JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **202,** 727–746 (1996)
ARTICLE NO. 0344

Measure Driven Differential Inclusions*

G. N. Silva and R. B. Vinter[†]

Interdisciplinary Research Centre for Process Systems and Department of Electrical and Electronic Engineering, Imperial College, London SW7-2BT, England

Submitted by Hélène Frankowska

Measure driven differential inclusions arise when we attempt to derive necessary conditions of optimality for optimal impulsive control problems with nonsmooth data. We introduce the concept of a robust solution to a measure driven inclusion, which extends to a multifunction setting interpretations of solutions to measure driven differential equations provided by Dal Maso and Rampazzo and others. Closure properties of sets of robust solutions are established, and notions of relaxation investigated. Implications for optimality conditions for impulsive control problems are pursued in a companion paper.

1. INTRODUCTION

In this paper we study measure driven differential inclusions (MDIs) of the type:

$$\begin{cases} dx(t) \in F_1(t, x(t))\, dt + F_2(t, x(t))\, \mu(dt) & \text{on } [0,1] \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Robust solutions are defined and their closure properties are investigated.

In (1.1) F_1: $[0,1] \times \Re^n \Rightarrow \Re^n$ and F_2: $[0,1] \times \Re^n \Rightarrow \Re^n$ are given multifunctions. The "driving measure," μ, is some non-negative, scalar valued measure on the Borel subsets of $[0,1]$, and the "initial value," x_0, is a point in $\Re^n$.

* The support of EPSERC and the Brazilian National Science Council is gratefully acknowledged.

[†] E-mail: gsilva@nimitz.ibilce.unesp.br, r.vinter@ps.ic.ac.uk

0022-247X/96 $18.00

The significance of dynamical descriptions of this nature has long been recognized in the singleton valued case

$$\begin{cases} dx(t) = f_1(t, x(t))\, dt + f_2(t, x(t))\, \mu(dt) \\ x(0) = x_0 \end{cases} \tag{1.2}$$

in which f_1: $[0,1] \times \Re^n \to \Re^n$ and f_2: $[0,1] \times \Re^n \to \Re^n$ are given functions. In engineering applications, arising principally in flight mechanics [5, 6], the driving measure is an idealization of a non-negative, scalar valued control which enters ‘‘linearly’’ into the dynamics and which takes large values over small intervals (an ‘‘impulse control’’). Such controls are applied in midcourse guidance, and correspond to ‘‘rapid fuel burn’’ to redirect the motion.

For the measure μ to be so regarded, we require that solutions to (1.2), corresponding to μ, are ‘‘close’’ to solutions of a conventional differential equation in which μ is approximated by a conventional control $u(t)$, interpreted as the measure $u(t)\, dt$.

These considerations have given rise to concepts of robust solutions of measure driven differential equations, developed by Dal Maso and Rampazzo [3] and having their roots in reparameterization techniques of Rishel [7] and Warga [10]. Our goal is to show that there is an analogous concept of robust solutions for measure driven differential inclusions with associated closure properties. The primary motivation for considering ‘‘set valued’’ dynamics is their applications to optimal control [8]. If we are to derive first order optimality conditions for impulse control problems with nondifferentiable data, we must give meaning to the relationship governing the costate function which is a measure driven differential inclusion and also be able to analyse the effects of limit taking. But measure driven differential inclusions also provide a convenient framework for formulating optimal control problems involving conventional and impulse controls when we wish to focus attention on state trajectories rather than on the controls from which they originate.

There follows a brief description of some notational conventions adhered to in this paper.

B denotes the open unit ball in Euclidean space. $C([0,1]; \Re^n)$ denotes the vector space of continuous $\Re^n$-valued functions on $[0,1]$ with supremum norm, and $C^*([0,1]; \Re^n)$ its topological dual.

$C^+([0,1]; \Re^n) \subset C^*([0,1]; \Re^n)$ is the cone of functionals taking non-negative values on non-negative functions.

$AC([0,1]; \Re^n)$ is the space of absolutely continuous $\Re^n$-valued functions on $[0,1]$.

$BV^+([0,1]; \Re^n)$ denotes the vector space of $\Re^n$-valued functions on $[0,1]$ of bounded variation and which are continuous from the right on

(0, 1). The total variation of an element $x \in BV^+([0,1]; \Re^n)$ is written $\|x\|_{T.V.}$. The Borel measure associated with some $x \in BV^+([0,1]; \Re^n)$ is denoted dx.

For the sake of brevity we often do not distinguish between elements in $C^*([0,1]; \Re^n)$ and the Borel measures which represent them.

The weak* topology on $BV^+([0,1]; \Re^n)$ refers to the weak* topology on $(\Re^n \times C([0,1]; \Re^n))^*$ under the isomorphism

$$x \to (x(0), dx).$$

Thus "$x_i \to x$ (weakly*)" indicates that $x_i(0) \to x(0)$ and $dx_i \to dx$ (weakly* in $C^*([0,1]; \Re^n)$). For simplicity we write $C(0,1)$ in place of $C([0,1]; \Re^1)$, $C^*(0,1)$ in place of $C^*([0,1]; \Re^1)$, and so on.

$\mathscr{L}$ denotes the Lebesgue subsets of $[0,1]$ and $\mathscr{B}$ the Borel sets in $\Re^k$ and $\mathscr{L} \times \mathscr{B}$ the product σ-field.

2. CHANGE OF VARIABLES

We describe a change of variables technique, previously used in Rishel [7], Warga [10], Dal Maso and Rampazzo [3], and elsewhere, which will provide a representation of robust solutions to measure driven differential inclusions in terms of solutions to conventional differential inclusions.

Fix a measure $\mu \in C^+(0,1)$. Let D be its distribution function

$$D(t) := \begin{cases} \int_{[0,t]} \mu(d\tau), & t \in (0,1] \\ 0 & \text{if } t = 0. \end{cases}$$

Define the *reparameterization function* η corresponding to μ to be

$$\eta(t) := \begin{cases} \left(t + \int_{[0,t]} \mu(d\tau)\right)/(1 + \mu([0,1])), & t \in (0,1] \\ 0 & \text{if } t = 0. \end{cases}$$

Evidently, η is an element in $BV^+([0,1]; \Re^n)$ which is non-negative and strictly increasing on $[0,1]$. Now define θ: $[0,1] \to [0,1]$ to be

$$\theta(s) := \sup_{t \in [0,1]} \{t : s \geq \eta(t)\} \qquad \forall s \in [0,1].$$

Let $\{t_i\}$ be an enumeration of the atoms of μ, and let $S_i (= [\sigma_i', \sigma_i''])$ be the subintervals $S_i := \theta^{-1}(\{t_i\})$ for $i = 1, 2, \ldots$. Now define the function γ:

$[0,1] \to \Re^+$ to be

$$\gamma(s) := \begin{cases} D(\theta(s)) \\ \qquad \text{if } s \in [0,1] \setminus \cup_{i=1}^{\infty} S_i \\ D(t_i) + \dfrac{(s - \sigma_i')}{(\sigma_i'' - \sigma_i')}(D(t_i) - D(t_i^-)) \\ \qquad \text{for } s \in S_i,\ i = 1, 2, \dots . \end{cases}$$

(In this formula $D(t_i) - D(t_i^-)$ is interpreted as $D(0^+) - D(0)$ if $t_i = 0$.)

Following Dal Maso and Rampazzo [3], we call the function (θ, γ): $[0,1] \to (\Re^+)^2$ the *graph completion* of the measure μ. This is because it results from "filling in" with straight line segments the graph of D and reparameterizing the resulting curve in $\Re^2$.

Properties of the graph completion, listed in the following proposition, will be required. Items (i) and (iv) are proved in [3]. Items (ii) and (iii) are consequences of the standard change of variables lemma [4, Theorem 6.9, p. 155]; the connection is provided by the observation that μ can be interpreted as the measure on the Borel subsets of $[0,1]$ induced by $\dot{\gamma}(s)\,ds$ under the mapping θ.

PROPOSITION 2.1. *Let $\theta, \gamma)$ be the graph completion of $\mu \in C^+(0,1)$. Then*

(i) *θ and γ are Lipschitz continuous, non-negative functions and*

$$\dot{\theta}(s) + \dot{\gamma}(s) = 1 + \mu([0,1]) \qquad \mathcal{L}\text{-a.e.}$$

(ii) *For any Borel measurable, μ integrable function h and Borel set $T \subset [0,1]$ we have*

$$\int_{\theta^{-1}(T)} h(\theta(s))\dot{\gamma}(s)\,ds = \int_T h(\tau)\mu(d\tau).$$

(iii) *For any $\mathcal{L}$-integrable function g and Borel set $S \subset [0,1]$, $\theta(S)$ is also a Borel set and*

$$\int_S g(\theta(s))\dot{\theta}(s)\,ds = \int_{\theta(S)} g(\tau)\,d\tau.$$

(iv) *Let μ_i be a sequence of elements in $C^+(0,1)$ and let $\{(\theta_i, \gamma_i)\}$ be the corresponding graph completions. Suppose that $\mu_i \to \mu$ (weakly*), then $(\theta_i, \gamma_i) \to (\theta, \gamma)$ uniformly and $(\dot{\theta}_i, \dot{\gamma}_i) \to (\dot{\theta}, \dot{\gamma})$ weakly in L^1.*

3. ROBUST SOLUTIONS

Consider the measure driven differential inclusion of Section 1:

$$\begin{cases} dx(t) \in F_1(t, x(t))\, dt + F_2(t, x(t))\, \mu(dt) & \text{on } [0,1] \\ x(0) = x_0 \end{cases} \tag{3.1}$$

in which, as before, F_1: $[0,1] \times \Re^n \Rightarrow \Re^n$ and F_2: $[0,1] \times \Re^n \Rightarrow \Re^n$ are given malfunctions.

It is natural to define solutions to (3.1) via the related integral inclusion

$$x(t) \in x_0 + \int_0^t F_1(\tau, x(\tau))\, d\tau + \int_{[0,t)} F_2(\tau, x(\tau))\, \mu(d\tau)$$

(taken in a "selector" sense). However, a choice must be made regarding interpretation of the final term on the right; what this should be is not immediately clear, since x is possibly discontinuous at the atoms of μ. A simple choice is

$$\int_{[0,t]} F_2(\tau, x(\tau^-))\, \mu(d\tau)$$

involving the "left limit" $x(\tau^-)$. This however does not provide a concept of solution which, in general, has the sought-for closure properties. An alternative approach is to substitute in place of $F_2(\tau, x(\tau^-))$ a multifunction which more effectively takes account of the interaction between the instantaneous changes in the state and the atoms of μ. Bearing this in mind we define $\tilde{F}_2$: $[0,1] \times \Re^n \times [0,\infty) \Rightarrow \Re^n$ to be the multifunction

$$\tilde{F}_2(t, v; \alpha) := \Big\{ \alpha^{-1}[\xi(1) - \xi(0)] : \xi \in \mathrm{AC}^1([0,1]; \Re^n),$$
$$\dot{\xi}(\sigma) \in \alpha F_2(t, \xi(\sigma)) \text{ a.e., and } \xi(0) = v \Big\}$$

if $\alpha > 0$ and

$$\tilde{F}_2(t, v; 0) = F_2(t, v).$$

DEFINITION 3.1. We say that a function $x \in \mathrm{BV}^+([0,1]; \Re^n)$ is a robust solution to (3.1) (corresponding to $\mu \in C^+(0,1)$ and $x_0 \in \Re^n$) if $x(0) = x_0$ and there exists an $\mathscr{L}$-integrable function ϕ_1 and μ-integrable function ϕ_2 such that

$$\phi_1(t) \in F_1(t, x(t)) \qquad \mathscr{L}\text{-a.e.}$$

$$\phi_2(t) \in \tilde{F}_2(t, x(t^-); \mu(\{t\})) \qquad \mu\text{-a.e.}$$

and

$$x(t) = x(0) + \int_0^t \phi_1(\tau) d(\tau) + \int_{[0,t]} \phi_2(\tau) \mu(d\tau) \qquad \text{for all } t \in (0,1].$$

4. REPARAMETERIZATION

The following theorem provides the link between robust solutions of the measure driven differential inclusions (3.1) and ordinary differential inclusions.

THEOREM 4.1. *Suppose that the data for* (3.1) *satisfy the hypotheses*:

- F_1 *has values closed sets and is* $\mathscr{L} \times \mathscr{B}$ *measurable and*
- F_2 *has values closed sets and is Borel measurable*.

Fix a measure $\mu \in C^+(0,1)$ *and an initial state* x_0. *Let* (θ, γ) *be the graph completion of* μ *and* η *the reparameterization function*.

(i) *Suppose* $x(\cdot) \in \mathrm{BV}^+([0,1]; \Re^n)$ *is a robust solution to* (3.1) (*corresponding to* μ *and* x_0). *Then there exists a solution* $y(\cdot) \in \mathrm{AC}([0,1]; \Re^n)$ *to*

$$\begin{cases} \dot{y}(s) \in F_1(\theta(s), y(s))\dot{\theta}(s) + F_2(\theta(s), y(s))\dot{\gamma}(s) \\ y(0) = x_0 \end{cases} \tag{4.1}$$

for which

$$x(t) = y(\eta(t)) \qquad \textit{for all } t \in [0,1]. \tag{4.2}$$

Conversely,

(ii) *Suppose* $y(\cdot) \in \mathrm{AC}([0,1]; \Re^n)$ *is a solution to* (4.1). *Then there exists a robust solution* $x(\cdot) \in \mathrm{BV}^+([0,1]; \Re^n)$ *to* (3.1) *for which* (4.2) *is satisfied*.

(iii) *Take any robust solution* x *to* (3.1). *Let* y *be a solution to* (4.1) *such that* (4.2) *is satisfied. Then*

$$\|x\|_{\text{T.V.}} \le \|y\|_{\text{T.V.}}$$

In the above, solutions to (4.1) are taken in the ‘‘selector’’ sense, i.e., there exist Lebesgue measurable functions ψ_1, ψ_2 such that $\psi_1(s) \in$

$F_1(\theta(s), y(s))$ and $\psi_2(s) \in F_2(\theta(s), y(s))$, $\mathscr{L}$-a.e., $\psi_1 \dot{\theta}$ and $\psi_2 \dot{\gamma}$ are integrable, and

$$\dot{y}(s) = y(0) + \int_0^s \big(\psi_1(\sigma) \dot{\theta}(\sigma) + \psi_2(\sigma) \dot{\gamma}(\sigma) \big)\, d\sigma \qquad \text{for all } s \in [0,1].$$

A proof of the theorem is given in Section 6.

The significance of this result is that known properties of solutions to differential inclusions translate into analogous properties of solutions to measure driven differential inclusions via the "reparameterized" differential inclusion (4.1). We have, for example, an existence theorem applying under the following hypotheses:

(H1) *F_1 and F_2 have values closed convex sets, $F_1(\cdot, x)$ is $\mathscr{L}$-measurable, and $F_2(\cdot, \cdot)$ is Borel measurable.*

(H2) *There exist $c_1(\cdot) \in L^1$ and $c_2 \in \Re$ such that*

$$F_1(t,x) \subset c_1(t)(1 + |x|) B \text{ and } F_2(t,x) \subset c_2(1+|x|) B \qquad \text{for all } x \in \Re^n,\ t \in [0,1].$$

(H3) *There exist $k_1(\cdot) \in L^1$ and $k_2 \in \Re$ such that*

$$F_1(t,x) \subset F_1(t,y) + k_1(t)|x-y| B \qquad \text{for all } x, y \in \Re^n$$

and

$$F_2(t,x) \subset F_2(t,y) + k_2 |x-y| B \qquad \text{for all } x, y \in \Re^n.$$

COROLLARY 4.2. *Suppose that the multifunctions F_1, F_2: $[0,1] \times \Re^n \to \Re^n$ satisfy hypotheses* (H1)–(H3). *Fix a measure $\mu \in C^+(0,1)$ and an initial state x_0. Then there exists a robust solution to* (3.1) (*corresponding to μ and x_0*).

Proof. The data for the reparameterized inclusion (4.1) are easily shown to satisfy the hypotheses under which existence of solutions is assured (see, e.g., [1]). This yields a solution y to the reparameterized inclusion. But then (3.1) has a robust solution by Theorem 4.1. ∎

5. CLOSURE PROPERTIES

In this section conditions are given under which robust solutions to perturbations of a nominal measure driven differential inclusion (3.1) yield a solution to the nominal differential inclusion in the limit. Under these conditions, then, robust solutions are truly "robust." The perturbations we allow include changes to the driving measure and also to the multifunction F_1. Perturbations of this nature need to be considered in the derivation of necessary conditions of optimality for optimal control problems involving measure driven differential equations.

Consider a sequence of measure driven differential inclusions

$$\begin{cases} dx_i(t) \in F_1^{(i)}(t, x_i(t))\, dt + F_2(t, x_i(t))\, \mu_i(dt) & \text{on } [0,1] \\ x_i(0) = x_0^i, \end{cases} \tag{5.1}$$

$i = 1, 2, \ldots$ approximating a nominal measure differential inclusion

$$\begin{cases} dx(t) \in F_1(t, x(t))\, dt + F_2(t, x(t))\, \mu(dt) & \text{on } [0,1] \\ x(0) = x_0. \end{cases} \tag{5.2}$$

Here $F_1^{(i)}$: $[0,1] \times \Re^n \Rightarrow \Re^n$, $i = 1, 2, \ldots,$ F_1: $[0,1] \times \Re^n \Rightarrow \Re^n$, and F_2: $[0,1] \times \Re^n \to \Re^n$ are given multifunctions. μ_i, $i = 1, 2, \ldots,$ and μ are elements in $C^+(0,1)$, and x_0^i, $i = 1, 2, \ldots,$ and x_0 are n-vectors.

The associated reparameterized equations are

$$\begin{cases} \dot{y}_i(s) \in F_1^{(i)}(\theta_i(s), y_i(s))\dot{\theta}_i(s) + F_2(\theta_i(s), y_i(s))\dot{\gamma}_i(s) & \text{a.e.} \\ y_i(0) = x_0^i \end{cases} \tag{5.3}$$

and

$$\begin{cases} \dot{y}(s) \in F_1(\theta(s), y(s))\dot{\theta}(s) + F_2(\theta(s), y(s))\dot{\gamma}(s) & \text{a.e.} \\ y(0) = x_0. \end{cases} \tag{5.4}$$

Here, (θ_i, γ_i) is the graph completion of μ_i, $i = 1, 2, \ldots,$ and (θ, γ) is the graph completion of μ. We shall refer also to the reparameterization function η_i: $[0,1] \to [0,1]$ of μ_i, $i = 1, 2, \ldots,$ and the reparameterization function η of μ.

THEOREM 5.1. *Consider multifunctions F_1, $F_1^{(i)}$, $i = 1, 2, \ldots,$ and F_2 with domain $[0,1] \times \Re^n$ and taking value compact subsets of $\Re^n$. Assume that*

- *$F_1^{(i)}(t, \cdot)$, $i = 1, 2, \ldots,$ and $F_1(t, \cdot)$ have closed graphs and $F_1^{(i)} i = 1, 2, \ldots,$ and F_1 are $\mathscr{L} \times \mathscr{B}$ measurable.*
- *$F_1(t, x)$ is convex for all $(t, x) \in [0,1] \times \Re^n$.*
- *$F_2(\cdot, \cdot)$ has closed graph and values convex sets.*

Assume further that

- *$\mathscr{L}$-measure $\{t: F_1^{(i)}(t, \cdot) \neq F_1(t, \cdot)\} \to 0$, as $i \to \infty$.*

Take a sequence $\{x_0^i\}$ *in* $\Re^n$ *and a sequence* $\{\mu_i\}$ *in* $C^+(0,1)$, *and elements* $x_0 \in \Re^n$ *and* $\mu \in C^+(0,1)$. *Take also a sequence* $\{x_i\} \in \mathrm{BV}^+([0,1];\Re^n)$ *such that* x_i *is a robust solution to* (5.1) *for each* i, *and*

$$x_0^{(i)} \to x_0 \text{ and } \mu_i \to \mu \text{ (weakly*)} \qquad \text{as } i \to \infty.$$

Assume that there exists $\beta(t) \in L^1$ *and* $c > 0$ *such that* $F_1^{(i)}(t, x_i(t)) \subset \beta(t)B$ *a.e. and* $F_2(t, x_i(t)) \subset cB$ *for all* t.

Then there exists a sequence $\{y_i\} \subset \mathrm{AC}([0,1];\Re^n)$ *such that* y_i *is a solution to* (5.3) *for each* i, *a solution* y *to* (5.4), *and a solution* x *to* (5.2) *such that*

$$x_i(t) = y_i(\eta_i(t)) \qquad \text{for all } t \in [0,1]$$

and

$$x(t) = y(\eta(t)) \qquad \text{for all } t \in [0,1].$$

Along a subsequence we have

$$x_i \to x \text{ (weakly*)}$$

$$x_i(t) \to x(t) \qquad \text{for all } t \in \big([0,1] \setminus \mathscr{M}_\mu\big) \cup \{0,1\}$$

(*where* $\mathscr{M}_\mu$ *denotes the atoms of* μ) *and*

$$y_i \to y \text{ strongly in } C([0,1];\Re^n).$$

A proof of Theorem 5.1 is given in Section 7.

The above theorem, specialized to the case when perturbations only in the driven measures and initial condition are considered, permits us to investigate the closure properties of sets of robust solutions.

Consider again the measure driven differential inclusion (now labelled “S”):

$$dx(t) \in F_1(t, x(t))\, dt + F_2(t, x(t))\, \mu(dt) \qquad \text{on } [0,1]. \tag{S}$$

By analogy with known properties of differential inclusions, we can expect that limits of robust solutions to (S) are identified with robust solutions of a “relaxed” measure driven inclusion in which $F_1(t,x)$ and $F_2(t,x)$ are replaced by their convex hulls:

$$dx(t) \in \operatorname{co} F_1(t, x(t))\, dt + \operatorname{co} F_2(t, x(t))\, \mu(dt) \qquad \text{on } [0,1]. \tag{$\mathrm{S_{relaxed}}$}$$

Such an identification is possible, as is now shown.

Fix a compact set $C \in \Re^n$ (an endpoint constraint set) and a weak* compact set $M \subset C^+(0,1)$ (a constraint set for the driving measures), and define

$$\mathscr{S} := \left\{ \begin{array}{l} x \in \mathrm{BV}^+([0,1];\Re^n) \colon x(\cdot) \text{ is a robust solution to (S)} \\ \text{corresponding to some } \mu \in M \text{ such that } x(0) \in C \end{array} \right\}$$

and

$$\mathscr{S}_{(\text{relaxed})} := \left\{ \begin{array}{l} x \in \mathrm{BV}^{+}([0,1]; \Re^{n}) \colon x(\cdot) \text{ is a robust solution to } (\mathrm{S}_{\text{relaxed}}) \\ \text{corresponding to some } \mu \in M \text{ such that } x(0) \in C \end{array} \right\}.$$

COROLLARY 5.2. *Assume that the multifunctions* F_1, F_2: $[0,1] \times \Re^n \Rightarrow \Re^n$ *satisfy hypotheses* (H1)–(H3) *of Section* 4. *Then*

(i) $\mathscr{S}_{(\text{relaxed})}$ *is a weak* compact subset of* $\mathrm{BV}^{+}([0,1]; \Re^{n})$

(ii) *weak* closure* $\{\mathscr{S}\} = \mathscr{S}_{(\text{relaxed})}$.

(*We refer the reader to Section* 1 *for a definition of the weak* topology on* $\mathrm{BV}^{+}([0,1]; \Re^{n})$.)

Proof. We note from the outset that co F_1 and co F_2 inherit from F_1 and F_2 the measurability, "closedness," and linear growth hypotheses required for the application of Theorems 4.1 and 4.2. These theorems may therefore be applied to the relaxed inclusion $(\mathrm{S}_{\text{relaxed}})$. Since M is a weak* compact set, the μ's generating elements in $\mathscr{S}_{(\text{relaxed})}$ are bounded in total variation. The admissible initial states too are bounded, because of the compactness hypothesis on C. Using these properties and the growth hypothesis (H2), and applying Gronwall's lemma to the reparameterized relaxed differential inclusions, we deduce that solutions y to reparameterized relaxed differential inclusions corresponding to elements in $\mathscr{S}_{(\text{relaxed})}$ are uniformly bounded in the supremum norm. It follows from the growth hypothesis that the velocities $\dot{y}$ are uniformly bounded in total variation. We conclude from Theorem 4.1 that elements in $\mathscr{S}_{(\text{relaxed})}$ themselves are uniformly bounded in the supremum norm and in total variation. In view of the growth hypothesis, this implies in particular the existence of $\beta(t) \in L^1$ and $c > 0$ such that

$$\text{co } F_1(t, x(t)) \subset \beta(t) B \text{ a.e.} \qquad \text{and} \qquad \text{co } F_2(t, x(t)) \subset cB,$$

for all $x \in \mathscr{S}_{(\text{relaxed})}$. This is the critical condition which we must check for application of Theorem 5.1 below.

(i) Since, as we have seen, elements in $\mathscr{S}_{(\text{relaxed})}$ are uniformly bounded in total variation, and the weak* topology relativized to subsets of $\mathrm{BV}^{+}([0,1]; \Re^{n})$ bounded in total variation is metrizable, we need only check sequential compactness. However this is proved by applying Theorem 5.1, when co F_1 is substituted in place of $F_1^{(i)}$, $i = 1, 2, \ldots,$ and of F_1, and co F_2 in place of F_2, and by appealing to the compactness hypotheses on C and M.

(ii) Take any $x \in \mathscr{S}_{(\text{relaxed})}$. Since the relativized weak* topology is metrizable and $\mathscr{S} \subset \mathscr{S}_{(\text{relaxed})}$, to check that $\mathscr{S}_{(\text{relaxed})}$ is the weak* closure of $\mathscr{S}$ we have only to construct a sequence $\{x_i\} \subset \mathscr{S}$ such that

$$x_i \to x \qquad \text{weakly* in } \mathrm{BV}^+([0,1]; \Re^n).$$

Let μ $(\in M)$ be the driving measure associated with x, and denote by (θ, γ) its graph completion. Define the multifunction

$$\tilde{F}(s, y) := F_1(\theta(s), y)\dot{\theta}(s) + F_2(\theta(s), y)\dot{\gamma}(s).$$

Since F_1 and F_2 take value compact sets,

$$\operatorname{co} \tilde{F}(s, y) = \operatorname{co}\big(F_1(\theta(s), y)\big)\dot{\theta}(s) + \operatorname{co}\big(F_2(\theta(s), y)\big)\dot{\gamma}(s)$$

for all $(s, y) \in [0,1] \times \Re^n$. By Theorem 4.1 however there exists a solution $y \in \mathrm{AC}([0,1]; \Re^n)$ to

$$\begin{cases} \dot{y}(s) \in \operatorname{co} \tilde{F}(\theta(s), y(s)) \\ y(0) = x(0) \end{cases}$$

such that

$$x(t) = y(\eta(t)) \qquad \text{for all } t \in [0,1].$$

Now $\tilde{F}$ satisfies the measurability, growth, and Lipschitz continuity hypotheses under which y can be approximated uniformly by solutions to the nonconvexified differential inclusion (see, e.g., [2]). To be precise, there exists a sequence $\{y_i\}$ in $\mathrm{AC}([0,1]; \Re^n)$ such that y_i solves

$$\begin{cases} \dot{y}_i(s) \in F_1(\theta(s), y_i(s))\dot{\theta}(s) + F_2(\theta(s), y_i(s))\dot{\gamma}(s) \\ y_i(0) = x_0 \end{cases}$$

for $i = 1, 2, \dots$ and

$$y_i \to y \text{ uniformly}.$$

By Theorem 4.1, $\{x_i\}$, defined by

$$\begin{aligned} x_i(t) &= y_i(\eta(t)) \qquad \text{for all } t \in (0,1] \\ x_i(0) &= x(0), \end{aligned} \tag{5.5}$$

is a sequence of robust solutions to (S) (corresponding to μ). The sequence lies in $\mathscr{S}$, since $x_i(0) = x(0)$ for each i. We deduce from the weak* compactness of $\mathscr{S}_{(\text{relaxed})}$ and the fact that (by (5.5)) $x_i(t) \to x(t)$ for all $t \in [0,1] \setminus \mathscr{M}_\mu$ that $x_i \to x$ weakly* in $\mathrm{BV}^+([0,1]; \Re^n)$. This confirms that $\mathscr{S}_{(\text{relaxed})}$ is the weak* closure of $\mathscr{S}$. ∎

6. PROOF OF THEOREM 4.1

(i) We take $x(\cdot)$ to be a robust solution to (3.1) corresponding to μ and x_0. Let $\{t_i\}$ be an enumeration of the atoms of μ, and set $I_i = [s_i', s_i''] = \theta^{-1}(\{t_i\})$, for $i = 1, 2, \dots$. Let ϕ_1, ϕ_2 be selectors corresponding to x (see Definition 3.1). By definition of $\tilde{F}_2$, for each i there exists $\xi_i \in AC([0,1]; \Re^n)$ such that

$$\begin{cases} \dot{\xi}_i(s) \in \mu(\{t_i\}) F_2(t_i, \xi_i(s)) & \text{a.e. on } [0,1] \\ \xi_i(0) = x(t_i^-) \end{cases}$$

and

$$\phi_2(t_i) = \left(\mu(\{t_i\}) \right)^{-1} \left[\xi_i(1) - \xi_i(0) \right].$$

Define

$$y(s) := \begin{cases} x(\theta(s)) & \text{when } s \in [0,1] \setminus \bigcup_i I_i \\ \xi_i((s - s_i')/(s_i'' - s_i')) & \text{when } s \in I_i,\ i = 1, 2, \dots. \end{cases} \tag{6.1}$$

Choose $e(\cdot)$ to be a measurable selector of $F_1(\theta(s), y(s))$. Set $A := \{s \colon \dot{\theta}(s) = 0\}$. We now define

$$\psi_1(s) := \begin{cases} e(s) & \text{if } s \in (\bigcup_i I_i) \cup A \\ \phi_1(\theta(s)) & \text{if } s \in [0,1] \setminus ((\bigcup_i I_i) \cup A). \end{cases}$$

We claim that the $\mathscr{L}$-measurable function ψ_1 satisfies

$$\psi_1(s) \in F_1(\theta(s), y(s)) \qquad \text{a.e. } s \in [0,1]. \tag{6.2}$$

By the nature of e, this inclusion holds a.e. on $(\bigcup_i I_i) \cup A$. To complete the verification we note that (by the properties of ϕ_1)

$$\int_{[0,1]} \chi_{F_1(\tau, x(\tau))}(\phi_1(\tau))\, d\tau = 0,$$

in which χ_A denotes the indicator function of the set A. The change of variables lemma now gives

$$\int_{[0,1]} \chi_{F_1(\theta(s), x(\theta(s)))}(\phi_1(\theta(s))\, \dot{\theta}(s)\, ds = 0.$$

Since $y(s) = x(\theta(s))$ a.e. on the complement of $\bigcup_i I_i$, we conclude that $\phi_1(\theta(s))$ $(= \psi_1(s)) \in F_1(\theta(s), y(s))$ for almost every $s \in [0,1] \setminus [(\bigcup_i I_i) \cup A]$. We have shown (6.2) to be true.

Set $S_\gamma := \{s: \dot{\gamma}(s) = 0\}$ and $J_i := \{s \in (s_i', s_i''): d/ds\xi_i((s - s_i')/(s_i'' - s_i'))$ exists$\}$. Let $\phi(\cdot)$ be any measurable selector of $F_2(\theta(s), y(s))$ and set

$$\psi_2(s) := \begin{cases} d/ds\xi_i((s - s_i')/(s_i'' - s_i'))(1 + \mu([0,1]))^{-1} \\ \qquad\qquad\qquad\qquad \text{if } s \in J_i,\ i = 1, 2, \dots \\ \phi(s) \qquad\qquad\quad \text{if } s \in S_\gamma \setminus \bigcup_i J_i \\ \phi_2(\theta(s)) \qquad\quad \text{if } s \in [0,1] \setminus \left(S_\gamma \cup (\bigcup_i J_i)\right). \end{cases}$$

Now we claim that the $\mathscr{L}$-measurable function ψ_2 satisfies

$$\psi_2(s) \in F_2(\theta(s), y(s)) \qquad \text{a.e. } s \in [0,1]. \tag{6.3}$$

This is clearly the case for almost every $s \in (S_\gamma \cup (\bigcup_i J_i))$. Note however that

$$\int_{[0,1]} \chi_{\tilde{F}_2(\tau,\, x(\tau^-);\, \mu(\{\tau\}))}(\phi_2(\tau))\, d\tau = 0.$$

By the change of variables lemma, Proposition 2.1,

$$\int_{[0,1]} \chi_{\tilde{F}_2(\theta(s),\, x(\theta(s)^-);\, \mu(\{\theta(s)\}))}(\phi_2(\theta(s)))\dot{\gamma}(s)\, ds = 0.$$

But for almost all points s in the set $[0,1] \setminus (S_\gamma \cup (\bigcup_i J_i))$ we have that $\mu(\{\theta(s)\}) = 0$ and $x(\theta(s)^-) = y(s)$; consequently $\tilde{F}_2(\theta(s), y(s); \mu(\{\theta(s)\})) = F_2(\theta(s), y(s))$. For almost all points in $[0,1] \setminus (S_\gamma \cup (\bigcup_i J_i))$ then $\psi_2(s)$ $(= \phi_2(\theta(s))) \in F_2(\theta(s), y(s))$. The relationship (6.3) is established.

It is not difficult to show that $y(\cdot)$ defined by (6.1) is Lipschitz continuous. We now show that y solves the differential equation

$$\dot{y}(s) = \psi_1(s)\dot{\theta}(s) + \psi_2(s)\dot{\gamma}(s), \qquad \text{a.e. on } [0,1]. \tag{6.4}$$

It suffices to check that equality holds at points s in the set of full measure $\mathscr{D} \cup (\bigcup_i E_i)$, where

$$\mathscr{D} := \left\{ \begin{matrix} s' \in (0,1) \setminus \bigcup_i I_i\text{: } s' \text{ is a Lebesgue point of } s \to \psi_1(\theta(s))\dot{\theta}(s) \\ \text{and of } s \to \psi_2(\theta(s))\dot{\gamma}(s), \text{ and } \dot{y}(s') \text{ exists} \end{matrix} \right\},$$

and

$$E_i := \left\{ s' \in \text{int}\{I_i\}\text{: } \dot{y}(s') \text{ exists}, \dot{\theta}(s') = 0, \dot{\gamma}(s') = 1 + (\mu([0,1]) \right\}.$$

Take first a point $s \in \mathscr{D}$. We may choose $h_j \downarrow 0$ such that $s + h_j \in (0,1) \setminus \bigcup_i I_i$ for all j. For $s' = s$ and $s' = s + h_j$, $j = 1, 2, \dots,$ we have

$y(s') = x(\theta(s'))$. By the change of variables lemma, Proposition 2.1, then

$$\begin{aligned}\dot{y}(s) &= \lim_j h_j^{-1}\big[y(s+h_j) - y(s)\big] = \lim_j h_j^{-1}\Big[x\big(\theta(s+h_j)\big) - x(\theta(s))\Big]\\ &= \lim_j \left[h_j^{-1}\int_{\theta(s)}^{\theta(s+h_j)} \phi_1(\tau)\,d\tau + h_j^{-1}\int_{[\theta(s),\,\theta(s+h_j)]} \phi_2(\tau)\,d\mu(\tau)\right]\\ &= \lim_j h_j^{-1}\left[\int_s^{s+h_j} \phi_1(\theta(\sigma))\dot{\theta}(\sigma)\,d\sigma + \int_s^{s+h_j} \phi_2(\theta(\sigma))\dot{\gamma}(\sigma)\,d\sigma\right]\\ &= \phi_1(\theta(s))\dot{\theta}(s) + \phi_2(\theta(s))\dot{\gamma}(s) = \psi_1(s)\dot{\theta}(s) + \psi_2(s)\dot{\gamma}(s)\end{aligned}$$

as required.

Take next any i and $s \in E_i$. We now have

$$\begin{aligned}\dot{y}(s) &= d/ds\xi_i\big((s - s_i')/(s_i'' - s_i')\big) = \psi_1(s)\cdot 0 + \big(1 + \mu([0,1])\big)\psi_2(s)\\ &= \psi_1(s)\dot{\theta}(s) + \psi_2(s)\dot{\gamma}(s).\end{aligned}$$

We have shown that y satisfies the differential equation (6.4).

It remains to show that $x(\tau) = y(\eta(\tau))$ for all $\tau \in [0,1]$. This is certainly true for $\tau \in (0,1] \setminus \cup_i\{t_i\}$, by (6.1), and $\tau = 0$ by definition. On the other hand, for any integer i such that $t_i > 0$ we have $x(t_i^-) = y(s_i')$. According to (6.1) however

$$\begin{aligned}x(t_i) &= x(t_i^-) + \phi_1(t_i)\mu(\{t_i\}) = y(s_i') + \big(\xi_i(1) - \xi_i(0)\big)\\ &= y(s_i') + \big[y(s_i'') - y(s_i')\big] = y(s_i'') = y(\eta(t_i)).\end{aligned}$$

This shows that equality holds also on the atoms of μ, and therefore everywhere on $[0,1]$.

(ii) Now take a solution y to (4.1). There exist bounded Borel measurable functions ψ_1, ψ_2 such that

$$y(s) = x_0 + \int_{[0,s]} \psi_1(\sigma)\dot{\theta}(\sigma)\,d\sigma$$

$$+ \int_{[0,s]} \psi_2(\sigma)\dot{\gamma}(\sigma)\,d\sigma \qquad \text{for all } s \in [0,1],$$

$$\psi_1(\sigma) \in F_1(\theta(\sigma), y(\sigma)) \qquad \text{a.e.,} \tag{6.5}$$

$$\psi_2(\sigma) \in F_2(\theta(\sigma), y(\sigma)) \qquad \text{a.e.} \tag{6.6}$$

Define

$$x(t) := y(\eta(t)) \qquad \text{for all } t \in [0,1]$$

$$\phi_1(\tau) := \psi_1(\eta(\tau)) \qquad \text{for all } \tau = [0,1]$$

and

$$\phi_2(\tau) := \begin{cases} \psi_2(\eta(\tau)) & \text{for } \tau \in [0,1] \setminus \bigcup_i \{t_i\} \\ (s_i'' - s_i')^{-1} \int_{I_i} \psi_2(\sigma)\, d\sigma & \text{for } \tau = t_i,\ i = 1, 2, \ldots \end{cases}$$

From (6.5), and since $\dot{\theta} = 0$ a.e. on $\bigcup_i I_i$,

$$0 = \int_{[0,1]} \chi_{F_1(\theta(\sigma), x(\theta(s)))}(\phi_1(\theta(\sigma)))\dot{\theta}(\sigma)\, d\sigma$$

$$= \int_{[0,1]} \chi_{F_1(\tau, x(\tau))}(\phi_1(\tau))\, d\tau.$$

This tells that $\phi_1(\tau) \in F_1(\tau, x(\tau),)$ a.e.

It is easy to check that, for each i,

$$\phi_2(t_i) = \mu(\{t_i\})^{-1}(\xi_i(1) - \xi_i(0))$$

for some ξ_i satisfying $\dot{\xi}_i \in \mu(\{t_i\})F_2(t_i, \xi)$ and $\xi_i(0) = x(t_i^-)$. Otherwise expressed,

$$\phi_2(t) \in \tilde{F}_2(t, x(t^-); \mu(\{t\})) \qquad \text{for } t \in \bigcup_i \{t_i\}. \tag{6.7}$$

We note also that, by (6.6),

$$\int_{[0,1]\setminus \bigcup_i I_i} \chi_{F_2(\theta(\sigma), x(\theta(\sigma)))}(\phi_2(\theta(\sigma)))\dot{\gamma}(\sigma)\, d\sigma = 0.$$

A further change of variables yields

$$\int_{[0,1]\setminus\{t_i\}} \chi_{F_2(\tau, x(\tau))}(\phi_2(\tau))\mu(d\tau) = 0.$$

By (6.7), and since $F_2(\tau, x(\tau)) = \tilde{F}_2(\tau, x(\tau^-); \mu(\{\tau\})$ on $[0,1] \setminus \bigcup_i \{t_i\}$, we arrive at

$$\phi_2(\tau) \in \tilde{F}_2(\tau, x(\tau^-); \mu(\{\tau\}) \qquad \mu\text{-a.e.}$$

Now take any $t > 0$. We have

$$
\begin{aligned}
x(t) - x_0 &\\
&= y(\eta(t)) - x_0 \\
&= \int_{[0,\eta(t)]} \psi_1(\sigma)\dot{\theta}(\sigma)\,d\sigma + + \int_{[0,\eta(t)]} \psi_2(\sigma)\dot{\gamma}(\sigma)\,d\sigma \\
&= \int_{[0,\eta(t)]} \phi_1(\theta(\sigma))\dot{\theta}(\sigma)\,d\sigma + \int_{[0,\eta(t)]\setminus\bigcup_i I_i} \phi_2(\theta(\sigma))\dot{\gamma}(\sigma)\,d\sigma \\
&\quad + \int_{\bigcup_i I_i} \phi_2(\theta(\sigma))\dot{\gamma}(\sigma)\,d\sigma
\end{aligned}
$$

(since $\dot{\gamma}(\cdot) \equiv 1 + \mu([0,1])$ on $\bigcup_i I_i$)

$$
= \int_{[0,t]} \phi_1(\tau)\,d\tau + \int_{[0,t]} \phi_2(\tau)\mu(d\tau).
$$

This confirms that $x(\cdot)$, defined by $x(t) := y(\eta(t))$, is a robust solution to (3.1).

(iii) Take a robust solution $x(\cdot)$ to (3.1) and a solution $y(\cdot)$ of (4.1) satisfying (4.2).

Let $\{\tau_0, \tau_1, \dots, \tau_N\}$ be an arbitrary finite partition of $[0,1]$. Then, by (4.2).

$$
\begin{aligned}
\sum_{i=1}^{N} |x(\tau_i) - x(\tau_{i-1})| &= \sum_{i=1}^{N} |y(\eta(\tau_i)) - y(\eta(\tau_{i-1}))| \\
&\le \sum_{i=1}^{N} \int_{(\eta(\tau_{i-1}),\,\eta(\tau_i))} |\dot{y}(s)|\,ds \\
&\le \int_0^1 |\dot{y}(s)|\,ds
\end{aligned}
$$

(since the intervals $(\eta(\tau_{i-1}), \eta(\tau_i))$ are non-overlapping)

$$
= \|y\|_{\text{T.V.}}
$$

But the partition was arbitrary. So

$$
\|x(\cdot)\|_{\text{T.V.}} \le \|y(\cdot))\|_{\text{T.V.}}
$$

7. PROOF OF THEOREM 5.1

Since $\{\mu_i\}$ is a weak* convergent sequence, there is a number d such that

$$(1 + \|\mu_i\|_{\text{T.V.}}) \leq d \qquad \text{for all } i. \tag{7.1}$$

By Lemma 2.1, $(\theta_i, \gamma_i) \to (\theta, \gamma)$ uniformly as $i \to \infty$. We also know that $\eta_i(t) \to \eta(t)$ as $i \to \infty$, for all points $t \in ([0,1] \setminus \mathscr{M}_\mu) \cup \{0,1\}$. ($\mathscr{M}_\mu$, we recall, is the set of atoms of μ.)

According to Theorem 4.1 we can choose a solution y_i to (5.3) such that $x_i(t) = y_i(\eta_i(t))$ for all $t \in [0,1]$, for $i = 1, 2, \dots$. We claim that the y_i's are uniformly bounded both in the L^1 norm and in total variation, and their derivatives $\{\dot{y}_i\}$ are uniformly integrably bounded. To see this, we conclude first from condition (7.1) and Proposition 2.1 that for any Borel set $S \subset [0,1]$ and any i,

$$\begin{aligned} \int_S |\dot{y}_i(s)|\, ds &\leq \int_S \beta(\theta_i(s))\dot{\theta}_i(s)\, ds + c\int_S \dot{\gamma}_i(s)\, ds \\ &= \int_{\theta_i(S)} \beta(t)\, dt + c(1 + \|\mu_i\|_{\text{T.V.}}) \cdot (\mathscr{L}\text{-meas}\{S\}). \end{aligned} \tag{7.2}$$

Setting $S = [0,1]$ (in which case $\theta_i(S) = [0,1]$), we see that the $\dot{y}_i$'s are uniformly bounded in the L^1 norm. That the y_i's are uniformly bounded both in total variation and in the supremum norm follows from this estimate and the uniform boundedness of the $y_i(0)$'s. On the other hand, estimate (7.2), coupled with the observation that

$$\mathscr{L}\text{-meas}\{\theta_i(S)\} \leq (1 + \|\mu_i\|_{\text{T.V.}}) \cdot (\mathscr{L}\text{-meas}\{S\}),$$

tells us that for any $\epsilon > 0$ we can choose $\delta > 0$ such that $\int_S |\dot{y}_i(s)|\, ds < \epsilon$, for $i = 1, 2, \dots,$ whenever $\mathscr{L}$-meas$\{S\} < \delta$. In other words, $\{\dot{y}_i\}$ is a uniformly integrably bounded sequence.

According to the Dunford–Pettis Theorem, then, there exists an absolutely continuous function y such that, following extraction of a subsequence,

$$y_i \to y \text{ uniformly}.$$

Since the y_i's are bounded in total variation it follows from Theorem 4.1, part (iii), that the x_i's too are bounded in total variation. The x_i's are also bounded in the supremum norm. These properties imply the existence of a countable set $\mathscr{A}$ and an element $x \in BV^+([0,1]; \mathfrak{R}^n)$ such that, following a further subsequence extraction, we have

$$x_i \to x \text{ weakly*}$$

and

$$x_i(t) \to x(t) \qquad \text{for all } t \in ([0,1] \setminus \mathscr{A}) \cup \{0,1\}.$$

We now show that we can substitute $\mathscr{M}_\mu$ (the atoms of μ) in place of $\mathscr{A}$ in the above relationship. We have $x_i(t) = y_i(\eta_i(t))$ for $i = 1, 2, \dots$. Since convergence of the y_i's is uniform and $\eta_i(t) \to \eta(t)$ as $i \to \infty$ for t taken to be 0, 1, or any element in the complement of some countable set, it follows that $x(t) = y(\eta(t))$ on a dense subset of points containing $\{0\} \cup \{1\}$. We conclude from the continuity from the right of $x(\cdot)$ and $y(\eta(\cdot))$ that

$$x(t) = y(\eta(t)) \qquad \text{for } t \in [0,1].$$

But then, for each i and $t \in [0,1]$, we have

$$\|x(t) - x_i(t)\| \le \|y(\eta(t) - y(\eta_i(t))\| + \|y(\eta_i(t)) - y_i(\eta_i(t))\|.$$

Since $y_i \to y$ uniformly and $\eta_i(t) \to \eta(t)$ for all $t \in ([0,1] \setminus \mathscr{M}_\mu) \cup \{0,1\}$, we conclude that

$$x_i(t) \to x(t) \qquad \text{for all } t \in \left([0,1] \setminus \mathscr{M}_\mu\right) \cup \{0,1\}$$

as claimed.

For $i = 1, 2, \dots$ there exists a Borel measurable function ψ_1^i and a bounded, Borel measurable function ψ_2^i such that $\psi_1^i \dot{\theta}_i$ is $\mathscr{L}$-integrable,

$$y_i(s) = x_0 + \int_{[0,s]} \psi_1^i(\sigma)\dot{\theta}_i(\sigma)\, d\sigma$$

$$+ \int_{[0,s]} \psi_2^i(\sigma)\dot{\gamma}_i(\sigma)\, d\sigma \qquad \text{for all } s \in [0,1], \tag{7.3}$$

$$\begin{aligned} \psi_1^i(\sigma) &\in F_1^{(i)}(\theta_i(\sigma), y_i(\sigma))\dot{\theta}_i(\sigma), \qquad \mathscr{L}\text{-a.e.} \\ \psi_2^i(\sigma) &\in F_2^{(i)}(\theta_i(\sigma), y_i(\sigma))\dot{\gamma}_i(\sigma), \qquad \mathscr{L}\text{-a.e.} \end{aligned} \tag{7.4}$$

by Theorem 4.1. Noting that $\sigma = \eta_i(\theta_i(\sigma))$ a.e. on $\{\sigma\colon \dot{\theta}_i(\sigma) \neq 0\}$ and applying Lemma 2.1 we deduce that for each $s \in [0,1]$

$$\int_{[0,s]} \psi_1^i(\sigma)\dot{\theta}_i(\sigma)\, d\sigma = \int_{[0,s]} \psi_1^i(\eta_i(\sigma))\dot{\theta}_i(\sigma)\, d\sigma = \int_{[0,\theta_i(s)]} \phi_1^i(\tau)\, d\tau \tag{7.5}$$

where $\phi_1^i(\tau) := \psi_1^i((\eta_i(\tau))$. Arguing as in the proof of Theorem 4.1 (part (ii)) we conclude from (7.4) that

$$\phi_1^i(\tau) \in F_1^{(i)}(\tau, x_i(\tau)) \qquad \mathscr{L}\text{-a.e.}$$

We know however that $x_i(\tau) \to x(\tau)$ on a subset of full Lebesgue measure and $\mathscr{L}$-meas$\{\tau: F_1^{(i)}(\tau, \cdot) \neq F_1(\tau, \cdot)\} \to 0$ as $i \to \infty$. Since $F_1(\tau, \cdot)$ has closed graph and F is integrably bounded and has values closed convex sets, we deduce via standard "weak* convergence/separating hyperplane techniques" (cf. [2, Proof of Theorem 3.1.7]) that, following extractions of subsequences,

$$\int_{[0,\,\theta_i(s)]} \phi_1^i(\tau)\, d\tau \to \int_{[0,\,\theta(s)]} \phi_1(\tau)\, d\tau \qquad \text{for all } s \in [0,1], \tag{7.6}$$

for some Lebesgue integrable function ϕ_1 satisfying

$$\phi_1(\tau) \in F_1(\tau, x(\tau)) \qquad \mathscr{L}\text{-a.e.}$$

Next, arguing as in the proof of Theorem 4.1 (part (i)), we show that their exists a Borel measurable function ψ_1 such that $\psi_1 \dot{\theta}$ is $\mathscr{L}$-integrable,

$$\psi_1(\sigma) \in F_1(\theta(\sigma), y(\sigma)) \qquad \mathscr{L}\text{-a.e.}$$

and

$$\int_{[0,\,\theta(s)]} \phi_1(\tau)\, d\tau = \int_{[0,\,s]} \psi_1(\sigma)\dot{\theta}(\sigma)\, d\sigma \qquad \text{for all } s \in [0,1].$$

By (7.5) and (7.6), then, for all $s \in [0,1]$ we have

$$\int_{[0,\,s]} \psi_1^i(\sigma)\dot{\theta}_1(s)\, d\sigma \to \int_{[0,\,s]} \psi_1(\sigma)\dot{\theta}(s)\, d\sigma \qquad \text{as } i \to \infty. \tag{7.7}$$

The usual "weak* convergence/separating hyperplane" arguments (cf. [9, Lemma 4.5], in this case) also establish that, following a further subsequence extraction,

$$\int_{[0,\,s]} \psi_2^i(\sigma)\dot{\gamma}_i(s)\, d\sigma \to \int_{[0,\,s]} \psi_2(\sigma)\dot{\gamma}(s)\, d\sigma \qquad \text{as } i \to \infty, \tag{7.8}$$

for all $s \in [0,1]$, for some bounded, Borel measurable function ψ_2 satisfying

$$\psi_2(\sigma) \in F_2(\theta(\sigma), y(\sigma)) \qquad \mathscr{L}\text{-a.e.}$$

(The fact that $F_2(\cdot, \cdot)$ is upper-semicontinuous is crucial here.) Passing to the limit across (7.3) with the help of (7.7) and (7.8) we obtain

$$y(s) = x_0 + \int_{[0,s]} \psi_1(\sigma)\dot{\theta}(s)\, d\sigma$$
$$+ \int_{[0,s]} \psi_2(\sigma)\dot{\gamma}(s)\, d\sigma, \qquad \text{for all } s \in [0,1].$$

We have shown that y is a solution of the ‘‘limiting’’ reparameterized inclusion (5.4). Since $x = y \circ \eta$ and by Theorem 4.1, x is a robust solution of the limiting measure driven differential inclusion (5.2), the proof of the theorem is complete.

REFERENCES

1. J. P. Aubin and A. Celina, ‘‘Differential Inclusions,’’ Springer-Verlag, Berlin, 1984.
2. F. H. Clarke, ‘‘Optimization and Nonsmooth Analysis,’’ Wiley, New York, 1983.
3. G. Dal Maso and F. Rampazzo, On systems of ordinary differential equations with measures as controls, *Differential Integral Equations* **4** (1991), 739–765.
4. J. F. C. Kingman and S. J. Taylor, ‘‘Introduction to Measure and Probability,’’ Cambridge Univ. Press, Cambridge, 1966.
5. D. F. Lawden, ‘‘Optimal Trajectories for Space Navigation,’’ Butterworth, London, 1993.
6. J. P. Marec, ‘‘Optimal Space Trajectories,’’ Elsevier, Amsterdam/New York, 1979.
7. R. W. Rishel, An extended Pontryagyn principle for control systems whose control laws contain measures, *SIAM J. Control* **3** (1965), 191–205.
8. G. N. Silva and R. B. Vinter, Necessary conditions for optimal impulsive control problems, *Siam J. Control Optim.*, to appear.
9. R. B. Vinter and G. Pappas, A maximum principle for nonsmooth optimal control problems with state constraints, *J. Math. Anal. Appl.* **89**, No. 1 (1982), 212–232.
10. J. Warga, ‘‘Optimal Control of Differential and Functional Equations,’’ Academic Press, New York, 1972.