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Degenerations for the Representations of a Quiver of Type $\mathscr{A}_m$

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1. INTRODUCTION

We consider the Dynkin diagram $\mathscr{A}_m$, whose consecutive vertices are labelled $1, 2, ..., m$, and we denote by $\mathbb{Q}_m \equiv (\mathscr{A}_m, \Omega)$ the graph $\mathscr{A}_m$ with a given orientation Ω for its edges. We call $\mathbb{Q}_m$ a "quiver" of type $\mathscr{A}_m$.

Let K be a field. For every $d \equiv (d_1, ..., d_m) \in \mathbb{N}^m$ we consider the variety L_d of all the representations over K of the quiver $\mathbb{Q}_m$ of dimension d. Let V_i, $i = 1, ..., m$, be a vector space over K of dimension d_i; the group $G = \prod_{i=1}^m GL(V_i)$ acts naturally on L_d and the number of orbits of this action is finite, each orbit $\mathscr{O}_A$, $(A \in L_d)$ corresponding to an isomorphism class $[A]$ of the previous representations (cf. [2–5]).

Let $A = (A_1, ..., A_m) \in L_d$. In this paper we introduce a set of non-negative integers $N^A = \{N^A_{uv}\}$, $1 \leqslant u \leqslant v \leqslant m$ which are ranks of maps deduced from the A_i's and depend on the orbit $\mathscr{O}_A$ (cf. Proposition 2.2).

First we prove that, through the set N^A, we can compute the indecomposable representations appearing in A and their multiplicities (cf. (2.6)). Moreover we find a system of inequalities which give a necessary and sufficient condition for a set of non-negative integers $N = \{N_{uv}\}$ to determine an isomorphism class of representations of $\mathbb{Q}_m$ (cf. (2.7)).

Next we study the problem of the degenerations for the representations of $\mathbb{Q}_m$ of given dimension. Given any orbit $\mathscr{O}_A \subset L_d$ we want to characterize the orbits $\mathscr{O}_B \subset L_d$ such that $\mathscr{O}_B \subset \bar{\mathscr{O}}_A$ ($\bar{\mathscr{O}}_A$ the closure of $\mathscr{O}_A$), i.e., the degenerations of $\mathscr{O}_A$.

We prove that $\mathscr{O}_B \subset \bar{\mathscr{O}}_A$ if and only if $N^B_{uv} \leqslant N^A_{uv}$ for every u, v, $1 \leqslant u \leqslant v \leqslant m$ (cf. Theorem 5.2, the part that states the equality of the orderings $\leqslant_g$ and $\leqslant_r$).

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Moreover we prove that if $\mathcal{O}_B \subset \bar{\mathcal{O}}_A$ and $\mathcal{O}_B$ is open in $\bar{\mathcal{O}}_A - \mathcal{O}_A$ then there exists a submodule $A' \subset A$ (we think of the representations as modules) such that $B \simeq A' \oplus A/A'$, (cf. Theorem 5.2, the part that states the equality of the orderings $\leqslant_c$ and $\leqslant_g$).

The simple case where the Dynkin diagram $\mathscr{A}$ is equioriented, has been treated in [1].

2. A Set of Rank Parameters for the Orbit $\mathcal{O}_A$

Let $\mathbb{Q}_m = (\mathscr{A}_m, \Omega)$ as in Section 1. The orientation Ω determines an increasing sequence of integers $1 = s_0 < s_1 < \cdots < s_\nu < s_{\nu+1} = m$, i.e., the sequence of sources and sinks of $\mathbb{Q}_m$ and, as soon as we know s_0 to be a source (or a sink) then s_t is a source or a sink according to the parity of the index t.

Conversely an increasing sequence $\{s_i\}$, $i = 0,..., \nu + 1$, $s_0 = 1$, $s_{\nu+1} = m$ determines the orientation of $\mathscr{A}_m$ up to duality, i.e., reversing all the arrows. As we will not need to know if s_0 is a source or a sink we identify Ω with the sequence $\{s_i\}$ and we will call the s_i's "critical points" for the orientation.

Let $A = (A_1,..., A_{m-1}) \in L_d$ be a given representation of $\mathbb{Q}_m$ and consider any pair of indices u, v such that $1 \leqslant u \leqslant v \leqslant m$. For the induced oriented graph starting at u and ending at v, u and v are either sources or sinks, and between u and v there will be a subsequence (possibly empty) of the sequence $\{s_i\}$.

Let φ^A_{uv} denote the linear map going from the direct sum of the spaces relative to all the sources to the one relative to all the sinks between u and v in the induced representation, (i.e., included u and v), whose components are

$$V_{s_{t-1}} \oplus V_{s_{t+1}} \to V_{s_t}$$
$$(z, z') \to (\bar{A}_{t-1,t} z - \bar{A}_{t+1,t} z')$$

where $\bar{A}_{pt}$, $p = t-1$, $t+1$, is the composition of all the maps A_i going from the sources s_{t-1} or s_{t+1} to the sink s_t.

To each representation A we associate the set of non-negative integers $N^A = \{N^A_{uv}\}_{1 \leqslant u \leqslant v \leqslant m}$ as follows:

Definition 2.0.

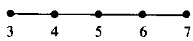
$$N^A_{uv} = rk\varphi^A_{uv} \qquad \text{if} \quad u < v,$$
$$N^A_{uu} = \dim V_u = d_u.$$

In the rest of this section we want to show some properties of the set N^A; in particular we want to show that if $A, B \in L_d$ are representations of $\mathbb{Q}_m$ and $\mathcal{O}_A = [A]$, $\mathcal{O}_B = [B]$ denote the corresponding orbits (i.e., the isomorphism classes of representations $[A]$, $[B]$) then we have $N^A_{uv} = N^B_{uv}$ for every (u, v) if and only if $\mathcal{O}_A = \mathcal{O}_B$.

We recall first the fact that the indecomposable representations of $\mathbb{Q}_m = (\mathscr{A}_m, \Omega)$ are in $1-1$ corresponence with the positive roots of the Dynkin diagram $\mathscr{A}_m$, independently from the orientation Ω (cf. [2–5]). It follows that we have an indecomposable representation, denoted by E_{pq}, for each pair (p, q) with $1 \leqslant p \leqslant q \leqslant m$, i.e., for each dimension $d = (d_j) \in N^m$ with $d_j = 1$ for $p \leqslant j \leqslant q$ and $d_j = 0$ otherwise.

We can visualize E_{pq} as the integer segment $[p, q]$ on which we have put a dot for each integer j, $p \leqslant j \leqslant q$; each dot j representing a base vector in the one dimensional vector space V_j.

EXAMPLE. E_{37}:



If we consider E_{pq} as an indecomposable representation of $\mathbb{Q}_m = (\mathscr{A}_m, \Omega)$, $\Omega = \{s_i\}$, then the pair (p, q) uniquely determines the pair of integers (a, b) such that

$$s_{a-1} < p \leqslant s_a, \qquad s_b \leqslant q < s_{b+1}$$

or, equivalently, the interval $[p, q]$ determines the subsequence $\{s_a, s_{a+1}, ..., s_{b-1}, s_b\}$ (possibly empty) of the critical points of Ω $\{s_a, s_{a+1}, ..., s_{b-1}, s_b\} = [p, q] \cap \Omega$.

From this point of view the indecomposables E_{pq} of $\mathbb{Q}_m$ are of two types:

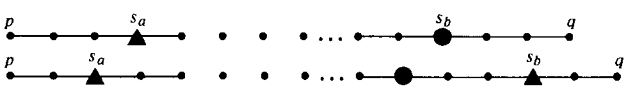
(1) $[p, q]$ contains an even number of critical points, and we will say that E_{pq} or $[p, q]$ is of "even type."

(2) $[p, q]$ contains an odd number of critical points, and we will say that E_{pq} or $[p, q]$ is of "odd type."

If we refer ourselves to the pair of integers (a, b) then we have that the even type corresponds to a pair (a, b) of integers with different parity (i.e., one of the two is odd and the other is even); the odd type corresponds to a pair (a, b) of integers with the same parity (both odd or both even numbers).

EXAMPLE.

(1)
(2)



(we have denoted by ▲, ● critical points of different nature, therefore in

Example (1), a and b have opposite parity; in Example (2), a and b have the same parity).

Remark 2.1. As soon as we know if s_a is a source or a sink and the type of the indecomposable E_{pq}, we can read from the corresponding segment if the base vector j, $p \leqslant j \leqslant q$ is sent to $j-1$ or $j+1$ or to zero, and if j is or is not the image of $j+1$ or $j-1$.

Let $A \in L_d$ be a given representation of $\mathbb{Q}_m = (\mathscr{A}_m, \Omega)$. Then the isomorphism class $[A] = \mathscr{O}_A$ determines and is determined by the set of non-negative integers $e^A = \{e^A_{pq}\}_{1 \leqslant p \leqslant q \leqslant m}$ such that

$$[A] = \bigoplus_{1 \leqslant p \leqslant q \leqslant m} e^A_{pq} E_{pq}. \tag{2.2}$$

If we represent each E_{pq} via the segment $[p, q]$, then $[A]$ is represented by a collection of segments, each segment $[p, q]$ having multiplicity e^A_{pq}. We call this collection of segments the "diagram" of the isomorphism class of the representation A.

Let us introduce now the set of non-negative integers $n^A = \{n^A_{rs}\}_{1 \leqslant r \leqslant s \leqslant m}$ associated to A and defined by

$$n^A_{rs} := \sum_{p \leqslant r \leqslant s \leqslant q} e^A_{pq}. \tag{2.3}$$

n^A_{rs} is the number of the segments of the diagram of $[A]$ which contain the integers r, s. It follows that we have

$$e^A_{pq} = n^A_{pq} - n^A_{p-1,q} - n^A_{p,q+1} + n^A_{p-1,q+1} \tag{2.4}$$

where we set $n^A_{rs} = 0$ if $r < 0$ or $s > m+1$.

As (2.3) and (2.4) are one the inverse of the other we deduce that the set n^A is determined by the orbit $\mathscr{O}_A$ and it determines an orbit if an only if the numbers n^A_{pq} satisfy the inequalities obtained by setting the right hand side of (2.4) bigger or equal to 0.

PROPOSITION 2.5. *Let $u \leqslant v$ and $s_{\alpha-1} \leqslant u < s_\alpha$, $s_\beta < v \leqslant s_{\beta+1}$, then we have*

$$\begin{aligned} N^A_{uv} = {} & n^A_{us_\alpha} + n^A_{s_\alpha s_{\alpha+1}} + \cdots + n^A_{s_{\beta-1}s_\beta} + n^A_{s_\beta v} \\ & - n^A_{us_{\alpha+1}} - n^A_{s_\alpha s_{\alpha+2}} - \cdots - n^A_{s_{\beta-1}v} \\ & + n^A_{u,s_{\alpha+2}} + b^A_{s_\alpha s_{\alpha+3}} + \cdots + n^A_{s_{\beta-2}v} \\ & \vdots \\ & + (-1)^{\beta-\alpha+1} n^A_{uv}. \end{aligned} \tag{2.5}$$

Proof. If $u=v$ or $\alpha=\beta+1$ the proof is trivial as (2.5) reduces to

$$N_{uv}^A = n_{uv}^A = \begin{cases} rk\varphi_{uv} & \text{if} \quad u<v \\ d_u & \text{if} \quad u=v. \end{cases}$$

Suppose $\alpha \neq \beta+1$, i.e., $s_{\alpha-1} \leqslant u < s_\alpha$, $s_\beta < v \leqslant s_{\beta+1}$ and assume s_α is a sink, i.e., u is a source in the induced representation (a similar argument holds if s_α is a source). Consider

$$V_u \oplus \bar{V} = V_u \oplus (V_{s_{\alpha+1}} \oplus V_{s_{\alpha+3}} \oplus \cdots) \xrightarrow{\varphi_{uv}^A} V_{s_\alpha} \oplus V_{s_{\alpha+2}} \oplus \cdots = W$$

we have

$$\varphi_{uv}^A(z, \bar{z}) = (\bar{A}_{us_\alpha}(z) + \varphi_{s_\alpha v}^A(\bar{z}), \qquad z \in V_u, \quad \bar{z} \in \bar{V}.$$

Note that

$$\begin{aligned} N_{uv}^A &= rk\bar{A}_{us_\alpha} + rk\varphi_{s_\alpha v}^A - \dim(\operatorname{Im} \bar{A}_{us_\alpha} \cap \operatorname{Im} \varphi_{s_\alpha v}^A) \\ &= n_{us_\alpha}^A + N_{s_\alpha v}^A - \dim(\operatorname{Im} \bar{A}_{us_\alpha} \cap \operatorname{Im} \varphi_{s_\alpha v}^A). \end{aligned}$$

By induction assume

$$\begin{aligned} N_{s_\alpha v}^A = n_{s_\alpha s_{\alpha+1}}^A &+ \cdots + n_{s_\beta v}^A \\ &- n_{s_\alpha s_{\alpha+2}}^A - \cdots - n_{s_{\beta-1} v}^A \\ &\vdots \\ &+ (-1)^{\beta-\alpha} n_{s_\alpha v}. \end{aligned}$$

Then we only need to prove that

$$\dim(\operatorname{Im} \bar{A}_{us_\alpha} \cap \operatorname{Im} \varphi_{s_\alpha v}^A) = \sum_{s_{\alpha+1} \leqslant s_{\alpha+t} < v} (-1)^{t-1} n_{us_{\alpha+t}} + (-1)^{\beta-\alpha} n_{uv}.$$

As $\operatorname{Im} \bar{A}_{us_\alpha} \subset V_{s_\alpha} \subset W$ we only need to count the number of base vectors in $\operatorname{Im} \bar{A}_{us_\alpha} \cap (\operatorname{Im} \varphi_{s_\alpha v}^A \cap V_{s_\alpha})$.

The number of base vectors in $\operatorname{Im} \varphi_{s_\alpha v}^A \cap V_{s_\alpha}$ is counted by the number of indecomposable E_{pq} with $p \leqslant s_\alpha < s_{\alpha+1} \leqslant q < s_{\alpha+2}$, $p \leqslant s_\alpha < s_{\alpha+3} \leqslant q < s_{\alpha+4}$, and so on (cf. Remark 2.1). Therefore the number of base vectors in $\operatorname{Im} \bar{A}_{us_\alpha} \cap \operatorname{Im} \varphi_{s_\alpha v}^A$ is counted by the number of E_{pq} in A with $p \leqslant u < s_{\alpha+1} \leqslant q < s_{\alpha+2}$; $p \leqslant u < s_{\alpha+3} \leqslant q < s_{\alpha+4}$,..., i.e., $n_{us_{\alpha+1}}^A - n_{us_{\alpha+2}}^A$; $n_{us_{\alpha+3}}^A - n_{us_{\alpha+4}}^A$;... and the claim is proved.

The linear system of equations (2.5) for $1 \leqslant u \leqslant v \leqslant m$ consists of recursive relations, therefore it is invertible over the integers. By substituting

(2.3) in (2.5), we express the N^A_{uv}'s as linear functions of the multiplicities e^A_{pq} and the linear system is invertible over the integers, i.e., we have

$$e^A_{pq} = f_{pq}(N^A_{uv}) \tag{2.6}$$

and explicitly

$$\begin{aligned}
e^A_{pq} &= (-1)^{b-a}(N^A_{pq} - N^A_{p-1,q} - N^A_{p,q+1} + N^A_{p-1,q+1}) \\
&\qquad \text{if} \quad s_a < p < s_{a+1}, \quad s_b < q < s_{b+1}, \\
e^A_{s_a q} &= (-1)^{b-a+1}(N^A_{s_{a+1},q} - N^A_{s_a-1,q} - N^A_{s_{a+1},q+1} + N^A_{s_a-1,q+1}) \\
&\qquad \text{if} \quad s_b < q < s_{b+1}, \\
e^A_{p s_b} &= (-1)^{b-a}(N^A_{p,s_{b-1}} - N^A_{p-1,s_{b-1}} - N^A_{p,s_b+1} + N^A_{p-1,s_b+1}) \\
&\qquad \text{if} \quad s_a < p < s_{a+1}, \\
e^A_{s_a s_b} &= (-1)^{b-a+1}(N^A_{s_{a+1},s_{b-1}} - N^A_{s_a-1,s_{b-1}} - N^A_{s_{a+1},s_b+1} + N^A_{s_a-1,s_b+1}).
\end{aligned} \tag{2.6}$$

If A is a given representation of $\mathbb{Q}_m$, i.e., in suitable bases, A is assigned through the set of matrices $(A_1, ..., A_{m-1})$, then we can compute the ranks N^A_{uv} and from (2.6) we deduce the multiplicities of the indecomposable factors of $[A] = \mathcal{O}_A$. Conversely if A is given through (2.2) then (2.3) and (2.5) allow us to find the set of rank parameters N^A.

Moreover we have the following:

Proposition 2.7. *A set of non-negative integers* $N = \{N_{uv}\}$, $1 \leqslant u \leqslant v \leqslant m$, *is the set of rank parameters for an isomorphism class of representations of* $\mathbb{Q}_m = (\mathscr{A}, \Omega)$ *if and only if they satisfy the inequalities obtained setting the right hand side of* (2.6) *bigger or equal to* 0.

This last proposition allows us to parametrize bijectively the isomorphism classes of representations of $\mathbb{Q}_m$ by the sets of rank parameters $N = \{N_{uv}\}$ subject to the stated conditions.

Proposition 2.8. *If* $A, B \in L_d$ *are such that* $\mathcal{O}_B \subseteq \bar{\mathcal{O}}_A$, *then* $N^B_{uv} \leqslant N^A_{uv}$ *for every* $1 \leqslant u < v \leqslant m$.

Proof. It is trivial; in fact, in a degeneration ranks cannot increase.

Remark 2.9. We have displayed the n^A_{pq}'s appearing in the expressions (2.5) of N^A_{uv} on rows and columns; on each column we have an alternated sign starting with $+$, and the column index is p and $p \in \{u, s_\alpha, ..., s_\beta\}$.

Remark 2.10. From now on we will display the rank parameters N^A_{uv} ($1 \leqslant u \leqslant m$, $1 \leqslant v \leqslant m$) of the representation A in a matrix (u and v are respectively the row and the column index), which will still be denoted by $N^A = \{N^A_{uv}\}$.

3. Elementary Degenerations

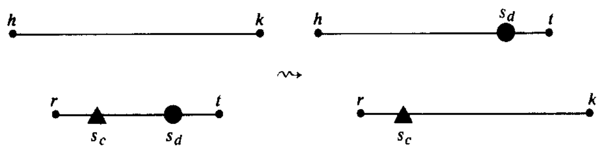
We introduce here some operations on the indecomposables E_{pq} of $\mathbb{Q}_m = (\mathscr{A}_m, \Omega)$ called "elementary degenerations" which will generate a preorder relation in the set of orbits of given dimension.

We will use the definition of indecomposable E_{pq} (or segment $[p, q]$) of even or odd type given in Section 2.

(*e*) For each pair of indecomposables E_{hk}, E_{rt} such that $h < r \leqslant t < k$ and $[r, t]$ is of even type we associate the pair E_{ht}, E_{rk}, i.e., we consider the operation

$$D^{e}_{hrtk} : E_{hk} \oplus E_{rt} \mapsto E_{ht} \oplus E_{rk}.$$

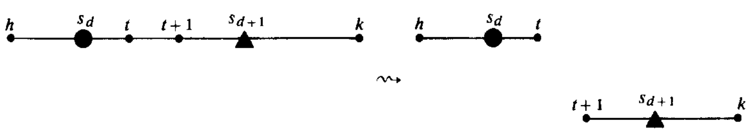
Example.



(*e'*) For each indecomposable E_{hk} and each integer t such that $h \leqslant t < k$ we consider the operation

$$D^{e'}_{htk} : E_{hk} \to E_{ht} \oplus E_{t+1k}.$$

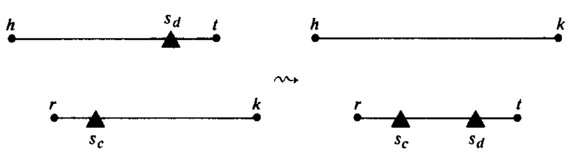
Example.



(*o*) For each pair of indecomposables E_{ht}, E_{rk} with $h < r \leqslant t < k$ and $[r, t]$ of odd type, we consider the operation

$$D^{o}_{hrtk} : E_{ht} \oplus E_{rk} \mapsto E_{hk} \oplus E_{rt}.$$

Example.



Remark 3.1. The operation (*e'*) can be considered as a special case of

(e) if we introduce the convention that $E_{t+1,t}$ is the 0 representation for every t and in (e) we allow the index r to be equal to $t+1$ (note that with our conventions $E_{t+1,t}$ is of even type). Therefore we will refer, from now on, to the elementary operations of types (e) and (o), and we call them resp. even or odd operation.

These three types of operations generate a preorder relation in the set of isomorphism classes of representations of $\mathbb{Q}_m$ of given dimension d. We will see in (3.3) that this is in fact an ordering which we call the "combinatorial ordering" and denote by $\leqslant_c$. The definition is the following:

DEFINITION 3.2. Given $A, B \in L_d$ we say that $\mathcal{O}_B \leqslant_c \mathcal{O}_A$ if and only if the set of indecomposable factors of B is obtained from the one of A with a finite number of elementary operations of types (e) and (o).

Let us denote by $\leqslant$ the geometrical ordering of the orbits given by $\mathcal{O}_B \leqslant_g \mathcal{O}_A$ if and only if $\mathcal{O}_B \subseteq \overline{\mathcal{O}_A}$ (i.e., if $\mathcal{O}_B$ is a degeneration of $\mathcal{O}_A$).

PROPOSITION 3.3. *For $A, B \in L_d$, if $\mathcal{O}_B \leqslant_c \mathcal{O}_A$ then $\mathcal{O}_B \leqslant_g \mathcal{O}_A$.*

Proof. Recall that if $0 \to M' \to M$ is an injection of modules, then the module $N = M' \oplus M/M'$ is a degeneration of M (i.e., N belongs to the closure of the isomorphism class of M). Therefore we only need to find such injections or projections for the elementary operations (e), (o).

Case (e). Suppose $s_{a-1} < r \leqslant s_a$, $s_b \leqslant t < s_{b+1}$ and s_{a-1} is a sink. As $\mathrm{Hom}(E_{ht}, E_{hk}) \simeq K$, let us denote by $\varphi: E_{ht} \to E_{hk}$ the morphism corresponding to 1 in the previous isomorphism. Let $\varphi': E_{ht} \to E_{rt}$ the analogous morphism. Then the morphism $\varphi - \varphi': E_{ht} \hookrightarrow E_{hk} \oplus E_{rt}$ is an injection and $(E_{hk} \oplus E_{rt})/E_{ht} \simeq E_{rk}$. If s_{a-1} is a source we have an injective map $E_{rk} \hookrightarrow E_{hk} \oplus E_{rt}$ and the quotient is E_{ht}.

Case (o). Can be treated with the same kind of argument.

According to what we have just proved in (3.3), we will refer to the result of an elementary operation as to an "elementary degeneration" (e), (o).

4. Obstructions to an Elementary Degeneration

Let A, $B \in L_d$ and suppose B is obtained from A performing one elementary degeneration. Propositions 3.3 and 2.7 imply that $N^B_{uv} \leqslant N^A_{uv}$ for every $1 \leqslant u < v \leqslant m$.

We first want to describe a way to compute for which pairs (u, v) the corresponding N_{uv} has in fact decreased its value, i.e., $N^B_{uv} < N^A_{uv}$.

Note that if B is obtained from A performing the operation $D^e_{hrtk}: E_{hk} \oplus E_{rt} \to E_{ht} \oplus E_{rk}$ then $n^B_{r-1,t+1} = n^A_{r-1,t+1} - 1$ and therefore

$$(4.1) \qquad \begin{array}{ll} n^B_{pq} = n^A_{pq} - 1 & \text{for } \; h \leqslant p \leqslant r-1, \quad t+1 \leqslant q \leqslant k \\ n^B_{pq} = n^A_{pq} & \text{otherwise.} \end{array}$$

If the operation is $D^o_{hrtk}: E_{ht} \oplus E_{rk} \mapsto E_{hk} \oplus E_{rt}$ then $n^B_{r-1,t+1} = n^A_{r-1,t+1} + 1$ and therefore

$$(4.2) \qquad \begin{array}{ll} n^B_{pq} = n^A_{pq} + 1 & \text{for } \; h \leqslant p \leqslant r-1, \quad t+1 \leqslant q \leqslant k \\ n^B_{pq} = n^A_{pq} & \text{otherwise.} \end{array}$$

Then from the expression (2.5) of N_{uv} (cf. Remark 2.9) and (4.1) or (4.2) we deduce that $N^B_{uv} = N^A_{uv} - 1$ if in (2.5) there is an odd number of n_{pq} altered by the degeneration; $N^B_{uv} = N^A_{uv}$ otherwise. With the notation of (2.5) we have:

PROPOSITION 4.3. *Let z, w be respectively the number of elements of the sequence $\{u, s_\alpha, \ldots, s_\beta, v\}$ which lie in the intervals $[h, r-1]$ and $[t+1, k]$, then have $N^B_{uv} = N^A_{uv} - 1$ if and only if the product $z \cdot w$ is odd; $N^B_{uv} = N^A_{uv}$ otherwise.*

Proof. It is trivial since z represents the number of columns in (2.5) on which some n_{pq} has changed its value, and in each such a column exactly w consecutive elements present a variation.

Let $A \in L_d$ and suppose we perform an elementary degeneration on its indecomposable factors. We will soon see that the N^A_{uv} which "change" under the effect of the operation are the elements of a submatrix of all rank parameters N^A (cf. Remark 2.10). We call this submatrix the "obstruction matrix" of a A relative to the performed operation and we use the notation $ob^A(D^e_{hrtk})$ or $ob^A(D^o_{hrtk})$. The reason we use the term "obstruction" is the following: suppose A is given via the set of its rank parameters (satisfying the inequalities stated in (2.7)). Can we perform on A, for example, the operation $E_{hk} \mapsto E_{ht} \oplus E_{t+1,k}$? The answer cannot be positive if some of the entries of $ob^A(D^{e'}_{htk})$ are zero (which means that there are no factors E_{hk} in A on which the operation can be performed).

Next we use Proposition 4.3 to list explicitly in four different cases the row and column indices of the various obstruction matrices one can get, as we will need them in the proof of (5.3).

Assume h, r, t, k are integers such that $1 \leqslant h < r \leqslant t < k$ (or $r = t+1$ and

$h \leqslant t < k$ as we want to consider simultaneously the operations (e) and (e')). In the given orientation $\Omega = \{s_i\}_{i=0,\dots,v+1}$ they satisfy the inequalities

$$s_{a-1} < h \leqslant s_a; \quad s_{c-1} < r \leqslant s_c; \quad s_d \leqslant t < s_{d+1}; \quad s_b \leqslant k < s_{b+1}$$

for suitable a, c, d, b.

Case (I). If $[h, r-1]$ and $[t+1, k]$ are of odd type, then for the corresponding obstruction matrix $ob^A(D^e_{hrtk})$ or $ob^A(D^o_{hrtk})$ we have

$\{1,\dots, h-1; s_a,\dots, s_{a+1}-1;\dots; s_{c-1},\dots, r-1\}$ row indices

$\{t+1,\dots, s_{d+1}; s_{d+2}+1,\dots, s_{d+3};\dots; s_{b-1}+1,\dots, s_b; k+1,\dots, m\}$ column indices.

Case (II). If $[h, r-1]$ is of even type and $[t+1, k]$ is of odd type then we have

$\{h,\dots, s_a-1; s_{a+1},\dots, s_{a+2}-1;\dots; s_{c-1},\dots, r-1\}$ row indices

$\{t+1,\dots, s_{d+1}; s_{d+2}+1,\dots, s_{d+3};\dots; s_{b-1}+1,\dots, s_b; k+1,\dots, m\}$ column indices.

Case (III). If $[h, r-1]$ is of odd type and $[t+1, k]$ is of even type, then we have

$\{1,\dots, h-1; s_a,\dots, s_{a+1}-1;\dots; s_{c-1},\dots, r-1\}$ row indices

$\{t+1,\dots, s_{d+1}; s_{d+2}+1,\dots, s_{d+3};\dots; s_b+1,\dots, k\}$ column indices.

Case (IV). If $[h, r-1]$ and $[t+1, k]$ are of even type, then we have

$\{h,\dots, s_a-1; s_{a+1},\dots, s_{a+2}-1;\dots; s_{c-1},\dots, r-1\}$ row indices

$\{t+1,\dots, s_{d+1}; s_{d+2}+1,\dots, s_{d+3};\dots; s_b+1,\dots, k\}$ column indices.

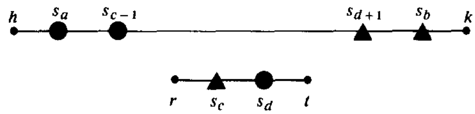
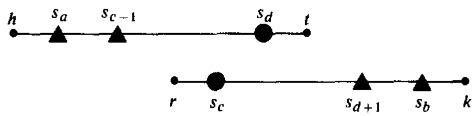
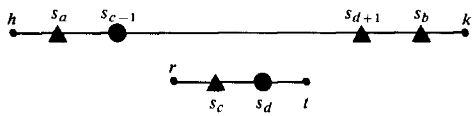
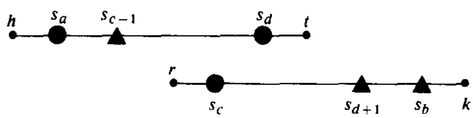
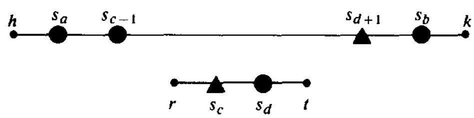
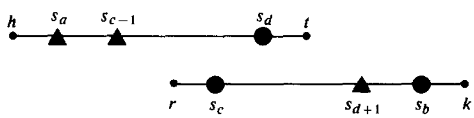
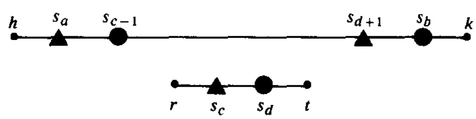
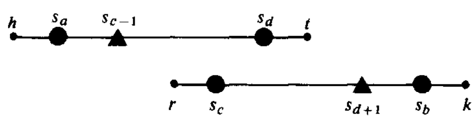
Note that we are prescribing the type of $[h, r-1]$ and $[t+1, k]$, therefore, as the operations D^e_{hrtk} or D^o_{hrtk} prescribe the type of $[r, t]$ we can deduce the type of the indecomposables on which we perform the operation. Each case contains therefore two different situations (up to duality) each one corresponding to an operation of type (e), (o).

All the possible situations concentrated in Cases I, II, III, IV are listed in the following table. In the last column we use the following:

NOTATION 4.4. Let $\{x_i\}_{i=1,2,\ldots,p}$, $\{y_j\}_{j=1,2,\ldots,q}$ be two sets of non-negative integers. The symbol

$$(x_1, x_2, \ldots, x_p \mid y_1, y_2, \ldots, y_q)$$

denotes that the x_i's have the same parity, different from the one of the y_j's.

Case	D	Indecomposables on which D acts	Parities
I	D^e_{hrtk}		$(a, d \mid b, c)$
I	D^o_{hrtk}		$(a, b \mid c, d)$
II	D^e_{hrtk}		$(a, b, c \mid d)$
II	D^o_{hrtk}		$(a, c, d \mid b)$
III	D^e_{hrtk}		$(a, b, d \mid c)$
III	D^o_{hrtk}		$(b, c, d \mid a)$
IV	D^e_{hrtk}		$(a, c \mid b, d)$
IV	D^o_{hrtk}		$(a, b, c, d \mid)$

Remark 4.5. (i) If in Case III or IV we have $k = m = s_b$, the index m does not appear as a column index, the last index of the list is in fact s_{b-1}. This fact will be used in the proof of (5.3) (cf. Section 7).

(ii) If $[h, r-1]$ (resp. $[t+1, k]$) does not contain any point of the sequence $\{s_i\} = \Omega$, then the corresponding row indices (resp. column indices) need to be contracted to $\{h,..., r-1\}$ (resp. $\{t+1,..., k\}$).

5. Strategy and Sketch of the Proof of the Main Theorem

For the quiver $\mathbb{Q}_m = (\mathscr{A}_m, \Omega)$ we consider the space $L := L_d$ of the isomorphism classes of representations of fixed dimension d.

As we have recalled in the Introduction, an isomorphism class $[A]$, $A \in L_d$, corresponds to the orbit $\mathcal{O}_A$ under the action of G on L.

In (3.2) we have defined an ordering on the orbit set, denoted by $\leqslant_c$ and in (3.3) we have compared it with the geometrical ordering denoted by $\leqslant_g$. Next we define a "rank ordering" in the same set, denoted by $\leqslant_r$, as follows:

Definition 5.1. Given A, $B \in L$, we say $\mathcal{O}_B \leqslant_r \mathcal{O}_A$ if and only if $N^B_{uv} \leqslant N^A_{uv}$ for $1 \leqslant u \leqslant v \leqslant m$.

Proposition 2.8 says that if $\mathcal{O}_B \leqslant_g \mathcal{O}_A$ then $\mathcal{O}_B \leqslant_r \mathcal{O}_A$. The theorem we want to prove is the following:

Theorem 5.2. *The three orderings $\leqslant_c$, $\leqslant_g$, $\leqslant_r$ coincide.*

We only need to compare the orderings $\leqslant_r$ and $\leqslant_c$ and the strategy we will use is described by Proposition 5.3.

To simplify to notations, from now on we will write $B \leqslant A$ instead of $\mathcal{O}_B \leqslant_r \mathcal{O}_A$.

Proposition 5.3. *Let A, $B \in L$ such that $B < A$. Then there exists a $C \in L$ obtained from A via an elementary degeneration and such that $B \leqslant C < A$.*

Note that Proposition 5.3 (and Theorem 5.2) has been proved in [1] in the case the Dynkin diagram is equioriented, i.e., in the case $v = 0$ (cf. [1, Theorem 3.2). Therefore, from now on we will assume $v > 0$.

We will say that "an elementary operation on A is allowed by B" if the operation on A gives rise to a C such that $B \leqslant C < A$.

If $B < A$ and we perform an elementary degeneration D on A, we do get a $C < A$, but if the entries N^A_{uv} of the obstruction matrix $ob^A(D)$ are not strictly greater then the corresponding N^B_{uv}'s, we do not have $B \leqslant C < A$. This explains once more the term "obstruction matrix."

Let $\mathbb{Q}'_{m-1}$ be the quiver obtained from $\mathbb{Q}_m$ erasing the last vertex. Therefore the vertices of $\mathbb{Q}'_{m-1}$ are labelled by $1,..., m-1$; the sequence of sources and sinks is $1 = s_0 < s_1 < \cdots < s_v \leqslant s'_{v+1} = m-1$. Similarly we define $\mathbb{Q}''_{m-1}$, erasing the first vertex.

To any representation $A = (A_1, A_2,..., A_{m-2}, A_{m-1})$ of $\mathbb{Q}_m$ corresponds the representation $A' = (A_1, A_2,..., A_{m-2})$ of $\mathbb{Q}'_{m-1}$ (resp. $A'' = A_2,..., A_{m-1})$ of $\mathbb{Q}''_{m-1}$). In particular to the indecomposable representation E_{pq}, $1 \leqslant p \leqslant q \leqslant m-1$, of $\mathbb{Q}_m$ corresponds the indecomposable representation E_{pq} of $\mathbb{Q}'_{m-1}$; to the indecomposable E_{pm}, $1 \leqslant p \leqslant m-1$, of $\mathbb{Q}_m$ corresponds the indecomposable $E_{p,m-1}$ of $\mathbb{Q}'_{m-1}$. It follows that to an elementary degeneration D_{hrtk} on A (of odd or even type, cf. Section 3) corresponds the same elementary degeneration on A' (resp. odd or even), if $k \leqslant m-1$; if $k = m$ and $t < m-1$ to D_{hrtm} (odd or even) corresponds D_{hrtm-1} (resp. odd or even).

Let $A = \sum_{1 \leqslant p \leqslant q \leqslant m} e^A_{pq} E_{pq}$ be a representation of $\mathbb{Q}_m$ and $A' = \sum_{1 \leqslant r \leqslant s \leqslant m-1} e^{A'}_{rs} E_{rs}$ the corresponding one in $\mathbb{Q}'_{m-1}$, then for the rank parameters and for the multiplicities we have the relations

(5.4) $N^{A'}_{uv} = N^A_{uv}$, $\quad 1 \leqslant u \leqslant v \leqslant m-1$ (resp. $N^{A''}_{uv} = N^A_{uv}$, $2 \leqslant u \leqslant v \leqslant m$),

(5.4)'
$$e^{A'}_{pq} = e^A_{pq}, \qquad 1 \leqslant p \leqslant q < m-1$$
$$e^{A'}_{p,m-1} = e^A_{p,m-1} + e^A_{pm}, \qquad 1 \leqslant p \leqslant m-1.$$

Remarks and Terminology 5.5. If $e^{A'}_{hk} > 0$, i.e., A' contains a factor (a direct summand), E_{hk}, and $k < m-1$ then from (5.4)' we deduce that $e^A_{hk} = e^{A'}_{hk} > 0$, i.e., A contains a factor E_{hk}. In this case we will say that "the factor E_{hk} of A' (of odd or even type, cf. Section 2), lifts to the factor E_{hk} of A (resp. of odd or even type)" (we will also say that "E_{hk} is the lifting to A of the same factor in A'").

If $e^{A'}_{h,m-1} > 0$, i.e., $E_{h,m-1}$ is a factor of A', then $e^A_{h,m-1} + e^A_{hm} > 0$ (cf. (5.4)'). It follows that either $e^A_{h,m-1} > 0$ and $E_{h,m-1}$ is a factor of A, or $e^A_{hm} > 0$ and E_{hm} is a factor of A. In this case we will say that "the factor $E_{h,m-1}$ of A' lifts either to a factor $E_{h,m-1}$ or to a factor E_{hm} of A" (the lifting need not be unique!). Note that $E_{h,m-1}$ has the same type odd or even (cf. Section 2) in $\mathbb{Q}'_{m-1}$ and $\mathbb{Q}_m$ if and only if $s = m-1$; $E_{h,m-1}$ in $\mathbb{Q}'_{m-1}$ and E_{hm} in $\mathbb{Q}_m$ have the same type odd or even if and only if $s_v < m-1$.

In any case a factor E_{hk} can be lifted to A. Therefore if in A' we can perform the elementary degeneration $D' = D_{hrtk}$, $k < m-1$, then in A we can perform $D = D_{hrtk}$; if in A' we can perform $D' = D_{hrtm-1}$ then in A we can perform either $D = D_{hrtm-1}$ or $D = D_{hrtm}$ or both. In any case we will say that "the elementary degeneration D' performed on A' lifts to an elementary degeneration D performed on A" (or equivalently "D is a lifting of D' from A' to A"). Note that if D' is an even operation (resp. odd) then D is an even operation (resp. odd) but, if D' is of type I (resp. II, III, IV) (cf. Section 4), then D need not to be of type I (resp. II, III, IV).

Remark 5.6. Suppose we can perform an elementary degeneration D' on A' allowed by B', i.e., there are no obstructions in $\mathbb{Q}'_{m-1}$; when we lift D' to a degeneration D on A new obstructions can arise, if we require the degeneration to be allowed by B, and these correspond to the column index m in $ob^A(D)$.

DEFINITION 5.7. We will say that the lifting D of D' from A' to A is trivial if $ob^A(D) = ob^{A'}(D')$.

To see if a lifting is trivial or not it is enough to look at the lists of row and column indices given in Section 4 (Cases I to IV).

Sketch of the proof of 5.3. Given A, $B \in L$, if $B < A$ then $B' \leqslant A'$ and $B'' \leqslant A''$. The proof 5.3 will be done in two steps.

Step 1. If $B' < A'$ (or $B'' < A''$) we proceed by induction on the length of the quiver, the initial case being trivial. We know by induction that there exists an elementary degeneration on A' allowed by B' (resp. A'' allowed by B''), and we show that we can lift this degeneration to one on A allowed by B.

Step 2. If $B' = A'$ and $B'' = A''$ then we are in the case $N^B_{1m} < N^A_{1m}$ and all the other rank parameters for B and A are equal; in this case we directly exhibit an elementary degeneration on A allowed by B.

In order ro realize this program we will need some lemmas.

6. LEMMAS

Let $A \in L$ be a given representation. We want to produce here some inequalities satisfied by the set of rank parameters N^A which we will need for the proof of (5.3). As the representation A is supposed to be fixed, we omit it in our notations.

Let $w = s_z + v$ be any index such that $s_z \leqslant w < s_{z+1}$ and let d be such that $z \leqslant d < v$ (where $m = s_{v+1}$). In the expression (2.5) of N_{wm} we can collect first the terms corresponding to $N_{w,s_{d+1}}$ and $N_{s_{d+1},m}$, i.e., we write

$$N_{wm} = N_{w,s_{d+1}} + N_{s_{d+1},m} - \text{(some other terms)}.$$

If the number of columns in $N_{w,s_{d+1}}$ is even we can collect the other terms in pairs of consecutive columns; otherwise we will collect the terms relative to the column w alone and all the other ones paired together. With this idea in mind we introduce the following notations:

NOTATION 6.1. (i) Let $j < h$ and $s_e \leqslant h < s_{e+1}$, we set

$$R_{jh} = n_{jh} - n_{j,s_{e+1}} + \sum_{t=2}^{v-e+1} (-1)^t n_{j,s_{e+t}} = (n_{jh} - n_{j,s_{e+1}}) + (n_{j,s_{e+2}} - n_{j,s_{e+3}}) + \dots$$

(ii) Let $i < j < h$ and $s_e \leqslant h < s_{e+1}$, we set

$$S_{ijh} = (n_{jh} - n_{ih} - n_{j,s_{e+1}} + n_{i,s_{e+1}}) + \sum_{t=2}^{v-e+1} (-1)^t (n_{j,s_{e+t}} - n_{i,s_{e+t}})$$
$$= (n_{jh} - n_{ih} - n_{j,s_{e+1}} + n_{i,s_{e+1}}) + (n_{j,s_{e+2}} - n_{i,s_{e+2}} - n_{j,s_{e+3}} + n_{i,s_{e+3}}) + \dots$$

(iii) Let $i < j < f$, we set

$$T_{ijf} = n_{jf} - n_{if}.$$

We deduce the following decompositions for $N_{s_z+v,m}$

$$N_{s_z+v,m} = N_{s_z+v,s_{d+1}} + N_{s_{d+1},m} - \left(R_{s_z+v,s_{d+2}} + \sum_{\alpha=z+2,z+4,\dots,d} S_{s_{\alpha-1},s_\alpha,s_{d+2}}\right) \tag{6.2$_1$}$$

if z and d have the same parity, i.e., s_z and s_d are both sources or sinks.

$$N_{s_z+v,m} = N_{s_z+v,s_{d+1}} + N_{s_{d+1},m} - \left(S_{s_z+v,s_{z+1},s_{d+2}} + \sum_{\alpha=z+3,z+5,\dots,d} S_{s_{\alpha-1},s_\alpha,s_{d+2}}\right) \tag{6.2$_2$}$$

if $z+1$ and d have the same parity, i.e., if s_{z+1} and s_d are both sources or sinks.

For any pair of indices $s_z + v$ and e such that $s_z + v < e$ and $s_v < e < m$, we have instead the following decompositions:

$$N_{s_z+v,m} = N_{s_z+v,e} - \left(R_{s_z+v,e} + \sum_{\alpha=z+2,z+4,\dots,v} S_{s_{\alpha-1},s_\alpha,e}\right) \tag{6.3$_1$}$$

if z and v have same parity

or

$$(6.3)_2 \qquad N_{s_z+v,m} = N_{s_z+v,e} - \left(S_{s_{z+v},s_{z+1},e} + \sum_{a=z+3,z+5,\ldots,v} S_{s_{a-1},s_a,e}\right)$$

if $z+1$ and v have the same parity.

Remark 6.4. Clearly we have: (i) $R_{jh} \geqslant 0$ since each summand in parenthesis for its expression is non-negative. In fact $n_{j,s_{e+t}} - n_{i,s_{e+t+1}}$ counts the number of factors E_{pq} (in A) with $p \leqslant j$, $s_{e+t} \leqslant q < s_{e+t+1}$; $n_{jh} - n_{j,s_{e+1}}$ counts the number of factors E_{pq} with $p \leqslant j$, $h \leqslant q < s_{e+1}$.

(ii) $S_{ijh} \geqslant 0$ since each summand of its expression is non-negative. Note that $n_{j,s_{e+t}} - n_{i,s_{e+t}} - n_{j,s_{e+t+1}} + n_{i,s_{e+t+1}}$ counts the number of factors E_{pq} wth $i < p \leqslant j$, $s_{e+t} \leqslant q < s_{e+t+1}$; similarly for $n_{jh} - n_{ih} - n_{j,s_{e+1}} + n_{i,s_{e+1}}$.

(iii) $T_{ijf} \geqslant 0$. In fact T_{ijf} counts the number of factors E_{pq} with $i < p \leqslant j$, $q \geqslant f$.

We deduce the following:

LEMMA 6.5. (i) $R_{jh} > 0$ *if and only if there exists* (*in* A) *a factor* E_{pq} *with* $p \leqslant j$ *and* $q \in \{h,\ldots, s_{e+1}-1;\ s_{e+2},\ldots,s_{e+3}-1;\ldots\}$.

(ii) *If* $j < f \leqslant h$ *then* $n_{jf} \geqslant R_{jh}$ *and the equality holds if and only if there are no factors* E_{pq} (*in* A) *with* $p \leqslant j$, $q \in \{f,\ldots, h-1;\ s_{e+1},\ldots, s_{e+2}-1;\ s_{e+3},\ldots, s_{e+4}-1;\ldots\}$.

(iii) $S_{ijh} > 0$ *if and only if there exists* (*in* A) *a factor* E_{pq} *with* $i < p \leqslant j$ *and* $q \in \{h,\ldots, s_{e+1}-1;\ s_{e+2},\ldots, s_{e+3}-1;\ldots\}$.

(iv) *If* $i \leqslant j < f \leqslant h$ *then* $T_{ijf} \geqslant S_{ijh}$ *and the equality holds if and only if there are no factors* E_{pq} (*in* A) *with* $i < p \leqslant j$, $q \in \{f,\ldots, h-1;\ s_{e+1},\ldots,s_{e+2}-1;\ldots\}$.

LEMMA 6.6. *Consider the indices* $s_a + r$, $s_b + t$, $s_c + w$ *such that* $1 \leqslant s_a + r < s_b + t < s_c + w < m$ *with* $s_a \leqslant s_a + r < s_{a+1}$; $s_b \leqslant s_b + t < s_{b+1}$; $s_c < s_c + w \leqslant s_{c+1}$. *We have the following inequalities*:

(i) $N_{s_a+r,m} - N_{s_b+t,m} \geqslant N_{s_a+r,s_c+w} - N_{s_b+t,s_c+w}$ *if* b *and* c *have the same parity.*

(ii) $N_{s_a+r,m} - N_{s_b+t,m} \leqslant N_{s_a+r,s_c+w} - N_{s_b+t,s_c+w}$ *if* b *and* c *have opposite parity.*

Moreover

(iii) *The equality holds both in* (i) *and* (ii) *if and only if there are no* E_{pq} (*in* A) *with*: $p \in \{1,\ldots, s_a + r;\ s_{a+1}+1,\ldots, s_{a+2};\ldots;\ s_{b-2}+1,\ldots, s_{b-1};\ s_b+1,\ldots, s_b+t\}$ *if* a *and* b *have different parity*; *or* $p \in \{s_a + r + 1,\ldots,\ s_{a+1};$

$s_{a+2}+1,\ldots, s_{a+3};\ldots; s_{b-2}+1,\ldots, s_{b-1}; s_b+1,\ldots, s_b+t\}$ *if* a *and* b *have the same parity*; *and* $q \in \{s_c+w,\ldots, s_{c+1}-1; s_{c+2},\ldots, s_{c+3}-1;\ldots\}$.

Proof. (i) The following computations are based directly on (2.5) (where we collect together the contribution of $N_{s_a+r,m}-N_{s_a+r,s_c+w}$ and the one of $N_{s_b+t,s_c+w}-N_{s_b+t,m}$), and on the notation (6.1).

Suppose the parity of a is different from the one of b and c:

$$\begin{aligned}
&(N_{s_a+r,m}-N_{s_b+t,m})-(N_{s_a+r,s_c+w}-N_{s_b+t,s_c+w})\\
&\quad=-R_{s_a+r,s_{c+1}}-\sum_{i=a+1,a+3,\ldots,b-2} S_{s_i,s_{i+1},s_{c+1}}-S_{s_b,s_b+t,s_{c+1}}\\
&\qquad+n_{s_a+r,s_c+w}+\sum_{i=a+1,a+3,\ldots,b-2} T_{s_i,s_{i+1},s_c+w}+T_{s_b,s_b+t,s_c+w}\\
&\quad=(n_{s_a+r,s_c+w}-R_{s_a+r,s_{c+1}})+\sum_{i=a+1,\ldots,\,b-2}(T_{s_i,s_{i+1},s_c+w}\\
&\qquad-S_{s_i,s_{i+1},s_{c+1}})+(T_{s_b,s_b+t,s_c+w}-S_{s_b,s_b+t,s_{c+1}}).
\end{aligned}$$

The statement follows now from Lemma 6.5(ii), (iv). If a has the same parity as b and c we have

$$\begin{aligned}
&(N_{s_a+r,m}-N_{s_b+t,m})-(N_{s_a+r,s_c+w}-N_{s_b+t,s_c+w})\\
&\quad=(T_{s_a+r,s_{a+1},s_c+w}-S_{s_a+r,s_{a+1},s_{c+1}})\\
&\qquad+\sum_{i=a+2,a+4,\ldots,b-2}(T_{s_i,s_{i+1},s_c+w}-S_{s_i,s_{i+1},s_{c+1}})\\
&\qquad+(T_{s_b,s_b+t,s_c+w}-S_{s_b,s_b+t,s_{c+1}})
\end{aligned}$$

and we get the same conclusion.

(ii) Is similar to (i) once one notices that under the new assumptions the signs in the various summands have been changed into their opposite.

(iii) We just read Lemma 6.5(ii), (iv) in the case when equalities hold.

7. Proof of Proposition 5.3, Step 1

For Step 1 (cf. Section 5) our assumption will be:

(*) There is no elementary degeneration on A' allowed by B' which admits a trivial lifting to A.

Otherwise the required C we are looking for is trivially found.

Suppose $B' < A'$. By induction we know that there is an elementary degeneration on A' allowed by B' which can be either

$$D^e_{hrtk}: E_{hk} \oplus E_{rt} \mapsto E_{ht} \oplus E_{rk} \quad \text{or} \quad D^o_{hrtk}: E_{ht} \oplus E_{rk} \mapsto E_{hk} \oplus E_{rt},$$

with $1 \leqslant h < r \leqslant t < k \leqslant m-1$ (or $r = t+1$ and $1 \leqslant h < t < k \leqslant m-1$) and $s_{a-1} < h \leqslant s_a$; $s_{c-1} < r \leqslant s_c$, $s_d \leqslant t < s_{d+1}$, $s_b \leqslant k < s_{b+1}$. We only need to analyze and prove Step 1 for all possible elementary degeneration on A' which cannot be trivially lifted to A; clearly they are listed in Section 4 Cases I and II where the column index m do appear, and some "limit case" for Cases III and IV, i.e., when $k = m-1$ (cf. Remark 4.5(i)).

7.I$_e$. *The Case* I *for an Even Degeneration*

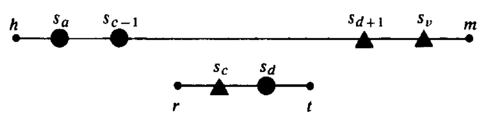
We assume that the elementary degeneration on A' allowed by B' is

$$D^e_{httk}: E_{hk} \oplus E_{rt} \to E_{ht} \oplus E_{rk}, \quad k \leqslant m-1 \qquad \text{(Case I)}.$$

From Section 4 Case I we read that the row indices in $ob^{A'}(D^e_{hrtk})$ are $\{1,..., h-1;\ s_a,..., s_{a+1}-1;...;\ s_{c-1},..., r-1\} =: H$, the column indices are $\{t+1,..., s_{d+1};\ s_{d+2}+1,...,\ s_{d+3};...;\ s_{b-1}+1,..., s_b;\ k+1,..., m-1\} =: K$ (where $K = \{t+1,..., m-1\}$ if $k = m+1$ and $d = v$).

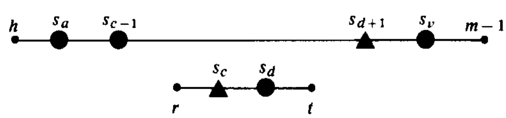
We know that D^e_{hrtk} can be lifted to A (cf. Remark 5.5) and we collect in Table I$_e$ the non-trivial liftings which can occur in A.

Remark I$_e$. In Table I$_e$ we have not listed all the possible liftings of $E_{h,m-1}$ from A' to A, but the missing ones give rise to degenerations which are trivial liftings, against (*). In fact if in (ii) we assume that the degeneration lifts to



(Case III)

or if in (iii) we assume that the degeneration lifts to

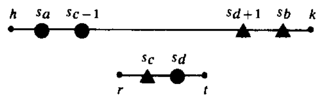
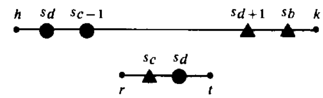
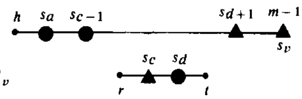
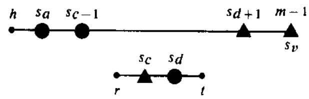
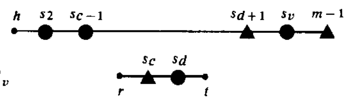
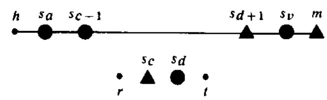
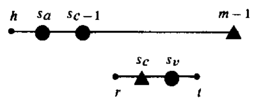
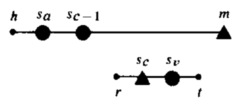


(Case III)

by Remarks 4.5(i), (5.6) and (5.7) we have a trivial lifting.

Moreover if $d = v$ we must have $k = m-1$, otherwise the degeneration in A' does not belong to Case I and in (iv) we can repeat the same argument as in (ii) or (iii).

TABLE I_e

	Column 1 D^e_{hrtk} in A': Case I	Column 2 Non-trivial lifting of D^e_{hrtk} in A	Case
(i) $d \neq v$ $k < m - 1$			I
(ii) $d \neq v$ $k = m - 1 = s_v$			I
(iii) $d \neq v$ $k = m - 1 \neq s_v$			I
(iv) $d = v$ $k = m - 1$			I

For all the possibilities listed in column 2 of Table I_e we have new obstruction indices relative to the column index m and row indices $\rho \in H$.

We claim that

$$(**) \qquad N^A_{\rho,m} > N^B_{\rho,m} \qquad \text{for every } \rho \in H.$$

From this claim, once proved, it will follow that if we assume the representation C to be obtained from A via the degeneration of column 2, then $A > C \geqslant B$ and the Proposition 5.3 is proved.

To prove $(**)$ assume by contradiction that there exists an index $\bar{\rho} = s_z + v \in H$ $(s_z \leqslant s_z + v < s_{z+1})$, and such that

$$(7.1) \qquad N^A_{s_z+v,m} = N^B_{s_z+v,m}.$$

The index $\bar{\rho} = s_z + v$ can be of two types:

(1) $s_z + v$ is such that z has the same parity as d,

(2) $s_z + v$ is such that $z + 1$ has the same parity as d.

We analyze first the case $d \neq v$, i.e., (i), (ii) and (iii) of Table I_e. We have

$$(7.2)\qquad \begin{aligned} &N^A_{s_z+v,s_{d+1}} > N^B_{s_z+v,s_{d+1}} &&\text{as the degeneration on } A' \text{ is allowed by } B' \text{ (cf. 5.4),}\\ &N^A_{s_{d+1},m} \geqslant N^B_{s_{d+1},m} &&\text{as } B < A. \end{aligned}$$

We discuss separately the cases $\bar{p} = s_z + v$ of type (1) or (2).

If $\bar{p} = s_z + v$ is of type (1), we use the decomposition $(6.2)_1$ for both sides of (7.1) and from (7.2) and (6.4) we deduce

$$(7.3)_1 \qquad R^A_{s_z+v,s_{d+2}} + \sum_{\alpha = z+2, z+4, \ldots, d} S^A_{s_{\alpha-1},s_\alpha,s_{d+2}} > 0.$$

From Lemma 6.5(i), (iii) it follows that A contains a factor (a direct summand) $E_{\bar{p}\bar{q}}$ with

$$\bar{p} \in \{1,\ldots, s_z + v; s_{z+1} + 1,\ldots, s_{z+2};\ldots; s_{d-1} + 1,\ldots, s_d\} =: P_1,$$
$$\bar{q} \in \{s_{d+2},\ldots, s_{d+3} - 1; s_{d+4},\ldots, s_{d+5} - 1;\ldots\}$$

and a fortiori with

$$\bar{q} \in \{t + 1,\ldots, s_{d+1} - 1; s_{d+2},\ldots, s_{d+3} - 1;\ldots\} =: Q.$$

We set $s_{i-1} < \bar{p} \leqslant s_i$, $s_j \leqslant \bar{q} < s_{j+1}$ for every $\bar{p} \in P_1$, $\bar{q} \in Q$. Note that j and d have the same parity, therefore if $\bar{q} = m = s_{v+1}$ then $v + 1$ and d have the same parity. We do not know a priori the parity of the index i, therefore the known parities are (we use the Notation 4.4)

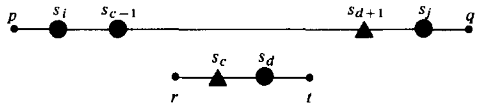
$$(a, d, z, j \mid c).$$

We choose now a factor E_{pq} of A such that $p \in P_1$, $q \in Q$ and p is minimum in P_1. Note that $p \neq r$, in fact if $p \leqslant s_z + v$ we have $p < r$; if $p > s_z + v$ then i and c have opposite parity.

If $p < r$ we have $p < r \leqslant t < q$ and we can perform on A the degeneration $D^e_{prtq}: E_{pq} \oplus E_{rt} \mapsto E_{pt} \oplus E_{rq}$. We claim that i and d have the same parity, otherwise the degeneration D^e_{prtq} is of type IV and we have a trivial lifting, against (*) (the argument holds also for $q = m$, cf. Remark 4.5(i)). It follows that the parities are

$$(a, d, z, j, i \mid c)$$

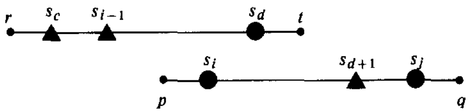
and we can perform on A the degeneration

D^e_{prtq}: 

(Case III)

and $h < p$ (not to contradict (*)).

If $p > r$ then i and d have the same parity; on A we can perform the degeneration

D^0_{rptq}: 

(Case (III)

and the parities are

$$(a, d, z, j, i \mid c).$$

We compare now the two obstruction matrices

$$ob^A(D^e_{prtq}) \qquad \text{and} \qquad ob^{A'}(D^e_{hrrk}) \qquad \text{if} \quad h < p < r$$

or

$$ob^A(D^o_{rptq}) \qquad \text{and} \qquad ob^{A'}(D^e_{hrtk}) \qquad \text{if} \quad p > r.$$

In ob^A there are new entries relative to the row indices $\{h,..., s_a - 1; s_{a+1},..., s_{a+2} - 1;...; s_{i-1},..., p - 1\} =: W$ and column indices $\{t + 1,..., s_{d+1}; s_{d+2} + 1,..., s_{d+3};...; s_j + 1,..., q\} =: V$ (cf. Cases III and I), and there must be a pair of indices $s_f + u \in W$, $s_g + w \in V$, $s_f \leqslant s_f + u < s_{f+1}$, $s_g < s_g + w \leqslant s_{g+1}$, such that

$$N^A_{s_f+u, s_g+w} = N^B_{s_f+u, s_g+w} \qquad (s_g + w < m) \tag{7.4}$$

(otherwise the degeneration D^e_{prtq} (resp. D^o_{rptq}) on A will be a trivial lifting of the same operation performed on A'). Note that $s_g + w < m$ even if $q = m$, as in this case $s_j = s_{v+1} = m$. The parities of the indices involved are

$$(a, d, z, j, i, g \mid c, f)$$

and $s_f + u \neq s_z + v$ (as z and f have different parity).

We claim that

$$N^A_{s_z+v, m} - N^A_{s_f+u, m} = N^A_{s_z+v, s_g+w} - N^A_{s_f+u, s_g+w}, \tag{7.5}$$

as a consequence of Lemma 6.6(iii).

In fact if $s_z + v < s_f + u$ in A there is no factor E_{xy}, $s_{\alpha-1} < x \leqslant s_\alpha$, $s_\beta \leqslant y < s_{\beta+1}$, with

$$x \in \{1,..., s_z + v; s_{z+1} + 1,..., s_{z+2};...; s_f + 1,..., s_f + u\},$$
$$y \in \{s_g + w,..., s_{g+1} - 1; s_{g+2},..., s_{g+3} - 1;...\}$$

(note that if in A we have $E_{x,y}$, with α and f of the same parity, then $x < s_z + v < r$, we can perform D^e_{xrty} (Case IV) and we contradict (*), if in A we have E_{xy}, with α and f of opposite parity then $x \leqslant s_f + u < p$ and we contradict the minimality of p). If $s_z + v > s_f + u$ we interchange the role of these two indices and again (7.5) holds.

We also have

$$N^A_{s_z+v,s_g+w} > N^B_{s_z+v,s_g+w}, \tag{7.6}$$

in fact $s_z + u \in H$, $s_g + w \in K$ (note that if $g = m$ then $s_j = s_{v+1} = m$, the last index in V is s_{j-1} and $V \subset K$).

We use now (7.4), (7.1) and (7.6) to deduce from (7.5) and Lemma 6.6(i) or (ii) applied to B the following

$$\begin{aligned} N^A_{s_z+v,m} - N^A_{s_f+u,m} = N^A_{s_z+v,s_g+w} - N^A_{s_f+u,s_g+w} > N^B_{s_z+v,s_g+w} \\ - N^B_{s_f+u,s_g+w} \geqslant N^B_{s_z+v,m} - N^B_{s_f+u,m}. \end{aligned} \tag{7.7}$$

It follows

$$N^A_{s_f+u,m} < N^B_{s_f+u,m}, \tag{7.8}$$

a contradiction to the assumption $B < A$.

If $\bar{p} = s_z + v$ is of type (2) then $s_z + v < h$; we use the decomposition $(6.2)_2$ for both sides of (7.1) and from (7.2) and (6.4) we deduce

$$S^A_{s_z+v,s_{z+1},s_{d+1}} + \sum_{\alpha=z+3,z+5,...,d} S^A_{s_{\alpha-1},s_\alpha,s_{d+2}} > 0. \tag{7.3$_2$}$$

From Lemma 6.5(i), and (iii) it follows that A contains at least a factor $E_{\bar{p},\bar{q}}$, $s_{i-1} < \bar{p} \leqslant s_i$, $s_j \leqslant \bar{q} < s_{j+1}$, with

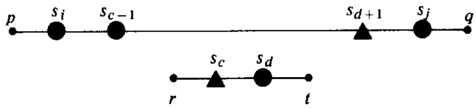
$$\bar{p} \in \{s_z + v + 1,..., s_{z+1}; s_{z+2} - 1,..., s_{z+3};...; s_{d-1} + 1,..., s_d\} =: P_2,$$
$$\bar{q} \in \{t + 1,..., s_{d+1} - 1; s_{d+2},..., s_{d+3} - 1;...\} = Q$$

and the parities are

$$(a, d, j, i \mid c, z).$$

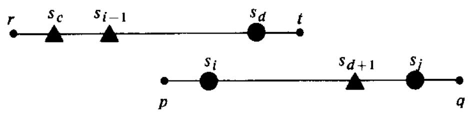
Also in this case we choose a factor E_{pq} of A such that $p \in P_2$, $q \in Q$ and p is minimum in P_2. $p \neq r$ as i and c have opposite parity.

If $p < r$, we can perform on A the degeneration

D^e_{prtq}:  (Case III)

and $h < p$ (not to contradict (*)).

If $p > r$ we can perform on A the degeneration

D^o_{rptq}:  (Case III)

We proceed now as for $\bar{\rho} = s_z + v$ of type (1). Using the same argument and the same notations, we see that (7.4) must hold for $s_f + u \in W$, $s_g + w \in V$. The parities are

$$(a, d, j, i, g \mid c, z, f)$$

and we have $s_z + v < h \leqslant s_f + u$. Again we claim (7.5), as a consequence of Lemma 6.6(iii). In fact in A there is no factor $E_{x,y}$, $s_{\alpha-1} < x \leqslant s_\alpha$, $s_\beta \leqslant y < s_{\beta+1}$, with

$$x \in \{s_z + v + 1,..., s_{z+1}; s_{z+2} + 1,..., s_{z+3};...; s_f + 1,..., s_f + u\},$$

$$y \in \{s_g + w,..., s_{g+1} - 1; s_{g+2},..., s_{g+3} - 1;...\}$$

as $x \in P_2$, $y \in Q$ and $x \leqslant s_f + u < p$.

Using again (7.4), (7.1) and (7.6) from (7.5) we deduce (7.7) and the contradiction (7.8) to $B < A$.

It follows that (**) is proved for $d \neq v$.

If $d = v$, we examine (iv) of Table I_e, and still claim (**). By contradiction we assume (7.1), but (7.2) does not hold. We use instead the following

$$(7.2)' \qquad N^A_{s_z+v,t+1} > N^B_{s_z+v,t+1}$$

as $t + 1 < m$ and the operation on A' is allowed by B'. We cannot use the decompositions (6.2), we use instead (setting $e = t + 1$) $(6.3)_1$ or $(6.3)_2$, according to the type (1) or (2) for the index $\bar{\rho} = s_z + v$, for both sides of (7.1) and from (7.2)′ we deduce

$$(7.3)'_1 \qquad R^A_{s_z+v,t+1} + \sum_{\alpha = z+2, z+4,..., v} S^A_{s_{\alpha-1}, s_\alpha, t+1} > 0,$$

or

$$(7.3)_2' \qquad S^A_{s_z+v,s_{z+1},t+1} + \sum_{\alpha=z+3,z+5,\ldots,v} S^A_{s_{\alpha-1},s_\alpha,t+1} > 0.$$

From now on the discussion is the same as for the case $d \neq v$, we only point out that

$$Q = \{t+1,\ldots,m-1\}, \qquad V = \{t+1,\ldots,q\}$$

and $s_g + w \in V$ is such that $g = v$. From (7.4), (7.5) and (7.6) we deduce (7.7) and the contradiction (7.8). (**) is now completely proved.

Remark. Note that the line of this proof of Case I_e goes like this: By contradiction we assume (7.1) and from (6.2) (or (6.3)) and (7.2) (or 7.2)′) we deduce (7.3) (or (7.3)′). This gives us in A a subset of indecomposables $\{E_{\bar{p},\bar{q}}\}$ in which we choose $E_{p,q}$ with some properties (p minimum, the parity of some indices, etc.) in such a way that (7.4) holds. Next we prove (7.5) and from the assumptions (7.2) and (7.6) we deduce (7.7) and (7.8), i.e., a contradiction to $B < A$.

In all the remaining cases we will follow the same line, using the same notations of case I_e when possible.

7.I_o *The Case* I *for an Odd Degeneration*

We assume that the elementary degeneration on A' allowed by B' is

$$D^o_{hrtk}: E_{ht} \oplus E_{rk} \to E_{hk} \oplus E_{rt}, \qquad k \leqslant m-1 \qquad \text{(Case I)}.$$

The row and column indices in $ob^{A'}(D^o_{hrtk})$ are the same as for the even operation (cf. Subsection 7.I_e), i.e., the row indices are $\{1,\ldots,h-1;\ s_a,\ldots,s_{a+1}-1;\ldots;\ s_{c-1},\ldots,r-1\} = H$, the column indices are $\{t+1,\ldots,s_{d+1};\ s_{d+2}+1,\ldots,s_{d+3};\ldots;\ s_{b+1}+1,\ldots,s_b;\ k+1,\ldots,m-1\} = K$.

We collect in Table I_o the non-trivial liftings which can occur in A.

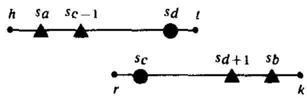
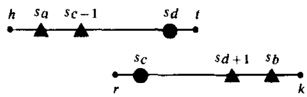
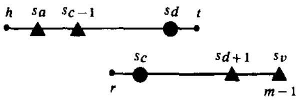
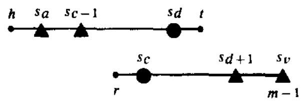
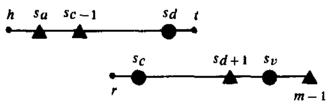
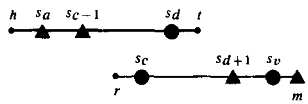
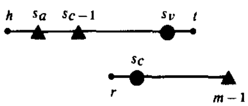
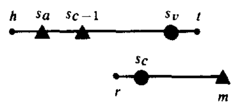
Remark I_o. In Table I_o we have not listed all the possible liftings of $E_{r,m-1}$ from A' to A, but the missing ones give rise to trivial liftings which are against (*). Moreover if $d = v$ then $k = m-1$. The argument is exactly the same as in Remark I_e.

For all the possibilities listed in column 2 of Table I_o, we have new obstruction indices relative to the column index m, and the row index $p \in H$, where H is the same set defined for the Case I_e, as the obstruction indices do not depend on the type even or odd of the operation.

Again we claim (**) (cf. Subsection 7.I_e).

To prove it we use the same notations as for the even operation of type I and by contradiction we assume (7.1). Then again the index $\bar{p} = s_z + v$ can by of type (1) or (2).

TABLE I_o

	Column 1 D^o_{hrtk} in A': Case I	Column 2 Non-trivial lifting of D^o_{hrtk} in A	Case
(i) $d \neq v$ $k \neq m-1$			I
(ii) $d \neq v$ $k = m-1 = s_v$			I
(iii) $d \neq v$ $k = m-1 \neq s_v$			I
(iv) $d = v$ $k = m-1$			I

Assume first $d \neq v$, then we can reproduce the same argument as in Subsection 7.I_e, up to the choice of a factor $E_{p,q}$ of A with $(p, q) \in P_\tau \times Q$ ($\tau = 1, 2$), and p minimum in P_τ. At this point the argument is similar, but not equal, to the one given in Subsection 7.I_e, and we develop it for the convenience of the reader.

We claim that $p \neq h$. In fact, for $\tau = 1$, if $p < s_z + v$ we have $p < h$; if $p > s_z + v$ then i and a have opposite parity; for $\tau = 2$ again i and a have opposite parity.

We cannot have $p < h$, otherwise we could perform on A the degeneration D^e_{phtq}: $E_{pq} \oplus E_{ht} \to E_{pt} \oplus E_{hq}$ which is a trivial lifting (if $\bar{p} = s_z + v$ is of type (1) the operation is of type III or IV, if $\bar{p} = s_z + v$ is of type (2) the operation is of type III), against the assumption (*).

If $p > h$ then i and d have the same parity and we can perform on A the degeneration D^o_{hptq} which is of type III. Not to contradict (*) we must also have $p > r$. We compare the two obstruction matrices:

$$ob^A(D^o_{hptq}) \qquad \text{and} \qquad ob^{A'}(D^o_{hrtk}) \qquad (p > r).$$

In $ob^A(D^o_{hptq})$ there are new entries relative to the row indices $\{r, \ldots, s_c - 1;\ s_{c+1}, \ldots,\ s_{c+2} - 1; \ldots;\ s_{i-1}, \ldots, p-1\} =: W'$ and the column indices

$\{t+1,..., s_{d+1}; s_{d+2}+1,..., s_{d+3};...; s_j+1,..., q\} = V$ and there must be a pair of indices $s_f+u \in W'$, $s_g+w \in V$ such that (7.5) holds. Note that here we have $s_z+v < s_f+u$ and we can proceed, as for the even operation, up to the inequality (7.8), against the assumption $B < A$. Therefore the claim (**) is proved for $d \neq v$. If $d = v$ we use $(7.2)'$ and deduce $(7.3)_1'$ or $(7.3)_2'$ and the proof of (**) is the same as for the case $d \neq v$, with obvious changes (compare also the cases $d \neq v$ and $d = v$ for the even operation).

7.II$_e$ *The Case* II *for an Even Degeneration*

We assume that the degeneration on A' allowed by B' is D^e_{hrtk}: $E_{hk} \oplus E_{rt} \to E_{ht} \oplus E_{rk}$, $k \leqslant m-1$ (Case II).

From Section 4, Case II we read that the row indices in $ob^{A'}(D^e_{hrtk})$ are

$$\{h,..., s_a - 1; s_{a+1},..., s_{a+2} - 1;...; s_{c-1},..., r-1\} =: L,$$

the column indices are

$$\{t+1,..., s_{d+1}; s_{d+2}+1,..., s_{d+3};...; s_{b-1}+1,..., s_b;$$
$$k+1,..., m-1\} = K$$

(where $K = \{t+1,..., m+1\}$ if $k = m-1$ and $d = v$).

In Table II$_e$ we collect all the non-trivial liftings of D^e_{qrtk} which can occur in A.

Note that if $d = v$ then $k = m-1$, otherwise the degeneration on A' is not of type II. Moreover all the liftings which do not appear in column 2 are of type IV, and have been eliminated, as we assume (*).

For all the possibilities listed in column 2 we have new obstruction indices relative to the column index m and row indices $\rho \in L$.

As in Case I we claim (**), i.e.,

$$(**) \qquad N^A_{\rho,m} > N^B_{\rho,m} \qquad \text{for every } \rho \in L.$$

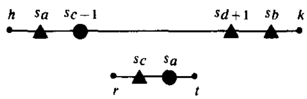
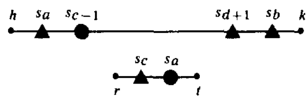
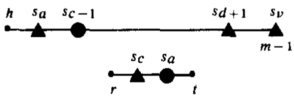
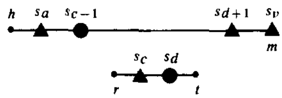
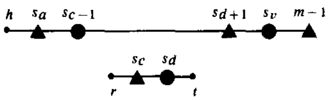
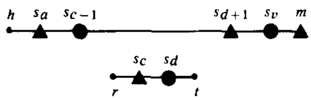
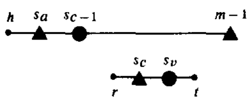
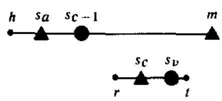
To prove (**) assume by contradiction that there exists an index $\bar{\rho} = s_z + v \in L$, $s_z \leqslant s_z + v < s_{z+1}$, such that

$$(7.1) \qquad N^A_{s_z+v,m} = N^B_{s_z+v,m}.$$

The index $\bar{\rho} = s_z + v \in L$ is such that z and a have the same parity. Therefore, from Table II$_e$ we deduce the parities

$$(a, c \mid d, z).$$

TABLE II_e

	Column 1 D^e_{hrtk} in A': Case II	Column 2 Non-trivial lifting of D^e_{hrtk} in A	Case
(i) $d \neq v$ $k \neq m-1$			II
(ii) $d \neq v$ $k = m-1 = s_v$			II
(iii) $d \neq v$ $k = m-1 \neq s_v$			II
(iv) $d = v$ $k = m-1$			II

We analyze first the case $d \neq v$. We have

$$N^A_{s_z+v,s_{d+1}} > N^B_{s_z+v,s_{d+1}}, \quad N^A_{s_{d+1},m} \geqslant N^B_{s_{d+1},m}. \tag{7.2}$$

We use $(6.2)_1$ for both sides of (7.1) and from (7.2) and (6.4) we deduce

$$R^A_{s_z+v,s_{d+2}} + \sum_{\alpha=z+2,z+4,\ldots,d} S_{s_{\alpha-1},s_\alpha,s_{d+2}} > 0. \tag*{$(7.3)_1$}$$

From Lemma 6.5(i) and (iii) it follows that A contains at least a factor $E_{\bar{p},\bar{q}}$ with

$$\bar{p} \in \{1,\ldots, s_z+v; s_{z+1}+1,\ldots, s_{z+2};\ldots; s_{d-1}+1,\ldots, s_d\} = P_1,$$
$$\bar{q} \in \{t+1,\ldots, s_{d+1}-1; s_{d+2},\ldots, s_{d+3}-1;\ldots\} = Q.$$

We set $s_{i-1} < \bar{p} \leqslant s_i$; $s_j \leqslant \bar{q} < s_{j+1}$. Clearly we have

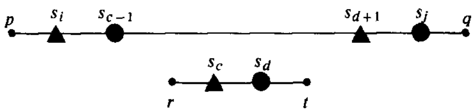
$$(a, c \mid d, z, j).$$

(*o*) Assume that in A we have factors $E_{\bar{p},\bar{q}}$, $\bar{p} \in P_1$, $\bar{q} \in Q$, such that

$$(a, c, i \mid d, z, j).$$

Then $\bar{p} < s_z + v$ and among the factors $E_{\bar{p},\bar{q}}$ satisfying (*o*) we choose a factor $E_{p,q}$ with p maximum in P_1. We have $p < s_z + v < r$ and we can perform on A degeneration

D^e_{prtq}:



(Case IV)

We compare the two obstruction matrices

$$ob^A(D^e_{prtq}) \qquad \text{and} \qquad ob^{A'}(D^e_{hrtk})$$

(Cases IV and II); not to go against (*) we must have $p < h$. Then in $ob^A(D^e_{prtq})$ there are new entries relative to the row indices

$$\{p,..., s_i - 1; s_{i+1},..., s_{i+2} - 1;...; s_{a-1},..., h - 1\} =: U$$

and the column indices

$$\{t + 1,..., s_{d+1}; s_{d+2} + 1,..., s_{d+3};...; s_j + 1,..., q\} = V$$

and not to go against (*) there must be a pair of indices $s_f + u \in U$, $s_g + w \in V$, $s_f \leqslant s_f + u < s_{f+1}$; $s_g < s_g + w \leqslant s_{g+1}$ such that

$$(7.4) \qquad N^A_{s_f+u, s_g+w} = N^B_{s_f+u, s_g+w} \qquad (s_g + w < m).$$

The parities now are

$$(a, c, i \mid d, z, j, f, g)$$

and $s_f + u < h \leqslant s_z + v$.

We claim that (7.5) hold:

$$(7.5) \qquad N^A_{s_z+v, m} - N^A_{s_f+u, m} = N^A_{s_z+v, s_g+w} - N^A_{s_f+u, s_g+w}.$$

The claim follows from Lemma 6.6(iii); in fact in A there is no factor E_{xy}, $s_{\alpha-1} < x \leqslant s_\alpha$, $s_\beta \leqslant y < s_{\beta+1}$, with

$$x \in \{s_f + u + 1,..., s_{f+1}; s_{f+2} + 1,..., s_{f+3}; ...; s_z + 1,..., s_z + v\},$$

$$y \in \{s_g + w,..., s_{g+1} - 1; s_{g+2},..., s_{g+3} - 1;...\}$$

as $x \in P_1$, $y \in Q$, $x > s_f + u \geqslant p$, α has the parity of i, β has the parity of j and p has been chosen maximum. We also have (7.6)

$$N^A_{s_z + v, s_g + w} > N^B_{s_z + v, s_g + w}, \tag{7.6}$$

in fact $s_z + v \in L$ and $s_g + w \in K$ (note that if $q = m$, then $s_j = s_{v+1} = m$, the last index in V is s_{j-1} and in any case $V \subset K$). As in 7.I we get now the contradiction (7.8) to $B < A$ and $(**)$ is proved under the assumption (o).

We may now assume that:

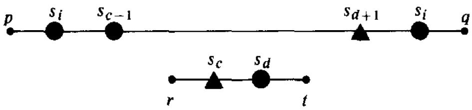
(oo) All the factors $E_{\bar{p},\bar{q}}$ of A such that $\bar{p} \in P_1$, $\bar{q} \in Q$ satisfy also the parity condition

$$(a, c \mid d, z, i, j).$$

Among them we choose $E_{p,q}$, with p minimum in P_1. We have $p \neq r$ and $p \neq h$ as i has parity different from the one of a and c.

If $h < p < r$ we can perform on A the degeneration

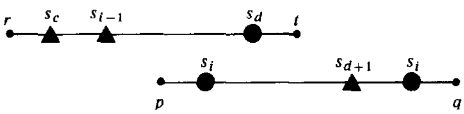
D^e_{prtq}:



(Case III)

If $p > r$ on A we can perform the degeneration

D^o_{rptq}:



(Case III)

We compare $ob^{A'}(D^e_{hrtk})$ with $ob^A(D^e_{prtq})$ and with $ob^A(D^o_{rptq})$ (Cases II and III); for both cases in ob^A we have new entries relative to the row indices

$$\{1, \ldots, h-1; s_a, \ldots, s_{a+1} - 1; \ldots, s_{i-1}, \ldots, p-1\} =: U'$$

and to the column indices

$$\{t+1, \ldots, s_{d+1}; s_{d+2} + 1, \ldots, s_{d+3}; \ldots; s_j + 1, \ldots, q\} =: V$$

and not to go against $(*)$ there must be a pair of indices $s_f + u \in U'$, $s_g + w \in V$, $s_f \leqslant s_f + u < s_{f+1}$, $s_g < s_g + w \leqslant s_{g+1}$ for which (7.4) holds.

We have the parities

$$(a, c \mid d, z, i, j, g)$$

and we claim that (7.5) holds, independently from the parity of f, as a consequence of Lemma 6.6(iii). In fact if f and a have opposite parity there $s_f + u \leqslant h - 1 < s_z + v$ and in A there is no factor $E_{x,y}$, $s_{\alpha-1} < x \leqslant s_\alpha$, $s_\beta \leqslant y < s_{\beta+1}$, with

$$x \in \{s_f + u + 1,..., s_{f+1}; s_{f+2} + 1,..., s_{f+3}; ...; s_z + 1,..., s_z + v\},$$

$$y \in \{s_g + w,..., s_{g+1} - 1; s_{g+2},..., s_{g+3} - 1;...\},$$

as $x \in P_1$, $y \in Q$ and α and a have the same parity (cf. (oo)).

If f and a have the same parity and $s_f + < s_z + v$, then in A there is no factor $E_{x,y}$, $s_{\alpha-1} < x \leqslant s_\alpha$, $s_\beta \leqslant y < s_{\beta+1}$, with

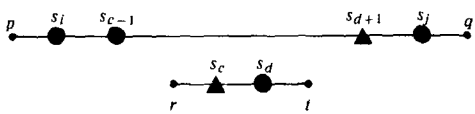
$$x \in \{1,..., s_f + u; s_{f+1} + 1,..., s_{f+2};...; s_z + 1,..., s_z + v\},$$

$$y \in \{s_g + w,..., s_{g+1} - 1; s_{g+2},..., s_{g+3} - 1;...\},$$

as $x \in P_1$, $y \in Q$, α and a need to have opposite parity (cf. (oo), which implies $x < s_f + u < p$ and p has been chosen minimum in P_1. If $s_f + u > s_z + v$ we interchange the role of these two indices and the argument is the same.

If $p < h$ we can perform on A the degeneration

D^e_{prtq}:



(Case III)

Proceeding as before we have new obstruction indices

$$\{1,..., p - 1; s_i,..., s_{i+1} - 1;...; s_{a-1},..., h - 1\} = U'',$$

$$\{t + 1,..., s_{d+1}, s_{d+2} + 1,..., s_{d+3};...; s_j + 1,..., q\} = V,$$

and a pair if indices $s_f + v \in U''$, $s_g + w \in V$ such that (7.4) holds. We have the parities

$$(a, c \mid d, z, i, j, g)$$

and we claim that (7.5) holds, independently from the parity of f, as a consequence of Lemma 6.6(iii). In fact if f and a have opposite parity, then in A there is no factor $E_{x,y}$, $s_{\alpha-1} < x \leqslant s_\alpha$, $s_\beta \leqslant y < s_{\beta+1}$, with

$$x \in \{s_f + u + 1,..., s_{f+1};...; s_z + 1,..., s_z + v\},$$

$$y \in \{s_g + w,..., s_{g+1};...\},$$

as $x \in P_1$, $y \in Q$ and α and a have the same parity (cf. (oo)). If f and a have the same parity then $s_f + u \leqslant p - 1$ and in A the is no factor $E_{x,y}$, $s_{\alpha-1} < x \leqslant s_\alpha$, $s_\beta \leqslant y < s_{\beta+1}$

$$x \in \{1,..., s_f + u; s_{f+1} + 1,..., s_{f+2};...; s_z + 1,..., s_z + v\},$$

$$y \in \{s_g + w,..., s_{g+1};...\},$$

as $x \in P_1$, $y \in Q$, α and a need to have opposite parity (cf (oo)) which implies $x \leqslant s_f + u < p$ and p has been chosen minimum in P_1.

We also have (7.6), independently from the fact that $h < p < r$, or $p > r$, or $p < h$, as $s_z + v \in L$ and $s_g + w \in K$. Therefore we get a contradiction to $B < A$ (cf. 7.I$_e$) and (**) is fully proved in the case $d \neq v$.

If $d = v$ we still claim (**); to prove it we assume (7.1) for an index $\bar{p} = s_z + v \in L$ and we have (7.2)$'$ (cf. 7.I$_e$). Using $(6.3)_1$ we also get $(7.3)_1'$ and from Lemma 6.5(i) and (ii) it follows that A contains at least a factor $E_{\bar{p},\bar{q}}$, with $\bar{p} \in P_1$ and $\bar{q} \in \{t - 1,..., m - 1\} = Q$. At this point we proceed exactly as in the case $d \neq v$ we have just treated, the only difference being that the set V in the actual case is $V = \{t + 1,..., q\}$ and $s_g + w \in V$ is such that $g = v$.

7.II$_o$ *The Case* II *for an Odd Degeneration*

We assume that the degeneration on A' allowed by B' is D^o_{hrtk}: $E_{ht} \oplus E_{rk} \to E_{hk} \oplus E_{rt}$, $k \leqslant m - 1$ (Case II). The row and column indices of $ob^{A'}(D^o_{hrtk})$ belong respectively to the sets L and K (cf. Subsection II$_e$).

We list in Table II$_o$ all the non-trivial liftings of D^o_{hrtk} which can occur in A (the liftings which do not appear in column 2 belong to Case IV and are trivial).

For all the possibilities listed in column 2 we have new obstruction indices relative to the column index m and the row index $\rho \in L$, and as in the even case we claim (**) (cf. Subsection II$_e$). To prove (**) we assume, by contradiction, (7.1), and the parities are

$$(a, c, d \mid z).$$

Assume $d \neq v$, then we have (7.2). We use $(6.2)_2$ for both sides of (7.1) and from (7.2) and (6.4) we deduce

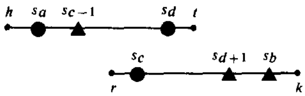
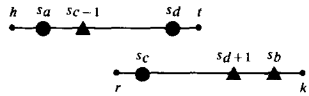
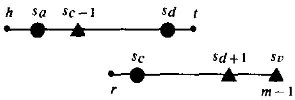
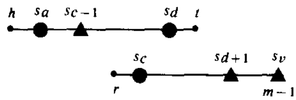
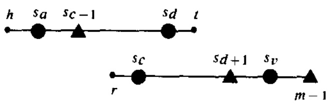
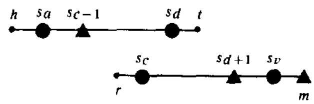
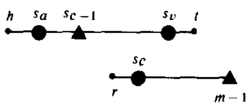
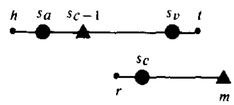
$$(7.3)_2 \qquad S^A_{s_z+v, s_{z+1}, d+2} + \sum_{\alpha = z+3, z+5, ..., d} S^A_{s_{\alpha-1}, s_\alpha, s_{d+2}} > 0.$$

From Lemma 6.5(iii) it follows that A contains at least a factor $E_{\bar{p},\bar{q}}$, $s_{i-1} < \bar{p} \leqslant s_i$, $s_j \leqslant \bar{q} < s_{j+1}$, with

$$\bar{p} \in \{s_z + v + 1,..., s_{z+1}; s_{z+2} + 1,..., s_{z+3};...; s_{d-1} + 1,..., s_d\} = P_2,$$

$$\bar{q} \in \{t + 1,..., s_{d+1} - 1; s_{d+2},..., s_{d+3} - 1,...\} = Q.$$

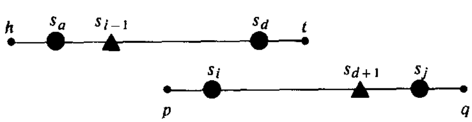
TABLE II_o

	Column 1 D^o_{hrtk} in A': Case II	Column 2 Non-trivial lifting of D^o_{hrtk} in A	Case
(i) $d \neq v$ $k \neq m-1$			II
(ii) $d \neq v$ $k = m-1 = s_v$			II
(iii) $d \neq v$ $k = m-1 \neq s_v$			II
(iv) $d = v$ $k = m-1$			II

The parities are

$$(a, c, d, i, j \mid z)$$

and among these factors we choose $E_{p,q}$, with p minimum in P_2. We note that $p > s_z + v \geqslant h$. We can perform on A the degeneration

D^o_{hptq}:  (Case IV)

and not to contradict (*) we must have $p > r$. It follows that in $ob^A(D^o_{hptq})$ there are new entries relative to the new indices

$$\{r,\ldots, s_c - 1; s_{c+1},\ldots, s_{c+2} - 1;\ldots; s_{i-1},\ldots, p - 1\} =: U'''$$

and column indices

$$\{t + 1,\ldots, s_{d+1}; s_{d+2} + 1,\ldots, s_{d+3};\ldots; s_j + 1,\ldots, q\} = V$$

and not to go against (*) there must be a pair of indices $s_f + u \in U'''$, $s_g + w \in V$ such that (7.4) holds. The parities are

$$(a, c, d, i, j, g \mid z, f)$$

and we claim (7.5) (cf. Subsection 7.II$_e$) s a consequence of Lemma 6.6(iii). In fact we have $s_z + v < r < s_f + u$ and in A there is no factor $E_{x,y}$ with

$$x \in \{s_z + v + 1, ..., s_{z+1}; s_{z+2} + 1, ..., s_{z+3}; ...; s_f + 1, ..., s_f + u\},$$

$$y \in \{s_g + w, ..., s_{g+1} - 1; s_{g+2}, ..., s_{g+3} - 1; ...\},$$

as $x \in P_2$, $y \in Q$, $x \leqslant s_f + u < p$ and p has been chosen minimum in P_2. We proceed now as in Subsection 7.I or 7.II$_e$ up to the contradiction (7.8), and (**) is proved for $d \neq v$.

If $d = v$ we have to use (7.2)$'$ and (6.3)$_2$ and the proof of (**) is the same.

7.III. *The Case* III *for an Even or Odd Degeneration*

Assume that the elementary degeneration on A' allowed by B' is of type III and either it is

$$D^e_{hrtk}: E_{hk} \oplus E_{rt} \to E_{ht} \oplus E_{rk} \qquad \text{(Case III)}$$

or it is

$$D^o_{hrtk}: E_{ht} \oplus E_{rk} \to E_{hk} \oplus E_{rt} \qquad \text{(Case III)}.$$

We claim that $k = m - 1$, otherwise both operations on A' lift to A to the same operation performed on the same indecomposables, and we have trivial liftings, agains (*).

We collect the only possible non-trivial liftings which can occur in A in Table III$_e$ for the even operation and in Table III$_o$ for the odd one.

TABLE III$_e$

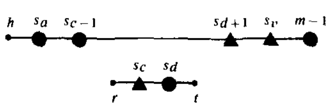
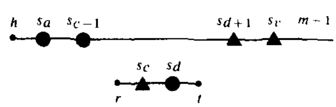
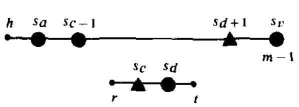
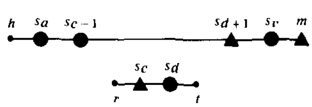
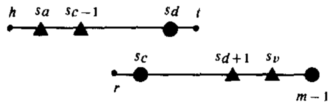
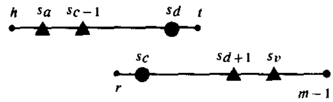
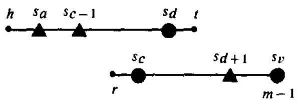
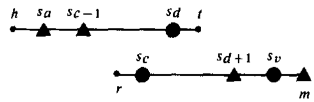
	Column 1 D^e_{hrtm-1} in A': Case III	Column 2 Non-trivial lifting of D^e_{hrtm-1} in A	Case
(i) $d \neq v$ $k = m - 1 \neq s_v$			I
(ii) $k = m - 1 = s_v$			I

TABLE III_o

	Column 1 $D^o_{hrtm-1}A'$: Case III	Column 2 Non-trivial lifting of D^o_{hrtm-1} in A	Case
(i) $d \neq v$ $k = m - 1 \neq s_v$			I
(ii) $d \neq v$ $k = m - 1 = s_v$			I

In $ob^{A'}(D_{hrtm-1})$ (for the even or odd operation) the row indices are

$$\{1,\dots, h-1; s_a,\dots, s_{a+1}-1;\dots; s_{c-1},\dots, r-1\} = H,$$

the column indices are

$$\{t+1,\dots, s_{d+1}; s_{d+2}+1,\dots, s_{d+3};\dots; s_{v-1}+1,\dots, s_v\} =: K'$$
$$\text{if} \quad k = m-1 \neq s_v,$$
$$\{t+1,\dots, s_{d+1}; s_{d+2}+1,\dots, s_{d+3};\dots; s_{v-2}+1,\dots, s_{v-1}\} =: K''$$
$$\text{if} \quad k = m-1 = s_v,$$

and for the liftings listed in column 2 (for the even or odd operation) we have new obstruction indices relative to the column index m and the row indices $\rho \in H$ (compare Cases III and I).

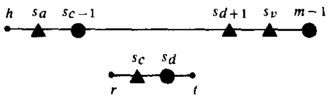
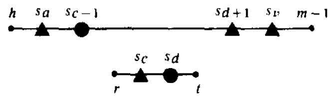
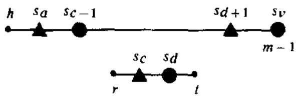
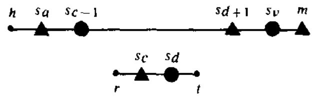
As in Subsection $7.\mathrm{I}_e$ (resp. $7.\mathrm{I}_o$) we claim (**) and the proof of it is exactly the same, as the argument in Subsection $7.\mathrm{I}_e$ (resp. $7.\mathrm{I}_o$) is independent from the value of k, which in Case III need to be $k = m-1$ (note also that in the actual case $d \neq v$).

7.IV. *The Case* IV *for an Even or Odd Degeneration*

Assume that the degeneration on A' allowed by B' is of type IV. As in Subsection 7.III we claim that $k = m-1$, independently from the fact the degeneration is even or odd. In fact if $k = m-1$ the operations on A' lift to A to the same operation performed on the same indecomposables and we have trivial liftings, against (*).

We collect the only possible non-trivial liftings in A in Table IV_e for the even operation, and in Table IV_o for the odd one.

TABLE IV_e

	Column 1 D^e_{hrtm-1} on A': Case IV	Column 2 Non-trivial lifting of D^e_{hrtm-1} on A	Case
(i) $d \neq v$ $k = m-1 \neq s_v$			II
(ii) $d \neq v$ $k = m-1 = s_v$			II

In $ob^{A'}(D_{hrtm-1})$ (for the even or odd operation) the row indices are

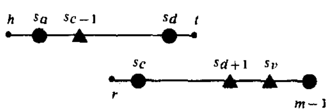
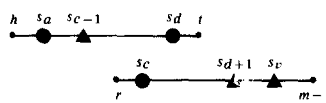
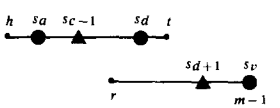
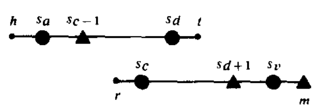
$$\{h,..., s_a - 1; s_{a+1},..., s_{a+2} - 1;...; s_{c-1},..., r-1\} = L;$$

the column indices are

$$\{t+1,..., s_{d+1}; s_{d+2}+1,..., s_{d+3};...; s_{v-1}+1,..., s_v\} = K'$$
$$\text{if} \quad k = m-1 \neq s_v,$$
$$\{t+1,..., s_{d+1}; s_{d+2}+1,..., s_{d+3};...; s_{v-2}+1,..., s_{v-1}\} = K''$$
$$\text{if} \quad k = m-1 = s_v.$$

For the liftings listed in column 2 (for the even or odd operation) we have new obstruction indices relative to the column index m and the row indices $\rho \in L$ (compare Case IV and II).

TABLE IV_o

	Column 1 D^o_{hrtm-1} on A': Case IV	Column 2 Non-trivial lifting of D^o_{hrtm-1} on A	Case
(i) $d \neq v$ $k = m-1 \neq s_v$			II
(ii) $d \neq v$ $k = m-1 = s_v$			II

As in Subsection 7.II$_e$ (resp. 7.II$_o$) we claim (**) and the proof of it is the same, as the argument in Subsection 7.II$_e$ (resp. 7.II$_o$) is independent from the value of k.

Step 1 is now completely proved.

8. Proof of Proposition 5.3, Step 2

We are in the case $A > B$, $A' = B'$, $A'' = B''$, i.e., $N^A_{1m} > N^B_{1m}$ and $N^A_{ij} = N^B_{ij}$ for $(i,j) \neq (1, m)$. As both A and B are the direct sum of indecomposables and the ranks are additive, we can assume: (**): no indecomposable $E_{p,q}$ appears simultaneously in A and B. We deduce that in the decompositions of A and B no $E_{h,k}$, with $k < m-1$ can appear as for these indices we have $e^A_{hk} = e^{A'}_{hk} = e^{B'}_{hk} = e^B_{hk}$; no $E_{r,s}$ can appear with $r > 2$ as we have $e^A_{rs} = e^{A''}_{rs} = e^{B''}_{rs} = e^B_{rs}$. It follows that in A or B the only indecomposables which can appear are

$$E_{1m-1}, \qquad E_{1m}, \qquad E_{2m-1}, \qquad E_{2m}. \tag{8.1}$$

It is easily seen using the assumptions $A' = B'$, $A'' = B''$ and (**), that only two configurations are possible:

(j) B is the direct sum of ρ copies of $(E_{1m-1} \oplus E_{2m})$ and A is the direct sum of ρ copies of $(E_{1m} \oplus E_{2m-1})$,

(jj) B is the direct sum of ρ copies of $(E_{1m} \oplus E_{2m-1})$ and A is the direct sum of ρ copies of $(E_{1m-1} \oplus E_{2m})$. If ν is even we have the decomposition of type $(6.2)_2$:

$$N_{1m} = N_{1s_\nu} + N_{s_\nu m} - \sum_{t=1,3,\ldots,\nu-1} S_{s_{t-1}s_t m}.$$

Moreover both for A and B we have

$$\sum_{t=3,5,\ldots,\nu-1} S_{s_{t-1}s_t m} = 0, \qquad S_{1s_1 m} = e_{2m}$$

(cf. Lemma 6.5(iii), and (8.1)).

As $N^A_{1m} > N^B_{1m}$, $N^A_{1s_\nu} = N^B_{1s_\nu}$, $N^A_{s_\nu m} = N^B_{s_\nu m}$ it follows $e^B_{2m} > e^A_{2m} \geqslant 0$ and we have the configuration (j). In A we can now perform the degeneration $E_{1m} \oplus E_{2,m-1} \mapsto E_{1,m-1} \oplus E_{2m}$ (cf. Case I). If ν is odd we have the decomposition

$$N_{1m} = N_{1s_\nu} + N_{s_\nu m} - \left(e_{1m} + \sum_{t=2,4,\ldots,\nu-1} S_{s_{t-1}s_t m}\right).$$

Moreover we have $\sum_{t=2,4,\ldots,v-1} S_{s_{t-1}s_t m} = 0$ both for A and B (cf. Lemma 6.5(iii) and (8.1)). From the assumptions it follows $e^B_{1m} > e^A_{1m} \geqslant 0$ and we have the configuration (jj). In A we can now perform the degeneration $E_{1m-1} \oplus E_{2m} \mapsto E_{1m} \oplus E_{2m-1}$ (cf. Case I).

References

1. S. Abeasis and A. Del Fra, Degenerations for the representations of an equioriented quiver of type $\mathscr{A}_m$, *Boll. Un. Mat. Ital. Suppl. Algebra Geom.* (1980), 157–171.
2. I. N. Bernstein, I. M. Gelfand, and V. A. Ponomarev, Coxeter functors and Gabriel's theorem., *Uspékhi Mat. Nauk* **28** (1973), 19–33; *Russian Math. Surveys* **28** (1973), 17–32 (transl).
3. V. Dlab and G. M. Ringel, Indecomposable representations of graphs and algebras, *Mem. Amer. Math. Soc.* **6**, No. (173) (1976), 1 – 57.
4. P. Gabriel, Unzerlegbare Darstellungen, I, *Manuscripta Math.* **6** (1972), 71–103.
5. P. Gabriel, Représentations indécomposables, *Sém. Bourbaki 26e année*, No. 444 (1973/'74), 1–27.