Algorithms for the Gauss–Manin Connection

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We give an introduction to the theory of the Gauss–Manin connection of an isolated hypersurface singularity and describe an algorithm to compute the V-filtration on the Brieskorn lattice. We use an implementation in the computer algebra system SINGULAR to prove C. Hertling’s conjecture about the variance of the spectrum for Milnor number \( \mu \leq 16 \).

1. Introduction

Let \( f : U \rightarrow \mathbb{C} \) be a holomorphic map defined in a neighbourhood \( 0 \in U \subset \mathbb{C}^{n+1} \) with isolated critical point 0 and critical value \( f(0) = 0 \). By Milnor (1968), for an appropriately chosen restriction

\[ f : X \rightarrow T \]

of \( f \) over a disc \( T \subset \mathbb{C} \) around 0, the non-singular fibres are homotopy equivalent to a bouquet of \( \mu \) \( n \)-spheres and form a \( \mathbb{C}^\infty \) fibre bundle over the punctured disc \( T' := T \setminus \{0\} \) called the Milnor fibration (see Figure 1). Hence, the cohomology of the general fibre \( X_t := f^{-1}(t) \), \( t \in T' \), the so-called Milnor fibre, is given by

\[ \tilde{H}^k(X_t, \mathbb{Z}) = \begin{cases} \mathbb{Z}^\mu, & k = n \\ 0, & \text{else.} \end{cases} \]

The local product structure of the Milnor fibration translates to the structure of a flat complex vector bundle on the \( n \)th complex cohomology groups of the non-singular fibres

\[ H^n := \bigcup_{t \in T'} H^n(X_t, \mathbb{C}) \]

called the cohomology bundle. The flatness of the cohomology bundle means that it can be described by local frames with constant transition functions. Hence, there is a well-defined notion of a holomorphic section in the cohomology bundle being constant. Algebraically, this translates to the existence of a flat connection on the cohomology bundle, the so-called Gauss–Manin connection, meaning that sections can be differentiated by the covariant derivative along vector fields defined on the base \( T \). The covariant derivative along \( \partial_t \), where \( t \) is the coordinate on \( T \), defines a differential operator which we also denote by \( \partial_t \).

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Moving an integer cohomology class along constant sections once around the critical point in counterclockwise direction, defines an automorphism
\[ M \in \text{Aut}_\mathbb{Z}(H(X_t, \mathbb{Z})) \]
defined over \( \mathbb{Z} \) which is called the algebraic monodromy. Since the monodromy is not the identity, flat sections are multivalued which means that they are global flat sections in the pullback of the cohomology bundle to the universal covering of the punctured disc. But one can multiply such a flat multivalued section by appropriate holomorphic twists, inverse to the action of the monodromy, in order to obtain a global holomorphic section. The sections arising in this way, defined over arbitrary small punctured neighbourhoods of \( 0 \in T \), span a regular \( \mathbb{C}[t][\partial_t] \)-module \( \mathcal{G}_0 \) which we will call the Gauss–Manin connection. The regularity of the Gauss–Manin connection means that the sections have moderate growth towards \( 0 \in T \). As Brieskorn (1970) has shown, the monodromy of the Gauss–Manin connection as \( \mathbb{C}[t][\partial_t] \)-module coincides with the complex monodromy and its eigenvalues are roots of unity. Up to this point, it is totally unclear how to approach this object by methods of computer algebra in order to obtain an algorithm to compute it.

Brieskorn (1970) gave an algebraic description of the complex monodromy and an algorithm to compute it. Using the holomorphic De Rham theorem, the cohomology of the Milnor fibre can be described in terms of integrals of holomorphic differential \( n \)-forms over vanishing cycles. The Gelfand–Leray form \( \frac{df}{f} \) of a holomorphic differential \( (n+1) \)-form \( \omega \) on \( X \) defines a holomorphic section in the cohomology bundle. This gives a map \( \Omega^{n+1}_{X,0} \to \mathcal{G}_0 \) which actually factors through an inclusion of the Brieskorn lattice
\[ \mathcal{H}_0'' = \Omega^{n+1}_{X,0}/df \wedge d\Omega^{n-1}_{X,0} \]
into the Gauss–Manin connection. The Leray residue formula gives the formula
for the action of $\partial_t$. This is the key to an algorithmic approach towards the Gauss–Manin connection. But it is still a non-trivial task to compute the monodromy.

The Brieskorn lattice is a free $\mathbb{C}\{t\}$-module of rank $\mu$ (Sebastiani, 1970) and $t^{n+1}\partial_t$ acts on it. E. Brieskorn explained how the computation of this action up to sufficiently high order allows one to compute the complex monodromy. Based on the work of Gérard and Leveit (1973), Nacken (1990) first implemented this algorithm in the computer algebra system MAPLE V. A later implementation by the author in the computer algebra system SINGULAR (Greuel et al., 2001) in the library mondromy.lib (Schulze, 1999, 2001) turned out to be more efficient.

An appropriate restriction of $\partial_t$ is invertible and $\partial_t^{-1}$ acts on the Brieskorn lattice. This extends to a structure over the ring of microdifferential operators with constant coefficients $\mathbb{C}\{\partial_t^{-1}\}$, a power series ring with a certain growth condition. As we will see, the Brieskorn lattice is a free $\mathbb{C}\{\partial_t^{-1}\}$-module of rank $\mu$ (Pham, 1977). We will explain how this structure leads to more efficient algorithms allowing us to compute more than just the monodromy.

The $V$-filtration on the Gauss–Manin system is defined by the generalized eigenspaces of $t\partial_t$ which are logarithms of the eigenvalues of the monodromy. The induced $V$-filtration on the Brieskorn lattice reflects its embedding in the Gauss–Manin connection and defines the spectrum, which is an important and deep invariant coming from the mixed Hodge-structure on the cohomology of the Milnor fibre (Steenbrink, 1977; Varchenko, 1982; Scherk and Steenbrink, 1985). Based on M. Saito’s result (Saito, 1988) saying that, for Newton non-degenerate singularities, the $V$-filtration coincides with the Newton filtration defined on $\mathbb{C}\{x_0, \ldots, x_n\}$ by the Newton polyhedron of $f$ at 0, Endrass (2001) implemented an algorithm for computing the spectrum of Newton non-degenerate singularities in the SINGULAR library spectrum.lib. We will present the first algorithm to compute the spectrum of arbitrary singularities.

The weight filtration on the Gauss–Manin connection is defined by the nilpotent part of $t\partial_t$, which is the logarithm of the unipotent part of the monodromy. This gives a refinement of the $V$-filtration defining the spectral pairs corresponding to the Hodge numbers of the mixed Hodge-structure on the cohomology of the Milnor fibre.

By the methods we are going to explain, one can actually compute all of the above invariants, namely the $V$- and weight filtration, the spectrum and spectral pairs, and the Hodge numbers, for not necessarily Newton non-degenerate singularities. Most of the algorithms are implemented in the SINGULAR library gaussman.lib (Schulze, 2001).

The spectrum consists of $\mu$ rational so called spectral numbers $\alpha_1, \ldots, \alpha_\mu$ in the interval $(-1, n)$, which are symmetric with mean value $\frac{n-1}{2}$. Hertling (2001) conjectured that their variance is bounded by

$$\gamma := \frac{1}{4} \sum_{i=1}^{\mu} \left( \alpha_i - \frac{n-1}{2} \right)^2 + \frac{\alpha_\mu - \alpha_1}{48} \mu \geq 0,$$

and proved that equality holds for quasihomogeneous singularities. Saito (2000) proved the conjecture for irreducible plane curve singularities. As an application, we use our implementation to prove C. Hertling’s conjecture for singularities with Milnor number $\mu \leq 16$, which were classified by Arnold et al. (1985).
This paper is based on the work with Schulze and Steenbrink (2001), Schulze (2000). In addition to Schulze and Steenbrink (2001), we give an introduction to the theory of the Gauss–Manin connection, a detailed description of the algorithm and its implementation, including a pseudocode, and an application to C. Hertling’s conjecture.

The methods presented in this paper are based on the interplay of the $\mathcal{D}$-module structure and the microlocal structure. They may serve as an example for symbolic $\mathcal{D}$-module computations with a computer algebra system.

2. Milnor Fibration

We consider an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ with Milnor number

$$\mu := \dim_{\mathbb{C}} \mathbb{C}\{x\}/(\partial_x f) < \infty,$$

where we denote $x = (x_0, \ldots, x_n)$, $\partial_x f = (\partial_{x_0} f, \ldots, \partial_{x_n} f)$. Let

$$X \xrightarrow{f} T$$

be a good representative (Looijenga, 1984) of $f$. This means that $T \subset \mathbb{C}$ is an open disc around 0 and $X$ is the intersection of $f^{-1}(T)$ with an open ball $B \subset \mathbb{C}^{n+1}$ around 0 such that the singular fibre $f^{-1}(0)$ intersects arbitrarily small spheres in $B$ around 0 transversally. We denote

$$T' := T \setminus \{0\},$$

$$X' := f^{-1}(T') \cap X.$$

Then $f : X' \rightarrow T'$ is a $\mathcal{C}^\infty$ fibre bundle with fibres $X_t := f^{-1}(t)$, $t \in T'$ homotopy equivalent to the bouquet of $\mu$ $n$-spheres (Milnor, 1968). Recall that the bouquet of a set of pointed topological spaces is the topological space which arises from gluing these spaces at their base points. Note that this implies that the cohomology of the Milnor fibre $X_t$ is given by

$$\tilde{H}^k(X_t, \mathbb{Z}) = \begin{cases} \mathbb{Z}^\mu, & k = n \\ 0, & \text{else}. \end{cases}$$

3. Gauss–Manin Connection

The cohomology bundle

$$H^n := \bigcup_{t \in T'} H^n(X_t, \mathbb{C}) = \bigcup_{t \in T'} H^n(X_t, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \supset \bigcup_{t \in T'} H^n(X_t, \mathbb{Z})$$

is a flat complex vector bundle of rank $\mu$ on $T'$. This means that it can be described by local frames with constant transition functions. Hence, the sheaf $\mathcal{H}^n$ of holomorphic sections in $H^n$ is a complex local system in the sense of Deligne (1970). By Deligne (1970, Proposition 2.16), there is a natural flat connection on $H^n$ and we denote its covariant derivative with respect to $\partial_t$ by

$$\mathcal{H}^n \xrightarrow{\partial_t} \mathcal{H}^n.$$
It induces a differential operator $\partial_t$ on $(i_*H^n)_0$ where $i: T' \hookrightarrow T$ denotes the inclusion and the lower index 0 denotes germs at 0. Note that an element of $(i_*H^n)_0$ is represented by a section in a punctured neighbourhood of $0 \in T$.

Let $u: T^\infty \to T$, $u(\tau) := \exp(2\pi i \tau)$, be the universal covering of $T'$. By $\tau$ we denote the coordinate on $T^\infty$. The pullback

$$X^\infty := X' \times_{T'} T^\infty$$

is called the canonical Milnor fibre. Since $T^\infty$ is contractible, the natural maps $X_{u(\tau)} \cong X^\infty \hookrightarrow X^\infty$, $\tau \in T^\infty$, are homotopy equivalences. Hence, $H^n(X^\infty, \mathbb{C})$ is a trivial complex vector bundle on $T^\infty$. We consider $A \in H^n(X^\infty, \mathbb{C})$ as a global flat multivalued section $A(t)$ in $H^n$. Note that

$$\partial_t A(t) = 0$$

for $A \in H^n(X^\infty, \mathbb{C})$.

There is a natural action of the fundamental group $\Pi_1(T', t)$, $t \in T'$, on $H^n(X_t, \mathbb{C}) \cong H^n(X^\infty, \mathbb{C})$ by lifting paths along flat sections in the cohomology bundle. A positively oriented generator operates via the monodromy operator $M$ defined by

$$(M s)(\tau) := s(\tau + 1)$$

for $s \in H^n(X^\infty, \mathbb{C})$. Let $M = M_s M_u$ be the decomposition of $M$ into semisimple $M_s$ and unipotent $M_u$ and

$$N := \log M_u.$$ By the monodromy theorem (Brieskorn, 1970), the eigenvalues of $M_s$ are roots of unity and $N^{n+1} = 0$. Let

$$H^n(X^\infty, \mathbb{C}) \cong \bigoplus_{\lambda} H^n(X^\infty, \mathbb{C})_{\lambda}$$

be the decomposition of $H^n(X^\infty, \mathbb{C})$ into generalized eigenspaces

$$H^n(X^\infty, \mathbb{C})_{\lambda} := \ker(M_s - \lambda)$$

of $M$ and $M_{\lambda} := M_{|H^n(X^\infty, \mathbb{C})_{\lambda}}$. For $A \in H^n(X^\infty, \mathbb{C})_{\lambda}$, $\lambda = \exp(-2\pi i \alpha)$, $\alpha \in \mathbb{Q}$, the elementary section $s(A, \alpha)$ defined by

$$s(A, \alpha)(t) := t^\alpha \exp\left(-\frac{N}{2\pi i} \log t\right) A(t)$$

is monodromy invariant and hence $s(A, \alpha)$ defines a holomorphic section in $H^n$. Note that the twist $t^\alpha \exp\left(-\frac{N}{2\pi i} \log t\right)$ is inverse to the action of the monodromy on $A(t)$. The elementary sections $i_* s(A, \alpha)$ span a $\partial_t$-invariant free $\mathcal{O}_T [t^{-1}]$-submodule $\mathcal{G} \subset i_* H^n$ of rank $\mu$. The Gauss–Manin connection is the regular $\mathcal{O}_0$-module $\mathcal{G}_0$ (Brieskorn, 1970; Pham, 1979) where $\mathcal{G} := \mathcal{G}_0[\partial_t]$ and the lower index 0 denotes germs at 0.

4. V-filtration

We want to use the $\mathcal{G}$-module structure of the Gauss–Manin connection to define the $V$-filtration.
Since the twist \( t^\alpha \exp(-\frac{N}{2\pi i} \log t) \) is invertible, \( \psi_\alpha(A) := (i_s(A, \alpha))_0 \) defines an inclusion
\[
H^0(X^\infty, \mathbb{C}) \xrightarrow{\psi_\alpha} \mathcal{G}_0
\]
which fulfills \( t \circ \psi_\alpha = \psi_{\alpha+1} \) and \( \partial_t \circ \psi_\alpha = \psi_{\alpha-1} \circ (\alpha - \frac{N}{2\pi i}) \) by definition of \( s(A, \alpha) \). Hence,
\[
(t\partial_t - \alpha) \circ \psi_\alpha = \psi_\alpha \circ \left(-\frac{N}{2\pi i}\right),
\]
\[
\exp(-2\pi it\partial_t) \circ \psi_\alpha = \psi_\alpha \circ \Lambda_\alpha.
\]
Equality (1) implies that the image
\[
C_\alpha := \text{im} \psi_\alpha = \ker(t\partial_t - \alpha)^{\alpha+1}
\]
of \( \psi_\alpha \) is the generalized \( \alpha \)-eigenspace of \( t\partial_t \), that \( t: C_\alpha \to C_{\alpha+1} \) is bijective, and that \( \partial_t: C_\alpha \to C_{\alpha-1} \) is bijective for \( \alpha \neq 0 \). Equality (2) gives a relation between the Gauss–Manin connection and the monodromy. The \( V \)-filtration \( V \) on \( \mathcal{G}_0 \) is defined by
\[
V^\alpha := V^\alpha \mathcal{G}_0 := \sum_{\alpha \leq \beta} \mathbb{C}\{t\}C_\beta,
\]
\[
V^{>\alpha} := V^{>\alpha} \mathcal{G}_0 := \sum_{\alpha < \beta} \mathbb{C}\{t\}C_\beta.
\]
\( V^\alpha \) and \( V^{>\alpha} \) are free \( \mathbb{C}\{t\} \)-modules of rank \( \mu \) with \( V^\alpha/V^{>\alpha} \cong C_\alpha \).

5. Saturation and Non-resonance

We want to use equality (2) to express the monodromy in terms of the Gauss–Manin connection.

A \( t\partial_t \)-stable \( \mathbb{C}\{t\} \)-lattice \( \mathcal{L} \subset \mathcal{G}_0 \) is called saturated. The notion of regularity is defined by the existence of a saturated \( \mathbb{C}\{t\} \)-lattice. Note that the \( V^\alpha \) (resp. \( V^{>\alpha} \)) are saturated. Since \( \mathbb{C}\{t\} \) is a discrete valuation ring, for any two \( \mathbb{C}\{t\} \)-lattices \( \mathcal{L}, \mathcal{L}' \subset \mathcal{G}_0 \), there is a \( k \in \mathbb{Z} \) such that \( t^k \mathcal{L} \subset \mathcal{L}' \). Hence, for any \( \mathbb{C}\{t\} \)-lattice \( \mathcal{L} \), there are \( \alpha_1 < \alpha_2 \) such that
\[
V^{\alpha_2} \subset \mathcal{L} \subset V^{>\alpha_1}.
\]
Since the \( V^\alpha \) (resp. \( V^{>\alpha} \)) are saturated and noetherian, this implies that the saturation
\[
\mathcal{L}_\infty := \bigoplus_{k=0}^{\infty} (t\partial_t)^k \mathcal{L}
\]
of a \( \mathbb{C}\{t\} \)-lattice \( \mathcal{L} \) is itself a \( \mathbb{C}\{t\} \)-lattice. Note that \( \mathcal{L}_\infty \) is saturated. One can actually show that \( \mathcal{L}_\infty = \sum_{k=0}^{\infty} (t\partial_t)^k \mathcal{L} \). Let \( V^{\alpha_2} \subset \mathcal{L} \subset V^{>\alpha_1} \) be a saturated \( \mathbb{C}\{t\} \)-lattice. Since \( t\partial_t \) operates on \( \mathcal{L} \), there is a decomposition into generalized eigenspaces
\[
\mathcal{L} = \left( \bigoplus_{\alpha_1 < \alpha < \alpha_2} \mathcal{L} \cap C_\alpha \right) \oplus V^{\alpha_2}
\]
and a residue endomorphism \( \overline{t\partial_t} \in \text{End}(\mathcal{L}/t\mathcal{L}) \) induced by \( t\partial_t \). If \( \overline{t\partial_t} \) has no positive integer differences of eigenvalues in any Jordan block of \( N \), \( \mathcal{L} \) is called non-resonant and
equality (2) implies that
\[ M = \exp(-2\pi i \overline{\partial}_t). \]

6. D-module Structure

We want to describe the \( D \)-module structure of the Gauss–Manin connection.

Let
\[ H^n(X^\infty, \mathbb{C}) \cong \bigoplus_{\lambda} H^n(X^\infty, \mathbb{C})_{\lambda,j} \]
be a decomposition of \( H^n(X^\infty, \mathbb{C})_{\lambda} \) into Jordan blocks \( H^n(X^\infty, \mathbb{C})_{\lambda,j} \) of size \( n_{\lambda,j} = \dim_{\mathbb{C}} H^n(X^\infty, \mathbb{C})_{\lambda,j} \) and \( C_{\alpha,j} := \text{im} \psi_\alpha|_{H^n(X^\infty, \mathbb{C})_{\lambda,j}} \). Then
\[ \mathcal{L}_\alpha := \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \mathbb{C}\{t\}C_{\alpha,j} \]
is a non-resonant saturated \( \mathbb{C}\{t\} \)-lattice. Let \( A_{\lambda,j} \) be a \( \mathbb{C}[N] \)-generator of \( H^n(X^\infty, \mathbb{C})_{\lambda,j} \).

Since \( \partial_t : C_\alpha \rightarrow C_{\alpha-1} \) is bijective for \( \alpha \neq 0 \),
\[ \mathcal{D}_0 \cong \mathbb{C}\{t\}[t^{-1}]\mathcal{L}_\alpha \]
\[ = \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \left( \mathbb{C}\{t\}[t^{-1}]s \left( \frac{-N}{2\pi i} \right)^k A_{\lambda,j}, \alpha_{\lambda,j} \right) \]
\[ = \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \mathbb{C}\{t\}[\partial_t](t\partial_t - \alpha_{\lambda,j})^k s(A_{\lambda,j}, \alpha_{\lambda,j}) \]
\[ \cong \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \mathcal{D}_0/\mathcal{D}_0(t\partial_t - \alpha_{\lambda,j})^{n_{\lambda,j}} \]
for \( \alpha_{\lambda,j} < 0 \) or \( \alpha_{\lambda,j} \notin \mathbb{Z} \).

7. Brieskorn Lattice

We want to define the Brieskorn lattice on which the action of \( \partial_t \) can be computed. The basic idea is to describe the cohomology of the Milnor fibre in terms of holomorphic differential forms via the De Rham isomorphism.

Since the Milnor fibre \( X_t, t \in T' \), is a Stein complex manifold, the De Rham homomorphism \( \rho : H^n_{DR} \Omega_{X_t} \rightarrow H^n(X_t, \mathbb{C}) \) defined by
\[ \rho([\omega])(\delta) := \int_{\delta} \omega \]
is an isomorphism. A geometrical section is a holomorphic section \( s(\omega) \) in \( H^n \) defined by
\[ s(\omega)(t) := \left[ \frac{\omega}{\overline{\partial}_t} \right]_{X_t} \]
where \( \frac{\omega}{\overline{\partial}_t} \) is the Gelfand–Leray form of the holomorphic differential form \( \omega \) in \( f_*\Omega^{n+1}_X \).
The map \( s : f_*\Omega_X^{n+1} \to i_*\mathcal{H}^n \) factors through the Brieskorn lattice
\[
\mathcal{H}'' := f_*\Omega_X^{n+1}/df \wedge d(f_*\Omega_X^{n-1})
\]
with image in \( \mathcal{G} \) inducing an isomorphism \( \mathcal{H}''|_{T^*} \cong \mathcal{H}_2 \) (Sebastiani, 1970). The Leray residue formula implies that
\[
\partial_t s([df \wedge \eta]) = s([d\eta]). \quad (3)
\]
This formula will allow us to compute the action of \( \partial_t \) on the Brieskorn lattice. Since \( \mathcal{H}'' \) is a free \( \mathcal{O}_T \)-module of rank \( \mu \) (Sebastiani, 1970),
\[
\mathcal{H}''_0 = \Omega_{X,0}^{n+1}/df \wedge \Omega_{X,0}^{n-1}
\]
is a torsion free \( \mathbb{C}\{t\}\)-module and hence \( s : \mathcal{H}''_0 \to \mathcal{B}_0 \) is an inclusion. We identify \( \mathcal{H}'' \) with its image in \( \mathcal{B}_0 \). By Malgrange (1974, Lemma 4.5), the growth of geometrical sections towards 0 is bounded by
\[
\mathcal{H}''_0 \subset V^{>1}. \quad (4)
\]
This will lead to estimations necessary for the computation.

8. Microlocal Structure

The isomorphism \( \partial_t : V^{>0} \cong V^{>1} \) induces an action of \( \partial_t^{-1} \) on the Brieskorn lattice. This action extends to the microlocal structure of the Brieskorn lattice and will be the key to an efficient computation.

The ring of microdifferential operators with constant coefficients
\[
\mathbb{C}\{\{\partial_t^{-1}\}\} := \left\{ \sum_{k \geq 0} a_k \partial_t^{-k} \in \mathbb{C}[\{\partial_t^{-1}\}] \middle| \sum_{k \geq 0} \frac{a_k}{k!} t^k \in \mathbb{C}\{t\} \right\}
\]
is a discrete valuation ring and \( t^n \mathbb{C}\{t\} \), \( \alpha \in \mathbb{Q} \), are free \( \mathbb{C}\{\{\partial_t^{-1}\}\}\)-modules of rank 1. For \( \alpha > -1 \), we identify \( \partial_t : \mathcal{C}_\alpha \to \mathcal{C}_\alpha \) with \( (\alpha + 1) - \frac{N}{2\pi i} \) via \( \psi_\alpha \). Then \( \partial_t \circ t^{\frac{N}{\pi i}} = (\alpha + 1)t^{\frac{N}{\pi i}} \) and \( \text{det} t^{\frac{N}{\pi i}} = t^{\nu} \) \( \frac{N}{\pi i} = 1 \). Hence,
\[
\mathbb{C}\{t\}\mathcal{C}_\alpha \cong t^n \mathbb{C}\{t\}^{\dim C_\alpha}
\]
as \( \mathbb{C}\{t\}[\partial_t] \)-modules and \( \mathbb{C}\{t\}\mathcal{C}_\alpha \) is a free \( \mathbb{C}\{\{\partial_t^{-1}\}\}\)-module of rank \( \dim C_\alpha \). In particular, \( V^{>\alpha} \) (resp. \( V^{\geq\alpha} \)) is a free \( \mathbb{C}\{\{\partial_t^{-1}\}\}\)-module of rank \( \mu \) for \( \alpha > -1 \) (resp. \( \alpha \geq -1 \)). Since \( \partial_t^{-1} \mathcal{H}_0'' \subset \mathcal{H}_0'' \) and \( \mathcal{H}_0'' \subset V^{>1} \),
\[
\mathcal{H}_0'' \cong \mathbb{C}\{\{\partial_t^{-1}\}\}\mu \quad (5)
\]
is a free \( \mathbb{C}\{\{\partial_t^{-1}\}\}\)-module of rank \( \mu \). Note that
\[
\mathcal{H}''/\partial_t^{-1} \mathcal{H}'' = \Omega f := \Omega^{n+1}/df \wedge \Omega^n \cong \mathbb{C}\{x\}/(\partial_x f)
\]
is the Jacobian algebra.

9. Singularity Spectrum

We want to define the singularity spectrum which is an important invariant of the singularity coming from the mixed Hodge structure on the cohomology of the Milnor fibre.
The Hodge filtration $F$ on $\mathcal{G}_0$ is defined by $F_k := F_k \mathcal{G}_0 := \partial^k_\tau \mathcal{H}''$ and $\partial_\tau$ induces isomorphisms

$$\text{Gr}^{\alpha+k}_V \mathcal{H}''/\partial^k_\tau \mathcal{H}'' = \text{Gr}^{\alpha+k}_V \mathcal{G}_0 \xrightarrow{\partial^k_\tau} \text{Gr}^k_\partial \text{Gr}^F_\mathcal{V} \mathcal{G}_0 \cong \text{Gr}^F_k \mathcal{C}_\alpha.$$ 

The singularity spectrum $\text{Sp}: \mathbb{Q} \rightarrow \mathbb{N}$ defined by

$$\text{Sp}(\alpha) := \dim \text{Gr}^\alpha \mathcal{V} \text{Gr}^F_0 \mathcal{G}_0 \sim \text{Gr}^F_k \text{Gr}^\alpha \mathcal{V} \mathcal{G}_0 \sim \text{Gr}^F_k \mathcal{C}_\alpha.$$ 

reflects the embedding of $\mathcal{H}''_0$ in $\mathcal{G}_0$ and has the symmetry property

$$\text{Sp}(n-1-\alpha) = \text{Sp}(\alpha). \quad (6)$$

Since $\mathcal{H}''_0 \subset V^{>-1}$, this implies that $V^{>-1} \supset \mathcal{H}''_0 \supset V^{n-1}$ or equivalently that $\text{Sp}(\alpha) = 0$ for $\alpha \leq -1$ or $\alpha \geq n$. This fact will be essential for the computation of the V-filtration on the Brieskorn lattice.

The spectral numbers $\alpha_1 \leq \cdots \leq \alpha_\mu$ are those $\alpha$ with multiplicity $\text{Sp}(\alpha) > 0$ and their mean value is $\frac{n-1}{2}$. Hertling (2001) conjectured that their variance is bounded by

$$\gamma := -\frac{1}{4} \sum_{i=1}^{\mu} \left( \alpha_i - \frac{n-1}{2} \right)^2 + \frac{\alpha_\mu - \alpha_1}{48} \mu \geq 0,$$

proved that $\gamma = 0$ for quasihomogeneous singularities, and gave the explicit formula

$$\gamma(T_{p,q,r}) = \frac{1}{24} \left( 1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} \right) \geq 0$$

for singularities of type $T_{p,q,r}$. Saito (2000) proved the conjecture for irreducible plane curve singularities.

10. Algorithm

Based on E. Brieskorn’s algebraic description of the Gauss–Manin connection (3), the microlocal structure of the Brieskorn lattice (5), B. Malgrange’s result (4), and the symmetry of the spectral numbers (6), we describe an algorithm to compute the V-filtration on the Brieskorn lattice. We abbreviate $\Omega := \Omega_{X,0}$, $\mathcal{H}'' := \mathcal{H}''_0$, $\mathcal{G} := \mathcal{G}_0$, and $s := \partial_\tau^{-1}$.

10.1. Idea

First, note that the commutator $[s^{-2}t, s] = \partial_\tau^2 t \partial_\tau^{-1} - \partial_\tau t = 1$ and hence $\mathcal{G}$ is a $\mathbb{C}\{s\}[\partial_s]$-module with $\partial_s$-action defined by

$$\partial_s := s^{-2} t = \partial_\tau^2 t.$$

Let us indicate the advantages of the $\mathbb{C}\{s\}[\partial_s]$-structure compared to the $\mathbb{C}\{t\}[\partial_t]$-structure in E. Brieskorn’s algorithm. Since $f^{n+1} \in \langle \partial_s f \rangle$ or equivalently $f^{n+1} \Omega^{n+1} \subset df \wedge \Omega^n$,

$$v^{n+1} \partial_\tau \mathcal{H}'' \subset \mathcal{H}''$$

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and hence $\partial_t$ has a $t$-pole of order of at least $n+1$ on $\mathcal{H}''$. But $s^2 H_s = \partial_t^{-2} \partial_t^2 t = t$ implies that

$$s^2 \partial_s \mathcal{H}'' \subset \mathcal{H}''$$

and hence $\partial_s$ has only an $s$-pole of at most 2 on $\mathcal{H}''$. But actually the lower pole order does not simplify the computation since

$$\partial_t t = s^{-1} t = s^{-2} s^2 \partial_s = s \partial_s.$$ 

The important point is that, in order to compute $\mathcal{H}''/t^k \mathcal{H}''$, one has to use Brieskorn (1970, Prop. 3.3) saying that for each $K$ there is an $N$ such that 

$$(x)^N \Omega^{n+1} \subset f^K \Omega^{n+1} + df \wedge \Omega^{n-1}.$$ 

An estimation for $N$ in terms of $K$ in not known to the author and we can only use linear algebra to compute $\mathcal{H}''/t^K \mathcal{H}''$. But

$$\mathcal{H}''/s\mathcal{H}'' = \Omega_f = \Omega^{n+1}/df \wedge \Omega^n \cong \mathbb{C}\{x\}/(\partial_t f)$$

is the Jacobian algebra which can be computed by standard basis methods.

We consider the $\mathbb{C}\{t\}$-lattices and $\mathbb{C}\{s\}$-lattices

$$\mathcal{H}_{s}'' := \sum_{0 \leq j \leq k} (\partial t)^j \mathcal{H}'' = \sum_{0 \leq j \leq k} (s \partial_s)^j \mathcal{H}''.$$ 

Since $V^{-1} \supset \mathcal{H}'' \supset V^{n-1}$, there is a minimal $k_\infty \leq \mu$ such that $\mathcal{H}_{s}'' = \mathcal{H}_{\mu \infty}'' = \mathcal{H}''$ is the saturation of $\mathcal{H}''$ for all $k > k_\infty$. As remarked before, one can actually show that $k_\infty \leq \mu - 1$.

For $N \geq n+1$, $V^{-1} \supset \mathcal{H}'' \supset V^{n-1}$ implies that

$$V^{-1} \supset \mathcal{H}'' \supset s\mathcal{H}'' \supset V^n \supset s^N V^{-1} \supset s^N \mathcal{H}''$$

and $\mathcal{H}_{s}''$ and $s^N \mathcal{H}''$ are $\partial_t$, $s\partial_s$-invariant. Hence, $\partial_t$ and $s\partial_s$ induce endomorphisms $\overline{\partial_t}, \overline{s\partial_s} \in \operatorname{End}_{\mathbb{C}}(\mathcal{H}_{s}''/s^N \mathcal{H}''_s)$ such that the V-filtration $\overline{V}_{s\partial_s} = V_{s\partial_s}^{\mu \infty}$ defined by $\overline{s\partial_s}$ on $\mathcal{H}_{s}''/s^N \mathcal{H}''$ induces the V-filtration on the subquotient $\mathcal{H}''/s\mathcal{H}'' = \operatorname{Gr}_0 \mathcal{G}$.

10.2. Computation

By the finite determinacy theorem, we may assume that $f \in \mathbb{C}\{x\}$ is a polynomial. Since $\mathbb{C}\{x\}_{(x)} \subset \mathbb{C}\{x\}$ is faithfully flat and all data will be defined over $\mathbb{C}\{x\}_{(x)}$, we may replace $\mathbb{C}\{x\}$ by $\mathbb{C}\{x\}_{(x)}$ and similarly $\mathbb{C}\{t\}$ by $\mathbb{C}\{t\}_{(t)}$ and $\mathbb{C}\{s\}$ by $\mathbb{C}\{s\}_{(s)}$ for the computation.

With the additional assumption $f \in \mathbb{Q}\{x\}$, all data will be defined over $\mathbb{Q}$ and we can apply methods of computer algebra.

The computer algebra system SINGULAR (Schönemann, 1996; Greuel et al., 2001) provides standard basis methods with respect to local monomial orderings for computations over localizations of polynomial rings over $\mathbb{Q}$.

From a standard basis of a zero-dimensional ideal, one can compute a monomial $\mathbb{C}$-basis of the quotient by the ideal. In SINGULAR, this is done by the commands std and kbase. Hence, one can compute a monomial $\mathbb{C}$-basis $m = (m_1, \ldots, m_\mu)$ of

$$\Omega_f = \Omega^{n+1}/df \wedge \Omega^n \cong \mathbb{C}\{x\}/(\partial_t f).$$

Since $\mathcal{H}''/s\mathcal{H}'' \cong \Omega_f$, $m$ represents a $\mathbb{C}\{s\}$-basis of $\mathcal{H}''$ and a $\mathbb{C}\{s\}$-basis of $\mathcal{G}$ by Nakayama’s lemma.
The matrix $A = \sum_{k \geq 0} A_k s^k$ of the operator $t$ with respect to $m$ is defined by $tm = Am$. Note that $t$ is not $C((s))$-linear and $A$ does not define the basis representation of $t$ with respect to $m$ just by matrix multiplication. But $t$ is a differential operator and $t = s^2 \partial_s$ implies that the basis representation of $t$ with respect to $m$ is given by

$$tg_m = (gA + s^2 \partial_s(g))m$$

for $g = (g_1, \ldots, g_n) \in C(s)^n$. If $U$ is a $C(s)$-basis transformation and $A'$ the matrix of $t$ with respect to $m' := Um$ then

$$A' = (UA + s^2 \partial_s(U))U^{-1}$$

is the basis transformation formula with respect to $U$.

A reduced normal form allows us to compute the projection to the quotient by a zero-dimensional ideal. In SINGULAR, this is done by the command `reduce`. Hence, one can compute the projection to the upper summand in

$$\frac{C(s)/\langle \partial_s f \rangle}{\langle \partial_s f \rangle} \cong \frac{\Omega_f}{df \wedge \Omega^n} \cong \frac{\Omega_f}{df \wedge \Omega^n} \cong \frac{\mathcal{H}''/s\mathcal{H}''}{s\mathcal{H}''}.$$

A syzygy computation allows us to express elements as a linear combination of generators of a module. In SINGULAR, this is done by the command `division`. Hence, one can compute $\eta = \sum_{i=0}^{n} (-1)^i \eta_i d \xi_0 \wedge \cdots \wedge d \xi_i$ from $df \wedge \eta = \sum_{i=0}^{n} \partial_{\xi_i} (f) \eta_i d \xi_0 \wedge \cdots \wedge d \xi_n$. Since $s^{-1}[df \wedge \eta] = \partial_{\xi_i}[df \wedge \eta] = [d\eta]$, one can compute basis representations with respect to $m$ inductively up to arbitrary order. Note that $t[\omega] = [f\omega]$ and $A$ is the basis representation of $tm$ with respect to $m$.

The basis representation $H_k$ of $\mathcal{H}''^m$ with respect to $m$ defined by $\mathcal{H}''^m := H_k m$ can be computed inductively by

$$H_k^0 := H_0 := C(s)^n,$n

$$H_k^1 := \text{jet}_{-1}(s^{-1}H_k \text{jet}_k (A) + s \partial_s H_k^1),$$

$$H_k^{k+1} := H_k + H_k^{k+1}.$$

Note that $H_{k+1}$ depends only on the $k$-jet of $A$ and that the coefficients of $H_k$ are in $s^{-k}C(s)^n$. A normal form allows us to test for module membership. In SINGULAR, this is done by the command `reduce`. Hence, one can check if $H_k = H_{k+1}$ to find $k_\infty$.

A syzygy computation gives a minimal set of generators of a module. In SINGULAR, this is done by the command `minbase`. Hence, one can compute a $C(s)^n$-basis $M$ of $H_{k_\infty} = H_\infty$ with

$$\delta(M) := \max\{\text{ord}(M_{i_1,j_1}) - \text{ord}(M_{i_2,j_2}) | M_{i_1,j_1} \neq 0 \neq M_{i_2,j_2} \} \leq k_\infty.$$

The matrix $A'$ of $t$ with respect to the $C(s)$-basis $Mm$ of $\mathcal{H}''^m$ is defined by $MA + s^2 \partial_s M =: A'M$ and $\text{jet}_k (A') = \text{jet}_k (A'_{\leq k})$ for $A'_{\leq k}$ defined by

$$M \text{jet}_{k+[M]}(A) + s^2 \partial_s M =: A'_{\leq k} M.$$

Note that $A'_{\leq k}$ depends only on a finite jet of $A$. Hence, one can compute $A'$ up to arbitrary order and the basis representation

$$s^{-1}A' + s \partial_s = s^{-1}A'_{\leq N} + s \partial_s = \text{End}_C(C(s)^n/s^N C(s)^n)$$

of $\partial_s \in \text{End}_C(\mathcal{H}''^m/\partial_s^N \mathcal{H}''^m)$ with respect to $Mm$ for arbitrary $N$. 

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As before, one can compute the basis representation $H''$ of $\mathcal{H}''$ with respect to $Mm$ defined by $H_0 =: H''M$. Choosing $N \geq n + 1$ as before, the $V$-filtration $V^{t+1}_{s^{-1}A_{\leq N} + s\partial_s}$ defined by $s^{-1}A_{\leq N} + s\partial_s$ on $\mathcal{C}\{\langle s \rangle \}^\mu / s^N \mathcal{C}\{\langle s \rangle \}^\mu$ induces the $V$-filtration $V^{t+1}_{s^{-1}A_{\leq N} + s\partial_s}(H'' / sH'')$ on the subquotient $H'' / sH''$ and

$$V^{t+1}_{s^{-1}A_{\leq N} + s\partial_s}(H'' / sH'')M$$

is the basis representation of the $V$-filtration $V(\mathcal{H}'' / s\mathcal{H}'')$ on $\mathcal{H}'' / s\mathcal{H}''$ with respect to $m$.

The matrix of $s^{-1}A_{\leq N} + s\partial_s$ with respect to the canonical $\mathcal{C}$-basis

$$\begin{pmatrix}
1 & s & s^2 & \cdots & s^{N-1} \\
\vdots & \iddots & \ddots & \ddots & \vdots \\
1 & s & s^2 & \cdots & s^{N-1}
\end{pmatrix}^t$$

of $\mathcal{C}\{\langle s \rangle \}^\mu / s^N \mathcal{C}\{\langle s \rangle \}^\mu$ is given by

$$\begin{pmatrix}
A'_{11} & A'_{12} & A'_{13} & \cdots & A'_{1N} \\
A'_{21} & A'_{22} & A'_{23} & \cdots & A'_{2N-1} \\
A'_{31} & A'_{32} & A'_{33} & \cdots & A'_{3N-2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
A'_{N1} & \cdots & \cdots & \cdots & A'_{N+N-1}
\end{pmatrix}$$

where $A' = \sum_{k \geq 0} A'_{k} s^k$.

Since the eigenvalues of $A'_{1}$ are rational by the monodromy theorem, they can be computed using univariate factorization. In SINGULAR, this can be done using the commands `det` and `factorize`. From the eigenvalues of $s^{-1}A_{\leq N} + s\partial_s$, one can compute $V^{t+1}_{s^{-1}A_{\leq N} + s\partial_s}(H'' / sH'')$ using methods of linear algebra. To compute only the spectrum, one can use its symmetry to simplify the computation. In SINGULAR, one can use the command `syz` to compute kernels and hence generalized eigenspaces and the commands `intersect`, `reduce`, and `std` for modules with constant coefficients to compute intersections, quotients, and bases of vector spaces.

### 10.3. Extensions

We indicate two possible extensions of our algorithm.

The $V$-filtration on the Jacobian algebra is defined by the $V$-filtration and the action of the Jacobian algebra $\mathcal{C}\{x\} / (\partial_x f)$ on $\Omega_f$ by multiplication and can be computed from the $V$-filtration on $\Omega_f$.

After a Jordan decomposition of the residue on $H_\infty$, one can use the basis transformation formula to replace $H_\infty$ by a non-resonant lattice with the same properties. Then $\exp(-2\pi i A'_{1})$ is a monodromy matrix and the weight filtration is defined by the nilpotent part of $A'_{1}$ on the graded parts of $H_\infty$. Hence, one can compute the monodromy and the spectral pairs.
10.4. Implementation

The Singular library `gaussman.lib` (Schulze, 2001) contains an implementation of the algorithm to compute the V-filtration on the Brieskorn lattice based on the following pseudocode:

```
proc vfiltration(f ∈ Q[x]) ≡
m := basis(Ω_f);
w := fm;
A := 0;
H'' := 0;
H := C{{s}}^n;
H' := H;
k := -1;
K := 0;
while k < K ∨ H'' ≠ H do
  Cm := w mod df ∧ Ω^n;
k := k + 1;
A := A + Cs^k;
if H'' ≠ H then
  H'' := H;
  H' := jet_{-1}(s^{-1}H'A + s∂,H');
  H := H + H';
  if H'' = H then
    M := basis(H'');
    K := delta(M) + n + 1;
  fi
fi
if k < K ∨ H'' ≠ H then w := d((w - Cm)/df) fi
od;
A'M := MA + s^2∂s,M;
H''M := C{{s}}^n;
V_{s^{-1}A'+s∂s}(H''/sH'')M, m.
```

10.5. Example

We use the Singular library `gaussman.lib` (Schulze, 2001) to compute an example. First, we have to load the library:

```
> LIB "gaussman.lib";
```

Then we define the ring \( R := \mathbb{Q}[x,y]_{(x,y)} \) and the polynomial \( f = x^5 + x^2y^2 + y^5 \in R \):

```
> ring R=0,(x,y),ds;
> poly f=x^5+x^2*y^2+y^5;
```
Note that $f$ defines a singularity of type $T_{2,5,5}$. Finally, we compute the $V$-filtration of the singularity defined by $f$ on $\Omega_f$:

```plaintext
> list l=vfiltration(f);
> print(matrix(l[1]));
-1/2,-3/10,-1/10,0,1/10,3/10,1/2
> l[2];
1,2,2,1,2,2,1
> l[3];
[1]:
   _[1]=gen(11)
[2]:
   _[1]=gen(10)
   _[2]=gen(6)
[3]:
   _[1]=gen(9)
   _[2]=gen(4)
[4]:
   _[1]=gen(5)
[5]:
   _[1]=gen(8)
   _[2]=gen(3)
[6]:
   _[1]=gen(7)
   _[2]=gen(2)
[7]:
   _[1]=gen(1)
> print(matrix(l[4]));
y5,y4,y3,y2,xy,y,x4,x3,x2,x,1
```

The result is a list with the following entries: The first contains the spectral numbers, the second, the corresponding multiplicities, the third, $\mathbb{C}$-bases of the graded parts of the $V$-filtration on $\Omega_f$ in terms of the monomial $\mathbb{C}$-basis in the fourth entry. In the third entry, $\text{gen}(i)$ represents the $i$th unit vector. A monomial $x^\alpha y^\beta$ in the fourth entry is considered as $[x^\alpha y^\beta dx \wedge dy] \in \Omega_f$. This result is presented in Table 1.

### 10.6. Application

We use the Singular library `gaussman.lib` (Schulze, 2001) to compute the spectrum and the $\gamma$-invariant for Milnor number $\mu \leq 16$ following the classification in Arnold (1988). C. Hertling’s conjecture, saying that $\gamma \geq 0$, holds for quasihomogeneous singularities and singularities of type $T_{p,q,r}$ (Hertling, 2001) and for irreducible plane curve singularities (Saito, 2000). Our results presented in Table 2 prove the conjecture for singularities with Milnor number $\mu \leq 16$. Most of the spectra computed occur already in the list of spectra of unimodal and bimodal singularities in Arnold et al. (1985).

The computation was done on a Pentium II 350 with Linux operating system. The choice of monomial ordering has a strong influence on the computation time.
Table 1. V-filtration of \( f = x^5 + x^2y^2 + y^7 \) on \( \Omega_f \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>(-\frac{1}{7})</th>
<th>(-\frac{3}{10})</th>
<th>(-\frac{1}{10})</th>
<th>0</th>
<th>\frac{1}{10}</th>
<th>\frac{1}{7}</th>
<th>\frac{1}{2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Gr}_V^\alpha \Omega_f / [dx \wedge dy] )</td>
<td>(1)</td>
<td>(x, y)</td>
<td>( (x^2, y^2) )</td>
<td>(xy)</td>
<td>( (x^3, y) )</td>
<td>( (x^4, y^3) )</td>
<td>( (y^4) )</td>
</tr>
</tbody>
</table>

Table 2. Spectrum and \( \gamma \)-invariant for Milnor number \( \mu \leq 16 \).

<table>
<thead>
<tr>
<th>Singularity</th>
<th>Polynomial</th>
<th>Singularity spectrum</th>
<th>( \gamma )-invariant</th>
<th>Computation time/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_{1,1} )</td>
<td>( y^7 + x^2y^3 + x^3y )</td>
<td>( -\frac{7}{7}, -\frac{1}{10}, -\frac{1}{10}, -\frac{2}{5}, -\frac{3}{10}, -\frac{1}{5}, \frac{7}{10}, \frac{1}{7} )</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>( W_{1,1} )</td>
<td>( y^7 + x^2y^3 + x^3 )</td>
<td>( -\frac{7}{7}, -\frac{3}{7}, -\frac{1}{10}, -\frac{3}{7}, -\frac{1}{7}, -\frac{2}{5}, \frac{7}{10}, \frac{1}{7} )</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( W_{1,1}^p )</td>
<td>( x^7 + 2xy^3 + xy^5 + y^6 )</td>
<td>( -\frac{7}{7}, -\frac{3}{7}, -\frac{1}{10}, -\frac{3}{7}, -\frac{1}{7}, -\frac{2}{5}, \frac{7}{10}, \frac{1}{7} )</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>( Q_{2,1} )</td>
<td>( y^2 + y^3 + x^2y^3 + x^3 )</td>
<td>( -\frac{7}{7}, \frac{2}{7}, \frac{3}{7}, \frac{11}{12}, \frac{13}{12}, \frac{13}{12}, \frac{7}{12} )</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>( Q_{2,2} )</td>
<td>( y^2 + y^3 + x^2y^3 + x^3 )</td>
<td>( -\frac{7}{7}, \frac{2}{7}, \frac{3}{7}, \frac{11}{12}, \frac{13}{12}, \frac{13}{12}, \frac{7}{12} )</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>( S_{1,1} )</td>
<td>( y^2 + y^3 + x^2z + x^3z^2 )</td>
<td>( -\frac{7}{7}, \frac{2}{7}, \frac{3}{7}, \frac{11}{12}, \frac{13}{12}, \frac{13}{12}, \frac{7}{12} )</td>
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<td></td>
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<tr>
<td>( V_{1,1} )</td>
<td>( y^2z + x^4 + x^2y^3 + 2y^5 )</td>
<td>( -\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8} )</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>( V_{1,1}^p )</td>
<td>( y^2z + x^3y + y^4 + x^3z )</td>
<td>( -\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8} )</td>
<td>9</td>
<td></td>
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References


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