$$

Hence the $\mathrm{Vir}[G_0]$ module $V_{\Lambda_0+g+G_0} = \bigoplus_{x \in G_0} V_{\Lambda_0+x+g}$ has a nontrivial irreducible sub-quotient. By Lemma 3.8, Theorem 2.5 and the inductive hypothesis, for any corank 1 subgroup G_0 and any $g \in G$ there exist a subgroup $G_{0,1}$ of G_0, $\lambda_0' \in \Lambda_0 + g + G_0$ and $g_{0,1} \in G_0 \setminus \{0\}$ with $G_0 = \mathbb{Z}g_{0,1} \oplus G_{0,1}$ such that

$$\lambda_0' + G_{0,1} - \mathbb{N}g_{0,1} \subset \operatorname{supp}(V). \tag{3.21}$$

Note that some other elements in $\lambda_0' + G_0$ can also be in $\operatorname{supp}(V)$. Next we are going to show that under the assumption (3.21) such a module V does not exist.

Claim 1. *There are no $\lambda_0 \in \mathrm{supp}(V)$, $t_0 \in \mathbb{Z}$, $g_0, g_1 \in G \setminus \{0\}$ or subgroups $G_1' \subset G_0' \subset G$ with $G = \mathbb{Z}g_0 \oplus G_0'$ and $G_0' = \mathbb{Z}g_1 \oplus G_1'$ satisfying*

$$\lambda_0 - \mathbb{Z}^+ g_1 + G_1',\ \lambda_0 + t_0 g_1 + \mathbb{Z}^+ g_1 + G_1' \ \subset\ \mathrm{supp}(V).$$

(If $t_0 \leqslant 0$, then $\lambda_0 + G_0' \subset \mathrm{supp}(V)$.)

Proof. Suppose that there exist $\lambda_0 \in \mathrm{supp}(V)$, $t_0 \in \mathbb{Z}$, $g_0, g_1 \in G \setminus \{0\}$ and subgroups $G_1' \subset G_0' \subset G$ with $G = \mathbb{Z}g_0 \oplus G_0'$ and $G_0' = \mathbb{Z}g_1 \oplus G_1'$ satisfying

$$\lambda_0 - \mathbb{Z}^+ g_1 + G_1',\ \lambda_0 + t_0 g_1 + \mathbb{Z}^+ g_1 + G_1' \ \subset\ \mathrm{supp}(V).$$

Choose $0 \neq (k_1, \ldots, k_N) \in \mathbb{Z}^N$, k_i relatively prime, such that

$$G_0' = \left\{ \sum_{i=1}^N x_i b_i \ \middle|\ \sum_{i=1}^N k_i x_i = 0 \right\}.$$

If there exist i, j such that $k_i k_j \leqslant 0$, then there exists $b' \in G_0' \setminus \{0\}$, $b' \geqslant 0$ with $\{x\lambda_0 - g_1 + xb' \in \mathrm{supp}(V)\} = (-\infty, m_0]$, a contradiction to the assumption (consider whether $b' \in G_1'$). Then $k_i k_j > 0$ for all $i, j \in [1, N]$. Hence we may assume that $k_i > 0$ for all $i \in [1, N]$. Let

$$g_0 = \sum_{i=1}^N s_i^{(N)} b_i.$$

Since $G_0' \oplus \mathbb{Z}g_0 = G$ we have

$$\sum_{i=1}^N s_i^{(N)} k_i = \pm 1.$$

By replacing g_0 with $-g_0$ if necessary, we may assume that $\sum_{i=1}^N s_i^{(N)} k_i = 1$. Choose a basis of G_1', say $\{b_1', b_2', \ldots, b_{N-2}'\}$. Take $b_{N-1}' = g_1$, $b_N' = g_0$, then $B' = \{b_1', b_2', \ldots, b_N'\}$ is a basis of G.

Subclaim. *For any $N_0 > 0$ there exists $m_0 \in \mathbb{N}$ such that $[m_0, \infty) \subset \{\sum_{i=1}^N k_i x_i \mid x_i \geqslant N_0, i = 1, 2, \ldots, N\}$.*

Proof. Note that $\sum_{i=1}^N s_i^{(N)} k_i = 1$. Choose $n_0 \in \mathbb{N}$ such that $n_0 + s_i^{(N)} \geqslant 0$ for all i. Note that $k_1 > 0$. Take $m_0 = \sum_{i=1}^N k_i (N_0 + k_1 n_0)$. Noting that

$$m_0 + t k_1 = \left(\sum_{i=1}^N k_i (N_0 + k_1 n_0) \right) + k_1 t, \quad \forall t > 0, \quad \text{and}$$

$$m_0 + tk_1 + i = \left(\sum_{i=1}^{N} k_i\big(N_0 + k_1 n_0 + i s_i^{(N)}\big)\right) + tk_1, \quad \text{for } 0 \leqslant i < k_1,$$

we have proved the subclaim. □

Denote $b'_{N-1} = g_1 = \sum_{i=1}^{N} s_i^{(N-1)} b_i$. Choose $N_0 \in \mathbb{N}$ such that $N_0 + t_0 s_i^{(N-1)} > 0$ for all i, then choose m_0 for this N_0 as in the subclaim above. By the subclaim for any $m \geqslant m_0$, there exists $(x_1, x_2, \ldots, x_n) \geqslant (N_0, N_0, \ldots, N_0)$, such that $m = \sum_{i=1}^{N} k_i x_i$. Then using the choice of $(k_1, \ldots, k_N)$ one can easily verify that

$$mb'_N - \sum_{i=1}^{N} x_i b_i \in G'_0.$$

Using this we can write $\lambda \in \lambda_0 - mb'_N + G'_0$ as

$$\lambda = \lambda_0 + h_0 - \sum_{i=1}^{N} x_i b_i, \qquad \lambda = \lambda_0 + h_0 + t_0 g_1 - \left(\left(\sum_{i=1}^{N} x_i b_i\right) + t_0 g_1\right),$$

where $h_0 \in G'_0$. Noting that

$$\sum_{i=1}^{N} x_i b_i, \; \left(\left(\sum_{i=1}^{N} x_i b_i\right) + t_0 g_1\right) \;\in\; \sum_{i=1}^{N} \mathbb{Z}^+ b_i,$$

and the fact that $\lambda_0 + h_0 \in \operatorname{supp}(V)$ or $\lambda_0 + h_0 + t_0 g_1 \in \operatorname{supp}(V)$, using Lemma 3.1(d5) we deduce

$$\lambda_0 - m_0 b'_N + G'_0 - \mathbb{Z}^+ b'_N \subset \operatorname{supp}(V). \tag{3.22}$$

Fix some $\lambda'_0 \in \Lambda_0 + (\sum_{i=1}^{N} \mathbb{N} b'_i)$ such that

$$\lambda'_0,\ \lambda'_0 \pm b'_i,\ \lambda'_0 \pm b'_i - b'_N \;\in\; \Lambda_0 + \sum_{i=1}^{N} \mathbb{Z}^+ b_i \quad \text{for all } i \in [1, N]. \tag{3.23}$$

Applying (3.21) to λ'_0 and G'_0 (replace G_0 by G'_0), since $N > 2$ we have $i_0 \in [1, N-1]$ and $s_0 \in \mathbb{Z}$ such that

$$\lambda'_0 + s_0 b'_{i_0} \in \operatorname{supp}(V). \tag{3.24}$$

Denote $b''_i = -s_0 b'_{i_0} - b'_i$ for all $i \in [1, N] \backslash i_0$ and $b''_{i_0} = -(s_0+1) b'_{i_0} - b'_N$. Fix a nonzero $v_{\lambda'_0 + s_0 b'_{i_0}} \in V_{\lambda'_0 + s_0 b'_{i_0}}$. It is easy to see that $\{b''_1, \ldots, b''_N\}$ forms a $\mathbb{Z}$-basis of G. By (3.23) we have

$$d_{b''_i} v_{\lambda'_0 + s_0 b'_{i_0}} = 0 \quad \text{for all } i \in [1, N].$$

So we have

$$d_b v_{\lambda_0'+s_0 b_{i_0}'} = 0 \quad \text{for all } b \in \left(\mathbb{Z}^+ b_1'' + \mathbb{Z}^+ b_2'' + \cdots + \mathbb{Z}^+ b_N''\right) \Big\backslash \left(\bigcup_{i=1}^{N} \mathbb{Z}^+ b_i''\right). \tag{3.25}$$

Hence $v_{\lambda_0'+s_0 b_{i_0}'}$ is a highest weight vector w.r.t.

$$B''' = \left\{2b_1'' + b_2'', b_1'' + b_2'', b_1'' + b_3'', \ldots, b_1'' + b_N''\right\}$$

which satisfies Lemma 3.1(d). Now by Lemma 3.1(b) there exists x_0 such that for any $x > x_0$,

$$\lambda_0' + s_0 b_{i_0}' + x\left(\left(2b_1'' + b_2''\right) + \left(b_1'' + b_2''\right) + \left(b_1'' + b_3''\right) + \cdots + \left(b_1'' + b_N''\right)\right) \notin \operatorname{supp}(V). \tag{3.26}$$

Write $(2b_1'' + b_2'') + (b_1'' + b_2'') + (b_1'' + b_3'') + \cdots + (b_1'' + b_N'') = h_0 - l' b_N'$, $\lambda_0' = \lambda_0 - m_0 b_N' + g_0 + l b_N'$ where $g_0, h_0 \in G_0'$, $l, l' \in \mathbb{Z}$, $l' > 0$ (since $G_0' = \sum_{i=1}^{N-1} \mathbb{Z} b_i'$). Then for sufficiently large x,

$$\begin{aligned} &\lambda_0' + s_0 b_{i_0}' + x\left(\left(2b_1'' + b_2''\right) + \left(b_1'' + b_2''\right) + \left(b_1'' + b_3''\right) + \cdots + \left(b_1'' + b_N''\right)\right) \\ &\quad \in \lambda_0 - m_0 b_N' + (l - l'x) b_N' + G_0' \subset \lambda_0 - m_0 b_N' + G_0' - \mathbb{N} b_N', \end{aligned} \tag{3.27}$$

which contradicts (3.22). Thus Claim 1 follows. □

Denote $\bar{G}_t = t b_1 + \mathbb{Z} b_2 + \mathbb{Z} b_3 + \cdots + \mathbb{Z} b_N$ for $t \in \mathbb{Z}$.

Claim 2. *If for $\lambda_0 \in \Lambda_0 + G$, $g_1, g_1' \in \bar{G}_0 \setminus \{0\}$, and subgroups of $\bar{G}_0$: G_1, G_1' with $\bar{G}_0 = \mathbb{Z} g_1 \oplus G_1$, $\bar{G}_0 = \mathbb{Z} g_1' \oplus G_1'$, we have*

$$\lambda_0 - \mathbb{N} g_1 + G_1,\ \lambda_0 - \mathbb{N} g_1' + G_1' \ \subset\ \operatorname{supp}(V),$$

then $G_1 = G_1'$.

Proof. Suppose that $G_1 \neq G_1'$. Fix $0 \neq f_1 = \sum_{i=1}^n u_i b_i \in \bar{G}_0$ (then $u_1 = 0$) satisfying

$$f_1 \in -\mathbb{N} g_1 + G_1 \quad \text{and} \quad \sum_{i=1}^n u_i x_i = 0$$

for all $(x_1, \ldots, x_N) \in \mathbb{Z}^n$ with $\sum_{i=1}^n x_i b_i \in G_1$, and fix $0 \neq f_1' = \sum_{i=1}^n u_i' b_i \in \bar{G}_0$ satisfying

$$f_1' \in -\mathbb{N} g_1' + G_1' \quad \text{and} \quad \sum_{i=1}^n u_i' x_i = 0$$

for all $(x_1, \ldots, x_N) \in \mathbb{Z}^n$ with $\sum_{i=1}^n x_i b_i \in G_1'$. (We simply write $f \perp G_1$, $f' \perp G_1'$.) Since $G_1 \neq G_1'$ we see that $\mathbb{Z}f_1' \cap \mathbb{Z}f_1 = \{0\}$. Hence we can choose a base $B' = \{b_1', b_2', \ldots, b_{N-1}'\}$ of $\bar{G}_0$ as follows: Fix $b_1' = \sum_{i=1}^N s_i^{(1)} b_i \in \bar{G}_0$ such that $s_1^{(1)}, s_2^{(1)}, \ldots, s_N^{(1)}$ are relatively prime,

$$\sum_{i=1}^N u_i s_i^{(1)} > 0 \quad \text{and} \quad \sum_{i=1}^N u_i' s_i^{(1)} < 0, \tag{3.28}$$

and extend it to a $\mathbb{Z}$ basis $B' = \{b_1', b_2', \ldots, b_{N-1}'\}$ of $\bar{G}_0$. By replacing $b_j' (j > 1)$ with $b_j' + m b_1', m \gg 0$ if necessary, we may assume that $b_j' = \sum_{i=1}^N s_i^{(j)} b_i$ satisfies

$$\sum_{i=1}^N s_i^{(j)} u_i > 0 \quad \text{and} \quad \sum_{i=1}^N s_i^{(j)} u_i' < 0 \quad \text{for all } j \in [1, N-1]. \tag{3.29}$$

Since $f \perp G_1$, $f' \perp G_1'$, we see that

$$b_i' \in -\mathbb{N}g_1 + G_1 \quad \text{and} \quad b_i' \in \mathbb{N}g_1' + G_1' \quad \text{for all } i \in [1, N-1]. \tag{3.30}$$

Take $b_N' = b_1$. Hence $B' = \{b_1', b_2', \ldots, b_N'\}$ is a basis of G. Fix

$$\lambda_0' = \lambda_0 + t_0 b_N' + \bar{g}_0, \tag{3.31}$$

where $t_0 > 0$, $\bar{g}_0 \in \bar{G}_0$ are such that

$$\lambda_0',\ \lambda_0' \pm b_i',\ \lambda_0' \pm b_i' - b_N' \ \in\ \Lambda_0 + \sum_{i=1}^N \mathbb{Z}^+ b_i \quad \text{for all } i \in [1, N].$$

So

$$\lambda_0',\ \lambda_0' \pm b_i',\ \lambda_0' \pm b_i' - b_N' \ \notin\ \operatorname{supp}(V) \quad \text{for all } i \in [1, N]. \tag{3.32}$$

Now applying (3.21) to λ_0' and $\bar{G}_0$, since $N > 2$ we see that there exist some $i_0 \in [1, N-1]$ and $s_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\lambda_0' + s b_{i_0}' \in \operatorname{supp}(V) \quad \text{for all } s \geqslant s_0, \quad \text{or} \\ &\lambda_0' + s b_{i_0}' \in \operatorname{supp}(V) \quad \text{for all } s \leqslant -s_0. \end{aligned} \tag{3.33}$$

We may assume that (3.33) holds (if $\lambda_0' + s b_{i_0}' \in \operatorname{supp}(V)$ for all $s \leqslant -s_0$, then the remaining arguments are exactly the same, using G_1). Denote $b_i'' = -s_0 b_{i_0}' - b_i'$ for all $i \in [1, N] \backslash i_0$ and $b_{i_0}'' = -(s_0 + 1) b_{i_0}' - b_N'$.

From (3.30) we see that

$$b_1'' + \cdots + b_N'' \in -\sum_{i=1}^{N} b_i' - s_0 n b_{i_0}' - \mathbb{N} b_1 \subset -\mathbb{N} g_1' + G_1' - \mathbb{N} b_1. \tag{3.34}$$

Fix a nonzero $v_{\lambda_0' + s_0 b_{i_0}'} \in V_{\lambda_0' + s_0 b_{i_0}'}$. It is easy to see that $\{b_1'', \ldots, b_N''\}$ forms a $\mathbb{Z}$-basis of G and

$$d_{b_i''} v_{\lambda_0' + s_0 b_{i_0}'} = 0 \quad \text{for all } i \in [1, N]. \tag{3.35}$$

So we have

$$d_b v_{\lambda_0' + s_0 b_{i_0}'} = 0 \quad \text{for all } b \in \left(\mathbb{Z}^+ b_1'' + \mathbb{Z}^+ b_2'' + \cdots + \mathbb{Z}^+ b_N''\right) \Big\backslash \left(\bigcup_{i=1}^{N} \mathbb{Z}^+ b_i''\right). \tag{3.36}$$

Hence $v_{\lambda_0' + s_0 b_{i_0}'}$ is a highest weight vector w.r.t.

$$B''' = \left\{2b_1'' + b_2'', b_1'' + b_2'', b_1'' + b_3'', \ldots, b_1'' + b_N''\right\}.$$

Now by Lemma 3.1(b) there exists some x_0 such that for any $x > x_0$ we have

$$\lambda_0' + s_0 b_{i_0}' + x\left(b_1'' + b_2'' + \cdots + b_N''\right) \notin \operatorname{supp}(V). \tag{3.37}$$

From (3.31) we can write $\lambda_0' = \lambda_0 + t_0 b_N' + l g_1' + h$ where $h \in G_1'$, $l \in \mathbb{Z}$. Using (3.34), for sufficiently large x we have

$$\lambda_0' + s_0 b_{i_0}' + x\left(b_1'' + b_2'' + \cdots + b_N''\right) \in \lambda_0 - \mathbb{N} g_1' + G_1' - \mathbb{N} b_1 \subset \operatorname{supp}(V) \tag{3.38}$$

since $\lambda_0 - \mathbb{N} g_1' + G_1' \in \operatorname{supp}(V)$. This is a contradiction to (3.37). Hence $G_1 = G_1'$ and Claim 2 follows. □

Denote $V_{\Lambda_0 + \bar{G}_t} = \bigoplus_{b \in \bar{G}_t} V_{\Lambda_0 + b}$ for $t \in \mathbb{Z}$. It is easy to see that $V_{\Lambda_0 + \bar{G}_t}$ is a $\mathrm{Vir}[b_2, \ldots, b_N]$-module. For any $0 \neq \lambda \in \operatorname{supp}(V_{\Lambda_0 + \bar{G}_t})$ (we refer to (3.21) for the existence), λ is a weight of a nontrivial irreducible $\mathrm{Vir}[b_2, \ldots, b_N]$-subquotient of $V_{\Lambda_0 + \bar{G}_t}$. From the inductive hypothesis and Claim 1, we know that such a nontrivial irreducible $\mathrm{Vir}[b_2, \ldots, b_N]$-module is isomorphic to $M(g_t, G_t, V'(\alpha_t, \beta_t, G_t))$ for suitable $\alpha_t, \beta_t \in \mathbb{C}$, and g_t, G_t with $\bar{G}_0 = \mathbb{Z} g_t \oplus G_t$. Thus from Lemma 3.8, if $0 \neq \lambda \in \operatorname{supp}(V) \cap (\Lambda_0 + g_t + \bar{G}_0)$, then there exists a corank 1 subgroup G_λ of $\bar{G}_0$ such that

$$\lambda + G_\lambda \subset \operatorname{supp}(V) \cup \{0\}. \tag{3.39}$$

Combining this with Claims 1 and 2, we deduce that for any $t \in \mathbb{Z}$ there exist a corank 1 subgroup G_t in $\bar{G}_0$, $\alpha_t \in \Lambda_0 + \bar{G}_t$ and $g_t \in \bar{G}_0$ such that $\bar{G}_0 = \mathbb{Z} g_t \oplus G_t$ and

$$\operatorname{supp}(V_{\Lambda_0 + \bar{G}_t}) \setminus \{0\} = \left(\alpha_t - \mathbb{Z}^+ g_t + G_t\right) \setminus \{0\}. \tag{3.40}$$

In particular,

$$\alpha_t - \mathbb{N}g_t + G_t \subset \operatorname{supp}(V_{\Lambda_0+\bar{G}_t}). \tag{3.41}$$

Lemma 3.1(d5) and Lemma 3.8 ensure that

$$\alpha_{t+1} - b_1 - \mathbb{N}g_{t+1} + G_{t+1},\ \alpha_t - \mathbb{N}g_t + G_t\ \subset\ \operatorname{supp}(V_{\Lambda_0+\bar{G}_t}).$$

It follows from Claim 2 that $G_t = G_{t+1}$ for all $t \in \mathbb{Z}$. Thus there exist a corank 1 subgroup G_0 in $\bar{G}_0$, $\alpha_t \in \Lambda_0 + \bar{G}_t$ and $g_0 \in \bar{G}_0$ with $\bar{G}_0 = \mathbb{Z}^+ g_0 \oplus G_0$ such that either

$$\begin{aligned} \operatorname{supp}(V_{\Lambda_0+\bar{G}_t}) \setminus \{0\} &= \big(\alpha_t + \mathbb{Z}^+ g_0 + G_0\big) \setminus \{0\}, \quad \text{or} \\ \operatorname{supp}(V_{\Lambda_0+\bar{G}_t}) \setminus \{0\} &= \big(\alpha_t - \mathbb{Z}^+ g_0 + G_0\big) \setminus \{0\}. \end{aligned} \tag{3.42}$$

If there exists $t \in \mathbb{Z}$ such that

$$\begin{aligned} \operatorname{supp}(V_{\Lambda_0+\bar{G}_t}) \setminus \{0\} &= \big(\alpha_t - \mathbb{Z}^+ g_0 + G_0\big) \setminus \{0\}, \\ \operatorname{supp}(V_{\Lambda_0+\bar{G}_{t+1}}) \setminus \{0\} &= \big(\alpha_{t+1} + \mathbb{Z}^+ g_0 + G_0\big) \setminus \{0\}. \end{aligned}$$

Similarly we have $\lambda_1, \lambda_2 \in \Lambda_0 + \bar{G}_t$ such that

$$\lambda_1 - \mathbb{N}g_0 + G_0,\ \lambda_2 + \mathbb{N}g_0 + G_0\ \subset\ \operatorname{supp}(V),$$

which contradicts Claim 1. So we may assume that

$$\operatorname{supp}(V_{\Lambda_0+\bar{G}_t}) \setminus \{0\} = \big(\alpha_t - \mathbb{Z}^+ g_0 + G_0\big) \setminus \{0\}, \quad \forall t \in \mathbb{Z}. \tag{3.43}$$

Hence we may assume that $\alpha_t \in \Lambda_0 + \mathbb{Z}g_0 + \mathbb{Z}b_1$. Then for any $\lambda \in \operatorname{supp}(V)$, we have

$$\lambda + G_0 \subset \operatorname{supp}(V) \cup \{0\}. \tag{3.44}$$

Consider the $\mathrm{Vir}[g_0, b_1]$-module $V_{\Lambda_0+\mathbb{Z}g_0+\mathbb{Z}b_1}$ where $g_0 \in G_0 \setminus \{0\}$ as before. From (3.43), $V_{\Lambda_0+\mathbb{Z}g_0+\mathbb{Z}b_1}$ has a nontrivial irreducible $\mathrm{Vir}[g_0, b_1]$-subquotient (we refer to the last paragraph in the proof of Lemma 3.2). Hence there exist some $\lambda_0' \in \Lambda_0 + \mathbb{Z}g_0 + \mathbb{Z}b_1$ and a basis b_0', g_0' of $\mathbb{Z}g_0 + \mathbb{Z}b_1$ such that

$$\lambda_0' + \mathbb{Z}g_0' \subset \operatorname{supp}(V_{\Lambda_0+\mathbb{Z}g_0+\mathbb{Z}b_1}).$$

From (3.43) with $t = 0$ we know that $g_0' \notin \mathbb{Z}g_0$. Hence by (3.44)

$$\lambda_0' + \mathbb{Z}g_0' + G_0 \subset \operatorname{supp}(V) \cup \{0\},$$

and $\mathbb{Z}b_0' + (\mathbb{Z}g_0' + G_0) = G$, which contradicts Claim 1. This completes the proof of the theorem. □

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